1 The First Fundamental Form

Let $\alpha(t) = \sigma(u(t), v(t))$ be a curve in a surface patch $\sigma$. The arc length of the curve starting at a point $\alpha(t_0)$ is given by

$$s = \int_{t_0}^{t_0} \|\dot{\alpha}(w)\| \, dw.$$  

From $\dot{\alpha} = \sigma_u \dot{u} + \sigma_v \dot{v}$ we evaluate the speed of the curve:

$$\|\dot{\alpha}\|^2 = (\sigma_u \dot{u} + \sigma_v \dot{v}) \cdot (\sigma_u \dot{u} + \sigma_v \dot{v}) = (\sigma_u \cdot \sigma_u) \dot{u}^2 + 2(\sigma_u \cdot \sigma_v) \dot{u} \dot{v} + (\sigma_v \cdot \sigma_v) \dot{v}^2.$$  

Introducing

$$E = \sigma_u \cdot \sigma_u, \quad F = \sigma_u \cdot \sigma_v, \quad G = \sigma_v \cdot \sigma_v,$$

the arc length becomes

$$s = \int_{t_0}^{t_0} \left( E \dot{u}^2 + 2F \dot{u} \dot{v} + G \dot{v}^2 \right)^{1/2} \, dt \quad (1)$$

$$= \int_{\alpha} \left( E \, du^2 + 2F \, du \, dv + G \, dv^2 \right)^{1/2}. \quad (2)$$

In the last step above, $dt$ is brought inside the square root so that $\dot{u}^2(du)^2 = du^2$, etc. The expression

$$ds^2 = E \, du^2 + 2F \, du \, dv + G \, dv^2$$

is called the first fundamental form of $\sigma$, which represents the principal part of the square of the increment on $\sigma(u,v)$ when $u$ and $v$ are increased by $du$ and $dv$, respectively. The matrix

$$\mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is known as the metric tensor of the surface.

In the case that the curve \( \alpha \) spans multiple surface patches, to compute its length we break the curve into segments, each of which lies in a surface patch. The first fundamental form will change when the surface patch is changed.

**Example 1.** For the unit sphere parametrized with latitude and longitude coordinates in Example 2:

\[
\sigma(\theta, \phi) = (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta),
\]

the two partial derivatives are

\[
\sigma_\theta = (-\sin \theta \cos \phi, -\sin \theta \sin \phi, \cos \theta), \\
\sigma_\phi = (-\cos \theta \sin \phi, \cos \theta \cos \phi, 0).
\]

Thus we have

\[
E = \|\sigma_\theta\|^2 = 1, \\
F = \sigma_\theta \cdot \sigma_\phi = 0, \\
G = \|\sigma_\phi\|^2 = \cos^2 \theta.
\]

So the first fundamental form is

\[
d\theta^2 + \cos^2 \theta \, d\phi^2.
\]

**Example 2.** Consider the generalized cylinder in Example 6: \( \sigma(u, v) = \gamma(u) + v a \). The partial derivatives are

\[
\sigma_u = \dot{\gamma} \quad \text{and} \quad \sigma_v = a.
\]

We assume that \( \gamma \) is unit-speed and lies in a plane perpendicular to \( a \), and \( a \) is a unit vector. Then

\[
E = 1, \quad F = \dot{\gamma} \cdot a = 0, \quad G = \|a\|^2 = 1.
\]

The first fundamental form of \( \sigma \) is

\[
du^2 + dv^2,
\]

which has no trace of the curve \( \gamma \). In the case \( \gamma(u) \) is a line, the generalized cylinder is indeed a plane containing \( \gamma \) and \( a \). So a plane has the same first fundamental form \( du^2 + dv^2 \).

## 2 Surface Area

Let \( \sigma : U \to \mathbb{R}^3 \) be a surface patch on a surface \( S \). The patch \( \sigma(u, v) \) is covered by two families of parameter curves with \( u \) and \( v \) being constant, respectively. Consider a point \( \sigma(u_0, v_0) \) and the area on the surface enclosed by the \( v \)-parameter curves \( \alpha(u_0, v) \) and \( \alpha(u_0 + \Delta u, v) \), and the \( u \)-parameter curves \( \alpha(u, v_0) \) and \( \alpha(u, v_0 + \Delta v) \). See the figure below.
As $\Delta u$ and $\Delta v$ tend to zero, the area approaches a parallelogram whose sides are given by $\sigma_u \Delta u$ and $\sigma_v \Delta v$. Its area is approximated as

$$\|\sigma_u \Delta u \times \sigma_v \Delta v\| = \|\sigma_u \times \sigma_v\| \Delta u \Delta v$$

Integrating over the domain $U$, we obtain the area of the surface patch:

$$\int \int_U \|\sigma_u \times \sigma_v\| \, dudv.$$

Next, we express the integrand using the coefficients $E, F, G$ of the first fundamental form. Here we make use of the following identity over four vectors $a, b, c, d \in \mathbb{R}^3$:

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c).$$

The square of the integrand is

$$(\sigma_u \times \sigma_v) \cdot (\sigma_u \times \sigma_v) = (\sigma_u \cdot \sigma_u)(\sigma_v \cdot \sigma_v) - (\sigma_u \cdot \sigma_v)^2 = EG - F^2.$$

Therefore, the area of the patch becomes

$$\int \int_U \sqrt{EG - F^2} \, dudv.$$

Here $\sqrt{EG - F^2} \, dudv$ is the area element.

## 3 Normal and Geodesic Curvatures

One way to examine how much a surface bends is to look at the curvature of curves on the surface. Let $\gamma(t) = \sigma(u(t), v(t))$ be a unit-speed curve in a surface patch $\sigma$. Thus, $\dot{\gamma}$ is a unit tangent vector to $\sigma$, and it is perpendicular to the surface normal $\hat{n}$ at the same point. The three vectors $\gamma$, $\hat{n} \times \gamma$, and $\hat{n}$ form a local coordinate frame by the right-hand rule.

Differentiating $\dot{\gamma} \cdot \gamma$ yields that $\ddot{\gamma}$ is orthogonal to $\dot{\gamma}$. Hence $\ddot{\gamma}$ is a linear combination of $\hat{n}$ and $\hat{n} \times \gamma$:

$$\ddot{\gamma} = \kappa_n \hat{n} + \kappa_g \hat{n} \times \gamma \quad (3)$$

Here $\kappa_n$ is called the **normal curvature** and $\kappa_g$ is the **geodesic curvature** of $\gamma$.

Since $\hat{n}$ and $\hat{n} \times \gamma$ are orthogonal to each other, (3) implies that

$$\kappa_n = \ddot{\gamma} \cdot \hat{n} \quad \text{and} \quad \kappa_g = \ddot{\gamma} \cdot (\hat{n} \times \gamma).$$
Since $\gamma$ is unit-speed, its curvature is
\[
\kappa = \|\dot{\gamma}\| = \|\kappa_n \hat{n} + \kappa_g \hat{\gamma} \times \dot{\gamma}\| = \sqrt{\kappa_n^2 + \kappa_g^2}. \tag{4}
\]
Let $\psi$ be the angle between the principal normal $\hat{n}_\gamma$ of the curve, in the direction of $\ddot{\gamma}$, and the surface normal $\hat{n}$. We have
\[
\kappa_n = \kappa \hat{n}_\gamma \cdot \hat{n} = \kappa \cos \psi. \tag{5}
\]
Therefore, equation (4) implies
\[
\kappa_g = \pm \kappa \sin \psi.
\]
If $\gamma$ is regular but arbitrary-speed, the normal and geodesic curvatures of $\gamma$ are defined to be those of a unit-speed reparametrization of the same curve.

A unit-speed curve $\gamma$ is a geodesic if $\kappa_g = 0$. By (3), its acceleration $\ddot{\gamma}$ is always normal to the surface. Geodesics have many applications that we will devote one lecture to the topic later on.

## 4 Darboux Frame

On the unit-speed surface curve $\gamma$, the frame formed by the unit vectors $\dot{\gamma}$, $\hat{n} \times \dot{\gamma}$, and $\hat{n}$ is called the Darboux frame on the curve. This frame is different from the Frenet frame on the curve defined by $\dot{\gamma}$, the principal normal $\dot{\gamma}/\|\dot{\gamma}\|$, and the binormal $\dot{\gamma} \times \ddot{\gamma}/\|\ddot{\gamma}\|$.

Let us rename the three unit vectors $\dot{\gamma}$, $\hat{n} \times \dot{\gamma}$, $\hat{n}$ as $T$, $V$, $U$, respectively. Their derivatives must be respectively orthogonal to themselves. Differentiation of $U \cdot T = 0$ yields
\[
\dot{U} \cdot T = -U \cdot \ddot{T} = -U \cdot (\kappa_n U + \kappa_g V) \quad \text{(by (3))}
\]
\[
= -\kappa_n U.
\]

The vector $\dot{U}$ has a second component — along $V$. The geodesic torsion, defined to be $\tau_g = -\dot{U} \cdot V$, describes the negative rate of change of $U$ in the direction of $V$. With $\tau_g$ we characterize this change rate completely as below:
\[
\dot{U} = -\kappa_g T - \tau_g V.
\]

Similarly, we express $\dot{V}$ in terms of $T, U, V$, and combine it with the above equation and (3) into the following compact form (where the vectors are viewed as "scalars"):
\[
\begin{pmatrix}
\dot{T} \\
\dot{V} \\
\dot{U}
\end{pmatrix} =
\begin{pmatrix}
0 & \kappa_g & \kappa_n \\
-k\kappa_g & 0 & \tau_g \\
-k\kappa_n & -\tau_g & 0
\end{pmatrix}
\begin{pmatrix}
T \\
V \\
U
\end{pmatrix} \tag{6}
\]

The formulas (6) describe the geometry of a curve at a point in a local frame suiting the curve as well as a surface on which it lies, whereas the Frenet formulas describe its geometry at the point in a local frame best suiting the curve alone.

The curve $\gamma$ is asymptotic provided its tangent $\dot{\gamma}$ always points in a direction in which $\kappa_n = 0$. In some sense, the surface bends less along $\gamma$ than it does along a general curve. In (6), $\dot{T} = \kappa_g V$. Thus, $\kappa = \kappa_g$ and $V$ is aligned with the principal normal $\dot{T}/\|\dot{T}\|$ (assuming $\kappa_g \neq 0$). Consequently, the Darboux frame $T-V-U$ coincides with the Frenet frame everywhere.
5 Independence of Normal Curvature to a Curve

Let $\gamma(t) = \sigma(u(t), v(t))$ be a unit-speed curve on the surface patch $\sigma$. Below we obtain the normal curvature of the curve:

$$
\kappa_n = \ddot{\gamma} \cdot \hat{n} = \frac{d}{dt} \dot{\gamma} \cdot \hat{n} = \dot{n} \cdot \frac{d}{dt} (\sigma_u \dot{u} + \sigma_v \dot{v}) = \dot{n} \cdot \left( \sigma_u \ddot{u} + \sigma_v \ddot{v} + (\sigma_u \dot{u} + \sigma_v \dot{v}) \ddot{u} + (\sigma_u \dot{u} + \sigma_v \dot{v}) \ddot{v} \right) = \dot{n} \cdot (\sigma_{uu} \dddot{u}^2 + 2(\sigma_{uv} \dddot{u} \dddot{v}) \dot{u} \dddot{v} + (\sigma_{vv} \dddot{v}^2).
$$

(7)

In the last step above, we used the fact that $\hat{n}$ is normal to both $\sigma_u$ and $\sigma_v$, i.e., $\hat{n} \cdot \sigma_u = \hat{n} \cdot \sigma_v = 0$.

Equation (7) states that the normal curvature of $\gamma(t)$ depends on $u, v, \dot{u}$, and $\dot{v}$. The first two quantities specify the location of the point on the surface $\sigma$, and are thus curve independent. Let $\dot{u}$ be the unit tangent vector so that $\dot{\gamma} = \dot{u}$. Take the dot products of the equation $\dot{u} = \sigma_u \dot{u} + \sigma_v \dot{v}$ with $\sigma_u$ and $\sigma_v$ separately, yielding

$$
E \dddot{u} + F \dddot{v} = \dot{u} \cdot \sigma_u,
$$

$$
F \dddot{u} + G \dddot{v} = \dot{u} \cdot \sigma_v,
$$

where $E, F, G$ are the coefficients of the first fundamental form of the surface patch. Since $\sigma_u$ and $\sigma_v$ are linearly independent, the coefficient matrix in the above linear system in $\dot{u}$ and $\dot{v}$ is non-singular. We solve the system to obtain

$$
\begin{pmatrix}
\dddot{u} \\
\dddot{v}
\end{pmatrix} = \begin{pmatrix} E & F \\
F & G \end{pmatrix}^{-1} \begin{pmatrix}
\dot{u} \cdot \sigma_u \\
\dot{u} \cdot \sigma_v
\end{pmatrix}
$$

This implies that $\dot{u}$ and $\dot{v}$ are independent of the parametrization of $\gamma$. By (7), we conclude that any two unit-speed curves passing through the same point in the same direction $\dot{u}$ must have the same normal curvature at this point.

Subsequently, we refer to $\kappa_n$ as the normal curvature of the surface $\sigma$ at the point $p = \sigma(u, v)$ in the tangent direction of $\dot{u}$. It measures the curving of the surface in that direction. Generally, the surface bends at different rates in different tangent directions. The tangent $\dot{u}$ and the normal $\hat{n}$ at $p$ defines a plane that cuts a curve $\alpha$ out of the patch. This curve is called the normal section of $\sigma$ in the $\dot{u}$ direction. Since the principal normal $\hat{n}_\gamma$ of the normal section is related to the surface normal $\hat{n}$ by $\hat{n}_\gamma = \pm \hat{n}$.

The normal curvature equation

$$
\kappa_n = \kappa \hat{n}_\gamma \cdot \hat{n}
$$

implies that the curvature of the normal section is the normal curvature $\kappa_n$ at the point or its opposite, depending on the choice of the surface normal $\hat{n}$ at the point.
6 The Second Fundamental Form

Let $\sigma$ be a surface patch in $\mathbb{R}^3$ with standard unit normal
\[
\hat{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{\sigma_u \times \sigma_v}{\sqrt{EG - F^2}}.
\] (8)

With an increase $(\Delta u, \Delta v)$ in the parameter values, the movement of the point is described by Taylor’s series below:
\[
\sigma(u + \Delta u, v + \Delta v) - \sigma(u, v) \approx \sigma_u \Delta u + \sigma_v \Delta v + \frac{1}{2} \left( \sigma_{uu} (\Delta u)^2 + 2 \sigma_{uv} \Delta u \Delta v + \sigma_{vv} (\Delta v)^2 \right) + O((\Delta u + \Delta v)^3).
\]

The first order terms are tangent to the surface, hence perpendicular to $\hat{n}$. The terms of order higher than two tend to zero as $(\Delta u)^2 + (\Delta v)^2$ does so.

The deviation of $\sigma$ from the tangent plane is determined by the dot product of the second order term with the surface normal $\hat{n}$, namely, it is
\[
\frac{1}{2} \left( L(\Delta u)^2 + 2M \Delta u \Delta v + N(\Delta v)^2 \right),
\] (9)

where
\[
L = \sigma_{uu} \cdot \hat{n} = \frac{\sigma_{uu} \cdot (\sigma_u \times \sigma_v)}{\|\sigma_u \times \sigma_v\|} \quad \text{by (8)}
\]
\[
M = \sigma_{uv} \cdot \hat{n} = \frac{\det(\sigma_{uv} \sigma_u \sigma_v)}{\sqrt{EG - F^2}}
\]
\[
N = \sigma_{vv} \cdot \hat{n} = \frac{\det(\sigma_{vv} \sigma_u \sigma_v)}{\sqrt{EG - F^2}}.
\] (10)

The form (7) of the normal curvature can be rewritten as
\[
\kappa_n = L\dot{u}^2 + 2M \dot{u} \dot{v} + N\dot{v}^2.
\]

Recall that a unit-speed space curve $\alpha(s)$ can be approximated around $s = 0$ up to the second order as $\alpha(0) + s \dot{t}(0) + \frac{1}{2} \kappa(0)s^2 \hat{n}(0)$, where $\dot{t}(0)$ and $\hat{n}(0)$ are the tangent and principal normal, respectively, and $\kappa(0)$ the curvature. The expression (9) for the surface is analogous to the curvature term $\frac{1}{2}\kappa(0)s^2 \hat{n}(0)$ for a curve. In particular, the expression
\[
Ldu^2 + 2M dudv + Ndv^2
\]
is the second fundamental form of $\sigma$.

While the first fundamental form permits the calculation of metric properties such as length and area on a surface patch, the second fundamental form captures how ‘curved’ a surface patch is. The roles of the two fundamental forms on describing the local geometry are analogous to those of speed and acceleration for a parametric curve. Just as a unit-speed space curve is determined up to a rigid motion by its curvature and torsion, a surface patch is determined up to a rigid motion by its first and second fundamental forms.
Example 3. Consider a surface of revolution
\[ \sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u)), \]
where \( f(u) > 0 \) always holds. The figure below plots a catenoid where \( f(u) = 2 \cosh\left(\frac{u}{2}\right) \) \text{ and } \( g(u) = u \) with \((u, v) \in [-2, 2] \times [0, 2\pi]\). Assume the profile curve \( u \rightarrow (f(u), 0, g(u)) \) is unit-speed, i.e., \( f^2 + \dot{g}^2 = 1 \), where a dot denotes differentiation with respect to \( u \). We perform the following calculations:

\[
\begin{align*}
\sigma_u &= (f \cos v, f \sin v, \dot{g}), \\
\sigma_v &= (-f \sin v, f \cos v, 0), \\
E &= \sigma_u \cdot \sigma_u = f^2 + \dot{g}^2 = 1, \\
F &= \sigma_u \cdot \sigma_v = 0, \\
G &= \sigma_v \cdot \sigma_v = f^2, \\
\sigma_u \times \sigma_v &= (-f \dot{g} \cos v, -f \dot{g} \sin v, f \dot{f}), \\
\|\sigma_u \times \sigma_v\| &= f, \\
\hat{n} &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (-\dot{g} \cos v, -\dot{g} \sin v, \dot{f}), \\
\sigma_{uu} &= (\dot{f} \cos v, \dot{f} \sin v, \ddot{g}), \\
\sigma_{uv} &= (-\dot{f} \sin v, \dot{f} \cos v, 0), \\
\sigma_{vv} &= (-\dot{f} \cos v, -\dot{f} \sin v, 0), \\
L &= \sigma_{uu} \cdot \hat{n} = \dot{f} \ddot{g} - \ddot{f} \dot{g}, \\
M &= \sigma_{uv} \cdot \hat{n} = 0, \\
N &= \sigma_{vv} \cdot \hat{n} = \dot{f} \ddot{g}.
\end{align*}
\]

Thus, the second fundamental form is
\[ (\dot{f} \ddot{g} - \ddot{f} \dot{g})du^2 + f \dot{g}dv^2. \]

A Invariance of the First Fundamental Form to Parametrization

Suppose we reparametrize the surface patch \( \sigma(u, v) \) with parameters \( \phi \) and \( \psi \) such that \( u = u(\phi, \psi) \) \text{ and } \( v = v(\phi, \psi) \). The new first fundamental form are then
\[ (ds)^2 = \tilde{E} \, d\phi^2 + 2\tilde{F} \, d\phi \, d\psi + \tilde{G} \, d\psi^2, \]
where
\[ \tilde{E} = \sigma_\phi \cdot \sigma_\phi, \quad \tilde{F} = \sigma_\phi \cdot \sigma_\psi, \quad \tilde{G} = \sigma_\psi \cdot \sigma_\psi. \]

The following differential relationships exist between the two sets of parameters:
\[ \begin{pmatrix} du \\ dv \end{pmatrix} = J \begin{pmatrix} d\phi \\ d\psi \end{pmatrix}, \quad (11) \]
where
\[ J = \begin{pmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \\ \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \psi} \end{pmatrix} \]

is the *Jacobian matrix*.

Applying the chain rule on differentiation, it is easy to verify that the new metric tensor is related to the old one as follows:
\[
\begin{pmatrix} \hat{E} & \hat{F} \\ \hat{F} & \hat{G} \end{pmatrix} = \left( \frac{\partial u}{\partial \phi} \frac{\partial u}{\partial \psi} \frac{\partial v}{\partial \phi} \frac{\partial v}{\partial \psi} \right)^\top \begin{pmatrix} E & F \\ F & G \end{pmatrix} \left( \frac{\partial u}{\partial \phi} \frac{\partial u}{\partial \psi} \frac{\partial v}{\partial \phi} \frac{\partial v}{\partial \psi} \right).
\]

That is, the metric tensor also transforms by the Jacobian matrix. Immediately, combining (11), we obtain
\[
ds^2 = (du \ dv) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = (d\phi \ d\psi) J^\top \begin{pmatrix} E & F \\ F & G \end{pmatrix} J \begin{pmatrix} du \\ dv \end{pmatrix} = (d\phi \ d\psi) \begin{pmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \\ \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \psi} \end{pmatrix}^\top \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \\ \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \psi} \end{pmatrix} \begin{pmatrix} d\phi \\ d\psi \end{pmatrix} = (d\tilde{s})^2.
\]

Hence, the first fundamental form of a surface patch is *invariant* under parametrization.

**References**
