Multiple Structures with Arbitrarily Large Projective Dimension

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RIDER UNIVERSITY

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(so $S(-d)_i = S_{i-d}$)
$I = (f_1, \ldots, f_t) \subset R$ a homogeneous ideal
(i.e. each f_j is in some R_i)

Stillman's Question

Question (Stillman)

Is there a bound on the projective dimension of ideals in $S = K[X_1, \ldots, X_n]$ which are generated by N homogeneous polynomials of given degrees $d_1, \ldots, d_N$?

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  - $\mathrm{pd}(S/(4 \text{ quadrics})) \leq 9.(6?)$. (HMMS)

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Note: $(I : h)$ has height 2 and multiplicity $\leq \deg(f)\deg(g)$.

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Yes, if $e = 1$, then I is a linear prime and $\mathrm{pd}(S/I) = h$.

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Yes, if $e = 1$, then I is a linear prime and $\mathrm{pd}(S/I) = h$.
Yes, if $h = e = 2$......

Engheta's Classification of h = e = 2 unmixed ideals

Proposition (Engheta)

Let $I \subset S$ be a homogeneous unmixed ideal with $ht(I) = e(S/I) = 2$.
Then $\mathrm{pd}(S/I) \leq 3$ and I is one of the following:

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- $(x, y)^2 + (ax + by)$ *such that* $ht(x, y, a, b) = 4$

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- (x, y^2) *(primary)*

Classification of Primes (minimal multiplicity)

Theorem (Swinnerton-Dyer)

Let $p \subset S$ be a homogeneous prime ideal with $ht(p) = 2$ and $e(S/p) = 3$. Then $\mathrm{pd}(S/I) = 2$ and p is one of the following:

- (x, c), $c =$ *irreducible cubic*
- $(ux - wv, uz - yv, wz - xy)$
- $(ux - v^2, uz - yv, vz - xy)$
- $(wy - x^2, wz - xy, xz - y^2)$

Manolache's Classification of CM Structures

Theorem (Manolache)

Let $I \subset S$ be homogeneous and primary to (x, y), Cohen-Macaulay on $\mathrm{Spec}(S) - \{\mathfrak{m}\}$ with $e(S/I) \leq 4$. Then I is one of the following:

- *Multiplicity 2:*
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- *Multiplicity 2:*
  - (x, y^2)
- *Multiplicity 3:*
  - (x^2, xy, y^2)
  - (x, y^3)
  - $(x^2 + ay, y^2, xy)$
  - $(ax^2 + c^2(bx + cy), axy - bc(bx + cy), ay^2 + b^2(bx + cy), x(bx + cy), y(bx + cy), x^3, x^2y, xy^2, y^3)$

Manolache's Classification of CM Structures (cont.)

Theorem

- *Multiplicity* 4*:*
  - (x^3, xy, y^2)
  - $(x^2 + y^2, xy)$
  - (x^2, y^2)
  - $(gx^2 - fxy, hx^2 - fy^2, hxy - gy^2, x^3, x^2y, xy^2, y^3)$, *where* $ht(f, g, h) \geq 2$
  - (x, y^4)
  - $(x^3 + ay, y^2, xy)$
  - $(x^2 + ay, y^2)$
  - $(x^2 + xy + ay, y^2)$
  - $(ax^3 + b(by + cx^2), axy - c(by + cx^2), y^2, x^4, x^2y, x(by + cx^2))$, *where* $ht(a, b, c) \geq 2$

Similar Classification for Unmixed ideals?

⋮	✓				
4	✓				
3	✓				
2	✓	✓			
1	✓	✓	✓	✓	✓
ht / e:	1	2	3	4	⋯

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Main Result

Theorem (Huneke-Mantero-M-Seceleanu)

Let K be an algebraically closed field. For any integers $h, e \geq 2$ with $(h, e) \neq (2, 2)$ and for any integer p, there exists a primary (unmixed) ideal $I = I_{h,e,p}$ in a polynomial ring S with

- $ht(I) = h$
- $e(S/I) = e$
- $\sqrt{I} = (x_1, x_2, \dots, x_h)$
- $\mathrm{pd}(S/I) \geq p$.

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Method: Direct construction of 4 unmixed ideals with fixed resolution. Use linkage to construct ideals in the main result.

Tools

Theorem (Buchsbaum-Eisenbud)

Let $\mathbf{F}$ *be a complex free* S*-modules of finite rank*

$$\mathbf{F} : F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} \cdots \xleftarrow{d_p} F_p \leftarrow 0.$$

Set $r_j = \sum_{i=j}^{p} (-1)^{p-i} \operatorname{rank} F_i$. *Then* $\mathbf{F}$ *is a resolution of* $M := \operatorname{Coker}(d_1)$ *if and only if*

$$\operatorname{ht}(I_{r_j}(d_j)) \geq j \qquad \text{for all } j = 1, \ldots, p.$$

Tools

As a corollary:

Proposition

Using the notation from the previous theorem, suppose **F** *is a minimal free resolution of M. Then M satisfies Serre's condition (S_k) if and only if*

$$\mathrm{ht}(I_{r_j}(d_j)) \geq \min\{\dim(S), j+k\} \qquad \text{for all } j = \mathrm{codim}(M)+1, \ldots, p.$$

Proof Sketch

Proposition

Suppose $f, g, h \in S_d$ *for some* $d \geq 1$ *such that* $\mathrm{ht}(x, y, f, g, h) \geq 4$. *Let*

$$L = (x, y)^3 + (y^2 f + xyg + x^2 h).$$

Then L *is* (x, y)*-primary,* $e(S/L) = 5$ *and* S/L *has the following free resolution:*

$$R \xleftarrow{d_1} R^5 \xleftarrow{d_2} R^5 \xleftarrow{d_3} R \leftarrow 0,$$

$$\text{where } d_3 = \begin{pmatrix} h \\ g \\ f \\ -y \\ x \end{pmatrix}.$$

Proof Sketch

Step 1: Note $\mathrm{pd}(\mathrm{Ker}(d_3^*)) = \mathrm{pd}(\mathrm{Ker}(h, g, f, -y, x)) = \mathrm{pd}(S/(x, y, f, g, h)) - 2 = \text{BIG}$.

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$pd:$	BIG	2	$BIG + 1$

Proof Sketch

Step 6:

Proposition

If I, I' are unmixed and in the same even linkage class (e.g. $I = (\mathbf{x}) : L, I' = (\mathbf{x}') : L$), then $\mathrm{pd}(S/I) = \mathrm{pd}(S/I')$.

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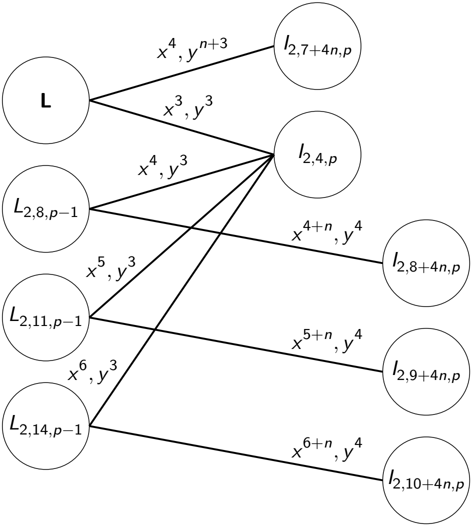
Note: I has several hundred generators.

Linkage Structure



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Open even for $(h, e) = (2, 3)$.

Thank you!

Reference:

- C Huneke, P. Mantero, J. McCullough and A. Seceleanu. "Multiple structures with arbitrarily large projective dimension supported on linear subspaces." submitted. arXiv:1301:4147