

Surface Curvatures*

(Com S 477/577 Notes)

Yan-Bin Jia

Oct 23, 2017

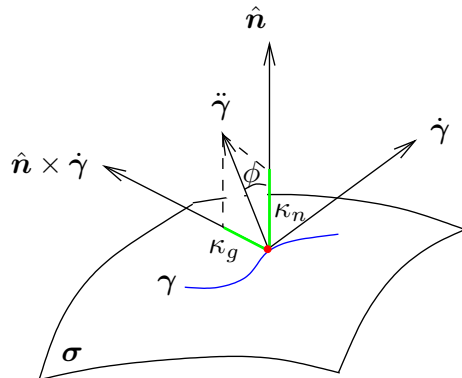
1 Curve on a Surface: Normal and Geodesic Curvatures

One way to examine how much a surface bends is to look at the curvature of curves on the surface. Let $\gamma(t) = \sigma(u(t), v(t))$ be a unit-speed curve in a surface patch σ . Thus, $\dot{\gamma}$ is a unit tangent vector to σ , and it is perpendicular to the surface normal \hat{n} at the same point. The three vectors $\dot{\gamma}$, $\hat{n} \times \dot{\gamma}$, and \hat{n} form a local coordinate frame by the right-hand rule.

Differentiating $\dot{\gamma} \cdot \dot{\gamma}$ yields that $\ddot{\gamma}$ is orthogonal to $\dot{\gamma}$. Hence $\ddot{\gamma}$ is a linear combination of \hat{n} and $\hat{n} \times \dot{\gamma}$:

$$\ddot{\gamma} = \kappa_n \hat{n} + \kappa_g \hat{n} \times \dot{\gamma} \quad (1)$$

Here κ_n is called the *normal curvature* and κ_g is the *geodesic curvature* of γ .



Since \hat{n} and $\hat{n} \times \dot{\gamma}$ are orthogonal to each other, (1) implies that

$$\kappa_n = \ddot{\gamma} \cdot \hat{n} \quad \text{and} \quad \kappa_g = \ddot{\gamma} \cdot (\hat{n} \times \dot{\gamma}).$$

Since γ is unit-speed, its curvature is

$$\begin{aligned} \kappa &= \|\ddot{\gamma}\| \\ &= \|\kappa_n \hat{n} + \kappa_g \hat{n} \times \dot{\gamma}\| \\ &= \sqrt{\kappa_n^2 + \kappa_g^2}. \end{aligned} \quad (2)$$

*The material is adapted from the book *Elementary Differential Geometry* by Andrew Pressley, Springer-Verlag, 2001.

Let ψ be the angle between the principal normal $\hat{\mathbf{n}}_\gamma$ of the curve, in the direction of $\dot{\gamma}$, and the surface normal $\hat{\mathbf{n}}$. We have

$$\kappa_n = \kappa \hat{\mathbf{n}}_\gamma \cdot \hat{\mathbf{n}} = \kappa \cos \psi. \quad (3)$$

Therefore, equation (2) implies

$$\kappa_g = \pm \kappa \sin \psi.$$

If γ is regular but arbitrary-speed, the normal and geodesic curvatures of γ are defined to be those of a unit-speed reparametrization of the same curve.

A unit-speed curve γ is a *geodesic* if $\kappa_g = 0$. By (1), its acceleration $\ddot{\gamma}$ is always normal to the surface. Geodesics have many applications that we will devote one lecture to the topic later on.

2 Darboux Frame on a Curve

On the unit-speed surface curve γ , the frame formed by the unit vectors $\dot{\gamma}$, $\hat{\mathbf{n}} \times \dot{\gamma}$, and $\hat{\mathbf{n}}$ is called the *Darboux frame* on the curve. This frame is different from the Frenet frame on the curve defined by $\dot{\gamma}$, the principal normal $\ddot{\gamma}/\|\ddot{\gamma}\|$, and the binormal $\dot{\gamma} \times \ddot{\gamma}/\|\ddot{\gamma}\|$.

Let us rename the three unit vectors $\dot{\gamma}$, $\hat{\mathbf{n}} \times \dot{\gamma}$, $\hat{\mathbf{n}}$ as T , V , U , respectively. Their derivatives must be respectively orthogonal to themselves. Differentiation of $U \cdot T = 0$ yields

$$\begin{aligned} U' \cdot T &= -U \cdot T' \\ &= -U \cdot (\kappa_n U + \kappa_g V) && \text{(by (1))} \\ &= -\kappa_n. \end{aligned}$$

The vector U' has a second component — along V . The *geodesic torsion*, defined to be $\tau_g = -U' \cdot V$, describes the negative rate of change of U in the direction of V . With τ_g we characterize this change rate completely as below:

$$U' = -\kappa_g T - \tau_g V.$$

Similarly, we express V' in terms of T, U, V , and combine it with the above equation and (1) into the following compact form (where the vectors are viewed as “scalars”):

$$\begin{pmatrix} T \\ V \\ U \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ V \\ U \end{pmatrix} \quad (4)$$

The formulas (4) describe the geometry of a curve at a point *in a local frame suiting the curve as well as a surface on which it lies*, whereas the Frenet formulas describe its geometry at the point in a local frame best suiting the curve alone.

The curve γ is *asymptotic* provided its tangent $\dot{\gamma}$ always points in a direction in which $\kappa_n = 0$. In some sense, the surface bends less along γ than it does along a general curve. In (4), $T' = \kappa_g V$. Thus, $\kappa = \kappa_g$ and V is aligned with the principal normal $T'/\|T'\|$ (assuming $\kappa_g \neq 0$). Consequently, the Darboux frame T - V - U coincides with the Frenet frame everywhere.

3 The Second Fundamental Form

Let σ be a surface patch in \mathbb{R}^3 with standard unit normal

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\sqrt{EG - F^2}}. \quad (5)$$

With an increase $(\Delta u, \Delta v)$ in the parameter values, the movement of the point is described by Taylor's series below:

$$\begin{aligned} & \boldsymbol{\sigma}(u + \Delta u, v + \Delta v) - \boldsymbol{\sigma}(u, v) \\ &= \boldsymbol{\sigma}_u \Delta u + \boldsymbol{\sigma}_v \Delta v + \frac{1}{2} \left(\boldsymbol{\sigma}_{uu} (\Delta u)^2 + 2\boldsymbol{\sigma}_{uv} \Delta u \Delta v + \boldsymbol{\sigma}_{vv} (\Delta v)^2 \right) + O\left((\Delta u + \Delta v)^3\right). \end{aligned}$$

The first order terms are tangent to the surface, hence perpendicular to $\hat{\mathbf{n}}$. The terms of order higher than two tend to zero as $(\Delta u)^2 + (\Delta v)^2$ does so.

The deviation of $\boldsymbol{\sigma}$ from the tangent plane is determined by the dot product of the second order term with the surface normal $\hat{\mathbf{n}}$, namely, it is

$$\frac{1}{2} \left(L (\Delta u)^2 + 2M \Delta u \Delta v + N (\Delta v)^2 \right), \quad (6)$$

where

$$\begin{aligned} L &= \boldsymbol{\sigma}_{uu} \cdot \hat{\mathbf{n}} \\ &= \frac{\boldsymbol{\sigma}_{uu} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v)}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} && \text{by (5)} \\ &= \frac{\det(\boldsymbol{\sigma}_{uu} \ \boldsymbol{\sigma}_u \ \boldsymbol{\sigma}_v)}{\sqrt{EG - F^2}}, \\ M &= \boldsymbol{\sigma}_{uv} \cdot \hat{\mathbf{n}} = \frac{\det(\boldsymbol{\sigma}_{uv} \ \boldsymbol{\sigma}_u \ \boldsymbol{\sigma}_v)}{\sqrt{EG - F^2}}, \\ N &= \boldsymbol{\sigma}_{vv} \cdot \hat{\mathbf{n}} = \frac{\det(\boldsymbol{\sigma}_{vv} \ \boldsymbol{\sigma}_u \ \boldsymbol{\sigma}_v)}{\sqrt{EG - F^2}}. \end{aligned} \quad (7)$$

Recall that a unit-speed space curve $\boldsymbol{\alpha}(s)$ can be approximated around $s = 0$ up to the second order as $\boldsymbol{\alpha}(0) + s\hat{\mathbf{t}}(0) + \frac{1}{2}\kappa(0)s^2\hat{\mathbf{n}}(0)$, where $\hat{\mathbf{t}}(0)$ and $\hat{\mathbf{n}}(0)$ are the tangent and principal normal, respectively, and $\kappa(0)$ the curvature. The expression (6) for the surface is analogous to the curvature term $\frac{1}{2}\kappa(0)s^2\hat{\mathbf{n}}(0)$ for a curve. In particular, the expression

$$Ldu^2 + 2Mdudv + Ndv^2$$

is the *second fundamental form* of σ .

While the first fundamental form permits the calculation of metric properties such as length and area on a surface patch, the second fundamental form captures how 'curved' a surface patch is. The roles of the two fundamental forms on describing the local geometry are analogous to those of speed and acceleration for a parametric curve. Just as a unit-speed space curve is determined up to a rigid motion by its curvature and torsion, a surface patch is determined up to a rigid motion by its first and second fundamental forms.

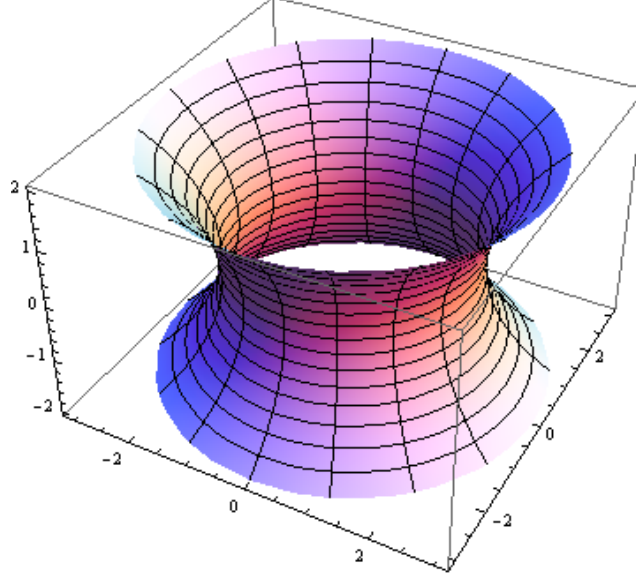
EXAMPLE 1. Consider a surface of revolution

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u)),$$

where $f(u) > 0$ always holds. The figure below plots a catenoid where

$$f(u) = 2 \cosh\left(\frac{u}{2}\right) \quad \text{and} \quad g(u) = u$$

with $(u, v) \in [-2, 2] \times [0, 2\pi]$.



Assume the profile curve $u \rightarrow (f(u), 0, g(u))$ is unit-speed, i.e., $\dot{f}^2 + \dot{g}^2 = 1$, where a dot denotes differentiation with respect to u . We perform the following calculations:

$$\begin{aligned} \sigma_u &= (\dot{f} \cos v, \dot{f} \sin v, \dot{g}), \\ \sigma_v &= (-f \sin v, f \cos v, 0), \\ E &= \sigma_u \cdot \sigma_u = \dot{f}^2 + \dot{g}^2 = 1, \\ F &= \sigma_u \cdot \sigma_v = 0, \\ G &= \sigma_v \cdot \sigma_v = f^2, \\ \sigma_u \times \sigma_v &= (-f \dot{g} \cos v, -f \dot{g} \sin v, f \dot{f}), \\ \|\sigma_u \times \sigma_v\| &= f, \\ \hat{n} &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (-\dot{g} \cos v, -\dot{g} \sin v, \dot{f}), \\ \sigma_{uu} &= (\ddot{f} \cos v, \ddot{f} \sin v, \ddot{g}), \\ \sigma_{uv} &= (-\dot{f} \sin v, \dot{f} \cos v, 0), \\ \sigma_{vv} &= (-f \cos v, -f \sin v, 0), \\ L &= \sigma_{uu} \cdot \hat{n} = \dot{f} \ddot{g} - \ddot{f} \dot{g}, \\ M &= \sigma_{uv} \cdot \hat{n} = 0, \\ N &= \sigma_{vv} \cdot \hat{n} = f \dot{g}. \end{aligned}$$

Thus, the second fundamental form is

$$(\dot{f} \ddot{g} - \ddot{f} \dot{g}) du^2 + f \dot{g} dv^2.$$

4 Principal Curvatures

Let $\gamma(t) = \sigma(u(t), v(t))$ be a unit-speed curve on a surface patch σ with the standard unit normal $\hat{\mathbf{n}}$. Below we obtain the normal curvature of the curve:

$$\begin{aligned}
 \kappa_n &= \ddot{\gamma} \cdot \hat{\mathbf{n}} \\
 &= \frac{d}{dt} \dot{\gamma} \cdot \hat{\mathbf{n}} \\
 &= \hat{\mathbf{n}} \cdot \frac{d}{dt} (\sigma_u \dot{u} + \sigma_v \dot{v}) \\
 &= \hat{\mathbf{n}} \cdot (\sigma_u \ddot{u} + \sigma_v \ddot{v} + (\sigma_{uu} \dot{u} + \sigma_{uv} \dot{v}) \dot{u} + (\sigma_{uv} \dot{u} + \sigma_{vv} \dot{v}) \dot{v}) \\
 &= L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2,
 \end{aligned} \tag{8}$$

where L, M, N are coefficients of the second fundamental form defined (7). In the last step above, we used the fact that $\hat{\mathbf{n}}$ is normal to both σ_u and σ_v , i.e., $\hat{\mathbf{n}} \cdot \sigma_u = \hat{\mathbf{n}} \cdot \sigma_v = 0$.

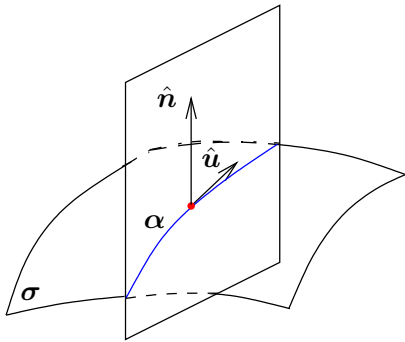
Equation (8) states that the normal curvature of $\gamma(t)$ depends on u, v, \dot{u} , and \dot{v} . The first two quantities specify the location of the point on the surface σ , and are thus curve independent. Let $\hat{\mathbf{u}}$ be the unit tangent vector so that $\dot{\gamma} = \hat{\mathbf{u}}$. Take the dot products of the equation $\hat{\mathbf{u}} = \sigma_u \dot{u} + \sigma_v \dot{v}$ with σ_u and σ_v separately, yielding

$$\begin{aligned}
 E\dot{u} + F\dot{v} &= \hat{\mathbf{u}} \cdot \sigma_u, \\
 F\dot{u} + G\dot{v} &= \hat{\mathbf{u}} \cdot \sigma_v,
 \end{aligned}$$

where E, F, G are the coefficients of the first fundamental form of the surface patch. Since σ_u and σ_v are linearly independent, the coefficient matrix in the above linear system in \dot{u} and \dot{v} is non-singular. We solve the system to obtain

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \hat{\mathbf{u}} \cdot \sigma_u \\ \hat{\mathbf{u}} \cdot \sigma_v \end{pmatrix}$$

This implies that \dot{u} and \dot{v} are independent of the parametrization of γ . By (8), we conclude that *any two unit-speed curves passing through the same point in the same direction $\hat{\mathbf{u}}$ must have the same normal curvature at this point.*



Subsequently, we refer to κ_n as the *normal curvature* of the surface σ at the point $\mathbf{p} = \sigma(u, v)$ in the tangent direction of $\hat{\mathbf{u}}$. It measures the curving of the surface in that direction. Generally, the surface bends at different rates in different tangent directions. The tangent $\hat{\mathbf{u}}$ and the normal $\hat{\mathbf{n}}$ at \mathbf{p} defines a plane that cuts a curve α out of the patch. This curve is called the *normal section* of σ in the $\hat{\mathbf{u}}$ direction. Since the principal normal $\hat{\mathbf{n}}_\gamma$ of the normal section is related to the surface normal $\hat{\mathbf{n}}$ by $\hat{\mathbf{n}}_\gamma = \pm \hat{\mathbf{n}}$. Equation (3) implies that the curvature of the normal section is the normal curvature κ_n at the point or its opposite, depending on the choice of the surface normal $\hat{\mathbf{n}}$ at the point.

To analyze the normal curvature κ_n further, we make use of the first and second fundamental forms:

$$Edu^2 + 2Fdudv + Gdv^2 \quad \text{and} \quad Ldu^2 + 2Mdudv + Ndv^2.$$

For convenience, we introduce two symmetric matrices

$$\mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad \mathcal{F}_2 = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

The tangent vector of the unit-speed curve $\gamma(t) = \sigma(u(t), v(t))$ is $\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$. Introducing $T = (\dot{u}, \dot{v})^t$, where t denotes the transpose operator, equation (8) can be rewritten as

$$\kappa_n = T^t \mathcal{F}_2 T. \quad (9)$$

Since σ_u and σ_v span the tangent plane, consider two tangent vectors:

$$\hat{t}_1 = \xi_1 \sigma_u + \eta_1 \sigma_v \quad \text{and} \quad \hat{t}_2 = \xi_2 \sigma_u + \eta_2 \sigma_v.$$

We obtain their inner product:

$$\begin{aligned} \hat{t}_1 \cdot \hat{t}_2 &= (\xi_1 \sigma_u + \eta_1 \sigma_v) \cdot (\xi_2 \sigma_u + \eta_2 \sigma_v) \\ &= E\xi_1\xi_2 + F(\xi_1\eta_2 + \xi_2\eta_1) + G\eta_1\eta_2 \\ &= T_1^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} T_2, \end{aligned} \quad (10)$$

where

$$T_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}.$$

The *principal curvatures* of the surface patch σ are the roots of the equation

$$\det(\mathcal{F}_2 - \kappa \mathcal{F}_1) = \begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{vmatrix} = 0. \quad (11)$$

From the linear independence of σ_u and σ_v , it is easy to show that matrix \mathcal{F}_1 is always invertible. Equation (11) essentially states that *the principal curvatures are the eigenvalues of $\mathcal{F}_1^{-1}\mathcal{F}_2$* .

Let κ be a principal curvature of $\mathcal{F}_1^{-1}\mathcal{F}_2$, and $T = (\xi, \eta)^T$ the corresponding eigenvector. That $(\mathcal{F}_1^{-1}\mathcal{F}_2)T = \kappa T$ implies

$$(\mathcal{F}_2 - \kappa \mathcal{F}_1)T = 0. \quad (12)$$

The unit tangent vector \hat{t} in the direction of $\xi\sigma_u + \eta\sigma_v$ is called the *principal vector* corresponding to the principal curvature κ .

Theorem 1 *Let κ_1 and κ_2 be the principal curvatures at a point \mathbf{p} of a surface patch σ . Then*

- (i) $\kappa_1, \kappa_2 \in \mathbb{R}$;
- (ii) if $\kappa_1 = \kappa_2 = \kappa$, then $\mathcal{F}_2 = \kappa \mathcal{F}_1$ and every tangent vector at \mathbf{p} is a principal vector.
- (iii) if $\kappa_1 \neq \kappa_2$, then the two corresponding principal vectors are perpendicular to each other

For a proof of the theorem, we refer the reader to [2, 133–135]. Intuitively, the principal vectors give the directions of maximum and minimum bending of the surface at the point, and the principal curvatures measure the bending rates. In case (ii), the point is *umbilic*. In this case, the surface bends the same amount in all directions at \mathbf{p} (thus all directions are principal).

EXAMPLE 2. A sphere bends the same amount in every direction. Take the unit sphere in Example 9 in the notes “Surfaces”, for instance, with the parametrization

$$\boldsymbol{\sigma}(\theta, \phi) = (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta),$$

We found previously that

$$E = 1, \quad F = 0, \quad G = \cos^2 \theta.$$

Since a sphere is a surface of revolution, we can plug in the result from Example 1, with $f(\theta) = \cos \theta$ and $g(u) = \sin \theta$:

$$\begin{aligned} L &= \dot{f}\ddot{g} - \ddot{f}\dot{g} \\ &= -\sin \theta(-\sin \theta) - (-\cos \theta)\cos \theta \\ &= 1, \\ M &= 0, \\ N &= f\dot{g} \\ &= \cos^2 \theta. \end{aligned}$$

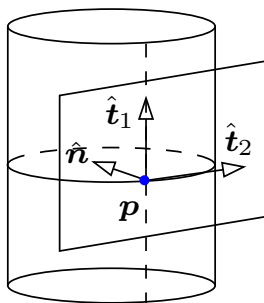
Hence the principal curvatures are the roots of

$$\det(\mathcal{F}_2 - \kappa \mathcal{F}_1) = \begin{vmatrix} 1 - \kappa & 0 \\ 0 & \cos^2 \theta - \kappa \cos^2 \theta \end{vmatrix} = 0.$$

Hence $\kappa = 1$. And every tangent direction is a principal vector.

EXAMPLE 3. Consider a cylinder with the z -axis as its axis and circular cross sections of unit radius. The parametrization is given as

$$\boldsymbol{\sigma}(u, v) = (\cos v, \sin v, u).$$



The coefficients of the first and second fundamental forms can be computed as

$$E = 1, \quad F = 0, \quad G = 1, \quad L = 0, \quad M = 0, \quad N = 1.$$

The principal curvatures are roots of

$$\begin{vmatrix} 0 - \kappa & 0 \\ 0 & 1 - \kappa \end{vmatrix}$$

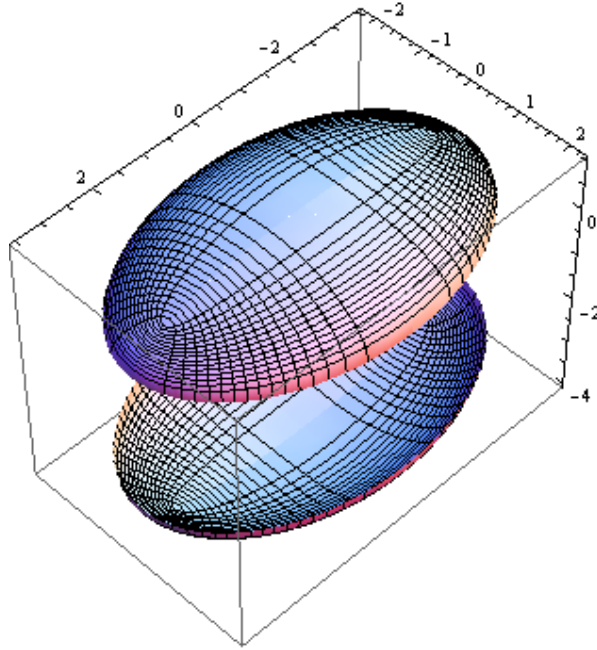
So we obtain $\kappa_1 = 0$ and $\kappa_2 = 1$. The eigenvectors $T_i = (\xi_i, \eta_i)$, $i = 1, 2$ of $\mathcal{F}_1^{-1}\mathcal{F}_2$ are found from solving the equation

$$(\mathcal{F}_2 - \kappa_i \mathcal{F}_1)T_i = 0.$$

The results are $T_1 = \lambda_1(1, 0)^t$ and $T_2 = \lambda_2(0, 1)^t$ for any non-zero $\lambda_1, \lambda_2 \in \mathbb{R}$. Hence the principal vector \hat{t}_1 is along the direction of $1\sigma_u + 0\sigma_v$, i.e., $\hat{t}_1 = (0, 0, 1)$. The principal vector \hat{t}_2 is along the direction of $0\sigma_u + 1\sigma_v = (-\sin v, \cos v, 0)$, i.e., $\hat{t}_2 = (-\sin v, \cos v, 0)$.

A curve γ on the surface σ is a *principal curve* if its velocity γ' always points in a *principal direction*, that is, the direction of a principal vector. At every point on a principal curve, the normal curvature is a maximum or minimum. The next figure shows some principal curves on the ellipsoid

$$\frac{x^2}{12} + \frac{y^2}{5} + z^2 = 1.$$



5 Euler's Formula

Suppose the two principal curvatures $\kappa_1 \neq \kappa_2$ at \mathbf{p} on the surface σ . Then by Theorem 1(iii), the two corresponding principal vectors $\hat{t}_1 = \xi_1\sigma_u + \eta_1\sigma_v$ and $\hat{t}_2 = \xi_2\sigma_u + \eta_2\sigma_v$ must be orthogonal to each other. Denote by $T_1 = (\xi_1, \eta_1)^t$ and $T_2 = (\xi_2, \eta_2)^t$. Replace the T in (12) with T_j , $j = 1, 2$, multiply both sides of the equation by T_i^t to the left, and move the second resulting term to the right hand side of the equation. This yields

$$T_i^t \mathcal{F}_2 T_j = \kappa_j T_i^t \mathcal{F}_1 T_j, \quad i, j = 1, 2.$$

Meanwhile, the orthogonality of the two principal vectors implies that

$$\hat{t}_1 \cdot \hat{t}_2 = T_1^t \mathcal{F}_1 T_2 = 0, \quad \text{from (10).}$$

To summarize, we have

$$T_i^t \mathcal{F}_2 T_j = \kappa_i T_i^t \mathcal{F}_1 T_j = \begin{cases} \kappa_i, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases} \quad (13)$$

With the principal curvatures and vectors at \mathbf{p} , we can evaluate the normal curvature in any direction.

Theorem 2 *Let κ_1, κ_2 be the principal curvatures, and $\hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2$ the two corresponding principal vectors of a patch σ at \mathbf{p} . The normal curvature of σ in the direction $\hat{\mathbf{u}} = \cos \theta \hat{\mathbf{t}}_1 + \sin \theta \hat{\mathbf{t}}_2$ is*

$$\kappa_n = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta.$$

Proof Let $\hat{\mathbf{u}} = \xi \sigma_u + \eta \sigma_v$ and $T = (\xi, \eta)^t$. We first look at the special case $\kappa_1 = \kappa_2 = \kappa$. By Theorem 1(ii), $\hat{\mathbf{u}} = \xi \sigma_u + \eta \sigma_v$ is a principal vector. The normal curvature in the direction $\hat{\mathbf{u}}$ is

$$\begin{aligned} \kappa_n &= T^t \mathcal{F}_2 T && \text{(by (9))} \\ &= \kappa T^t \mathcal{F}_1 T && \text{(by (12))} \\ &= \kappa \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \kappa. \end{aligned} \quad (14)$$

Meanwhile, we have

$$\kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta = \kappa (\cos^2 \theta + \sin^2 \theta) = \kappa.$$

So the theorem holds when the point is umbilic.

Assume $\kappa_1 \neq \kappa_2$. Therefore by Theorem 1(iii), $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$ are perpendicular to each other. Let

$$\hat{\mathbf{t}}_i = \xi_i \sigma_u + \eta_i \sigma_v, \quad \text{and} \quad T_i = (\xi_i, \eta_i)^t.$$

Thus,

$$\begin{aligned} \hat{\mathbf{u}} &= \cos \theta (\xi_1 \sigma_u + \eta_1 \sigma_v) + \sin \theta (\xi_2 \sigma_u + \eta_2 \sigma_v) \\ &= (\xi_1 \cos \theta + \xi_2 \sin \theta) \sigma_u + (\eta_1 \cos \theta + \eta_2 \sin \theta) \sigma_v. \end{aligned}$$

So we have $\hat{\mathbf{u}} = \xi \sigma_u + \eta \sigma_v$, where

$$\begin{aligned} \xi &= \xi_1 \cos \theta + \xi_2 \sin \theta, \\ \eta &= \eta_1 \cos \theta + \eta_2 \sin \theta, \end{aligned}$$

The above is written succinctly as $T = \cos \theta T_1 + \sin \theta T_2$. By equation (9) the normal curvature in the $\hat{\mathbf{u}}$ direction is

$$\begin{aligned} \kappa_n &= (\cos \theta T_1^t + \sin \theta T_2^t) \mathcal{F}_2 (\cos \theta T_1 + \sin \theta T_2) \\ &= \cos^2 \theta T_1^t \mathcal{F}_2 T_1 + \cos \theta \sin \theta (T_1^t \mathcal{F}_2 T_2 + T_2^t \mathcal{F}_2 T_1) + \sin^2 \theta T_2^t \mathcal{F}_2 T_2 \\ &= \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta. \end{aligned}$$

The last step above followed from the equation (13). \square

Theorem 2 implies that κ_1 and κ_2 are the maximum and minimum of any normal curvatures at the point. Equivalently, among all tangent directions at the point, the geometry varies the most in one principal direction while the least in the other.

References

- [1] B. O'Neill. *Elementary Differential Geometry*. Academic Press, Inc., 1966.
- [2] A. Pressley. *Elementary Differential Geometry*. Springer-Verlag London, 2001.