The Simplex Method*

(Com S 477/577 Notes).

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To understand how simplex solves a general linear programming problem, we need to know the answers to a number of questions:

- How to determine if a minimum feasible solution has been found?
- Which non-basic variable is to enter the basis at a pivot step?
- Which basic variable should become non-basic at a pivot step?
- How to find an initial basic feasible solution to start simplex?

We already had an answer to the first question, that is, checking if the coefficients of the new objective function are all nonnegative. Let us now look at the other three questions. For convenience, we assume that the following holds.

> *Nondegeneracy assumption*: In every basic feasible solution, no basic variable has zero value.[1]

Given a linear program in the standard form

$$\begin{array}{rl} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

Suppose after zero or a few steps we have a tableau of the following form:

$$\begin{array}{c|cccccccccc} & \boldsymbol{a}_1' & \boldsymbol{a}_2' & \cdots & \boldsymbol{a}_m' & \boldsymbol{a}_{m+1}' & \cdots & \boldsymbol{a}_k' & \cdots & \boldsymbol{a}_n' & \boldsymbol{b}' \\ \hline & 1 & 0 & \cdots & 0 & a_{1,m+1}' & \cdots & a_{1k}' & \cdots & a_{1n}' & b_1' \\ & 0 & 1 & \cdots & 0 & a_{2,m+1}' & \cdots & a_{2k}' & \cdots & a_{2n}' & b_2' \\ & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & \cdots & 1 & a_{m,m+1}' & \cdots & a_{mk}' & \cdots & a_{mn}' & b_m' \\ \hline \boldsymbol{c}'^T & c_1' & c_2' & \cdots & c_m' & c_{m+1}' & \cdots & c_k' & \cdots & c_n' & -\alpha \end{array}$$

*Most of the material is adapted from [1]

[1] When this assumption is violated, a degenerate basic variable (with zero value) occurs in a basic feasible solution. Often it can be handled as a nondegenerate basic feasible solution. However, it is possible that at pivoting the new variable will come in at zero value. This implies that the zero-valued basic variable is the one to go out. The objective will not decrease and the new basic feasible solution will also be degenerate. The result is a cycle that could be repeated indefinitely. Methods have been developed to avoid such cycles [1, pp. 78].

The tableau represents a solution with basic variables $x_1, x_2, \ldots, x_m$. We assume $b'_1, b'_2, \ldots, b'_m > 0$ so that the corresponding basic feasible solution $x_1 = b'_1, \ldots, x_m = b'_m$ satsifies the nondegeneracy assumption. Note that the tableau represents a general case where the coefficients $c'_1, c'_2, \ldots, c'_m$ associated with the basic variables are not necessarily zero.

1 Variable to Enter the Basis

The idea of the simplex method is to select a column to pivot so that the resulting new basic feasible solution will yield a lower value to the objective function than the previous one. Suppose the basic solution at the current pivot step is

$$(\boldsymbol{x}_B^T, \mathbf{0}^T) = (b'_1, b'_2, \ldots, b'_m, 0, \ldots, 0).$$

The objective function is

$$z = c'_1 x_1 + c'_2 x_2 + \cdots + c'_n x_n + \alpha. \tag{1}$$

For the basic solution, it has the value

$$\begin{aligned} z_0 &= \boldsymbol{c}_B^T \boldsymbol{x}_B + \alpha \\ &= c'_1 b'_1 + c'_2 b'_2 + \cdots + c'_m b'_m + \alpha, \end{aligned}$$

where $\boldsymbol{c}_B = (c'_1, c'_2, \ldots, c'_m)^T$.

If arbitrary values are assigned to $x_{m+1}, \ldots, x_n$, we can solve for the basic variables as

$$\begin{aligned} x_1 &= b'_1 - \sum_{j=m+1}^{n} a'_{1j} x_j, \\ x_2 &= b'_2 - \sum_{j=m+1}^{n} a'_{2j} x_j, \\ &\vdots \\ x_m &= b'_m - \sum_{j=m+1}^{n} a'_{mj} x_j. \end{aligned} \tag{2}$$

Substitute equations (2) into (1) to eliminate $x_1, x_2, \ldots, x_m$:

$$z = z_0 + (c'_{m+1} - w_{m+1}) x_{m+1} + \cdots + (c'_n - w_n) x_n, \tag{3}$$

where

$$w_j = a'_{1j} c'_1 + a'_{2j} c'_2 + \cdots + a'_{mj} c'_m, \qquad m+1 \le j \le n. \tag{4}$$

The above substitutions have equivalently transformed the tableau into the following:

$\boldsymbol{a}'_1$	$\boldsymbol{a}'_2$	$\cdots$	$\boldsymbol{a}'_m$	$\boldsymbol{a}'_{m+1}$	$\cdots$	$\boldsymbol{a}'_k$	$\cdots$	$\boldsymbol{a}'_n$	$\boldsymbol{b}'$
1	0	$\cdots$	0	$a'_{1,m+1}$	$\cdots$	a'_{1k}	$\cdots$	a'_{1n}	b'_1
0	1	$\cdots$	0	$a'_{2,m+1}$	$\cdots$	a'_{2k}	$\cdots$	a'_{2n}	b'_2
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\cdots$	1	$a'_{m,m+1}$	$\cdots$	a'_{mk}	$\cdots$	a'_{mn}	b'_m
0	0	$\cdots$	0	$c'_{m+1} - w_{m+1}$	$\cdots$	$c'_k - w_k$	$\ldots$	$c'_n - w_n$	$-z_0$

From (3) we can now determine if introducing one of the nonbasic variable would decrease the value of the objective function. More specifically, if $c'_k - w_k < 0$ for some k, $m+1 \leq k \leq n$, then making $x_k > 0$ would decrease the cost. Equations (2) will give the new values of $x_1, \ldots, x_m$ to accommodate the increase in x_k, as long as the new values of these variables do not go below zero.

2 Variable to Leave the Basis

Suppose that the nonbasic variable x_k, $m < k \leq n$, is to become basic and result in value changes to the other variables to maintain feasibility. We now determine which basic variable should become non-basic. From the previous tableau, we clearly see that

$$
\begin{aligned}
b'_1\boldsymbol{a}'_1 + b'_2\boldsymbol{a}'_2 + \cdots + b'_m\boldsymbol{a}'_m &= \boldsymbol{b}', && (5)\\
a'_{1k}\boldsymbol{a}'_1 + a'_{2k}\boldsymbol{a}'_2 + \cdots + a'_{mk}\boldsymbol{a}'_m &= \boldsymbol{a}'_k. && (6)
\end{aligned}
$$

We multiply (6) by some $\epsilon \geq 0$ and subtract from (5), obtaining

$$
(b'_1 - \epsilon a'_{1k})\boldsymbol{a}'_1 + (b'_2 - \epsilon a'_{2k})\boldsymbol{a}'_2 + \cdots + (b'_m - \epsilon a'_{mk})\boldsymbol{a}'_m + \epsilon\boldsymbol{a}'_k = \boldsymbol{b}'. \tag{7}
$$

Two cases will follow:

Case 1 : There exists $a'_{ik} > 0$ for some $1 \leq i \leq m$. Then we set

$$
\epsilon = \min_{1 \leq i \leq m} \left\{ \frac{b'_i}{a'_{ik}} \,\middle|\, a'_{ik} > 0 \right\}
$$

and pivot at a'_{jk}, where j is the minimizing index. Then we will have a new basic feasible solution with x_k replacing x_j such that the value of x_k increases from 0 to b'_j/a'_{jk}. Note that only one single index j can achieve ϵ. Otherwise, one basic variable would become zero after the pivoting is performed to bring x_k into the basis, thereby violating the nondegeneracy assumption.

Case 2: All $a'_{1k}, \ldots, a'_{mk}$ are negative. In this case, the coefficients in (7) increase with ϵ, and no new basic feasible solution is obtained. As a result, the solution of $A\boldsymbol{x} = \boldsymbol{b}$ can have arbitrarily large components. For instance, $x_1 = b'_1 - \epsilon a'_{1k}, \ldots$, $x_m = b'_m - \epsilon a'_{mk}$, $x_k = \epsilon$, and $x_j = 0$ for $m+1 \leq j \leq n$ and $j \neq k$, as ϵ becomes arbitrarily large. Consequently, x_k can have an arbitrarily large value. This results in an arbitrarily small value of the objective function $z = z_0 + (c'_k - w_k)x_k$, which is reduced from (3) since $x_j = 0$ for $m+1 \leq j \leq n$ and $j \neq k$.

Theorem 1 (Improvement of Basic Feasible Solution) *Given a nondegenerate basic feasible solution with corresponding objective value z_0, suppose $c'_k - z_k < 0$ for some k. Then there is a feasible solution with objective value $z < z_0$.*

1. *If the variable x_k can be substituted for some variable in the original basis to yield a new basic feasible solution, this new solution will have objective value $z < z_0$.*

2. *If x_k cannot be substituted to yield a basic feasible solution, then the solution set is unbounded and the objective function can be made arbitrarily small.*

In the above theorem, when x_k with the coefficient $c'_k - z_k < 0$ cannot be substituted into the basis, $a'_{1k}, \ldots, a'_{mk}$ must be negative from our reasoning.

Theorem 2 (Optimality Condition) *If for some basic feasible solution $c'_j - z_j \geq 0$ for all j, then that solution is optimal.*

3 Initializing the Simplex Method

For LP problems with constraints of the form

$$\begin{aligned} A\boldsymbol{x} &\leq \boldsymbol{b}, \qquad \text{with } \boldsymbol{b} \geq \boldsymbol{0}, \\ \boldsymbol{x} &\geq \boldsymbol{0}, \end{aligned}$$

a basic feasible solution to the corresponding standard form of the problem is provided by slack variables. This provides a means for initiating a simplex procedure. But initial basic feasible solutions are not always apparent for other types of LP problems. Interestingly, an auxiliary linear program will provide the required initial solution to the original linear program.

We consider the constraints of an LP problem in the standard form:

$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b}, \qquad \text{with } \boldsymbol{b} \geq \boldsymbol{0}, \\ \boldsymbol{x} &\geq \boldsymbol{0}, \end{aligned} \tag{8}$$

In order to find an initial basic feasible solution, consider the minimization problem

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} y_i \\ \text{subject to} \quad & A\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}, \\ & \boldsymbol{y} \geq \boldsymbol{0}, \end{aligned} \tag{9}$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)^T$ is a vector of *artificial variables*.

It is clear that

- if (8) has a feasible solution, then (9) can achieve minimum objective value zero with $\boldsymbol{y} = \boldsymbol{0}$;
- if (8) has no feasible solution, then (9) has a minimum objective value greater than zero.

Now (9) is an LP problem in variables $\boldsymbol{x}, \boldsymbol{y}$. It has a trivial basic feasible solution $\boldsymbol{y} = \boldsymbol{b}$. Use simplex to solve (8) and obtain a basic feasible solution at each step. If the minimum objective value in (9) is zero, then all y_i must be 0 in the final basic solution, which in the nondegenerate case will have no y_i in the basis. If some y_i are in the basis, they can be exchanged for nonbasic x_j variables (which have zero values) to yield a basic feasible solution involving x_j variables only.

To reiterate, a general LP problem can be solved by two phases:

Phase I : Introduce artificial variables and use simplex to find a basic feasible solution.

Phase II : Using the solution found in phase I, run simplex to minimize the original objective function.

EXAMPLE 1. Consider the problem

$$\begin{array}{rl} \min & 4x_1 + x_2 + x_3 \\ \text{subject to} & 2x_1 + x_2 + 2x_3 = 4 \\ & 3x_1 + 3x_2 + x_3 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

We introduce artificial variables $x_4 \geq 0$, $x_5 \geq 0$, and an objective function $x_4 + x_5$. The initial tableau is

$$\begin{array}{cccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\ 2 & 1 & 2 & 1 & 0 & 4 \\ 3 & 3 & 1 & 0 & 1 & 3 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

We first update the last row so that it has zero components under the artificial variables

$$\begin{array}{cccccc} 2 & 1 & 2 & 1 & 0 & 4 \\ \boxed{3} & 3 & 1 & 0 & 1 & 3 \\ \hline -5 & -4 & -3 & 0 & 0 & -7 \end{array}$$

Pivoting at $\boxed{3}$ yields

$$\begin{array}{cccccc} 0 & -1 & \boxed{\frac{4}{3}} & 1 & -\frac{2}{3} & 2 \\ 1 & 1 & \frac{1}{3} & 0 & \frac{1}{3} & 1 \\ \hline 0 & 1 & -\frac{4}{3} & 0 & \frac{5}{3} & -2 \end{array}$$

Next, we pivot at $\boxed{\frac{4}{3}}$:

$$\begin{array}{cccccc} 0 & -\frac{3}{4} & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{4} & 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

The final tableau above leads to a basic solution to the original problem:

$$x_1 = \frac{1}{2}, \qquad x_2 = 0, \qquad x_3 = \frac{3}{2}.$$

Beginning Phase II, we use the original cost function and delete the artificial variable columns in the final tableau of Phase I:

$$\begin{array}{cccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{b} \\ 0 & -\frac{3}{4} & 1 & \frac{3}{2} \\ 1 & \frac{5}{4} & 0 & \frac{1}{2} \\ \hline 4 & 1 & 1 & 0 \end{array}$$

Again, transform the last row so that zeros appear in the basic columns

$$\begin{array}{cccc} 0 & -\frac{3}{4} & 1 & \frac{3}{2} \\ 1 & \boxed{\frac{5}{4}} & 0 & \frac{1}{2} \\ \hline 0 & -\frac{13}{4} & 0 & -\frac{7}{2} \end{array}$$

Pivoting at $\boxed{\frac{5}{4}}$ yields the final tableau:

$$\begin{array}{cccc}
\frac{3}{5} & 0 & 1 & \frac{9}{5} \\
\frac{4}{5} & 1 & 0 & \frac{2}{5} \\
\hline
\frac{13}{5} & 0 & 0 & -\frac{11}{5}
\end{array}$$

The optimal solution is

$$x_1 = 0, \qquad x_2 = \frac{2}{5}, \qquad x_3 = \frac{9}{5}.$$

Example 2. In fact, we could also obtain the optimal solution for Example 1 by performing a sequence of "pivoting" on the initial tableau directly:

$$\begin{array}{cccc}
2 & 1 & 2 & 4 \\
3 & 3 & \boxed{1} & 3 \\
\hline
4 & 1 & 1 & 0
\end{array}$$

$$\begin{array}{cccc}
\boxed{-4} & -5 & 0 & -2 \\
3 & 3 & 1 & 3 \\
\hline
1 & -2 & 0 & -3
\end{array}$$

$$\begin{array}{cccc}
1 & \boxed{\frac{5}{4}} & 0 & \frac{1}{2} \\
0 & -\frac{3}{4} & 1 & \frac{3}{2} \\
\hline
0 & -\frac{13}{4} & 0 & -\frac{7}{2}
\end{array}$$

$$\begin{array}{cccc}
\frac{4}{5} & 1 & 0 & \frac{2}{5} \\
\frac{3}{5} & 0 & 1 & \frac{9}{5} \\
\hline
\frac{13}{5} & 0 & 0 & -\frac{11}{5}
\end{array}$$

To pivot from the second tableu to the third tableu, the simplex method would have multiply the first row by -1 first.

As we have seen, the above pivoting steps were not determined procedually (we were a little lucky in following a path to reach the optimal solution shortly). Therefore the two-phase pivoting should be applied on general linear programs.

A Duality

The linear program

$$\begin{array}{rl}
\min & \boldsymbol{c}^T\boldsymbol{x} \\
\text{subject to} & A\boldsymbol{x} \geq \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{array}$$

is referred to as *primal*. It has a *dual* linear program in the form of

$$\begin{array}{rl}
\max & \boldsymbol{b}^T\boldsymbol{\lambda} \\
\text{subject to} & A^T\boldsymbol{\lambda} \leq \boldsymbol{c} \\
& \boldsymbol{\lambda} \geq \boldsymbol{0}
\end{array}$$

Here A has dimension $m \times n$, and $\boldsymbol{x}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{\lambda}$ have dimensions $n \times 1$, $m \times 1$, $n \times 1$, and $m \times 1$, respectively.

The roles of primal and dual LPs can be reversed. To see this, we change the dual above to its equivalent formation

$$\begin{array}{rl} \min & (-\boldsymbol{b})^T \boldsymbol{\lambda} \\ \text{subject to} & (-A)^T \boldsymbol{\lambda} \geq -\boldsymbol{c} \\ & \boldsymbol{\lambda} \geq \mathbf{0} \end{array}$$

The dual of the above LP is clearly equivalent to the original primal.

The dual of any LP can be found by converting the problem to the primal form. For instance, given an LP in standard form:

$$\begin{array}{rl} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \end{array}$$

we write it in the equivalent form

$$\begin{array}{rl} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \geq \boldsymbol{b} \\ & -A\boldsymbol{x} \geq -\boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \end{array}$$

The corresponding dual is then

$$\begin{array}{rl} \max & \boldsymbol{b}^T \boldsymbol{u} - \boldsymbol{b}^T \boldsymbol{v} \\ \text{subject to} & A^T \boldsymbol{u} - A^T \boldsymbol{v} \leq \boldsymbol{c} \\ & \boldsymbol{u} \geq \mathbf{0} \\ & \boldsymbol{v} \geq \mathbf{0} \end{array}$$

We introduce $\boldsymbol{\lambda} = \boldsymbol{u} - \boldsymbol{v}$ to simplify the dual representation to

$$\begin{array}{rl} \max & \boldsymbol{b}^T \boldsymbol{\lambda} \\ \text{subject to} & A^T \boldsymbol{\lambda} \leq \boldsymbol{c} \end{array}$$

This is the asymmetric form of the duality. The dual vector $\boldsymbol{\lambda}$ is not restricted to be nonnegative.

Now we look at the underlying interpretation of a dual LP relative to the original LP. This is illustrated on the transportation problem.

Example 2. (Dual of the transportation problem) Recall that the transportation problem asks to select the pattern of product shipments between a number of origins and a number of destinations so as to minimize transportation cost while meeting the demand of each destination.

To interpret the dual problem, imagine an entrepreneur who, feeling that he can ship more efficiently, comes to the manufacturer with the offer to buy his product at the plant sites (origins) and sell it at the warehouses (destinations). The product prices to be used in these transactions vary from point to point. He must choose these prices so that his offer will be attractive to the manufacturer.

So he select prices $\lambda_1, \dots, \lambda_m$ for the m origins and $\mu_1, \dots, \mu_n$ for the n destinations. To compete with the usual transportation, his prices must satisfy the constraints $\mu_j - \lambda_i \leq c_{ij}$, for all i, j, which represents the net amount the manufacturer must pay to sell a unit of product at origin i and buy it back at destination

j. Then he needs to solve the LP problem

$$\begin{array}{lll} \max & \sum_{j=1}^{n} \mu_j b_j - \sum_{i=1}^{m} \lambda_i a_i & \\ \text{subject to} & \mu_j - \lambda_i \le c_{ij} & i = 1, \ldots, m \\ & & j = 1, \ldots, n \end{array}$$

whereas the original transportation problem is

$$\begin{array}{lll} \min & \sum_{i,j} c_{ij} x_{ij} & \\ \text{subject to} & \sum_{j=1}^{n} x_{ij} = a_i & i = 1, \ldots, m \\ & \sum_{i=1}^{m} x_{ij} = b_j & j = 1, \ldots, n \\ & x_{ij} \ge \mathbf{0} & i = 1, \ldots, m \\ & & j = 1, \ldots, n \end{array}$$

There exists deeper connection between a problem and its dual than just their forms and underlying interpretations. Let us consider the primal problem in standard form

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \ge 0 \end{array} \tag{10}$$

and it dual

$$\begin{array}{ll} \max & \boldsymbol{b}^T \boldsymbol{\lambda} \\ \text{subject to} & A^T \boldsymbol{\lambda} \le \boldsymbol{c} \end{array} \tag{11}$$

Lemma 3 *If $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are feasible for the primal problem (10) and dual problem (11), respectively, then $\mathbf{c}^T \boldsymbol{x} \ge \boldsymbol{b}^T \boldsymbol{\lambda}$.*

The above lemma directly follows from that

$$\boldsymbol{b}^T \boldsymbol{\lambda} \;=\; \boldsymbol{\lambda}^T A \boldsymbol{x} \;\le\; \mathbf{c}^T \boldsymbol{x}.$$

Corollary 4 *If $\boldsymbol{x}^*$ and $\boldsymbol{\lambda}^*$ are feasible solutions of the primal and dual problems, respectively, and if $\mathbf{c}^T \boldsymbol{x}^* = \boldsymbol{b}^T \boldsymbol{\lambda}^*$, then $\boldsymbol{x}^*$ and $\boldsymbol{\lambda}^*$ are optimal.*

Theorem 5 (Duality Theorem) *If either the primal problem (10) or the dual problem (11) has a finite optimal solution, so does the other and the corresponding values of the objective functions are equal. If either problem has an unbounded objective, the other problem has no feasible solution.*

The following theorem relates the duality theorem to the simplex procedure:

Theorem 6 *Let the primal problem (10) have an optimal basic feasible solution $\boldsymbol{x} = (\boldsymbol{x}_B, \mathbf{0})^T$, without loss of generality, and let the corresponding columns in A form a submatrix B, and the corresponding cost coefficients form a column vector $\boldsymbol{c}_B$. Then the vector $\boldsymbol{\lambda} = (B^{-1})^T \boldsymbol{c}_B$ is an optimal solution to the dual problem (11). The optimal values of both problems are equal.*

References

[1] D. G. Luenberger. *Linear and Nonlinear Programming*. Addison-Wesley, 2nd edition, 1984.

[2] V. Chvátal. *Linear Programming*. W. H. Freeman and Company, 1983.