Variational Problems*

(Com S 477/577 Notes)

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The variational derivative of a functional $J[y]$ can be defined [1, pp. 27–29] as

$$\delta J/\delta y = F_y(x, y, y') - \frac{d}{dx}F_{y'}(x, y, y').$$

Euler's equation essentially states that *the variational derivative of the functional must vanish at an extremum*. This is analogous to the well-known result from calculus that the derivative of a function must vanish at an extremum.

1 Variable End Point

In this section, we consider a simple case of the variable end point problem, which is stated as follows: Among all curves whose end points lie on two vertical lines $x = a$ and $x = b$, find the curve for which the functional

$$J[y] = \int_a^b F(x, y, y')\, dx \tag{1}$$

has an extremum.

We determine the variation of the functional (1), which is the linear component of the increment

$$\Delta J = J[y + h] - J[y] = \int_a^b \Big(F(x, y + h, y' + h') - F(x, y, y')\Big)\, dx,$$

due to an increment of h in y. The Taylor expansion immediately leads to

$$\delta J = \int_a^b (F_y h + F_{y'} h')\, dx.$$

Unlike the fixed end point problem, the function $h(x)$ no longer vanishes at the points $x = a$ and $x = b$. Integration by parts now yields

$$\begin{aligned} \delta J &= \int_a^b \left(F_y - \frac{d}{dx}F_{y'}\right) dx + F_{y'} h(x)|_{x=a}^{x=b} \\ &= \int_a^b \left(F_y - \frac{d}{dx}F_{y'}\right) dx + F_{y'}|_{x=b}\, h(b) - F_{y'}|_{x=a}\, h(a). \end{aligned} \tag{2}$$

*The material is adapted from the book *Calculus of Variations* by I. M. Gelfand and S. V. Fomin, Prentice Hall Inc., 1963; Dover, 2000.

Consider all functions $h(x)$ with $h(a) = h(b) = 0$ first. The condition $\delta J = 0$ implies that

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

This means that the solution y of the end point problem must be a solution of Euler's equation. Suppose y is a solution. The integral in (2) must therefore vanish, reducing $\delta J = 0$ to

$$F_{y'}|_{x=b}\, h(b) - F_{y'}|_{x=a}\, h(a) = 0.$$

Since h is arbitrary, it follows from the above equation that

$$F_{y'}|_{x=a} = 0 \qquad \text{and} \qquad F_{y'}|_{x=b} = 0. \tag{3}$$

In summary, to solve the variable end point problem, we first find a general solution to Euler's equation (2), and then use the conditions (3) to determine the constants in the general solution.

EXAMPLE 1. Starting at the origin, a particle slides down a curve in the vertical plane. Find the curve such that the particle reaches the vertical line $x = b$, where $b > 0$, in the shortest time.

The velocity v of motion equals $\sqrt{2gy}$ from the conservation of the total energy, where $g < 0$ is the gravitational acceleration. Meanwhile, it is also determined along the curve as

$$v = \frac{ds}{dt} = \sqrt{1 + y'^2}\, \frac{dx}{dt},$$

from which we immediately have

$$dt = \frac{\sqrt{1 + y'^2}}{v}\, dx = \frac{\sqrt{1 + y'2}}{\sqrt{2gy}}\, dx.$$

The above gives us the transit time as a functional

$$J = \int_0^b \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\, dx. \tag{4}$$

The integrand $F = \sqrt{1 + y'^2}/\sqrt{2gy}$ does not depend on x. So this is Case 2 where Euler's equation can be reduced to

$$F - y' F_{y'} = C. \tag{5}$$

Since

$$F_{y'} = \frac{1}{\sqrt{2gy}} \cdot \frac{y'}{\sqrt{1 + y'^2}},$$

several steps from Euler's equation (5) lead to

$$y'^2 = \frac{a - y}{y}, \qquad \text{for some } a < 0.$$

Given the downward sliding motion, we have that $y' \le 0$, thus

$$y' = -\sqrt{\frac{a - y}{y}},$$

which leads to

$$dx = -\sqrt{\frac{y}{a - y}}\, dy.$$

Next, make the substitution $y = a\sin^2\frac{\theta}{2}$ over $[0, \pi]$ with

$$\frac{dy}{d\theta} = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\frac{1}{2} = a\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

Under the above and the substitution,

$$\begin{aligned} dx &= -\sqrt{\frac{a\sin^2\frac{\theta}{2}}{a\cos^2\frac{\theta}{2}}}\cdot a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta \\ &= -a\sin^2\frac{\theta}{2}\,d\theta \\ &= -a\left(\frac{1-\cos\theta}{2}\right)\,d\theta \\ &= r(1-\cos\theta)\,d\theta, \end{aligned}$$

where a constant $r = -a/2$ is introduced in the last step. The solution to Euler's equation (5) is a cycloid:

$$x = r(\theta - \sin\theta), \tag{6}$$
$$y = r(\cos\theta - 1). \tag{7}$$

since the curve passes through the origin. The first boundary condition $F'_y|_{x=0} = 0$ in (3), which is essentially $y'(0) = 0$, is thus satisfied. To determine the value of r, we apply the second boundary condition $F_{y'}|_{x=b} = 0$, which reduces to $y' = 0$ at $x = b$. This states that the tangent to the curve at its right end point must be horizontal. From

$$y' = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-r\sin\theta}{r - r\cos\theta} = \frac{-\sin\theta}{1-\cos\theta}, \tag{8}$$

we obtain $\theta = \pi$. Substituting this value into $r(\theta - \sin\theta) = b$, we obtain

$$r = \frac{b}{\pi}.$$

Hence the curve trajectory is

$$(x, y) = \frac{b}{\pi}(\theta - \sin\theta, \cos\theta - 1).$$

Figure 1 shows the optimal trajectory of the sliding particle to reach the line $x = 2\pi$, represented by the cycloid $2(\theta - \sin\theta, \cos\theta - 1)$ over $[0, \pi]$.

Next, we caculate the time of travel according to (4), utilizing (8) and $dx/d\theta = r(1-\cos\theta)$. We have

$$\begin{aligned} 1 + y'^2 &= 1 + \frac{\sin^2\theta}{(1-\cos\theta)^2} \\ &= 1 + \frac{1-\cos^2\theta}{(1-\cos\theta)^2} \\ &= 1 + \frac{1+\cos\theta}{1-\cos\theta} \\ &= \frac{2}{1-\cos\theta}, \end{aligned} \tag{9}$$

$$\begin{aligned} J &= \int_0^\pi \frac{\sqrt{\frac{2}{1-\cos\theta}}}{\sqrt{-2gr(1-\cos\theta)}}\,r(1-\cos\theta)\,d\theta \\ &= \frac{r}{\sqrt{-gr}}\int_0^\pi 1\,d\theta \end{aligned}$$

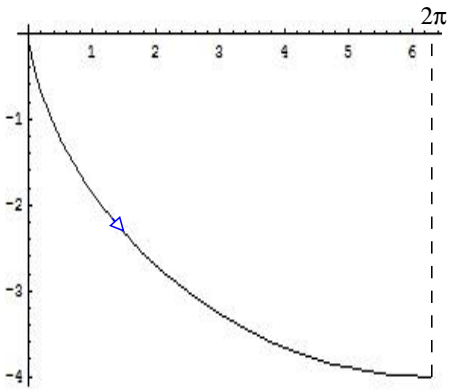

Figure 1: The cycloid $2(\theta - \sin\theta, \cos\theta - 1)$ with $0 \le \theta \le \pi$ is the minimum time trajectory along which a particle starting at the origin slides pass the vertical line $x = 2\pi$ under gravity.

$$\begin{aligned} &= \pi\sqrt{\frac{-r}{g}} \\ &= \sqrt{-\frac{b\pi}{g}}. \end{aligned} \tag{10}$$

Theorem 1 *On the clycloid $2(\theta - \sin\theta, \cos\theta - 1)$, a particle sliding down from any location with $\theta = \theta_0 \in [0, \pi)$ will reach the vertical line $x = b$, $b > 0$, in the same amount of time given in (10).*

Proof Let the starting location be $(x_0, y_0) = (x(\theta_0), y(\theta_0))$ on the cycloid (6)–(7). The velocity of motion now equals $\sqrt{2g(y - y_0)}$ and the transit time is therefore

$$\begin{aligned} J &= \int_{x_0}^{b} \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y - y_0)}}\, dx \\ &= \sqrt{-\frac{b}{\pi g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos\theta}{\cos\theta_0 - \cos\theta}}\, d\theta, \end{aligned} \tag{11}$$

after substitutions of (9) and (7).

It suffices to show that the value of the integral in (11) does not depend on θ_0. For this purpose, we introduce a new variable:

$$u = \frac{\cos\frac{\theta}{2}}{\cos\frac{\theta_0}{2}},$$

which decreases from 1 to 0 as θ increases from θ_0 to π. It follows that

$$d\theta = -2\frac{\cos\frac{\theta_0}{2}}{\sin\frac{\theta}{2}}\, du. \tag{12}$$

We have

$$\int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos\theta}{\cos\theta_0 - \cos\theta}}\, d\theta = \int_{\theta_0}^{\pi} \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\left(\cos\frac{\theta_0^2}{2} - \cos^2\frac{\theta}{2}\right)}}\, d\theta$$

$$
\begin{aligned}
&= \int_{\theta_0}^{\pi} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta_0}{2}\sqrt{1-u^2}}\,d\theta \\
&= \int_{\theta_0}^{\pi} \frac{1}{\sqrt{1-u^2}} \cdot \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta_0}{2}}\,d\theta \\
&= \int_{1}^{0} \frac{-2}{\sqrt{1-u^2}}\,du \\
&= \int_{0}^{1} \frac{2}{\sqrt{1-u^2}}\,du,
\end{aligned}
$$

where the last step uses (12). The last integral above is independent of θ_0, the starting location of the particle on the cycloid segment. Hence, the time of travel is independent of the starting location and the same as given in (10) for $\theta = 0$. □

Intuitively, we compare particle sliding at two different starting points on the cycloid. The slide starting at the higher point will reach the lower point with some velocity, whereas the slide starting at the lower point has to build up its velocity from zero. The first slide will travel the remaining curve segment in shorter time to make up for the time it has spent already.

2 Several Variables

Many problems involve functionals that depend on functions of several independent variables, for example, surfaces in 3D depending on two parameters. Here we only look at how the solution to the case of single-variable variational problems would carry over to the case of functionals depending on surfaces. We focus on the case of two independent variables but refer to [1] for the case of more than two variables.

Let $F(x, y, z, p, q)$ be twice continuously differentiable with respect to all five variables, and consider

$$
J[z] = \iint_{\mathcal{R}} F(x, y, z, z_x, z_y)\,dx\,dy, \tag{13}
$$

where $\mathcal{R}$ is some closed region, and z_x and z_y are the partial derivatives of $z(x, y)$ with respect to x and y. We are looking for a function $z(x, y)$ that

1. is twice continuously differentiable with respect to x and y in $\mathcal{R}$;
2. assumes given values on the boundary Γ of $\mathcal{R}$;
3. yields an extremum of the functional (13).

Theorem 1 in the notes titled *Calculus of Variations* does not depend on the form of the functional J. Therefore, a necessary condition for the functional (13) to have an extremum is that its variation vanishes. We must first calculate the variation δJ. Let $h(x, y)$ be an arbitrary function with continuous first and second derivatives in the region $\mathcal{R}$ and vanishes on it boundary Γ. Then, if the surface $z(x, y)$ satisfies conditions 1–3, so does $z(x, y) + h(x, y)$. We apply the Taylor series to the change in the functional as follows:

$$
\Delta J = J[z + h] - J[z]
$$

$$
\begin{aligned}
&= \iint_{\mathcal{R}} \Big(F(x, y, z + h, z_x + h_x, z_y + h_y) - F(x, y, z, z_x, z_y) \Big)\, dx\, dy \\
&= \iint_{\mathcal{R}} (F_z h + F_{z_x} h_x + F_{z_y} h_y)\, dx\, dy + \cdots . \qquad (14)
\end{aligned}
$$

Meanwhile, we have that

$$
\begin{aligned}
&\iint_{\mathcal{R}} (F_{z_x} h_x + F_{z_y} h_y)\, dx\, dy \\
&= \iint_{\mathcal{R}} \left(\frac{\partial}{\partial x}(F_{z_x} h) + \frac{\partial}{\partial y}(F_{z_y} h) \right) dx\, dy - \iint_{\mathcal{R}} \left(\frac{\partial}{\partial x} F_{z_x} + \frac{\partial}{\partial y} F_{z_y} \right) h\, dx\, dy \\
&= \int_{\Gamma} (F_{z_x} h\, dy - F_{z_y} h\, dx) - \iint_{\mathcal{R}} \left(\frac{\partial}{\partial x} F_{z_x} + \frac{\partial}{\partial y} F_{z_y} \right) h\, dx\, dy. \qquad (15)
\end{aligned}
$$

where the last step used Green's theorem:

$$
\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_{\Gamma} (P\, dx + Q\, dy).
$$

In (15), the integral along Γ is zero since h vanishes on the boundary. Substituting (15) into (14), we obtain

$$
\delta J = \iint_{\mathcal{R}} \left(F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} \right) h(x, y)\, dx\, dy. \qquad (16)
$$

Thus, the condition $\delta J = 0$ implies that the double integral above vanishes for any $h(x, y)$ with continuous derivatives up to the second order and vanishing on the boundary Γ. This leads to the second-order partial differential equation below [1], also known as *Euler's equation*:

$$
F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} = 0.
$$

Next, we generalize the functional (13) to allow second order partial derivatives of z to appear in the integrand function F. More specifically, let $F(x, y, z, p, q, r, s, t)$ be a function with continuous first and second partial derivatives with respective to all arguments. The function $z = z(x, y)$ has continuous derivatives up to the fourth order, and has given values on the boundary Γ of $\mathcal{R}$. The functional

$$
\tilde{J}[z] = \iint_{\mathcal{R}} F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy})\, dx\, dy
$$

has an extremum at $z = z(x, y)$ only if the following version of Euler's equation holds:

$$
F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} + \frac{\partial^2}{\partial x^2} F_{z_{xx}} + \frac{\partial^2}{\partial x \partial y} F_{z_{xy}} + \frac{\partial^2}{\partial y^2} F_{z_{yy}} = 0. \qquad (17)
$$

EXAMPLE 2. *Surface $z = z(x, y)$ of least area spanned by a given contour.* The problem reduces to finding the minimum of the functional

$$
J[z] = \iint_{\mathcal{R}} \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy,
$$

[1] To be rigorous, a simple proof can be derived from (16) and that h is arbitrary under stipulated conditions.

for which Euler's equation has the form

$$z_{xx}(1+z_y^2) - 2z_{xy}z_x z_y + z_{yy}(1+z_x^2) = 0. \tag{18}$$

To understand the geometric meaning of equation (18), we note that the mean curvature of the surface is given as

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)},$$

where E, F, G are the coefficients of the first fundamental form of the surface, and L, M, N are the coefficients of its second fundamental form. We obtain that

$$E = 1 + z_x^2, \qquad F = z_x z_y, \qquad G = 1 + z_y^2,$$
$$L = \frac{z_{xx}}{\sqrt{1+z_x^2+z_y^2}}, \quad M = \frac{z_{xy}}{\sqrt{1+z_x^2+z_y^2}}, \quad N = \frac{z_{yy}}{\sqrt{1+z_x^2+z_y^2}},$$

and hence,

$$H = \frac{z_{xx}(1+z_y^2) - 2z_{xy}z_x z_y + z_{yy}(1+z_x^2)}{2(1+z_x^2+z_y^2)^{3/2}}.$$

Thus, Euler's equation (18) implies that the mean curvature of the solution surface is zero everywhere.

A surface with zero mean curvature everywhere is thus called a *minimal surface*. Figure 2 plots a minimal surface — a catenoid $(2\cos u\cosh(v/2), 2\sin u\cosh(v/2), v)$ for $0 \le u \le 2\pi$, $-4 \le v \le 4$.

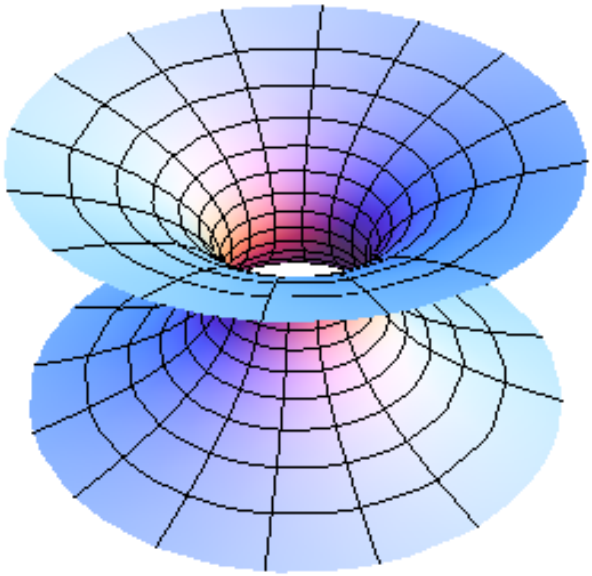

Figure 2: Catenoid $(2\cos u\cosh(v/2), 2\sin u\cosh(v/2), v)$ over $[0, 2\pi] \times [-4, 4]$.

3 Several Unknown Functions

Let $F(x, y_1, \ldots, y_n, z_1, \ldots, z_n)$ be a function with continuous first and second derivatives with respect to all its arguments. We now derive necessary conditions for an extremum of the functional

$$J[y_1, \ldots, y_n] = \int_a^b F(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx,$$

where $y_1(x), \ldots, y_n(x)$ are continuously differentiable and satisfy the boundary conditions

$$y_i(a) = a_i, \qquad y_i(b) = b_i, \qquad \text{for } i = 1, \ldots, n.$$

Namely, we are looking for an extremum of J defined on the set of smooth curves joining two points $(a, a_1, \ldots, a_n)$ and $(b, b_1, \ldots, b_n)$ in the $(n+1)$-dimensional space $\mathbb{R}^{n+1}$.

Similar to the derivation of Euler's equations for the variational problems studied before, we add a continuously differentiable function $h_i(x)$ to each $y_i(x)$ such that the resulting $y_i(x) + h_i(x)$ still satisfy the boundary conditions. Therefore, $h_i(a) = h_i(b) = 0$, for $i = 1, \ldots, n$. The variation of $J[y_1, \ldots, y_n]$ is found to be

$$\delta J = \int_a^b \sum_{i=1}^{n} (F_{y_i} h_i + F_{y_i'} h_i')\, dx.$$

At each step, setting all but one of the h_i to zero, we obtain the following system of Euler's equations that must be satisfied at an extremum:

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0, \qquad i = 1, \ldots, n. \tag{19}$$

Example 3. *Geodesics.* Given a parametric surface

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(u, v),$$

the variational problem here is to find the minimum distance between $\boldsymbol{p}$ and $\boldsymbol{q}$ on the surface.

Parametrize a curve from $\boldsymbol{p}$ and $\boldsymbol{q}$ as $\boldsymbol{\alpha}(t) = \boldsymbol{\sigma}(u(t), v(t))$ so that $\boldsymbol{p} = \boldsymbol{\alpha}(t_0)$ and $\boldsymbol{q} = \boldsymbol{\alpha}(t_1)$. We denote by '.' differentiation with respect to t. The arc length between the two points is

$$J[u, v] = \int_{t_0}^{t_1} \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt, \tag{20}$$

where E, F, G are the coefficients of the first fundamental form of the surface given as

$$E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u, \qquad F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v, \qquad G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v.$$

Euler's equations for the functional (20) become

$$\frac{E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2}{2\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}} - \frac{d}{dt}\left(\frac{E\dot{u} + F\dot{v}}{\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}}\right) = 0, \tag{21}$$

$$\frac{E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2}{2\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}} - \frac{d}{dt}\left(\frac{F\dot{u} + G\dot{v}}{\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}}\right) = 0. \tag{22}$$

Since the parametrization does not change the total length of the curve, we may use the arc length parametrization. From surface geometry, we know that this means

$$E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 = 1.$$

Then equations (21) and (22) reduce to

$$\begin{aligned}\frac{d}{dt}(E\dot{u}+F\dot{v}) &= \frac{1}{2}(E_u\dot{u}^2+2F_u\dot{u}\dot{v}+G_v\dot{v}^2),\\ \frac{d}{dt}(F\dot{u}+G\dot{v}) &= \frac{1}{2}(E_v\dot{u}^2+2F_v\dot{u}\dot{v}+G_v\dot{v}^2).\end{aligned}$$

They are essentially the *geodesic equations*. This proves what we have learned before that the shortest curve on $\boldsymbol{\sigma}$ connecting two surface points $\boldsymbol{p}$ and $\boldsymbol{q}$ is a geodesic between the two points.

Sometimes, it is more convenient to use one of the surface parameters to parametrize the geodesic. In such a case, we still need to fall back on equations (21) and (22).

4 Variational Problems in Parametric Form

We have considered functionals of curves given by equations of the form $y = y(x)$. Often, it is more convenient to consider functionals of curves in parametric form, say, $(x(t), y(t))$. Given a functional of such a curve

$$\int_{x_0}^{x_1} F(x, y, y')\, dx, \tag{23}$$

over $[t_0, t_1]$ such that $x(t_0) = x_0$ and $x(t_1) = x_1$, we can rewrite it as

$$\int_{t_0}^{t_1} F\left(x(t), y(t), \frac{\dot{y}(t)}{\dot{x}(t)}\right)\, \dot{x}\, dt = \int_{t_0}^{t_1} \Phi(x, y, \dot{x}, \dot{y})\, dt,$$

where

$$\Phi(x, y, \dot{x}, \dot{y}) \equiv F\left(x(t), y(t), \frac{\dot{y}(t)}{\dot{x}(t)}\right)\, \dot{x}.$$

Here the overdot denotes differentiation with respect to t. This becomes a functional depending on two unknown functions $x(t)$ and $y(t)$. The function Φ does not have t as an explicit argument. It is *positive-homogeneous of degree 1* in $\dot{x}$ and $\dot{y}$:

$$\Phi(x, y, c\dot{x}, c\dot{y}) = c\Phi(x, y, \dot{x}, \dot{y}),$$

for any $c > 0$.

Conversely, let

$$\int_{t_0}^{t_1} \Phi(x, y, \dot{x}, \dot{y})\, dt$$

be a functional whose integrand Φ is positive-homogeneous of degree 1 in $\dot{x}$ and $\dot{y}$. We now show that the value of the functional is independent of the curve parameterization. Suppose we reparametrize the curve with τ over $[\tau_0, \tau_1]$ such that $t = t(\tau)$, and $dt/d\tau > 0$ over $[\tau_0, \tau_1]$. Then

$$\begin{aligned}\int_{\tau_0}^{\tau_1} \Phi\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right)\, d\tau &= \int_{\tau_0}^{\tau_1} \Phi\left(x, y, \dot{x}\frac{dt}{d\tau}, \dot{y}\frac{dt}{d\tau}\right)\, d\tau\\ &= \int_{t_0}^{t_1} \Phi(x, y, \dot{x}, \dot{y})\frac{dt}{d\tau}\, d\tau\\ &= \int_{t_0}^{t_1} \Phi(x, y, \dot{x}, \dot{y})\, dt.\end{aligned}$$

Now, suppose the curve $y = y(x)$ has a parameterization in t that reduces the functional (23) to the form

$$\int_{t_0}^{t_1} \Phi(x, y, \dot{x}, \dot{y})\, dt.$$

Applying Euler's equation (19), we end up with two equations:

$$\begin{aligned} \Phi_x - \frac{d}{dt}\Phi_{\dot{x}} &= 0, \\ \Phi_y - \frac{d}{dt}\Phi_{\dot{y}} &= 0. \end{aligned} \tag{24}$$

We can solve them for x and y.

The two equations in (24) must be equivalent to the single Euler equation:

$$F_y - \frac{d}{dx}F_{y'} = 0,$$

which results from the variational problem for the original functional (23). Hence the two equations are not independent.

We refer to [1] for other variational problems such as involving higher derivatives, with subsidiary conditions, involving multiple integrals, etc. Also, we refer to [2] for application of the calculus of variations in theory of elasticity, quantum mechanics, and electrostatics.

References

[1] I. M. Gelfand and S. V. Fomin. *Calculus of Variations* Prentice-Hall, Inc., 1963; Dover Publications, Inc., 2000.

[2] R. Weinstock. *Calculus of Variations: With Applications to Physics and Engineering.* Dover Publications, Inc., 1974.