Surfaces*

(COMS 4770/5770 Notes)

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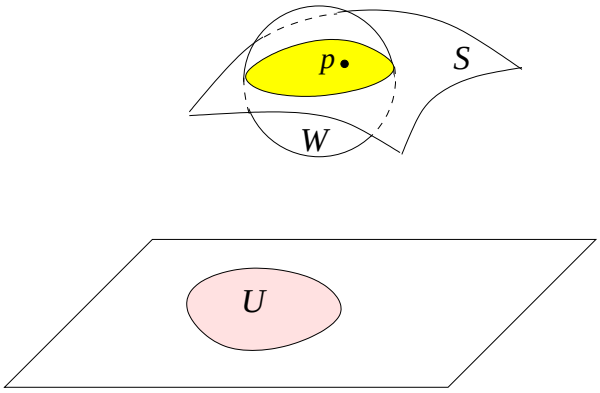
1 Definition

To define a surface, we need the concepts of continuity and homeomorphism of mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. A map $f : X \to Y$ is continuous at a point $\boldsymbol{p} \in X$ if points in the set near $\boldsymbol{p}$ are mapped by f onto points in Y near $f(\boldsymbol{p})$. More precisely, f is *continuous* at $\boldsymbol{p}$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\boldsymbol{u}) - f(\boldsymbol{p})\| < \epsilon$ for any $\boldsymbol{u} \in X$ with $\|\boldsymbol{u} - \boldsymbol{p}\| < \delta$. The function is *continuous* if it is continuous at every point of X.

A map $f : X \to Y$ is *one-to-one* (or *injective*) if $f(\boldsymbol{p}) \neq f(\boldsymbol{q})$ whenever $\boldsymbol{p} \neq \boldsymbol{q}$. The map is *onto* (or *surjective*) if for every $\boldsymbol{y} \in Y$ there exists some $\boldsymbol{x} \in X$ such that $f(\boldsymbol{x}) = \boldsymbol{y}$. A map is called *bijective* if it is both one-to-one and onto. If f is bijective, there exists a function from Y to X called the *inverse* of f and denoted by f^{-1}. It is defined as $f^{-1}(\boldsymbol{y}) = \boldsymbol{x}$ whenever $f(\boldsymbol{x}) = \boldsymbol{y}$.

If $f : X \to Y$ is continuous and bijective, and if its inverse map $f^{-1} : Y \to X$ is also continuous, then f is a *homeomorphism* and the two sets X and Y are *homeomorphic*.

A subset $\mathcal{S}$ of $\mathbb{R}^3$ is a *surface* if, for every point $\boldsymbol{p} \in \mathcal{S}$, there is an open set $U \subseteq \mathbb{R}^2$ and an open set $W \subseteq \mathbb{R}^3$ containing $\boldsymbol{p}$ such that $\mathcal{S} \cap W$ is *homeomorphic* to U.



A surface is thus viewed as a collection of homeomorphisms $\boldsymbol{\sigma} : U \to S \cap W$, which are called *surface patches*. The collection is called the *atlas* of the surface. Every point of the surface lies on at least one surface patch in the atlas.

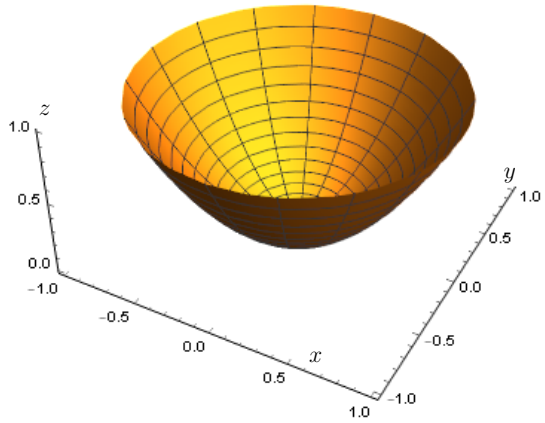
*Most of the material is adapted from [3].

EXAMPLE 1. **Plane** Every plane in $\mathbb{R}^3$ is a single surface patch. Let $\boldsymbol{p}$ be a point in the plane, and $\boldsymbol{a}$ and $\boldsymbol{b}$ two orthogonal unit vectors parallel to the plane. Then any point $\boldsymbol{q}$ in the plane can be represented as

$$\boldsymbol{q} = \boldsymbol{p} + u\boldsymbol{a} + v\boldsymbol{b}.$$

Hence, the surface patch is $\boldsymbol{\sigma}(u, v) = \boldsymbol{p} + u\boldsymbol{a} + v\boldsymbol{b}$ with the inverse map $\boldsymbol{\sigma}^{-1}(\boldsymbol{q}) = ((\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{a}, (\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{b})$.

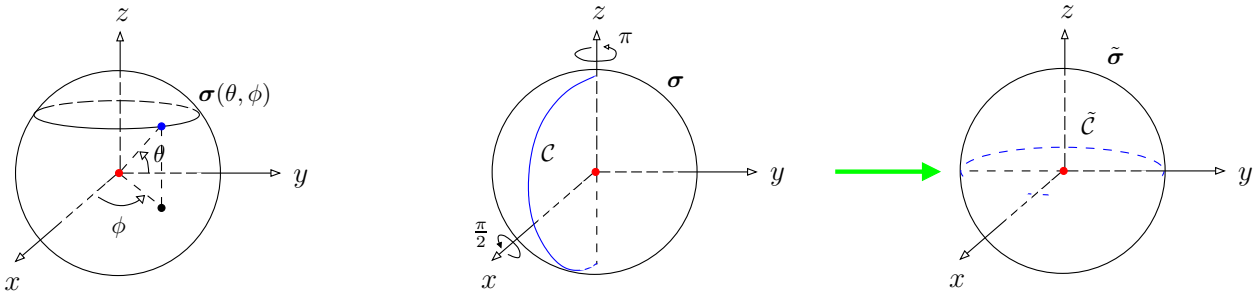
EXAMPLE 2. **Monge patch** Let f be a differentiable real-valued function on an open set $D \subseteq \mathbb{R}^2$. The graph of f, i.e., the set of all points in $\mathbb{R}^3$ whose coordinates satisfy the equation $z = f(x, y)$ is a surface referred to as the *Monge patch*. The figure below plots the paraboloid $z = x^2 + y^2$ over the domain $\{(x, y) \mid x^2 + y^2 \leq 1\}$.



EXAMPLE 3. **Sphere** The unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is a surface.



We first consider the most common parametrization using latitude $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and longitude $\phi \in [0, 2\pi]$:

$$\boldsymbol{\sigma}(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta). \tag{1}$$

This is shown in the left figure above. However, there are two issues. First, $\boldsymbol{\sigma}$ is not injective over $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi]$. Second, this Cartesian product is not an open subset of $\mathbb{R}^2$ and hence cannot be used as the domain of a surface patch. We consider the largest open set:

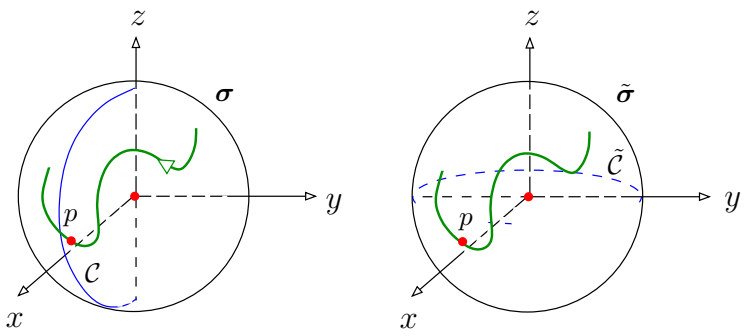
$$U = \left\{(\theta, \phi) \,\middle|\, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, 0 < \phi < 2\pi\right\}. \tag{2}$$

Now, the image $\boldsymbol{\sigma}(U)$ is the sphere minus the great semi-circle $\mathcal{C}$ which, shown in the middle figure above, consists of the points of the form $(x, 0, z)$ with $x \geq 0$. Thus $\boldsymbol{\sigma}$ defines only a patch of the sphere.

To show that the sphere is a surface, we need to construct at least one more surface patch to cover the semi-circle $\mathcal{C}$ omitted by $\boldsymbol{\sigma}$. Imagine rotating $\boldsymbol{\sigma}$ by π about the z-axis and then by $\pi/2$ about the x-axis. The new configuration of the great semi-circle, shown as $\tilde{\mathcal{C}}$ in the right figure above, will not overlap with its original configuration $\mathcal{C}$. Essentially, we define a new patch $\tilde{\boldsymbol{\sigma}} : U \to \mathbb{R}^3$ given by

$$\begin{aligned}
\tilde{\boldsymbol{\sigma}}(\theta, \phi)^\top &= \mathrm{Rot}_x\left(\frac{\pi}{2}\right) \mathrm{Rot}_z(\pi) \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix} \\
&= \begin{pmatrix} -\cos\theta\cos\phi \\ -\sin\theta \\ -\cos\theta\sin\phi \end{pmatrix}. \qquad (3)
\end{aligned}$$

It is the clear that the two great semi-circles $\mathcal{C}$ and $\tilde{\mathcal{C}}$ not covered by $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$, respectively, do not interest. The two patches thus form an atlas of the sphere.



(a) (b)

Often in an application, multiple patches need to be used. For instance, consider that the sphere from Example 3 is rolling on a horizontal plane. Its contact point with the plane will trace out a (green) curve on the sphere as shown in part (a) of the left figure, where the x-y-z frame is rigidly attached to the sphere. The first patch parametrization (1) is used to describe the motion $\boldsymbol{\sigma}(\theta(t), \phi(t))$ of the contact point in terms of time t on the sphere. When the contact reaches a position $\boldsymbol{p}$ near the half great circle $\mathcal{C}$, the value ϕ is close to 0. Namely, (θ, ϕ) is near the boundary of the domain U given in (2). At this point, we want to switch to the second patch $\tilde{\boldsymbol{\sigma}}$ defined in (3) by solving for the new values of θ and ϕ from the following equation:

$$\boldsymbol{p} = \begin{pmatrix} -\cos\theta\cos\phi \\ -\sin\theta \\ -\cos\theta\sin\phi \end{pmatrix}.$$

As shown in (b), the point $\boldsymbol{p}$ is very far from the half great circle $\tilde{\mathcal{C}}$ that is not described by the patch $\tilde{\boldsymbol{\sigma}}$.

A unit vector $\hat{\boldsymbol{u}} \in \mathbb{R}^3$ can be viewed as a point on the unit sphere, and therefore parameterized with ϕ and ψ to lie in one of the above two patches $\boldsymbol{\sigma}(\phi, \psi)$ and $\tilde{\boldsymbol{\sigma}}(\phi, \psi)$. This eliminates the redundancy of representing $\hat{\boldsymbol{u}}$ as a vector $(u_x, u_y, u_z)^\top$, on which the following $\sqrt{u_x^2 + u_y^2 + u_z^2} = 1$ needs to be imposed.

Recall from an earlier lecture that any rotation can be represented by a unit quaternion:

$$q = \cos\frac{\alpha}{2} + \hat{\boldsymbol{u}}\sin\frac{\alpha}{2},$$

where the unit vector $\hat{\boldsymbol{u}}$ gives the direction of the axis of rotation, and $\alpha \in [0, \pi)$ is the angle of rotation. We can thus parameterize q using three parameters (α, ϕ, ψ), where ϕ and ψ parametrize $\hat{\boldsymbol{u}}$ on the unit sphere. Clearly, such parametrization has no redundancy. Suppose $q(t)$ represents the changing orientation of a rigid body with time under the influence of some external force (e.g., the gravitational force). The orientation can be conveniently tracked by updating the three functions $\alpha(t), \phi(t)$, and $\psi(t)$ — often through integrations of some equations derived using rigid body dynamics.

2 Smooth Surface

A map $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$ from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^n$ is *smooth* if each component f_i, $1 \le i \le n$, has continuous partial derivatives of all orders. A surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ is *regular* if it is smooth and the vectors $\boldsymbol{\sigma}_u = \partial\boldsymbol{\sigma}/\partial u$ and $\boldsymbol{\sigma}_v = \partial\boldsymbol{\sigma}/\partial v$ are linearly independent at every point $(u, v) \in U$. Equivalently, the vector product $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \neq 0$ at every point of U. A *smooth surface* is a surface $\boldsymbol{\sigma}$ whose atlas consists of regular surface patches. It is clear that the plane in Example 1 is a smooth surface.

EXAMPLE 4. For the unit sphere in Example 3, it is obvious that $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are smooth functions. To verify regularity, we obtain

$$\boldsymbol{\sigma}_\theta = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta), \tag{4}$$

$$\boldsymbol{\sigma}_\phi = (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0). \tag{5}$$

Thus

$$\boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_\phi = (-\cos^2\theta\cos\phi, -\cos^2\theta\sin\phi, -\sin\theta\cos\theta),$$

which has norm $\|\boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_\phi\| = |\cos\theta|$. Since on this patch, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so $\cos\theta \neq 0$. Similarly, we can verify that $\tilde{\boldsymbol{\sigma}}$ is also regular.

3 Tangent Plane and Orientability

If a (smooth) curve $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ is contained in a surface patch $\boldsymbol{\sigma}(U)$ in the atlas of a surface $\mathcal{S}$, there exists a map $t \mapsto (u(t), v(t))$ such that

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t)). \tag{6}$$

Here the functions u and v are smooth. Conversely, if u and v are smooth, then (6) defines a curve lying in S.

The *tangent space* at a point $\boldsymbol{p}$ of a surface $\mathcal{S}$ consists of the tangent vectors of all curves in $\mathcal{S}$ that pass through $\boldsymbol{p}$. Suppose $\boldsymbol{p} = \boldsymbol{\sigma}(u_0, v_0)$.

Theorem 1 *The tangent space is spanned by the vectors $\boldsymbol{\sigma}_u(u_0, v_0)$ and $\boldsymbol{\sigma}_v(u_0, v_0)$.*

Proof Let $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ be a smooth curve in $\mathcal{S}$. Denoting differentiation with respect to t by dot, we have

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}.$$

Namely, $\dot{\boldsymbol{\gamma}}$ is a linear combination of $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$.

Next, we show that every vector spanned by $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ is the tangent vector at $\boldsymbol{p}$ of some curve in $\mathcal{S}$. Such a vector is of the form $\xi\boldsymbol{\sigma}_u + \eta\boldsymbol{\sigma}_v$, for some ξ and η. Consider the smooth curve

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u_0 + \xi t, v_0 + \eta t).$$

At $t = 0$, i.e., at the point $\boldsymbol{p}$, we have

$$\dot{\boldsymbol{\gamma}} = \xi\boldsymbol{\sigma}_u + \eta\boldsymbol{\sigma}_v.$$

□

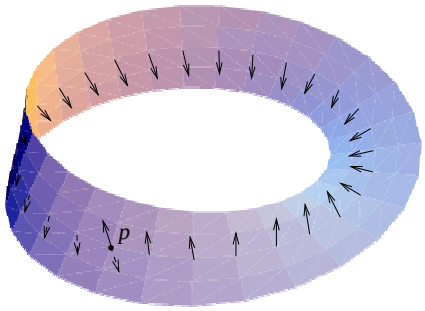
Since $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly independent, the tangent space is indeed the *tangent plane*, which by definition is independent of the patch choice. The tangent plane is uniquely determined by a *unit normal* to $\mathcal{S}$ at $\boldsymbol{p}$, which is perpendicular to the tangent plane. Though there are two such vectors, choosing a surface patch $\boldsymbol{\sigma}$ leads to a definite choice, that is,

$$\boldsymbol{n}_\sigma = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}.$$

The above normal is called the *standard unit normal*.

Unlike the tangent plane, the normal $\boldsymbol{n}_\sigma$ is not quite independent of the choice of the patch $\boldsymbol{\sigma}$. An *orientable* surface $\mathcal{S}$ has a canonical choice of unit normal at every point, obtained by taking the standard unit normal of each surface patch in the atlas of $\mathcal{S}$. Most of the surfaces we shall discuss are orientable. Example 5 describes one that is not.

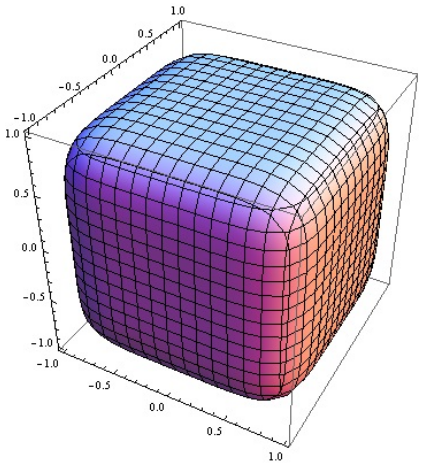
EXAMPLE 5. The Möbius band is obtained by cutting a closed band into a single strip, giving one of the two ends thus produced a half twist, and then reattaching the two ends.



We omit the details in showing that the Möbius band is not orientable according to the definition. An intuitive explanation is given by considering a closed path in the surface that starts at point $\boldsymbol{p}$, as shown in the above figure. Whichever of the two unit normal vectors at $\boldsymbol{p}$ is chosen at the start, the normal vector will be in the opposite direction when returning to $\boldsymbol{p}$ at the end of the loop.

4 Surface in Implicit Form

A surface may also be given in an implicit form $f(x, y, z) = 0$. As an example, Euler's quartic surface $x^4 + y^4 + z^4 = 1$ is plotted in the figure on the right.[1] The implicit function theorem [1] implies that the surface $f(x, y, z) = 0$ can be parametrized locally at a point $\boldsymbol{p}$ which has non-vanishing gradient, i.e.,



$$\nabla f|_p = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Bigg|_p \neq \mathbf{0}.$$

Let $\boldsymbol{\sigma}(u, v)$ be a local parametrization at $\boldsymbol{p}$. A smooth curve $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ in the surface passing by $\boldsymbol{p}$ satisfies (6) for some functions $u(t)$ and $v(t)$. Differentiation of $f(x, y, z) = 0$ with respect to t yields

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} \\ &= \nabla f \cdot (\dot{x}, \dot{y}, \dot{z}) \\ &= \nabla f \cdot \dot{\boldsymbol{\gamma}} \\ &= \nabla f \cdot (\boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}). \end{aligned}$$

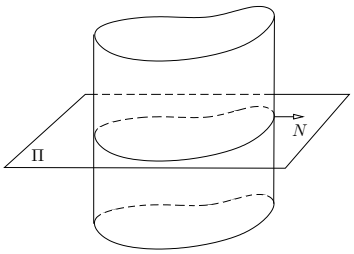
Since the curve $\boldsymbol{\gamma}(t)$ is arbitrary, the gradient ∇f is orthogonal to $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ at $\boldsymbol{p}$, following the reasoning used in the proof of Theorem 1. In other words, ∇f is orthogonal to the tangent plane; hence it is a surface normal at $\boldsymbol{p}$.

5 More Surface Examples

EXAMPLE 6. **Generalized cylinder** This surface is obtained by translating a curve $\boldsymbol{\gamma} : (c, d) \to \mathbb{R}^3$ in the direction given by the unit vector $\boldsymbol{a}$. Translating the point $\boldsymbol{\gamma}(u)$ by the vector $v\boldsymbol{a}$ parallel to $\boldsymbol{a}$ yields a new point

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{a},$$

which is the parametrization of the surface. Thus $\boldsymbol{\sigma} : (c, d) \times \mathbb{R} \to \mathbb{R}^3$. The two partial derivatives are $\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}}$ and $\boldsymbol{\sigma}_v = \boldsymbol{a}$.

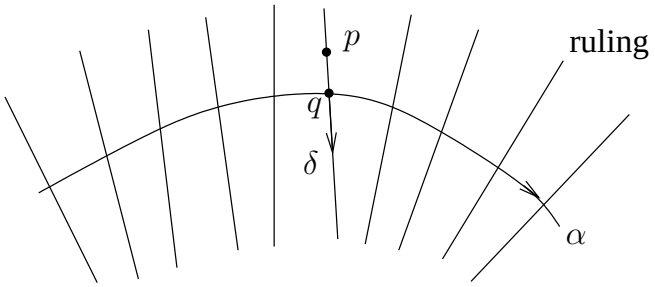


EXAMPLE 7. **Ruled surface** A *ruled surface* is a surface formed by straight lines, called the *rulings* of the surface. Let $\boldsymbol{\alpha}(u)$ be a curve in $\mathbb{R}^3$ that meets every such line. Thus every point $\boldsymbol{p}$ in the surface must lie on one of the straight lines that intersects $\boldsymbol{\alpha}$ at, say, $\boldsymbol{q}$. Suppose $\boldsymbol{q} = \boldsymbol{\alpha}(u)$ and let $\boldsymbol{\delta}(u)$ be a vector in the direction of the line through $\boldsymbol{\alpha}(u)$. Then $\boldsymbol{p}$ has the form

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\delta}(u).$$

for some $v \in \mathbb{R}$.

[1] This is an *algebraic surface* because f is a polynomial.



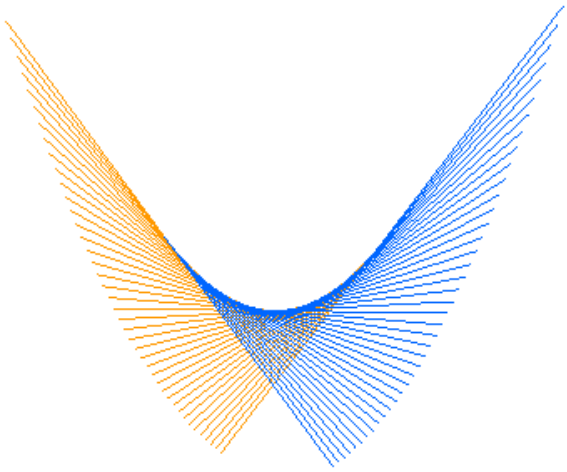
The hyperbolic paraboloid is defined implicitly as

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2},$$

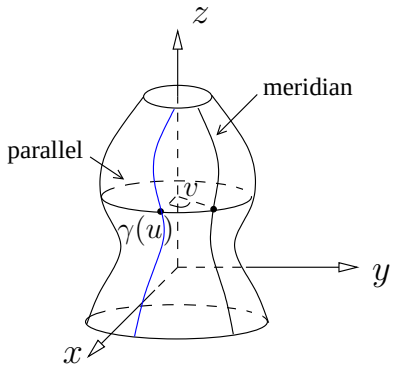
where $a, b > 0$. It can also be parametrized as

$$\boldsymbol{\gamma}(u, v) = (a(u + v), bv, u^2 + 2uv).$$

The next figure shows the rulings on the hyperbolic paraboloid with $a = 3$ and $b = 3$ when viewed from the point $(0, 9, 0)$ on the y-axis.



EXAMPLE 8. **Surface of Revolution** A *surface of revolution* is generated by rotating a plane curve around a straight line in the same plane. The curve is referred to as the *profile curve* and the line as the *axis of rotation*. Every circle generated by rotating a fixed point on the profile curve around the rotation axis is called a *parallel* of the surface. Every curve on the surface obtained by rotating the profile curve through a fixed angle is called a *meridian*.



Let the axis of rotation be the z-axis and the plane containing the profile curve to be the xz-plane. Any point $\boldsymbol{p}$ of the surface is obtained by rotating some point $\boldsymbol{q}$ on the profile curve through an angle v around the z-axis. Let the profile curve be parametrized as

$$\boldsymbol{\gamma}(u) = (f(u), 0, g(u)).$$

Then $\boldsymbol{p}$ is given as

$$\boldsymbol{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

Now we compute the two tangent vectors as partial derivatives:

$$\begin{aligned} \boldsymbol{\sigma}_u &= (\dot{f}\cos v, \dot{f}\sin v, \dot{g}), \\ \boldsymbol{\sigma}_v &= (-f\sin v, f\cos v, 0). \end{aligned}$$

We have

$$\begin{aligned} \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (f\dot{g}\cos v, -f\dot{g}\sin v, f\dot{f}), \\ \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| &= \sqrt{f^2(\dot{f}^2 + \dot{g}^2)}. \end{aligned}$$

Thus $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly independent if $f(u) \neq 0$, that is, if $\boldsymbol{\gamma}(u)$ does not lie on the z-axis, and if $\dot{f}(u)$ and $\dot{g}(u)$ are not simultaneously zero. If the two vectors are linearly independent at every point, then we might as well assume $f(u) > 0$ and view the function as the distance of $\boldsymbol{\sigma}(u, v)$ from the axis of rotation.

References

[1] J. R. Munkres. *Analysis on Manifolds.* Reading, MA: Addison-Wesley, 1991.

[2] B. O'Neill. *Elementary Differential Geometry.* Academic Press, Inc., 1966.

[3] A. Pressley. *Elementary Differential Geometry.* Springer-Verlag London, 2001.