On the Maximal Graded Shifts of Modules over a Polynomial Ring

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Notation

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M a finitely generated S-module

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$t_i(M) = \max\{j \mid \beta_{i,j} \neq 0\}$

An Example: $I = (x^2, xy, y^3, z^4)$

Betti Table of S/I:

	0	1	2	3
0:	1	-	-	-
1:	-	2	1	-
2:	-	1	1	-
3:	-	1	-	-
4:	-	-	2	1
5:	-	-	1	1

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$(t_0, t_1, t_2, t_3) = (0, 2, 3, 7)$

Maximal Graded Shifts

Question

Which sequences of maximal graded shifts are possible for a cyclic module S/I?

Some Examples

Caviglia: For $r \geq 1$ there exists an ideal with 3 generators in degree r and first syzygy in degree r^2.

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The maximal graded shifts are:
$(T_0, T_1, T_2, T_3, T_4) = (0, r, r^2, r^2 + 1, r^2 + 2)$

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2:	-	3	-	-	-
3:	-	-	-	-	-
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Maximal shifts: $(0, 3, 9, 10, 11)$

Some Examples

Theorem (Ullery)

Given any strictly increasing sequence of integers $2 \leq a_1 < a_2 < a_3 \cdots < a_n$, there exists an ideal $J \subseteq R$ such that

$$T_i(R/J) = a_i \text{ for } 1 \leq i \leq n.$$

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Caveat: Length of resolution is $\gg n$.

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Ullery example with $(T_1, T_2, T_3) = (2, 3, 4, 7)$

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	0	1	2	3	4	5	6	7	8	9	10	11	12
0:	1	-	-	-	-	-	-	-	-	-	-	-	-
1:	-	91	735	3080	8393	16170	22902	24255	19250	11319	4795	1386	245
2:	-	-	-	-	-	-	-	-	-	-	-	-	-
3:	-	-	-	-	1	10	45	120	210	252	210	120	45

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Theorem (Bourbaki)

Given any strictly increasing sequence of integers $2 \leq a_1 < a_2 < a_3 \cdots < a_n$, *there exists a positive integer* k *and an ideal* $J \subseteq S$ *such that*

$$T_i(S/J) = a_i + k \text{ for } 1 \leq i \leq n.$$

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Caveat: Large degree generators.

Restrictions on Maximal Graded Shifts

Theorem (Eisenbud-Huneke-Ulrich)

Let I be a homogeneous ideal in $S = k[x_1, \ldots, x_n]$ such that $\dim(S/I) \leq 1$*. Then*

$$T_n(S/I) \leq T_i(S/I) + T_{n-i}(S/I),$$

for all $0 \leq i \leq n$*.*

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Conjecture (M)

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Theorem (M)

Let $I \subset S = K[x_1, \dots, x_n]$ be a homogeneous ideal. Set $h = \lceil \frac{n}{2} \rceil$. Then

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Theorem (M (2017))

Let $I \subset S = K[x_1, \ldots, x_n]$ be a homogeneous ideal.

$$T_n(S/I) \leq t_1(S/I) + T_{n-1}(S/I).$$

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Theorem (Eisenbud-Huneke-Ulrich)

Let M be a finitely-generated graded S-module of codimension c and with $\delta := \dim(M) - \operatorname{depth}(M) \leq 1$. Let J be a homogeneous ideal contained in $\operatorname{Ann}(M)$. If $\operatorname{depth}(S/J) \geq \operatorname{depth}(M)$, then for $0 \leq q \leq \operatorname{codim}(J)$,

$$T_{c+\delta}(M) \leq T_{c+\delta-q}(M) + T_q(S/J).$$

In particular: If $\operatorname{Ann}(M)$ contains a regular sequence of degrees $d_1, \ldots, d_q$, then

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