Surface Curves and Fundamental Forms*

(Com S 477/577 Notes)

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Oct 15, 2020

1 The First Fundamental Form

Let $\boldsymbol{\alpha}(t) = \boldsymbol{\sigma}(u(t), v(t))$ be a curve in a surface patch $\boldsymbol{\sigma}$. The arc length of the curve starting at a point $\boldsymbol{\alpha}(t_0)$ is given by

$$s = \int_{t_0}^{t} \|\dot{\boldsymbol{\alpha}}(w)\|\, dw.$$

From $\dot{\boldsymbol{\alpha}} = \boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}$ we evaluate the speed of the curve:

$$\begin{aligned} \|\dot{\boldsymbol{\alpha}}\|^2 &= (\boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}) \cdot (\boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}) \\ &= (\boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u)\dot{u}^2 + 2(\boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v)\dot{u}\dot{v} + (\boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v)\dot{v}^2. \end{aligned}$$

Introducing

$$E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u, \qquad F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v, \qquad G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v,$$

the arc length becomes

$$\begin{aligned} s &= \int_{t_0}^{t} (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}\, dt && (1) \\ &= \int_{\alpha} (E\, du^2 + 2F\, du\, dv + G\, dv^2)^{1/2}. && (2) \end{aligned}$$

In the last step above, dt is brought inside the square root so that $\dot{u}^2(dt)^2 = du^2$, etc. The expression

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2$$

is called the *first fundamental form* of $\boldsymbol{\sigma}$, which represents the principal part of the square of the increment on $\boldsymbol{\sigma}(u, v)$ when u and v are increased by du and dv, respectively. The matrix

$$\mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is known as the *metric tensor* of the surface.

*The material is adapted from the book *Elementary Differential Geometry* by Andrew Pressley, Springer-Verlag, 2001.

In the case that the curve $\boldsymbol{\alpha}$ spans multiple surface patches, to compute its length we break the curve into segments, each of which lies in a surface patch. *The first fundamental form will change when the surface patch is changed.*

EXAMPLE 1. For the unit sphere parametrized with latitude and longitude coordinates in Example 2:

$$\boldsymbol{\sigma}(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta),$$

the two partial derivatives are

$$\begin{aligned} \boldsymbol{\sigma}_\theta &= (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta), \\ \boldsymbol{\sigma}_\phi &= (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0). \end{aligned}$$

Thus we have

$$\begin{aligned} E &= \|\boldsymbol{\sigma}_\theta\|^2 = 1, \\ F &= \boldsymbol{\sigma}_\theta \cdot \boldsymbol{\sigma}_\phi = 0, \\ G &= \|\boldsymbol{\sigma}_\phi\|^2 = \cos^2\theta. \end{aligned}$$

So the first fundamental form is

$$d\theta^2 + \cos^2\theta\, d\phi^2.$$

EXAMPLE 2. Consider the generalized cylinder in Example 6: $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{a}$. The partial derivatives are

$$\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}} \qquad \text{and} \qquad \boldsymbol{\sigma}_v = \boldsymbol{a}.$$

We assume that $\boldsymbol{\gamma}$ is unit-speed and lies in a plane perpendicular to $\boldsymbol{a}$, and $\boldsymbol{a}$ is a unit vector. Then

$$E = 1, \qquad F = \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{a} = 0, \qquad G = \|\boldsymbol{a}\|^2 = 1.$$

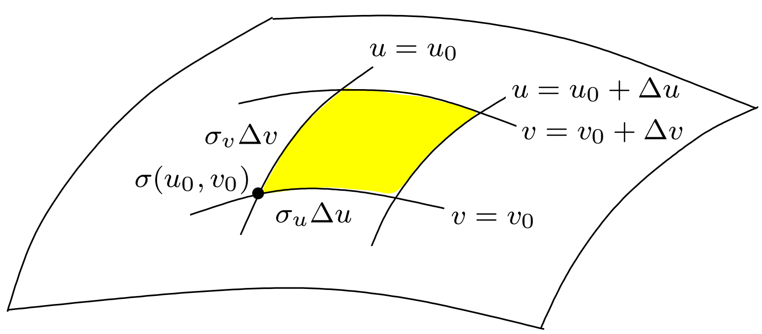
The first fundamental form of $\boldsymbol{\sigma}$ is

$$du^2 + dv^2,$$

which has no trace of the curve $\boldsymbol{\gamma}$. In the case $\boldsymbol{\gamma}(u)$ is a line, the generalized cylinder is indeed a plane containing $\boldsymbol{\gamma}$ and $\boldsymbol{a}$. So a plane has the same first fundamental form $du^2 + dv^2$.

2 Surface Area

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a surface patch on a surface $\mathcal{S}$. The patch $\boldsymbol{\sigma}(u,v)$ is covered by two families of parameter curves with u and v being constant, respectively. Consider a point $\boldsymbol{\sigma}(u_0, v_0)$ and the area on the surface enclosed by the *v-parameter curves* $\boldsymbol{\alpha}(u_0, v)$ and $\boldsymbol{\alpha}(u_0 + \Delta u, v)$, and the *u-parameter curves* $\boldsymbol{\alpha}(u, v_0)$ and $\boldsymbol{\alpha}(u, v_0 + \Delta v)$. See the figure below.



As Δu and Δv tend to zero, the area approaches a parallelogram whose sides are given by $\boldsymbol{\sigma}_u \Delta u$ and $\boldsymbol{\sigma}_v \Delta v$. Its area is approximated as

$$\|\boldsymbol{\sigma}_u \Delta u \times \boldsymbol{\sigma}_v \Delta v\| = \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| \Delta u \Delta v$$

Integrating over the domain U, we obtain the area of the surface patch:

$$\int\int_U \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| \, du dv.$$

Next, we express the integrand using the coefficients E, F, G of the first fundamental form. Here we make use of the following identity over four vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$:

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}).$$

The square of the integrand is

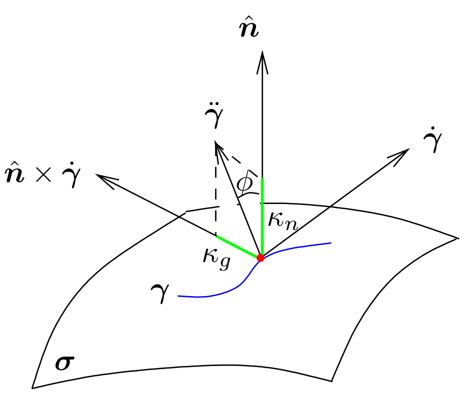
$$\begin{aligned} (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) &= (\boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u)(\boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v) - (\boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v)^2 \\ &= EG - F^2. \end{aligned}$$

Therefore, the area of the patch becomes

$$\int\int_U \sqrt{EG - F^2} \, du dv.$$

Here $\sqrt{EG - F^2} \, du dv$ is the area element.

3 Normal and Geodesic Curvatures



One way to examine how much a surface bends is to look at the curvature of curves on the surface. Let $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ be a unit-speed curve in a surface patch $\boldsymbol{\sigma}$. Thus, $\dot{\boldsymbol{\gamma}}$ is a unit tangent vector to $\boldsymbol{\sigma}$, and it is perpendicular to the surface normal $\hat{\boldsymbol{n}}$ at the same point. The three vectors $\dot{\boldsymbol{\gamma}}$, $\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\gamma}}$, and $\hat{\boldsymbol{n}}$ form a local coordinate frame by the right-hand rule.

Differentiating $\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}$ yields that $\ddot{\boldsymbol{\gamma}}$ is orthogonal to $\dot{\boldsymbol{\gamma}}$. Hence $\ddot{\boldsymbol{\gamma}}$ is a linear combination of $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\gamma}}$:

$$\ddot{\boldsymbol{\gamma}} = \kappa_n \hat{\boldsymbol{n}} + \kappa_g \hat{\boldsymbol{n}} \times \dot{\boldsymbol{\gamma}} \tag{3}$$

Here κ_n is called the *normal curvature* and κ_g is the *geodesic curvature* of $\boldsymbol{\gamma}$.

Since $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\gamma}}$ are orthogonal to each other, (3) implies that

$$\kappa_n = \ddot{\boldsymbol{\gamma}} \cdot \hat{\boldsymbol{n}} \qquad \text{and} \qquad \kappa_g = \ddot{\boldsymbol{\gamma}} \cdot (\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\gamma}}).$$

Since $\boldsymbol{\gamma}$ is unit-speed, its curvature is

$$\begin{aligned}\kappa &= \|\ddot{\boldsymbol{\gamma}}\| \\ &= \|\kappa_n \hat{\boldsymbol{n}} + \kappa_g \hat{\boldsymbol{n}} \times \dot{\boldsymbol{\gamma}}\| \\ &= \sqrt{\kappa_n^2 + \kappa_g^2}. \end{aligned} \tag{4}$$

Let ψ be the angle between the principal normal $\hat{\boldsymbol{n}}_\gamma$ of the curve, in the direction of $\ddot{\boldsymbol{\gamma}}$, and the surface normal $\hat{\boldsymbol{n}}$. We have

$$\kappa_n = \kappa \hat{\boldsymbol{n}}_\gamma \cdot \hat{\boldsymbol{n}} = \kappa \cos\psi. \tag{5}$$

Therefore, equation (4) implies

$$\kappa_g = \pm\kappa \sin\psi.$$

If $\boldsymbol{\gamma}$ is regular but arbitrary-speed, the normal and geodesic curvatures of $\boldsymbol{\gamma}$ are defined to be those of a unit-speed reparametrization of the same curve.

A unit-speed curve $\boldsymbol{\gamma}$ is a *geodesic* if $\kappa_g = 0$. By (3), its acceleration $\ddot{\boldsymbol{\gamma}}$ is always normal to the surface. Geodesics have many applications that we will devote one lecture to the topic later on.

4 Darboux Frame

On the unit-speed surface curve $\boldsymbol{\gamma}$, the frame formed by the unit vectors $\dot{\boldsymbol{\gamma}}$, $\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\gamma}}$, and $\hat{\boldsymbol{n}}$ is called the *Darboux frame* on the curve. This frame is different from the Frenet frame on the curve defined by $\dot{\boldsymbol{\gamma}}$, the principal normal $\ddot{\boldsymbol{\gamma}}/\|\ddot{\boldsymbol{\gamma}}\|$, and the binormal $\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}/\|\ddot{\boldsymbol{\gamma}}\|$.

Let us rename the three unit vectors $\dot{\boldsymbol{\gamma}}$, $\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\gamma}}$, $\hat{\boldsymbol{n}}$ as T, V, U, respectively. Their derivatives must be respectively orthogonal to themselves. Differentiation of $U \cdot T = 0$ yields

$$\begin{aligned}\dot{U} \cdot T &= -U \cdot \dot{T} \\ &= -U \cdot (\kappa_n U + \kappa_g V) \qquad \text{(by (3))} \\ &= -\kappa_n. \end{aligned}$$

The vector $\dot{U}$ has a second component — along V. The *geodesic torsion*, defined to be $\tau_g = -\dot{U} \cdot V$, describes the negative rate of change of U in the direction of V. With τ_g we characterize this change rate completely as below:

$$\dot{U} = -\kappa_g T - \tau_g V.$$

Similarly, we express $\dot{V}$ in terms of T, U, V, and combine it with the above equation and (3) into the following compact form (where the vectors are viewed as "scalars"):

$$\begin{pmatrix} \dot{T} \\ \dot{V} \\ \dot{U} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ V \\ U \end{pmatrix} \tag{6}$$

The formulas (6) describe the geometry of a curve at a point *in a local frame suiting the curve as well as a surface on which it lies*, whereas the Frenet formulas describe its geometry at the point in a local frame best suiting the curve alone.

The curve $\boldsymbol{\gamma}$ is *asymptotic* provided its tangent $\dot{\boldsymbol{\gamma}}$ always points in a direction in which $\kappa_n = 0$. In some sense, the surface bends less along $\boldsymbol{\gamma}$ than it does along a general curve. In (6), $\dot{T} = \kappa_g V$. Thus, $\kappa = \kappa_g$ and V is aligned with the principal normal $\dot{T}/\|\dot{T}\|$ (assuming $\kappa_g \neq 0$). Consequently, the Darboux frame T-V-U coincides with the Frenet frame everywhere.

5 Independence of Normal Curvature to a Curve

Let $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ be a unit-speed curve on the surface patch $\boldsymbol{\sigma}$. Below we obtain the normal curvature of the curve:

$$
\begin{aligned}
\kappa_n &= \ddot{\boldsymbol{\gamma}} \cdot \hat{\boldsymbol{n}} \\
&= \frac{d}{dt}\dot{\boldsymbol{\gamma}} \cdot \hat{\boldsymbol{n}} \\
&= \hat{\boldsymbol{n}} \cdot \frac{d}{dt}(\boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}) \\
&= \hat{\boldsymbol{n}} \cdot \Big(\boldsymbol{\sigma}_u \ddot{u} + \boldsymbol{\sigma}_v \ddot{v} + (\boldsymbol{\sigma}_{uu}\dot{u} + \boldsymbol{\sigma}_{uv}\dot{v})\dot{u} + (\boldsymbol{\sigma}_{uv}\dot{u} + \boldsymbol{\sigma}_{vv}\dot{v})\dot{v} \Big) \\
&= (\boldsymbol{\sigma}_{uu} \cdot \hat{\boldsymbol{n}})\dot{u}^2 + 2(\boldsymbol{\sigma}_{uv} \cdot \hat{\boldsymbol{n}})\dot{u}\dot{v} + (\boldsymbol{\sigma}_{vv} \cdot \hat{\boldsymbol{n}})\dot{v}^2. \qquad (7)
\end{aligned}
$$

In the last step above, we used the fact that $\hat{\boldsymbol{n}}$ is normal to both $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$, i.e., $\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}_u = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}_v = 0$.

Equation (7) states that the normal curvature of $\boldsymbol{\gamma}(t)$ depends on u, v, $\dot{u}$, and $\dot{v}$. The first two quantities specify the location of the point on the surface $\boldsymbol{\sigma}$, and are thus curve independent. Let $\hat{\boldsymbol{u}}$ be the unit tangent vector so that $\dot{\boldsymbol{\gamma}} = \hat{\boldsymbol{u}}$. Take the dot products of the equation $\hat{\boldsymbol{u}} = \boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}$ with $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ separately, yielding

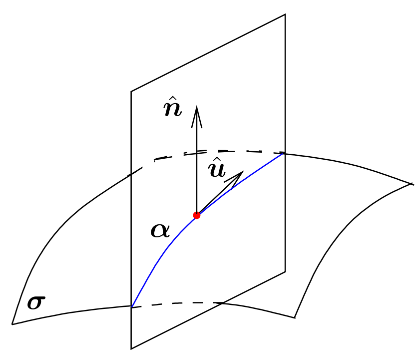
$$
\begin{aligned}
E\dot{u} + F\dot{v} &= \hat{\boldsymbol{u}} \cdot \boldsymbol{\sigma}_u, \\
F\dot{u} + G\dot{v} &= \hat{\boldsymbol{u}} \cdot \boldsymbol{\sigma}_v,
\end{aligned}
$$

where E, F, G are the coefficients of the first fundamental form of the surface patch. Since $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly independent, the coefficient matrix in the above linear system in $\dot{u}$ and $\dot{v}$ is non-singular. We solve the system to obtain

$$
\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \hat{\boldsymbol{u}} \cdot \boldsymbol{\sigma}_u \\ \hat{\boldsymbol{u}} \cdot \boldsymbol{\sigma}_v \end{pmatrix}
$$

This implies that $\dot{u}$ and $\dot{v}$ are independent of the parametrization of $\boldsymbol{\gamma}$. By (7), we conclude that *any two unit-speed curves passing through the same point in the same direction* $\hat{\boldsymbol{u}}$ *must have the same normal curvature at this point.*

Subsequently, we refer to κ_n as the *normal curvature* of the surface $\boldsymbol{\sigma}$ at the point $\boldsymbol{p} = \boldsymbol{\sigma}(u, v)$ in the tangent direction of $\hat{\boldsymbol{u}}$. It measures the curving of the surface in that direction. Generally, the surface bends at different rates in different tangent directions. The tangent $\hat{\boldsymbol{u}}$ and the normal $\hat{\boldsymbol{n}}$ at $\boldsymbol{p}$ defines a plane that cuts a curve $\boldsymbol{\alpha}$ out of the patch. This curve is called the *normal section* of $\boldsymbol{\sigma}$ in the $\hat{\boldsymbol{u}}$ direction. Since the principal normal $\hat{\boldsymbol{n}}_\gamma$ of the normal section is related to the surface normal $\hat{\boldsymbol{n}}$ by $\hat{\boldsymbol{n}}_\gamma = \pm\hat{\boldsymbol{n}}$. The normal curvature equation



$$
\kappa_n = \kappa \hat{\boldsymbol{n}}_\gamma \cdot \hat{\boldsymbol{n}}
$$

implies that the curvature of the normal section is the normal curvature κ_n at the point or its opposite, depending on the choice of the surface normal $\hat{\boldsymbol{n}}$ at the point.

6 The Second Fundamental Form

Let $\boldsymbol{\sigma}$ be a surface patch in $\mathbb{R}^3$ with standard unit normal

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\sqrt{EG - F^2}}. \tag{8}$$

With an increase $(\Delta u, \Delta v)$ in the parameter values, the movement of the point is described by Taylor's series below:

$$\begin{aligned} & \boldsymbol{\sigma}(u + \Delta u, v + \Delta v) - \boldsymbol{\sigma}(u, v) \\ = \ & \boldsymbol{\sigma}_u \Delta u + \boldsymbol{\sigma}_v \Delta v + \frac{1}{2}\Big(\boldsymbol{\sigma}_{uu}(\Delta u)^2 + 2\boldsymbol{\sigma}_{uv}\Delta u \Delta v + \boldsymbol{\sigma}_{vv}(\Delta v)^2\Big) + O\Big((\Delta u + \Delta v)^3\Big). \end{aligned}$$

The first order terms are tangent to the surface, hence perpendicular to $\hat{\boldsymbol{n}}$. The terms of order higher than two tend to zero as $(\Delta u)^2 + (\Delta v)^2$ does so.

The deviation of $\boldsymbol{\sigma}$ from the tangent plane is determined by the dot product of the second order term with the surface normal $\hat{\boldsymbol{n}}$, namely, it is

$$\frac{1}{2}\Big(L(\Delta u)^2 + 2M\Delta u \Delta v + N(\Delta v)^2\Big), \tag{9}$$

where

$$\begin{aligned} L &= \boldsymbol{\sigma}_{uu} \cdot \hat{\boldsymbol{n}} \\ &= \frac{\boldsymbol{\sigma}_{uu} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v)}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} \qquad \text{by (8)} \\ &= \frac{\det(\boldsymbol{\sigma}_{uu}\ \boldsymbol{\sigma}_u\ \boldsymbol{\sigma}_v)}{\sqrt{EG - F^2}}, \\ M &= \boldsymbol{\sigma}_{uv} \cdot \hat{\boldsymbol{n}} = \frac{\det(\boldsymbol{\sigma}_{uv}\ \boldsymbol{\sigma}_u\ \boldsymbol{\sigma}_v)}{\sqrt{EG - F^2}}, \\ N &= \boldsymbol{\sigma}_{vv} \cdot \hat{\boldsymbol{n}} = \frac{\det(\boldsymbol{\sigma}_{vv}\ \boldsymbol{\sigma}_u\ \boldsymbol{\sigma}_v)}{\sqrt{EG - F^2}}. \end{aligned} \tag{10}$$

The form (7) of the normal curvature can be rewritten as

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2.$$

Recall that a unit-speed space curve $\boldsymbol{\alpha}(s)$ can be approximated around $s = 0$ up to the second order as $\boldsymbol{\alpha}(0) + s\hat{\boldsymbol{t}}(0) + \frac{1}{2}\kappa(0)s^2\hat{\boldsymbol{n}}(0)$, where $\hat{\boldsymbol{t}}(0)$ and $\hat{\boldsymbol{n}}(0)$ are the tangent and principal normal, respectively, and $\kappa(0)$ the curvature. The expression (9) for the surface is analogous to the curvature term $\frac{1}{2}\kappa(0)s^2\hat{\boldsymbol{n}}(0)$ for a curve. In particular, the expression

$$Ldu^2 + 2Mdudv + Ndv^2$$

is the *second fundamental form* of $\boldsymbol{\sigma}$.

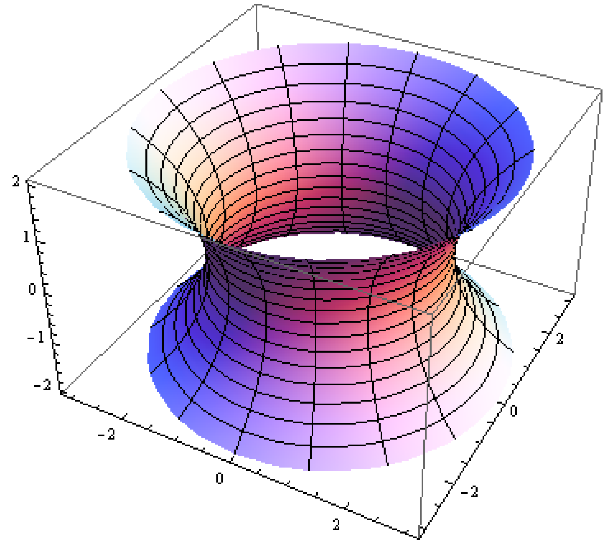
While the first fundamental form permits the calculation of metric properties such as length and area on a surface patch, the second fundamental form captures how 'curved' a surface patch is. The roles of the two fundamental forms on describing the local geometry are analogous to those of speed and acceleration for a parametric curve. Just as a unit-speed space curve is determined up to a rigid motion by its curvature and torsion, a surface patch is determined up to a rigid motion by its first and second fundamental forms.

Example 3. Consider a surface of revolution

$$\boldsymbol{\sigma}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where $f(u) > 0$ always holds. The figure below plots a catenoid where

$$f(u) = 2\cosh\left(\frac{u}{2}\right) \qquad \text{and} \qquad g(u) = u$$



with $(u,v) \in [-2,2] \times [0,2\pi]$. Assume the profile curve $u \to (f(u), 0, g(u))$ is unit-speed, i.e., $\dot{f}^2 + \dot{g}^2 = 1$, where a dot denotes differentiation with respect to u. We perform the following calculations:

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (\dot{f}\cos v, \dot{f}\sin v, \dot{g}), \\
\boldsymbol{\sigma}_v &= (-f\sin v, f\cos v, 0), \\
E &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = \dot{f}^2 + \dot{g}^2 = 1, \\
F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0, \\
G &= \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = f^2, \\
\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (-f\dot{g}\cos v, -f\dot{g}\sin v, f\dot{f}), \\
\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| &= f, \\
\hat{\boldsymbol{n}} &= \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} = (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f}), \\
\boldsymbol{\sigma}_{uu} &= (\ddot{f}\cos v, \ddot{f}\sin v, \ddot{g}), \\
\boldsymbol{\sigma}_{uv} &= (-\dot{f}\sin v, \dot{f}\cos v, 0), \\
\boldsymbol{\sigma}_{vv} &= (-f\cos v, -f\sin v, 0), \\
L &= \boldsymbol{\sigma}_{uu} \cdot \hat{\boldsymbol{n}} = \dot{f}\ddot{g} - \ddot{f}\dot{g}, \\
M &= \boldsymbol{\sigma}_{uv} \cdot \hat{\boldsymbol{n}} = 0, \\
N &= \boldsymbol{\sigma}_{vv} \cdot \hat{\boldsymbol{n}} = f\dot{g}.
\end{aligned}$$

Thus, the second fundamental form is

$$(\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + f\dot{g}dv^2.$$

A Invariance of the First Fundamental Form to Parametrization

Suppose we reparametrize the surface patch $\boldsymbol{\sigma}(u,v)$ with parameters ϕ and ψ such that $u = u(\phi,\psi)$ and $v = v(\phi,\psi)$. The new first fundamental form are then

$$(d\tilde{s})^2 = \tilde{E}\,d\phi^2 + 2\tilde{F}\,d\phi\,d\psi + \tilde{G}\,d\psi^2,$$

where

$$\tilde{E} = \boldsymbol{\sigma}_\phi \cdot \boldsymbol{\sigma}_\phi, \qquad \tilde{F} = \boldsymbol{\sigma}_\phi \cdot \boldsymbol{\sigma}_\psi, \qquad \tilde{G} = \boldsymbol{\sigma}_\psi \cdot \boldsymbol{\sigma}_\psi.$$

The following differential relationships exist between the two sets of parameters:

$$\begin{pmatrix} du \\ dv \end{pmatrix} = J \begin{pmatrix} d\phi \\ d\psi \end{pmatrix}, \tag{11}$$

where

$$J = \begin{pmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \\ \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \psi} \end{pmatrix}$$

is the *Jacobian matrix*.

Applying the chain rule on differentiation, it is easy to verify that the new metric tensor is related to the old one as follows:

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \\ \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \psi} \end{pmatrix}^{\top} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \\ \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \psi} \end{pmatrix}.$$

That is, the metric tensor also transforms by the Jacobian matrix. Immediately, combining (11), we obtain

$$\begin{aligned} ds^2 &= (du \;\; dv) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \binom{du}{dv} \\ &= (d\phi \;\; d\psi)\, J^{\top} \begin{pmatrix} E & F \\ F & G \end{pmatrix} J \binom{du}{dv} \\ &= (d\phi \;\; d\psi) \begin{pmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \\ \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \psi} \end{pmatrix}^{\top} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \\ \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \psi} \end{pmatrix} \binom{d\phi}{d\psi} \\ &= (d\phi \;\; d\psi) \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \binom{d\phi}{d\psi} \\ &= (d\tilde{s})^2. \end{aligned}$$

Hence, the first fundamental form of a surface patch is *invariant* under parametrization.

References

[1] B. O'Neill. *Elementary Differential Geometry.* Academic Press, Inc., 1966.

[2] A. Pressley. *Elementary Differential Geometry.* Springer-Verlag London, 2001.