Principal Curvatures*

(COMS 4770/5770 Notes)

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1 Definition

To further analyze the normal curvature κ_n, we make use of the first and second fundamental forms:

$$Edu^2 + 2Fdudv + Gdv^2 \qquad \text{and} \qquad Ldu^2 + 2Mdudv + Ndv^2.$$

For convenience, we introduce two symmetric matrices

$$\mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \text{and} \qquad \mathcal{F}_2 = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

The tangent vector of the unit-speed curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ is $\dot{\boldsymbol{\gamma}} = \dot{u}\boldsymbol{\sigma}_u + \dot{v}\boldsymbol{\sigma}_v$. Introducing $T = (\dot{u}, \dot{v})^\top$, where $\top$ denotes the transpose operator, the normal curvature equation

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$$

can be rewritten as

$$\kappa_n = T^\top \mathcal{F}_2 T. \tag{1}$$

Since $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ span the tangent plane, consider two tangent vectors:

$$\hat{\boldsymbol{t}}_1 = \xi_1\boldsymbol{\sigma}_u + \eta_1\boldsymbol{\sigma}_v \qquad \text{and} \qquad \hat{\boldsymbol{t}}_2 = \xi_2\boldsymbol{\sigma}_u + \eta_2\boldsymbol{\sigma}_v.$$

We obtain their inner product:

$$\begin{aligned} \hat{\boldsymbol{t}}_1 \cdot \hat{\boldsymbol{t}}_2 &= (\xi_1\boldsymbol{\sigma}_u + \eta_1\boldsymbol{\sigma}_v) \cdot (\xi_2\boldsymbol{\sigma}_u + \eta_2\boldsymbol{\sigma}_v) \\ &= E\xi_1\xi_2 + F(\xi_1\eta_2 + \xi_2\eta_1) + G\eta_1\eta_2 \\ &= \boldsymbol{\nu}_1^\top \begin{pmatrix} E & F \\ F & G \end{pmatrix} \boldsymbol{\nu}_2, \end{aligned} \tag{2}$$

where

$$\boldsymbol{\nu}_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\nu}_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}.$$

*The material is adapted from the book *Elementary Differential Geometry* by Andrew Pressley, Springer-Verlag, 2001.

The *principal curvatures* of the surface patch $\boldsymbol{\sigma}$ are the roots of the equation

$$\det(\mathcal{F}_2 - \kappa\mathcal{F}_1) = \begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{vmatrix} = 0. \tag{3}$$

From the linear independence of $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$, it is easy to show that matrix $\mathcal{F}_1$ is always invertible. Equation (3) essentially states that *the principal curvatures are the eigenvalues of* $\mathcal{F}_1^{-1}\mathcal{F}_2$.

Let κ be a principal curvature of $\mathcal{F}_1^{-1}\mathcal{F}_2$, and $T = (\xi, \eta)^\top$ the corresponding eigenvector. That $(\mathcal{F}_1^{-1}\mathcal{F}_2)T = \kappa T$ implies

$$(\mathcal{F}_2 - \kappa\mathcal{F}_1)T = 0. \tag{4}$$

The unit tangent vector $\hat{\boldsymbol{t}}$ in the direction of $\xi\boldsymbol{\sigma}_u + \eta\boldsymbol{\sigma}_v$ is called the *principal vector* corresponding to the principal curvature κ.

Theorem 1 *Let κ_1 and κ_2 be the principal curvatures at a point $\boldsymbol{p}$ of a surface patch $\boldsymbol{\sigma}$. Then*

(i) $\kappa_1, \kappa_2 \in \mathbb{R}$*;*

(ii) *if $\kappa_1 = \kappa_2 = \kappa$, then $\mathcal{F}_2 = \kappa\mathcal{F}_1$ and every tangent vector at $\boldsymbol{p}$ is a principal vector.*

(iii) *if $\kappa_1 \neq \kappa_2$, then the two corresponding principal vectors are perpendicular to each other*

For a proof of the theorem, we refer the reader to [2, 133–135]. Intuitively, the principal vectors give the directions of maximum and minimum bending of the surface at the point $\boldsymbol{p}$, and the principal curvatures measure the bending rates. In the case (ii), the point is *umbilic*, as the surface bends the same amount in all directions at $\boldsymbol{p}$ (thus all directions are principal). In the case (iii), because $\kappa_1 \neq \kappa_2$, the corresponding eigenvectors $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ of the matrix $\mathcal{F}_1^{-1}\mathcal{F}_2$ are orthogonal in addition to the principal vectors $\hat{\boldsymbol{t}}_1 = (\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v)\boldsymbol{\nu}_1$ and $\hat{\boldsymbol{t}}_1 = (\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v)\boldsymbol{\nu}_2$ being orthogonal.

Example 2. A sphere bends the same amount in every direction. Take the unit sphere in Example 9 in the notes "Surfaces", for instance, with the parametrization

$$\boldsymbol{\sigma}(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta),$$

We found previously that

$$E = 1, \qquad F = 0, \qquad G = \cos^2\theta.$$

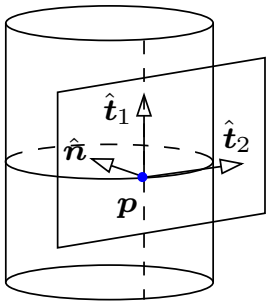
Since a sphere is a surface of revolution, we can plug in the result from Example 1, with $f(\theta) = \cos\theta$ and $g(u) = \sin\theta$:

$$\begin{aligned} L &= \dot{f}\ddot{g} - \ddot{f}\dot{g} \\ &= -\sin\theta(-\sin\theta) - (-\cos\theta)\cos\theta \\ &= 1, \\ M &= 0, \\ N &= f\dot{g} \\ &= \cos^2\theta. \end{aligned}$$

Hence the principal curvatures are the roots of

$$\det(\mathcal{F}_2 - k\mathcal{F}_1) = \begin{vmatrix} 1 - \kappa & 0 \\ 0 & \cos^2\theta - \kappa\cos^2\theta \end{vmatrix} = 0.$$

Hence $\kappa = 1$. And every tangent direction is a principal vector.



EXAMPLE 3. Consider a cylinder with the z-axis as its axis and circular cross sections of unit radius. The parametrization is given as

$$\boldsymbol{\sigma}(u, v) = (\cos v, \sin v, u).$$

We have

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (0, 0, 1), \\
\boldsymbol{\sigma}_v &= (-\sin v, \cos v, 0), \\
\boldsymbol{\sigma}_{uu} &= (0, 0, 0), \\
\boldsymbol{\sigma}_{uv} &= (0, 0, 0), \\
\boldsymbol{\sigma}_{vv} &= (-\cos v, -\sin v, 0), \\
\hat{\boldsymbol{n}} &= \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v / \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| \;=\; (-\cos v, -\sin v, 0).
\end{aligned}$$

The coefficients of the first and second fundamental forms can be computed as

$$E = 1, \qquad F = 0, \qquad G = 1, \qquad L = 0, \qquad M = 0, \qquad N = 1.$$

The principal curvatures are roots of

$$\begin{vmatrix} 0 - \kappa & 0 \\ 0 & 1 - \kappa \end{vmatrix}$$

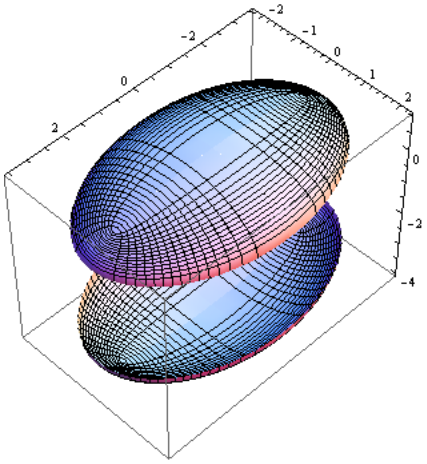
So we obtain $\kappa_1 = 0$ and $\kappa_2 = 1$. The eigenvectors $\boldsymbol{\nu}_i = (\xi_i, \eta_i)^\top$, $i = 1, 2$ of $\mathcal{F}_1^{-1}\mathcal{F}_2$ are found from solving the equation

$$(\mathcal{F}_2 - \kappa_i \mathcal{F}_1)\boldsymbol{\nu}_i = 0.$$

The results are $\boldsymbol{\nu}_1 = a_1(1, 0)^\top$ and $\boldsymbol{\nu}_2 = a_2(0, 1)^\top$ for any non-zero $a_1, a_2 \in \mathbb{R}$. Hence the principal vector $\hat{\boldsymbol{t}}_1$ is along the direction of $1\boldsymbol{\sigma}_u + 0\boldsymbol{\sigma}_v$, i.e., $\hat{\boldsymbol{t}}_1 = (0, 0, 1)$. The principal vector $\hat{\boldsymbol{t}}_2$ is along the direction of $0\boldsymbol{\sigma}_u + 1\boldsymbol{\sigma}_v = (-\sin v, \cos v, 0)$, i.e., $\hat{\boldsymbol{t}}_2 = (-\sin v, \cos v, 0)$.

Two orthogonal principal vectors $\hat{\boldsymbol{t}}_1$ and $\hat{\boldsymbol{t}}_2$, together with the unit surface normal $\hat{\boldsymbol{n}}$ such that $\hat{\boldsymbol{t}}_1 \times \hat{\boldsymbol{t}}_2 = \hat{\boldsymbol{n}}$, form the *Darboux frame* of the surface at $\boldsymbol{p}$. When $\kappa_1 \neq \kappa_2$, this frame is unique up to the two choices of $\hat{\boldsymbol{n}}$ and up to the choices of the principal directions (each out of two opposing directions). It is different from the earlier introduced Darboux frame at $\boldsymbol{p}$ that is attached to an embedded curve passing through the point. The latter frame varies with the choice of curve.

A curve $\boldsymbol{\gamma}$ on the surface $\boldsymbol{\sigma}$ is a *principal curve* or a *line of curvature* if its velocity $\boldsymbol{\gamma}'$ always points in a *principal direction*, that is, the direction of a principal vector. At every point on a principal curve, the normal curvature is a maximum or minimum. The figure on the right shows some principal curves on the ellipsoid $x^2/12 + y^2/5 + z^2 = 1$.



2 Euler's Formula

Suppose the two principal curvatures $\kappa_1 \neq \kappa_2$ at $\boldsymbol{p}$ on the surface $\boldsymbol{\sigma}$. Then by Theorem 1(iii), the two corresponding principal vectors $\hat{\boldsymbol{t}}_1 = \xi_1\boldsymbol{\sigma}_u + \eta_1\boldsymbol{\sigma}_v$ and $\hat{\boldsymbol{t}}_2 = \xi_2\boldsymbol{\sigma}_u + \eta_2\boldsymbol{\sigma}_v$ must be orthogonal to

each other. Denote by $\boldsymbol{\nu}_1 = (\xi_1, \eta_1)^\top$ and $\boldsymbol{\nu}_2 = (\xi_2, \eta_2)^\top$. Replace the T in (4) with $\boldsymbol{\nu}_j$, $j = 1, 2$, multiply both sides of the equation by $\boldsymbol{\nu}_i^\top$ to the left, and move the second resulting term to the right hand side of the equation. This yields

$$\boldsymbol{\nu}_i^\top \mathcal{F}_2 \boldsymbol{\nu}_j = \kappa_j \boldsymbol{\nu}_i^\top \mathcal{F}_1 \boldsymbol{\nu}_j, \qquad i, j = 1, 2.$$

Meanwhile, the orthogonality of the two principal vectors implies that

$$\hat{\boldsymbol{t}}_1 \cdot \hat{\boldsymbol{t}}_2 = \boldsymbol{\nu}_1^\top \mathcal{F}_1 \boldsymbol{\nu}_2 = 0, \qquad \text{from (2)}.$$

To summarize, we have

$$\boldsymbol{\nu}_i^\top \mathcal{F}_2 \boldsymbol{\nu}_j = \kappa_i \boldsymbol{\nu}_i^\top \mathcal{F}_1 \boldsymbol{\nu}_j = \begin{cases} \kappa_i, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

With the principal curvatures and vectors at $\boldsymbol{p}$, we can evaluate the normal curvature in any direction.

Theorem 2 *Let κ_1, κ_2 be the principal curvatures, and $\hat{\boldsymbol{t}}_1, \hat{\boldsymbol{t}}_2$ the two corresponding principal vectors of a patch $\boldsymbol{\sigma}$ at $\boldsymbol{p}$. The normal curvature of $\boldsymbol{\sigma}$ in the direction $\hat{\boldsymbol{u}} = \cos\theta \hat{\boldsymbol{t}}_1 + \sin\theta \hat{\boldsymbol{t}}_2$ is*

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

Proof Let $\hat{\boldsymbol{u}} = \xi \boldsymbol{\sigma}_u + \eta \boldsymbol{\sigma}_v$ and $\boldsymbol{\nu} = (\xi, \eta)^\top$. We first look at the special case $\kappa_1 = \kappa_2 = \kappa$. By Theorem 1(ii), $\hat{\boldsymbol{u}} = \xi \boldsymbol{\sigma}_u + \eta \boldsymbol{\sigma}_v$ is a principal vector. The normal curvature in the direction $\hat{\boldsymbol{u}}$ is

$$\begin{aligned} \kappa_n &= \boldsymbol{\nu}^\top \mathcal{F}_2 \boldsymbol{\nu} && \text{(by (1))} \\ &= \kappa \boldsymbol{\nu}^\top \mathcal{F}_1 \boldsymbol{\nu} && \text{(by (4))} \\ &= \kappa \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}} = \kappa. \end{aligned} \tag{6}$$

Meanwhile, we have

$$\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta = \kappa(\cos^2\theta + \sin^2\theta) = \kappa.$$

So the theorem holds when the point is umbilic.

Assume $\kappa_1 \neq \kappa_2$. Therefore by Theorem 1(iii), $\hat{\boldsymbol{t}}_1$ and $\hat{\boldsymbol{t}}_2$ are perpendicular to each other. Let

$$\hat{\boldsymbol{t}}_i = \xi_i \boldsymbol{\sigma}_u + \eta_i \boldsymbol{\sigma}_v, \qquad \text{and} \qquad \boldsymbol{\nu}_i = (\xi_i, \eta_i)^\top.$$

Thus,

$$\begin{aligned} \hat{\boldsymbol{u}} &= \cos\theta(\xi_1 \boldsymbol{\sigma}_u + \eta_1 \boldsymbol{\sigma}_v) + \sin\theta(\xi_2 \boldsymbol{\sigma}_u + \eta_2 \boldsymbol{\sigma}_v) \\ &= (\xi_1 \cos\theta + \xi_2 \sin\theta)\boldsymbol{\sigma}_u + (\eta_1 \cos\theta + \eta_2 \sin\theta)\boldsymbol{\sigma}_v. \end{aligned}$$

So we have $\hat{\boldsymbol{u}} = \xi \boldsymbol{\sigma}_u + \eta \boldsymbol{\sigma}_v$, where

$$\begin{aligned} \xi &= \xi_1 \cos\theta + \xi_2 \sin\theta, \\ \eta &= \eta_1 \cos\theta + \eta_2 \sin\theta, \end{aligned}$$

The above is written succinctly as $\boldsymbol{\nu} = \cos\theta\boldsymbol{\nu}_1 + \sin\theta\boldsymbol{\nu}_2$. By equation (1) the normal curvature in the $\hat{\boldsymbol{u}}$ direction is

$$\begin{aligned}\kappa_n &= (\cos\theta\boldsymbol{\nu}_1^\top + \sin\theta\boldsymbol{\nu}_2^\top)\mathcal{F}_2(\cos\theta\boldsymbol{\nu}_1 + \sin\theta\boldsymbol{\nu}_2)\\ &= \cos^2\theta\,\boldsymbol{\nu}_1^\top\mathcal{F}_2\boldsymbol{\nu}_1 + \cos\theta\sin\theta\,(\boldsymbol{\nu}_1^\top\mathcal{F}_2\boldsymbol{\nu}_2 + \boldsymbol{\nu}_2^\top\mathcal{F}_2\boldsymbol{\nu}_1) + \sin^2\theta\,\boldsymbol{\nu}_2^\top F_2\boldsymbol{\nu}_2\\ &= \kappa_1\cos^2\theta + \kappa_2\sin^2\theta.\end{aligned}$$

The last step above followed from the equation (5). □

Theorem 2 implies that κ_1 and κ_2 are the maximum and minimum of any normal curvatures at the point. Equivalently, among all tangent directions at the point, the geometry varies the most in one principal direction while the least in the other.

3 Geometric Interpretation of Principal Curvatures

In this section, we look at how the local shape at a surface point can be approximated using its principal curvatures and direction. The values of the principal curvatures and vectors at a point $\boldsymbol{p}$ on a surface patch $\boldsymbol{\sigma}$ tell us about the shape near $\boldsymbol{p}$. To see this, we apply a rigid motion followed by a reparametrization.[1] More specifically, we move the origin to $\boldsymbol{p}$ and let the tangent plane to $\boldsymbol{\sigma}$ at $\boldsymbol{p}$ be the xy-plane with the x-axis and y-axis along the directions of the two principal vectors, which correspond to principal curvatures κ_1 and κ_2, respectively. Furthermore, we let the values of both parameters at the origin be zero, that is,

$$\boldsymbol{\sigma}(0,0) = \mathbf{0}. \tag{7}$$

Without any ambiguity, we still denote the new parametrization by $\boldsymbol{\sigma}$.

Let us determine the function $z = z(x,y)$ that describes the local shape. The unit principal vectors can be expressed in terms of the partial derivatives:

$$\begin{aligned}(1,0,0) &= \xi_1\boldsymbol{\sigma}_u + \eta_1\boldsymbol{\sigma}_v,\\ (0,1,0) &= \xi_2\boldsymbol{\sigma}_u + \eta_2\boldsymbol{\sigma}_v.\end{aligned}$$

So can any point $(x,y,0)$ in the tangent plane:

$$\begin{aligned}(x,y,0) &= x(1,0,0) + y(0,1,0)\\ &= x(\xi_1\boldsymbol{\sigma}_u + \eta_1\boldsymbol{\sigma}_v) + y(\xi_2\boldsymbol{\sigma}_u + \eta_2\boldsymbol{\sigma}_v)\\ &= s\boldsymbol{\sigma}_u + t\boldsymbol{\sigma}_v,\end{aligned} \tag{8}$$

where

$$s = x\xi_1 + y\xi_2 \qquad \text{and} \qquad t = x\eta_1 + y\eta_2. \tag{9}$$

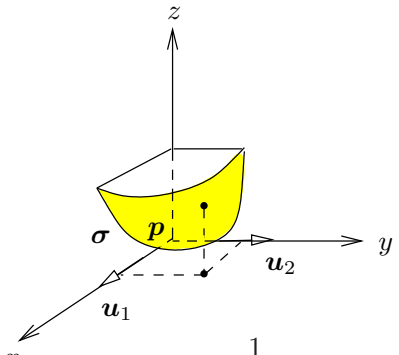
Let us evaluate $\boldsymbol{\sigma}(s,t)$ at the parameter values s and t, applying Taylor's theorem with higher order terms in s and t neglected:

$$\begin{aligned}\boldsymbol{\sigma}(s,t) &= \boldsymbol{\sigma}(0,0) + s\boldsymbol{\sigma}_u + t\boldsymbol{\sigma}_v + \frac{1}{2}(s^2\boldsymbol{\sigma}_{uu} + 2st\boldsymbol{\sigma}_{uv} + t^2\boldsymbol{\sigma}_{vv})\\ &= (x,y,0) + \frac{1}{2}(s^2\boldsymbol{\sigma}_{uu} + 2st\boldsymbol{\sigma}_{uv} + t^2\boldsymbol{\sigma}_{vv}), \qquad \text{(by (7) and (8))}\end{aligned}$$

[1] The shape does not change under any rigid motion or reparametrization.

All derivatives are evaluated at the origin $\boldsymbol{p}$. Neglecting the second order terms added to x and y, the coordinates of $\boldsymbol{\sigma}(s,t)$ is (x,y,z), where

$$\begin{aligned} z &= \boldsymbol{\sigma}(s,t)\cdot\hat{\boldsymbol{n}} \\ &= \frac{1}{2}(Ls^2+2Mst+Nt^2) \\ &= \frac{1}{2}(s\ \ t)\begin{pmatrix} L & M \\ M & N\end{pmatrix}\begin{pmatrix} s \\ t\end{pmatrix}. \end{aligned}$$



Writing

$$\boldsymbol{\nu}_1 = \begin{pmatrix}\xi_1 \\ \eta_1\end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\nu}_2 = \begin{pmatrix}\xi_2 \\ \eta_2\end{pmatrix},$$

we have from (9):

$$\begin{pmatrix} s \\ t\end{pmatrix} = x\boldsymbol{\nu}_1 + y\boldsymbol{\nu}_2.$$

Thus,

$$\begin{aligned} z &= \frac{1}{2}(x\boldsymbol{\nu}_1+y\boldsymbol{\nu}_2)^\top \mathcal{F}_2(x\boldsymbol{\nu}_1+y\boldsymbol{\nu}_2) \\ &= \frac{1}{2}\Big(x^2\boldsymbol{\nu}_1\mathcal{F}_2\boldsymbol{\nu}_1 + xy(\boldsymbol{\nu}_1^\top\mathcal{F}_2\boldsymbol{\nu}_2+\boldsymbol{\nu}_2^\top\mathcal{F}_2\boldsymbol{\nu}_1)+y^2\boldsymbol{\nu}_2^\top\mathcal{F}_2\boldsymbol{\nu}_2\Big) \\ &= \frac{1}{2}(\kappa_1x^2+\kappa_2y^2), \end{aligned}$$

since $\boldsymbol{\nu}_i^\top\mathcal{F}_2\boldsymbol{\nu}_j = \kappa_i$ if $i=j$ or 0 otherwise. Hence the shape of a surface near the point $\boldsymbol{p}$ has a *quadratic approximation* determined by its principal curvature κ_1 and κ_2. It is described by the equation $z=\frac{1}{2}(\kappa_1x^2+\kappa_2y^2)$.

4 Covariant Derivative

Next, we look at how to characterizes the rate of change of a vector defined on a surface with respect to a tangent vector. Let us slightly abuse the notation $\hat{\boldsymbol{n}}$ to represent a function that assigns to every point $\boldsymbol{p}$ on the surface S the normal $\hat{\boldsymbol{n}}(\boldsymbol{p})$ at the point. Since $\hat{\boldsymbol{n}}$ is continuous, it is a *vector field* on S, and referred to as the normal vector field. Similarly, $\hat{\boldsymbol{t}}_1$ and $\hat{\boldsymbol{t}}_2$ are also vector fields on S that continuously assign to every point two orthogonal principal vectors.

At the point $\boldsymbol{p}$, a vector field Z typically changes differently in different tangential directions. The rate of change along a tangent $\boldsymbol{w}$ is characterized by its *covariant derivative* along $\boldsymbol{w}$. More specifically, we let $\boldsymbol{\alpha}(t)$ be a curve on S that has initial velocity $\dot{\boldsymbol{\alpha}}(0)=\boldsymbol{w}$. Consider restriction of Z to $\boldsymbol{\alpha}$. Then, the covariant derivative of Z with respect to $\boldsymbol{w}$ is defined to be

$$\nabla_{\boldsymbol{w}}Z = \frac{dZ(\boldsymbol{\alpha}(t))}{dt}\Big|_{t=0}.$$

In particular, consider the u-curve $\boldsymbol{\alpha}(u) = \boldsymbol{\sigma}(u, v_0)$ passing through $\boldsymbol{p} = \boldsymbol{\sigma}(u_0, v_0)$ at velocity $\boldsymbol{w} = \boldsymbol{\sigma}_u(u_0, v_0)$. We have

$$\begin{aligned} \nabla_{\boldsymbol{w}} Z &= \frac{dZ(\boldsymbol{\alpha}(u))}{du}\Big|_{u=u_0} \\ &= \frac{dZ(\boldsymbol{\sigma}(u, v_0))}{du}\Big|_{u=u_0} \\ &= Z_u(u_0, v_0). \end{aligned}$$

Reparametrize $\boldsymbol{\alpha}(u)$ as a unit-speed curve $\boldsymbol{\beta}(s)$, where s is arc length. Clearly,

$$\frac{ds}{du}(0) = \|\dot{\boldsymbol{\alpha}}(u_0)\| = \|\boldsymbol{\sigma}_u(u_0, v_0)\|.$$

At $\boldsymbol{p}$, let $\hat{\boldsymbol{x}} = \dot{\boldsymbol{\beta}}(0) = \boldsymbol{\sigma}_u(u_0, v_0)/\|\boldsymbol{\sigma}_u(u_0, v_0)\|$ be the unit velocity of the u-curve. The covariant derivative with respect to $\hat{\boldsymbol{x}}$ is

$$\begin{aligned} \nabla_{\hat{\boldsymbol{x}}} Z &= \frac{dZ(\boldsymbol{\beta}(s))}{ds}\Big|_{s=0} \\ &= \frac{dZ(\boldsymbol{\alpha}(u(s)))/du}{ds/du}\Big|_{u=u_0} \\ &= \frac{Z_u(u_0, v_0)}{\|\boldsymbol{\sigma}_u(u_0, v_0)\|}. \end{aligned}$$

In the Darboux frame T-V-U at $\boldsymbol{p}$ of a unit-speed surface curve, where T is the curve tangent, U the unit surface normal $\hat{\boldsymbol{n}}$, and $V = U \times T$, it holds that $\dot{U} = -\kappa_n T - \tau_g V$, where κ_n and τ_g are the surface's normal curvature and curve's geodesic torsion at $\boldsymbol{p}$. Meanwhile, $\dot{U}$ is the covariant derivative along T, i.e., $\dot{U} = \nabla_T U$. The normal curvature at the point in the direction T is equivalently defined to be $k_n(T)$ [1, p. 196], for we have

$$\begin{aligned} \kappa_n(T) &= -\dot{U} \cdot T \\ &= -\nabla_T U \cdot T. \end{aligned}$$

The principal curvatures are the normal curvatures in the two principal directions, that is, the covariant derivatives of the normal with respect to the principal vectors:

$$\begin{aligned} \kappa_1 &= \kappa_n(\hat{\boldsymbol{t}}_1) = -\nabla_{\hat{\boldsymbol{t}}_1} U \cdot \hat{\boldsymbol{t}}_1 = -\nabla_{\hat{\boldsymbol{t}}_1} \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{t}}_1, \\ \kappa_2 &= \kappa_n(\hat{\boldsymbol{t}}_2) = -\nabla_{\hat{\boldsymbol{t}}_2} U \cdot \hat{\boldsymbol{t}}_2 = -\nabla_{\hat{\boldsymbol{t}}_2} \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{t}}_2. \end{aligned}$$

It can be shown that $-\nabla_{\hat{\boldsymbol{t}}_i} \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{t}}_j = 0$ if $i \neq j$.

References

[1] B. O'Neill. *Elementary Differential Geometry.* Academic Press, Inc., 1966.

[2] A. Pressley. *Elementary Differential Geometry.* Springer-Verlag London, 2001.