Tutorial on Robotic Manipulator Control

(Part III: Null Space Projection)

Yan-Bin Jia

Department of Computer Science

Iowa State University

Ames, IA 50011

Outline

I. Kinematic redundancy

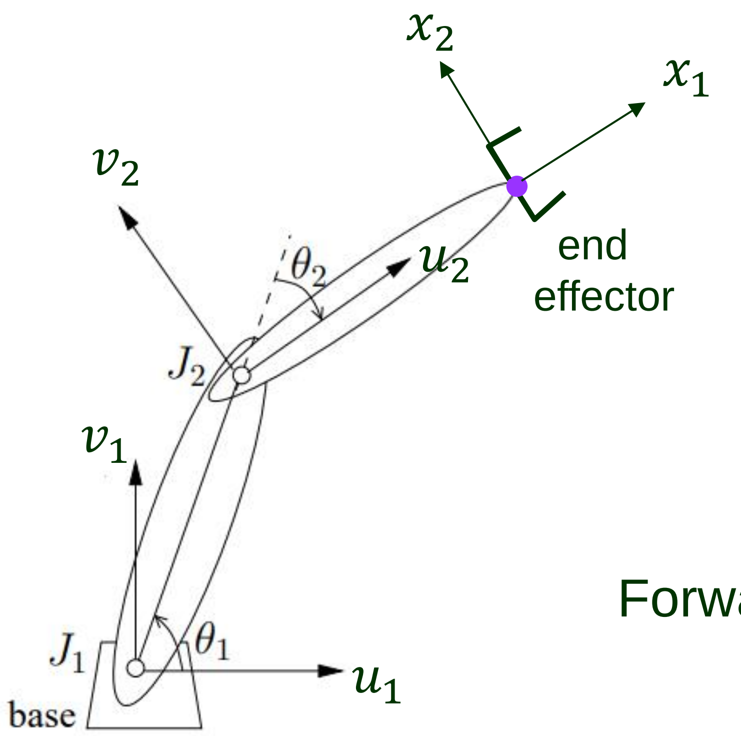
II. Dynamic force/torque relationship

III. Two-priority control at the velocity level

IV. Multi-priority control at the acceleration level

V. Torque projection

I. Manipulator Jacobian



$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}$$: joint angles

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$: task (or operational) coordinates

(often position & orientation of the end-effector frame; so $m = 6$ if a 3D task or 3 if a 2D one.)

Forward kinematics: $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta})$

$\Downarrow$

$$\dot{\boldsymbol{x}} = \underbrace{J(\boldsymbol{\theta})}\dot{\boldsymbol{\theta}}$$

Jacobian $J(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} \in \mathbb{R}^{m \times n}$

Assumption 1 The Jacobian $J(\boldsymbol{\theta})$ has full row rank. $\Longrightarrow$ $\text{rank}(J) = m \le n$

Manipulator Redundancy

Assumption 2 (redundancy) $n = \|\dot{\boldsymbol{\theta}}\| > 6. \;\Rightarrow\;$ null$(J) \neq \emptyset$

- $JJ^{\dagger} = I_6$ (6×6 identity matrix)

$$\dot{\boldsymbol{x}} = J(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\Downarrow$$

$$\dot{\boldsymbol{\theta}} = \underbrace{J^{\dagger}}\dot{\boldsymbol{x}} + \underbrace{\left(I_n - J^{\dagger}J\right)}\boldsymbol{\omega} \qquad \text{for any } \boldsymbol{\omega} \in \mathbb{R}^n.$$

Pseudoinverse
$J^{\dagger} = J^T(JJ^T)^{-1}$

Projection of $\boldsymbol{\omega}$ onto
null$(J) = \{(I_n - J^{\dagger}J)\boldsymbol{\omega} \mid \boldsymbol{\omega} \in \mathbb{R}^n\}$

$$\Downarrow \quad J\left(I_n - J^{\dagger}J\right) = J - JJ^{\dagger}J = J - J = 0$$

$$J\dot{\boldsymbol{\theta}} = JJ^{+}\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}$$

- null(J) characterizes all the joint motions beyond realizing the specified motion of the task frame located at the end effector.

Application 1: Position-Dependent Performance Optimization

Make use of the redundancy to minimize a position-dependent criterion $p(\boldsymbol{\theta})$.

- Projects the gradient of p onto the joint motion.

$$\boldsymbol{\omega} = -\lambda \left(\frac{\partial p}{\partial \boldsymbol{\theta}}\right)^T, \qquad \lambda > 0$$

$$\dot{\boldsymbol{\theta}} = J^\dagger \dot{\boldsymbol{x}} + (I_n - J^\dagger J)\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\theta}} = J^\dagger \dot{\boldsymbol{x}} - \underbrace{\lambda (I_n - J^\dagger J)\left(\frac{\partial p}{\partial \boldsymbol{\theta}}\right)^T}_{\text{Null space component}}$$

- Null space component (self-motion) of $\dot{\boldsymbol{\theta}}$ decreases p while having no effect on the end effector motion $\dot{\boldsymbol{x}}$.
- Let $p = -\sqrt{\det(JJ^T)}$ to maximize the *manipulability index* $\sqrt{\det(JJ^T)}$ and keep away from singularities.

II. Force/Torque Relationship

$\boldsymbol{\tau} \in \mathbb{R}^{\boldsymbol{n}}$: joint torque vector

$\boldsymbol{f} \in \mathbb{R}^{\boldsymbol{n}}$: force applied by the end effector

Virtual displacement in the task space

Virtual displacement in the joint space

$$\underbrace{\delta \boldsymbol{x}} = J(\boldsymbol{\theta})\underbrace{\delta \boldsymbol{\theta}}$$

- Non-redundant manipulator ($m = n$) $\Rightarrow J$ nonsingular under Assumption 1.

$$\boldsymbol{\tau}^T \delta\boldsymbol{\theta} = \boldsymbol{f}^T \delta\boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{\tau}^T \delta\boldsymbol{\theta} = \boldsymbol{f}^T J \delta\boldsymbol{\theta} \quad \xrightarrow{\text{For all } \delta\boldsymbol{\theta}} \quad \boldsymbol{\tau}^T = \boldsymbol{f}^T J$$

$$\Longrightarrow \quad \boldsymbol{\tau} = J^T \boldsymbol{f}$$

* Materials in Section II are from [2], [4]

Some Caveats

$$\boldsymbol{\tau} = J^T \boldsymbol{f}$$

- ♣ $\boldsymbol{\tau} \in$ row$(J) \leq m$. But the set of joint torques has n dimensions. The above force-torque relationship is not always true if (a) $n > m$ or if (b) $m = n$ but J is singular.

- ♣ The equation has also ignored the gravity effect of the arm:

$$\boldsymbol{\tau} = \boldsymbol{g}(\boldsymbol{\theta}) + J^T \boldsymbol{f}$$

- ♣ It has also assumed non-existence of any dynamic effect:

$$\boldsymbol{\tau} = M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + C(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) - \boldsymbol{\tau}_{\text{ext}}$$

Redundant Manipulator

$n > m$

Under Assumption 1, rank$\left(J(\boldsymbol{\theta})\right) = m$. Hence $J^{\#} = V \begin{pmatrix} \Sigma_m^{-1} \\ Y \end{pmatrix} U^T$ with the SVD $J = U(\Sigma_m, 0)V^T$.

- The generalized inverse $J^{\#}$ is reflexive.
- $JJ^{\#} = I_m$

$$\delta \boldsymbol{x} = J(\boldsymbol{\theta})\delta\boldsymbol{\theta} \quad \Longrightarrow \quad \delta\boldsymbol{\theta} = J^{\#}(\boldsymbol{\theta})\delta\boldsymbol{x} + \underbrace{\overbrace{\left(I_n - J^{\#}(\boldsymbol{\theta})J(\boldsymbol{\theta})\right)}^{\text{Null space of } J \text{ associated with } J^{\#}}\,\delta\boldsymbol{\theta}_0}_{}$$

Null space of J associated with $J^{\#}$

Arbitrary joint vector ($\delta\boldsymbol{\theta}_0$)

- $\left(I_n - J^{\#}(\boldsymbol{\theta})J(\boldsymbol{\theta})\right)\delta\boldsymbol{\theta}_0$ does not vary the position and orientation of the end-effector.

Infinitely many joint displacements exist without altering its pose!

- $\left(I_n - J^{\#}(\boldsymbol{\theta})J(\boldsymbol{\theta})\right)\delta\boldsymbol{\theta}_0$ varies the internal pose of the manipulator.

Force/Torque Relationship under Redundancy

A torque in addition to $J^T \boldsymbol{f} \in \text{row}(J)$ must exist to change the manipulator's pose.

$$\boldsymbol{\tau} = J^T \boldsymbol{f} + \boldsymbol{\tau}'$$

- ♣ $\boldsymbol{\tau}^T \delta\boldsymbol{\theta} = \boldsymbol{f}^T \delta\boldsymbol{x}$ does not hold because the applied joint torque $\boldsymbol{\tau}$ will no longer be supported by the reaction force $\boldsymbol{f}$ of the environment.
- ♦ $\boldsymbol{f}^T \delta\boldsymbol{x}$ only accounts for the part of δW done by $\boldsymbol{f}$ at the end-effector.
- ♦ The other part of δW, done by $\boldsymbol{\tau}'$, is attributed to causing the pose changes.

Total virtual displacement: $\delta\boldsymbol{\theta} = J^{\#}(\boldsymbol{\theta})\delta\boldsymbol{x} + \left(I_n - J^{\#}(\boldsymbol{\theta})J(\boldsymbol{\theta})\right)\delta\boldsymbol{\theta}_0$ for some $\delta\boldsymbol{\theta}_0$

Total virtual work:

$$\begin{aligned}\delta W &= \boldsymbol{\tau}^T \delta\boldsymbol{\theta} \\ &= \boldsymbol{\tau}^T J^{\#}(\boldsymbol{\theta})\delta\boldsymbol{x} + \boldsymbol{\tau}^T \left(I_n - J^{\#}(\boldsymbol{\theta})J(\boldsymbol{\theta})\right)\delta\boldsymbol{\theta}_0 \\ &= \left(\left(J^{\#}\right)^T \boldsymbol{\tau}\right)^T \delta\boldsymbol{x} + \left(\left(I_n - J^{\#}J\right)^T \boldsymbol{\tau}\right)^T \delta\boldsymbol{\theta}_0\end{aligned}$$

Static Consistency

$$\delta W = \Big(\underbrace{(J^{\#})^T \boldsymbol{\tau}}_{= \boldsymbol{f}} \Big)^T \delta \boldsymbol{x} + \Big((I_n - J^{\#} J)^T \boldsymbol{\tau} \Big)^T \delta \boldsymbol{\theta}_0 \qquad \boldsymbol{\tau} = J^T \boldsymbol{f} + \boldsymbol{\tau}'$$

- The additional torque $\boldsymbol{\tau}'$ does not yield work along $\delta \boldsymbol{x}$. So, it must hold that

$$\boldsymbol{f} = (J^{\#})^T \boldsymbol{\tau} = J^{\#T} (J^T \boldsymbol{f} + \boldsymbol{\tau}')$$

(static consistency)

$\Downarrow$ $J^{\#T} J^T = (J J^{\#})^T = I_m$

$$J^{\#T} \boldsymbol{\tau}' = 0 \ \text{ i.e., } \boldsymbol{\tau}' \in \text{null}\left(J^{\#T}\right).$$

$\Downarrow$ $J^{\#}$ is reflexive (i.e., $J^{\#} J J^{\#} = J^{\#}$) and thus $J^{\#\#} = J$.

$J^{T\#} = J^{\#T}$

$$\boldsymbol{\tau}' = \underbrace{\left(I_n - J^T J^{\#T}\right)}_{\text{Projection onto null}\left(J^{\#T}\right)} \boldsymbol{\tau}_0 \qquad \text{for some } \boldsymbol{\tau}_0 \in \mathbb{R}^n$$

Total Joint Torque

- Total joint torque:

$$\boldsymbol{\tau} = J^T \boldsymbol{f} + \left(I_n - J^T J^{\#^T}\right) \boldsymbol{\tau}_0$$

$$J^T \boldsymbol{f} = J^T \left(J^{\#^T} \boldsymbol{\tau}\right) = J^T J^{\#^T} \boldsymbol{\tau}$$

$$\left(I_n - J^T J^{\#^T}\right) \boldsymbol{\tau}_0 = \left(I_n - J^T J^{\#^T}\right) \boldsymbol{\tau}_0 + \left(I_n - J^T J^{\#^T}\right) J^T \boldsymbol{f}$$

since $J^T J^{\#^T} J^T = J^T$

$$= \left(I_n - J^T J^{\#^T}\right) (J^T \boldsymbol{f} + \boldsymbol{\tau}_0)$$

$$= \left(I_n - J^T J^{\#^T}\right) \boldsymbol{\tau}$$

- Decomposition of $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \underbrace{J^T J^{\#^T} \boldsymbol{\tau}}_{\text{Torque generating the end-effector force}} + \underbrace{\left(I_n - J^T J^{\#^T}\right) \boldsymbol{\tau}}_{\text{Torque acting in the null space}}$$

Torque generating the end-effector force

Torque acting in the null space

Manipulator Dynamics

$$\underbrace{M(\boldsymbol{\theta})}_{\text{Mass matrix}}\ddot{\boldsymbol{\theta}} + \underbrace{C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})}_{\text{Coriolis \& centrifugal terms}}\dot{\boldsymbol{\theta}} + \underbrace{\boldsymbol{g}(\boldsymbol{\theta})}_{\text{Gravity}} = \boldsymbol{\tau} + \underbrace{\boldsymbol{\tau}_{\text{ext}}}_{\text{External torque}} \Longrightarrow \ddot{\boldsymbol{\theta}} + M^{-1}C\dot{\boldsymbol{\theta}} + M^{-1}\boldsymbol{g} = M^{-1}(\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{ext}})$$

$$\Longrightarrow J\ddot{\boldsymbol{\theta}} + JM^{-1}C\dot{\boldsymbol{\theta}} + JM^{-1}\boldsymbol{g} = JM^{-1}(\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{ext}})$$

$$\dot{\boldsymbol{x}} = J\dot{\boldsymbol{\theta}} \Longrightarrow \ddot{\boldsymbol{x}} = J\ddot{\boldsymbol{\theta}} + \dot{J}\dot{\boldsymbol{\theta}}$$

$$\ddot{\boldsymbol{x}} + \left(JM^{-1}C - \dot{J}\right)\dot{\boldsymbol{\theta}} + JM^{-1}\boldsymbol{g} = JM^{-1}(\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{ext}})$$

$$\boldsymbol{\tau} = J^T\boldsymbol{f} + \left(I_{\boldsymbol{n}} - J^T(J^T)^{\#}\right)\boldsymbol{\tau}_0$$

$$\ddot{\boldsymbol{x}} + \left(JM^{-1}C - \dot{J}\right)\dot{\boldsymbol{\theta}} + JM^{-1}\boldsymbol{g} = JM^{-1}(J^T\boldsymbol{f} + \boldsymbol{\tau}_{\text{ext}}) + JM^{-1}\left(I_{\boldsymbol{n}} - J^T(J^T)^{\#}\right)\boldsymbol{\tau}_0$$

$\boldsymbol{\tau}_{\text{ext}}$ is exerted on the manipulator due to, say, the resultant force passed on from the environment through the object in contact with the end-effector ($\boldsymbol{\tau}_{\text{ext}} = J^T\boldsymbol{f}_{\text{ext}}$ often holds).

Dynamic Consistency

$$\ddot{\boldsymbol{x}} + (JM^{-1}C - \dot{J})\dot{\boldsymbol{\theta}} + JM^{-1}\boldsymbol{g} = JM^{-1}(J^T\boldsymbol{f} + \boldsymbol{\tau}_{\text{ext}}) + JM^{-1}\left(I_{\boldsymbol{n}} - J^T(J^T)^{\#}\right)\boldsymbol{\tau}_0$$

- The acceleration $\ddot{\boldsymbol{x}}$ at the operational point is not affected by any $\boldsymbol{\tau}_0$ if and only if

$$JM^{-1}\left(I_{\boldsymbol{n}} - J^T(J^T)^{\#}\right)\boldsymbol{\tau}_0 = 0 \quad \xRightarrow{\boldsymbol{\tau}_0 \text{ arbitrary}} \quad JM^{-1}\left(I_{\boldsymbol{n}} - J^T(J^T)^{\#}\right) = 0$$

$$\xRightarrow{(J^T)^{\#} = (J^{\#})^T} \quad JM^{-1}\left(I_{\boldsymbol{n}} - J^T(J^{\#})^T\right) = 0$$

$$\Longrightarrow \quad (J^{\#})^T = (JM^{-1}J^T)^{-1}\, JM^{-1}$$

$$\Longrightarrow \quad J^{\#} = J^{*} \equiv M^{-1}\, J^T(JM^{-1}J^T)^{-1}$$

- J^* is the unique *dynamically consistent* generalized inverse.

Dynamics Projected onto the End Effector

With $J^{\#} = J^{*} = M^{-1} J^{T}(JM^{-1}J^{T})^{-1}$, dynamics is simplified to

$$\ddot{\boldsymbol{x}} + \left(JM^{-1}C - \dot{J}\right)\dot{\boldsymbol{\theta}} + JM^{-1}\boldsymbol{g} = JM^{-1}(J^{T}\boldsymbol{f} + \boldsymbol{\tau}_{\text{ext}})$$

$$\Downarrow$$

$$\Lambda(\boldsymbol{\theta})\ddot{x} + \boldsymbol{\mu}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) + \boldsymbol{p}(\boldsymbol{\theta}) = \boldsymbol{f} + \underbrace{\Lambda(\boldsymbol{\theta})JM^{-1}\boldsymbol{\tau}_{\text{ext}}}_{= \boldsymbol{f}_{\text{ext}} \text{ if } \boldsymbol{\tau}_{\text{ext}} = J^{T}\boldsymbol{f}_{\text{ext}}}$$

(Dynamic behavior of the end effector in the workspace.)

where

- ♣ $\Lambda(\boldsymbol{\theta}) \equiv (JM^{-1}J^{T})^{-1}$: the *operational space kinetic energy matrix*.
- ♣ $\boldsymbol{\mu}(\boldsymbol{\theta}) \equiv \Lambda(\boldsymbol{\theta})JM^{-1}C\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} - \Lambda(\boldsymbol{\theta})\dot{J}\dot{\boldsymbol{\theta}}$

  $= {J^{*}}^{T}C\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} - \Lambda(\boldsymbol{\theta})\dot{J}\dot{\boldsymbol{\theta}}$ since $J^{*} = M^{-1} J^{T}(JM^{-1}J^{T})^{-1} = M^{-1} J^{T}\Lambda(\boldsymbol{\theta})$
- ♣ $\boldsymbol{p}(\boldsymbol{\theta}) \equiv {J^{*}}^{T}\boldsymbol{g}(\boldsymbol{\theta})$

♦ $\boldsymbol{f}$ includes all the dynamic forces reflected at the end effector.

Decoupled Controls

$$M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{ext}}$$

$$= J^T\boldsymbol{f} + \left(I_n - J^T J^{*T}\right)\boldsymbol{\tau}_0 + \boldsymbol{\tau}_{\text{ext}} \qquad = J^T J^{*T}\boldsymbol{\tau} + \left(I_n - J^T J^{*T}\right)\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{ext}}$$

$$= J^T\left(\Lambda(\boldsymbol{\theta})\ddot{\boldsymbol{x}} + \boldsymbol{\mu}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{p}(\boldsymbol{\theta}) - J^{*T}\boldsymbol{\tau}_{\text{ext}}\right) + \left(I_n - J^T J^{*T}\right)\boldsymbol{\tau}_0 + \boldsymbol{\tau}_{\text{ext}}$$

If $\boldsymbol{\tau}_{\text{ext}} = J^T\boldsymbol{f}_{\text{ext}}$, then

$$-J^T J^{*T}\boldsymbol{\tau}_{\text{ext}} + \boldsymbol{\tau}_{\text{ext}} = \left(-J^T J^{*T} J^T + J^T\right)\boldsymbol{f}_{\text{ext}} = \mathbf{0}$$

$$\Downarrow$$

$$M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) = J^T\left(\Lambda(\boldsymbol{\theta})\ddot{\boldsymbol{x}} + \boldsymbol{\mu}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{p}(\boldsymbol{\theta})\right) + \left(I_n - J^T J^{*T}\right)\boldsymbol{\tau}_0$$

Operational Space (i.e.,Task Space) Control

Rewrite dynamics:

$$\boldsymbol{\tau} = J^T\left(\Lambda(\boldsymbol{\theta})\ddot{\boldsymbol{x}} + \boldsymbol{\mu}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{p}(\boldsymbol{\theta}) - {J^*}^T \boldsymbol{\tau}_{\text{ext}}\right) + \left(I_n - J^T {J^*}^T\right)\boldsymbol{\tau}_0$$

Construct controller:

- Control the end-effector by designing $\boldsymbol{f}$ to compensate for centrifugal, Coriolis, and gravitational forces, i.e., $\boldsymbol{f} = \Lambda(\boldsymbol{\theta})\boldsymbol{\alpha} + \boldsymbol{\mu}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{p}(\boldsymbol{\theta})$.
  - For instance, in the above equation replace $\ddot{\boldsymbol{x}}$ with a servo $\boldsymbol{\alpha} = \ddot{\boldsymbol{x}}_d + K_v \dot{\boldsymbol{x}}_e + K_p \boldsymbol{x}_e$.
- Control the internal motion by designing $\boldsymbol{\tau}_0$, which will be responsible for the dynamics projected onto the null space. For example, $\boldsymbol{\tau}_0 = \nabla V$ for some potential function V.

$$\boldsymbol{\tau} = J^T\left(\Lambda(\boldsymbol{\theta})\boldsymbol{\alpha} + \boldsymbol{\mu}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{p}(\boldsymbol{\theta}) - {J^*}^T \boldsymbol{\tau}_{\text{ext}}\right) + \left(I_n - J^T {J^*}^T\right)\boldsymbol{\tau}_0$$

$$\boldsymbol{\tau} = J^T\left(\Lambda(\boldsymbol{\theta})\boldsymbol{\alpha} + \boldsymbol{\mu}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{p}(\boldsymbol{\theta}) - \boldsymbol{f}_{\text{ext}}\right) + \left(I_n - J^T {J^*}^T\right)\boldsymbol{\tau}_0 \qquad \text{if } \boldsymbol{\tau}_{\text{ext}} = J^T \boldsymbol{f}_{\text{ext}}$$

Application 2: Potential Field Based Minimization

$$\boldsymbol{\tau} = J^T \boldsymbol{f} + \left(I_n - J^T J^{*T}\right)\boldsymbol{\tau}_0$$

Choose $\boldsymbol{\tau}_0$ to realize a desired internal motion of the manipulator.

- Construct an *artificial potential field* function $V(\boldsymbol{q})$.
- Let $\boldsymbol{\tau}_0 = -\nabla V(\boldsymbol{q})$.
- Interference of $\boldsymbol{\tau}_0$ with the end effector dynamics is avoided by the projection $I_n - J^T J^{*T}$ onto null$\left(J^{*T}\right)$.

Example Keep every joint angle q_i in the middle of its range $[q_{imin}, q_{imax}]$.

$$V = k \sum_{i=1}^{n} \left(q_i - \frac{q_{imin} + q_{imax}}{2}\right)^2$$

Minimum Kinetic Energy

The *kinetic energy* of the manipulator is $\frac{1}{2}\dot{\boldsymbol{\theta}}^T M \dot{\boldsymbol{\theta}}$.

Let us determine the joint velocity to achieve a given end-effector velocity while minimizing the kinetic energy:

$$\min_{\dot{\boldsymbol{\theta}}} \frac{1}{2}\dot{\boldsymbol{\theta}}^T M \dot{\boldsymbol{\theta}} \qquad \text{subject to} \quad J\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{x}}$$

We convert the problem into an unconstrained one using m Lagrange multipliers represented by a vector $\boldsymbol{\lambda}$:

$$\min_{\dot{\boldsymbol{\theta}}} \frac{1}{2}\dot{\boldsymbol{\theta}}^T M \dot{\boldsymbol{\theta}} + \boldsymbol{\lambda}^T (J\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{x}}) \implies \dot{\boldsymbol{\theta}}^T M + \boldsymbol{\lambda}^T J = 0 \implies \dot{\boldsymbol{\theta}} = M^{-1} J^T \boldsymbol{\lambda}$$

$$\implies J\dot{\boldsymbol{\theta}} = J M^{-1} J^T \boldsymbol{\lambda} = \dot{\boldsymbol{x}} \implies \boldsymbol{\lambda} = (J M^{-1} J^T)^{-1} \dot{\boldsymbol{x}}$$

J fully ranked and
M non-singular

$$\implies \dot{\boldsymbol{\theta}} = M^{-1} J^T (J M^{-1} J^T)^{-1} \dot{\boldsymbol{x}} = J^* \dot{x}$$

The dynamically consistent generalized inverse J^* of the Jacobian also minimizes the kinetic energy.

III. Task Variables at Two Priority Levels

Task coordinates partitioned into two groups:

- $\boldsymbol{x}_1$: high priority variables, which must be realized; $(m_1 = \|\boldsymbol{x}_1\|)$
- $\boldsymbol{x}_2$: low priority variables, which should be realized if possible. $(m_2 = \|\boldsymbol{x}_2\|)$

$$\dot{\boldsymbol{x}}_1 = J_1 \dot{\boldsymbol{\theta}} \implies \dot{\boldsymbol{\theta}} = J_1^{\dagger} \dot{\boldsymbol{x}}_1 + (I_n - J_1^{\dagger} J_1)\boldsymbol{\omega}$$

$$\dot{\boldsymbol{x}}_2 = J_2 \dot{\boldsymbol{\theta}} \implies \boldsymbol{\omega} = \left(J_2(I_n - J_1^{\dagger} J_1)\right)^{\dagger} (\dot{\boldsymbol{x}}_2 - J_2 J_1^{\dagger} \dot{\boldsymbol{x}}_1)$$

(solution with the minimum norm by our choice)

$$\dot{\boldsymbol{\theta}} = J_1^{\dagger} \dot{\boldsymbol{x}}_1 + \underbrace{(I_n - J_1^{\dagger} J_1)}_{\equiv N_1} \Big(\underbrace{J_2(I_n - J_1^{\dagger} J_1)}_{\equiv \bar{J}_2}\Big)^{\dagger} (\dot{\boldsymbol{x}}_2 - J_2 J_1^{\dagger} \dot{\boldsymbol{x}}_1)$$

An Important Property

Theorem 1 Let $N \in \mathbb{R}^{n \times n}$ be symmetric and *idempotent* (i.e., $N^2 = N$). Then, $\text{row}(N) = \text{row}(BN)$ and $N(BN)^\# = (BN)^\#$ for any $B \in \mathbb{R}^{m \times n}$.

Proof

$$\boldsymbol{v} \in \text{null}(N) \implies N\boldsymbol{v} = 0 \implies BN\boldsymbol{v} = 0 \implies \boldsymbol{v} \in \text{null}(BN)$$

$$\implies \text{null}(N) \subseteq \text{null}(BN) \implies \text{row}(N) \supseteq \text{row}(BN)$$

Meanwhile, $\text{row}(BN) \subseteq \text{row}(N)$ because every row in BN is a linear combination of rows in N. Thus, $\text{row}(N) = \text{row}(BN)$.

For any $\boldsymbol{v} \in \mathbb{R}^n$,

$$(BN)^\#\boldsymbol{v} \in \text{col}\left((BN)^\#\right) = \text{row}(BN) = \text{row}(N) = \text{col}(N) \quad \text{since } N \text{ is symmetric}$$

$$\Downarrow$$

$$(BN)^\#\boldsymbol{v} = N\boldsymbol{w} \quad \text{for some } \boldsymbol{w} \in \mathbb{R}^n$$

$$\Downarrow$$

$$N(BN)^\#\boldsymbol{v} = NN\boldsymbol{w}$$

Because $\boldsymbol{v}$ is arbitrary, we conclude that $N(BN)^\# = (BN)^\#$. $\square$

Simplification

Corollary 2 Let $N_1 = I_n - J_1^\dagger J_1$ and $\bar{J}_2 = J_2 N_1$. Then $N_1 \bar{J}_2^\dagger = \bar{J}_2^\dagger$.

Proof Let $B = J_2$ and $N = N_1$ so $\bar{J}_2 = BN$. Since $J_1^\dagger$ is a pseudoinverse, N is symmetric and idempotent. We then apply Theorem 1 with that $J_1^\dagger$ is a generalized inverse. ■

$$\dot{\boldsymbol{\theta}} = J_1^\dagger \dot{\boldsymbol{x}}_1 + (I_n - J_1^\dagger J_1)\left(J_2(I_n - J_1^\dagger J_1)\right)^\dagger (\dot{\boldsymbol{x}}_2 - J_2 J_1^\dagger \dot{\boldsymbol{x}}_1)$$

⇩

$$\dot{\boldsymbol{\theta}} = J_1^\dagger \dot{\boldsymbol{x}}_1 + N_1 \bar{J}_2^\dagger (\dot{\boldsymbol{x}}_2 - J_2 J_1^\dagger \dot{\boldsymbol{x}}_1)$$

⇩ $N_1 \bar{J}_2^\dagger = \bar{J}_2^\dagger$

$$\dot{\boldsymbol{\theta}} = J_1^\dagger \dot{\boldsymbol{x}}_1 + \bar{J}_2^\dagger (\dot{\boldsymbol{x}}_2 - J_2 J_1^\dagger \dot{\boldsymbol{x}}_1)$$

$\bar{J}_2$ projects the rows of J_2 onto the null space of J_1 so task 2 does not interfere with task 1.

Application 3: Obstacle Avoidance

$\dot{\boldsymbol{x}}$: task velocity

$$\dot{\boldsymbol{x}} = J\dot{\boldsymbol{\theta}} \qquad \text{(high priority)}$$

$\boldsymbol{x}_0$: point on the arm closest to an obstacle (determined using geometry).

$\dot{\boldsymbol{x}}_0$: a velocity away from the obstacle

$$\dot{\boldsymbol{x}}_0 = J_0\dot{\boldsymbol{\theta}} \qquad \text{(low priority)}$$

Controlled joint velocity:

$$\dot{\boldsymbol{\theta}} = J^\dagger\dot{\boldsymbol{x}} + \left(J_0\left(I_n - J^\dagger J\right)\right)^\dagger\left(\dot{\boldsymbol{x}}_0 - J_0 J^\dagger\dot{\boldsymbol{x}}\right)$$

♠ Slow manipulator response at the velocity level.

* See [1] for more applications.

IV. Multi-Priority Control at Acceleration Level

$\boldsymbol{x}_1 \in \mathbb{R}^{m_1}$: 1st-priority task

$\vdots$

$\boldsymbol{x}_l \in \mathbb{R}^{m_l}$: lth-priority task

Problem Design a control $\boldsymbol{\tau}$ for the main task of tracking $\boldsymbol{x}_1$ while realizing the tasks $\boldsymbol{x}_2, \dots \boldsymbol{x}_l$, if possible, in the decreasing order.

$$\dot{\boldsymbol{x}}_k = J_k \dot{\boldsymbol{\theta}}, \qquad 1 \le k \le l \quad \Longrightarrow \quad \ddot{\boldsymbol{x}}_k = J_k \ddot{\boldsymbol{\theta}} + \dot{J}_k \dot{\boldsymbol{\theta}}$$

Define

$$\bar{J}_1 = J_1$$

$$\bar{J}_k = J_k N_{k-1} \qquad 2 \le k \le l$$ (projects rows of J_k onto the null space of J_{k-1})

$$N_k = \prod_{i=1}^{k} \left(I_n - \bar{J}_i^{\dagger} \bar{J}_i \right) \qquad 1 \le k \le l$$

* Section IV is expanded from [5].

Invariances of Multiplication by N_k

We will repeatedly use the following properties (the first of which generalizes Corollary 2) :

$$N_k \bar{J}_{k+1}^{\dagger} = \bar{J}_{k+1}^{\dagger}$$

$$\bar{J}_{k+1} N_k = \bar{J}_{k+1}$$

since

$$\begin{aligned} N_k \bar{J}_{k+1}^{\dagger} &= N_k \, (J_{k+1} N_k)^{\dagger} \\ &= (J_{k+1} N_k)^{\dagger} && \text{by Theorem 1} \\ &= \bar{J}_{k+1}^{\dagger} \end{aligned}$$

$$\begin{aligned} \bar{J}_{k+1} N_k &= J_{k+1} N_k N_k \\ &= J_{k+1} N_k && N_k \text{ is a projection matrix} \\ &= \bar{J}_{k+1} \end{aligned}$$

Summation Form of N_k

Lemma 3 The matrix $N_k = I_n - \sum_{j=1}^{k} \bar{J}_j^{\dagger} \bar{J}_j$ for $1 \le k \le l$, and it is symmetric and idempotent.

Proof By induction.

- $N_1 = I_n - J_1^{\dagger} J_1$ is symmetric and idempotent because $J_1^{\dagger}$ is a pseudoinverse, as already mentioned in the proof of Corollary 2.
- Assume that the statement holds for N_i.

$$N_{i+1} = N_i\left(I_n - \bar{J}_{i+1}^{\dagger} \bar{J}_{i+1}\right) = N_i - N_i (J_{i+1} N_i)^{\dagger} (J_{i+1} N_i)$$

$$= I_n - \sum_{j=1}^{i} \bar{J}_j^{\dagger} \bar{J}_j - (J_{i+1} N_i)^{\dagger} (J_{i+1} N_i),$$

by induction and by Theorem 1 with N_i being symmetric and idempotent

$$= I_n - \sum_{j=1}^{i} \bar{J}_j^{\dagger} \bar{J}_j - (\bar{J}_{i+1})^{\dagger} (\bar{J}_{i+1})$$

$$= I_n - \sum_{j=1}^{i+1} \bar{J}_j^{\dagger} \bar{J}_j$$

$\bar{J}_1 = J_1$

$\bar{J}_k = J_k N_{k-1} \quad 2 \le k \le l$

$N_k = \prod_{i=1}^{k} \left(I_n - \bar{J}_i^{\dagger} \bar{J}_i\right)$

Proposition 3 (cont'd)

Symmetry of N_{i+1} follows from that of $\bar{J}_1^\dagger \bar{J}_1, \ldots, \bar{J}_{i+1}^\dagger \bar{J}_{i+1}$.

Next, we establish idempotency of $N_{i+1} = N_i - \bar{J}_{i+1}^\dagger \bar{J}_{i+1}$:

$$N_{i+1}N_{i+1} = \underbrace{N_i N_i}_{= N_i} - \bar{J}_{i+1}^\dagger \underbrace{\bar{J}_{i+1} N_i}_{= \bar{J}_{i+1}} - \underbrace{N_i \bar{J}_{i+1}^\dagger}_{= \bar{J}_{i+1}^\dagger} \bar{J}_{i+1} + \underbrace{\bar{J}_{i+1}^\dagger \bar{J}_{i+1}}_{\text{Projection matrix}} \bar{J}_{i+1}^\dagger \bar{J}_{i+1}$$

$$= N_i - \bar{J}_{i+1}^\dagger \bar{J}_{i+1} - \bar{J}_{i+1}^\dagger \bar{J}_{i+1} + \bar{J}_{i+1}^\dagger \bar{J}_{i+1}$$

$$= N_i - \bar{J}_{i+1}^\dagger \bar{J}_{i+1}$$

$$= N_{i+1}$$

■

Joint Acceleration Derivation

$\bar{J}_1 = J_1$

$\bar{J}_k = J_k N_{k-1} \qquad 2 \le k \le l$

$N_k = \prod_{i=1}^{k}(I_n - \bar{J}_i^{\dagger}\bar{J}_i) \quad 1 \le k \le l$

$$\ddot{\boldsymbol{x}}_1 = J_1\ddot{\boldsymbol{\theta}} + \dot{J}_1\dot{\boldsymbol{\theta}}$$

$$\Downarrow$$

$$\ddot{\boldsymbol{\theta}} = \underbrace{\bar{J}_1^{\dagger}(\ddot{\boldsymbol{x}}_1 - \dot{J}_1\dot{\boldsymbol{\theta}})}_{\equiv\, \ddot{\boldsymbol{\theta}}_1} + \underbrace{N_1}_{= I_n - J_1^{\dagger}J_1}\boldsymbol{\eta}_1$$

$$\Downarrow$$

$$\ddot{\boldsymbol{x}}_2 = J_2\ddot{\boldsymbol{\theta}} + \dot{J}_2\dot{\boldsymbol{\theta}} = J_2(\ddot{\boldsymbol{\theta}}_1 + N_1\boldsymbol{\eta}_1) + \dot{J}_2\dot{\boldsymbol{\theta}}$$

$$\Downarrow$$

$$\boldsymbol{\eta}_1 = \underbrace{\bar{J}_2^{\dagger}}_{\bar{J}_2 = J_2 N_1}(\ddot{\boldsymbol{x}}_2 - \dot{J}_2\dot{\boldsymbol{\theta}} - J_2\ddot{\boldsymbol{\theta}}_1) + (I_n - \bar{J}_2^{\dagger}\bar{J}_2)\boldsymbol{\eta}_2$$

$$\Downarrow \quad \ddot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}}_1 + N_1\boldsymbol{\eta}_1$$

$$\ddot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}}_1 + \underbrace{N_1\bar{J}_2^{\dagger}}_{= \bar{J}_2^{\dagger}}(\ddot{\boldsymbol{x}}_2 - \dot{J}_2\dot{\boldsymbol{\theta}} - J_2\ddot{\boldsymbol{\theta}}_1) + \underbrace{N_2}_{= N_1(I_n - \bar{J}_2^{\dagger}\bar{J}_2)}\boldsymbol{\eta}_2$$

(cont'd)

$$\ddot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}}_1 + \underbrace{\bar{J}_2^{\dagger}\left(\ddot{\boldsymbol{x}}_2 - \dot{J}_2\dot{\boldsymbol{\theta}} - J_2\ddot{\boldsymbol{\theta}}_1\right)}_{\equiv\, \ddot{\boldsymbol{\theta}}_2} + N_2\boldsymbol{\eta}_2$$

$$\Downarrow$$

$$\ddot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}}_1 + \ddot{\boldsymbol{\theta}}_2 + N_2\boldsymbol{\eta}_2$$

$$\Downarrow$$

$$\ddot{\boldsymbol{x}}_3 = J_3\ddot{\boldsymbol{\theta}} + \dot{J}_3\dot{\boldsymbol{\theta}} = J_3\left(\ddot{\boldsymbol{\theta}}_1 + \ddot{\boldsymbol{\theta}}_2 + N_2\boldsymbol{\eta}_2\right) + \dot{J}_3\dot{\boldsymbol{\theta}}$$

$$\Downarrow$$

$$\boldsymbol{\eta}_2 = \bar{J}_3^{\dagger}\left(\ddot{\boldsymbol{x}}_3 - \dot{J}_3\dot{\boldsymbol{\theta}} - J_3\left(\ddot{\boldsymbol{\theta}}_1 + \ddot{\boldsymbol{\theta}}_2\right)\right) + \left(I_n - \bar{J}_3^{\dagger}\bar{J}_3\right)\boldsymbol{\eta}_3$$

$$\Downarrow \quad \ddot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}}_1 + \ddot{\boldsymbol{\theta}}_2 + N_2\boldsymbol{\eta}_2$$

$$\ddot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}}_1 + \ddot{\boldsymbol{\theta}}_2 + \underbrace{\underbrace{N_2\bar{J}_3^{\dagger}}_{=\bar{J}_3^{\dagger}}\left(\ddot{\boldsymbol{x}}_3 - \dot{J}_3\dot{\boldsymbol{\theta}} - J_3\left(\ddot{\boldsymbol{\theta}}_1 + \ddot{\boldsymbol{\theta}}_2\right)\right)}_{\equiv\, \ddot{\boldsymbol{\theta}}_3} + \underbrace{N_3}_{= N_2\left(I_n - \bar{J}_3^{\dagger}\bar{J}_3\right)}\boldsymbol{\eta}_3 \quad = \ddot{\boldsymbol{\theta}}_1 + \ddot{\boldsymbol{\theta}}_2 + \ddot{\boldsymbol{\theta}}_3 + N_3\boldsymbol{\eta}_3$$

$$\Downarrow$$

$$\vdots$$

Acceleration for All l Tasks

Terminate the steps at l by setting $\boldsymbol{\eta}_l = 0$.

$$\ddot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}}_1 + \cdots + \ddot{\boldsymbol{\theta}}_l$$

where

$$\ddot{\boldsymbol{\theta}}_1 = \bar{J}_1^{\dagger}\left(\ddot{\boldsymbol{x}}_1 - \dot{J}_1\dot{\boldsymbol{\theta}}\right)$$

$$\ddot{\boldsymbol{\theta}}_k = \bar{J}_k^{\dagger}\left(\ddot{\boldsymbol{x}}_k - \dot{J}_k\dot{\boldsymbol{\theta}} - J_k \sum_{i=1}^{k-1} \ddot{\boldsymbol{\theta}}_i\right), \;\; 2 \le k \le l$$

We first establish some properties of $N_k, \bar{J}_k, \bar{J}_k^{\dagger}$ in order to understand the meaning of the above form before designing a control policy.

Further Simplification of N_k

Define the *augmented Jacobian*:

$$L_k = \begin{pmatrix} J_1 \\ \vdots \\ J_k \end{pmatrix} \quad \Longrightarrow \quad \text{null}(L_k) = \text{null}(J_1) \cap \cdots \cap \text{null}(J_k)$$

$$M_k = I_n - L_k^\dagger L_k$$ projection onto $\text{null}(L_k)$

Recall by Lemma 3

$$N_k = I_n - \sum_{j=1}^{k} \bar{J}_j^\dagger \bar{J}_j$$

Proof of Theorem 4 (cont'd)

$M_k = I_n - L_k^\dagger L_k$

$N_k = I_n - \sum_{j=1}^{k} \bar{J}_j^\dagger \bar{J}_j$

Theorem 4 $N_k = M_k$ for $1 \le k \le l$.

Proof Use mathematical induction.

- For $k = 1$, $L_1 = J_1 = \bar{J}_1$. That $M_1 = N_1 = I_n - L_1^\dagger L_1$ trivially follows.
- Assume that $M_k = N_k$. Note that

$$\text{null}(L_k) = \text{row}(M_k) = \text{row}(N_k)$$

Below we prove $M_{k+1} = N_{k+1}$. Since both are projections that do not change any vector in their row spaces, it suffices to show $\text{row}(M_{k+1}) = \text{row}(N_{k+1})$.

First, we make use of $N_{k+1} = N_k - \bar{J}_{k+1}^\dagger \bar{J}_{k+1}$ established by Lemma 3:

$$\begin{aligned}
\text{row}(N_{k+1}) &= \text{row}\left(N_k - \bar{J}_{k+1}^\dagger \bar{J}_{k+1}\right) && \text{by definition} \\
&= \text{row}\left(N_k\left(I_n - \bar{J}_{k+1}^\dagger \bar{J}_{k+1}\right)\right) && \text{since } N_k \bar{J}_{k+1}^\dagger = \bar{J}_{k+1}^\dagger \\
&\subseteq \text{row}\left(I_n - \bar{J}_{k+1}^\dagger \bar{J}_{k+1}\right) && \text{since row}(AB) \subseteq \text{row}(B) \\
&= \text{null}(\bar{J}_{k+1})
\end{aligned}$$

Proof of Theorem (cont'd)

$$\text{row}(N_{k+1}) = \text{row}\left(\left(I_n - \bar{J}_{k+1}^{\dagger}\bar{J}_{k+1}\right)N_k\right) \qquad \text{since } \bar{J}_{k+1}N_k = \bar{J}_{k+1}.$$

$$\subseteq \text{row}(N_k) = \text{row}(M_k) \qquad \text{by assumption}$$

$$= \text{null}(L_k)$$

It then follows that

$$\text{row}(N_{k+1}) = \text{null}(\bar{J}_{k+1}) \cap \text{null}(L_k)$$

Finally, let us establish that

$\text{null}(\bar{J}_{k+1}) \cap \text{null}(L_k) = \text{null}(J_{k+1}) \cap \text{null}(L_k)$, which is the same as $\text{null}(L_{k+1})$.

This is shown below:

$$\text{null}(\bar{J}_{k+1}) \cap \text{null}(L_k) = \{\boldsymbol{v} \mid J_{k+1}N_k\boldsymbol{v} = 0 \wedge \boldsymbol{v} \in \text{row}(N_k)\} \qquad \text{since null}(L_k) = \text{row}(N_k)$$

$$= \{\boldsymbol{v} \mid J_{k+1}\boldsymbol{v} = 0 \wedge \boldsymbol{v} \in \text{row}(N_k)\} \qquad \text{since } N_k\boldsymbol{v} = \boldsymbol{v} \text{ for } \boldsymbol{v} \in \text{row}(N_k)$$

$$= \text{null}(J_{k+1}) \cap \text{null}(L_k)$$

We have thus shown $\text{row}(N_{k+1}) = \text{null}(L_{k+1}) = \text{row}(M_{k+1})$. ∎

Orthogonality of Lower Order J_i & Higher Order N_k

Corollary 5 $J_i N_k = 0$ for $1 \le i \le k \le l$ and $J_i J_k^\dagger = 0$ for $1 \le i < k \le l$.

Proof It follows from Theorem 4 that

$$N_k \equiv I_n - \begin{pmatrix} J_1 \\ \vdots \\ J_k \end{pmatrix}^\dagger \begin{pmatrix} J_1 \\ \vdots \\ J_k \end{pmatrix}$$

which implies

$$\text{col}(N_k) = \text{null}(J_1) \cap \cdots \cap \text{null}(J_k)$$

Hence, $J_i N_k = 0$ if $i \le k$.

When $i < k$, we have

$$\text{col}(J_k^\dagger) = \text{col}\left((J_k N_{k-1})^\dagger\right) = \text{row}(J_k N_{k-1})$$
$$\subseteq \text{row}(N_{k-1}) = \text{col}(N_{k-1})$$

Since $\text{row}(J_i) \perp \text{col}(N_{k-1})$ from $J_i N_{k-1} = 0$, we have $\text{row}(J_i) \perp \text{col}(J_k^\dagger)$ and thus $J_i J_k^\dagger = 0$. ■

Dynamics Summary

$$\boldsymbol{\tau} = M\ddot{\boldsymbol{\theta}} + C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) - \boldsymbol{\tau}_{\text{ext}}$$

under constraints $\dot{\boldsymbol{x}}_k = J_k\dot{\boldsymbol{\theta}}, 1 \le k \le l$ in the decreasing order as k increases.

where $$\ddot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}}_1 + \cdots + \ddot{\boldsymbol{\theta}}_l$$

with $$\ddot{\boldsymbol{\theta}}_1 = \bar{J}_1^{\dagger}(\ddot{\boldsymbol{x}}_1 - \dot{J}_1\dot{\boldsymbol{\theta}})$$

$$\ddot{\boldsymbol{\theta}}_k = \underbrace{\bar{J}_k^{\dagger}\left(\ddot{\boldsymbol{x}}_k - \dot{J}_k\dot{\boldsymbol{\theta}} - J_k\sum_{i=1}^{k-1}\ddot{\boldsymbol{\theta}}_i\right)}, \quad 2 \le k \le l$$

Projections onto the null spaces of $J_1, \dots, J_{k-1}$
so tasks at levels $1, \dots, k-1$ are not affected.

$$\bar{J}_1 = J_1, \quad \bar{J}_k = J_k N_{k-1}, \quad N_{k-1} = I_n - \begin{pmatrix} J_1 \\ \vdots \\ J_{k-1} \end{pmatrix}^{\dagger} \begin{pmatrix} J_1 \\ \vdots \\ J_{k-1} \end{pmatrix}, \quad 2 \le k \le l$$

Control Design

Apply torque to track some desired task trajectories $\boldsymbol{x}_{1d}, \dots, \boldsymbol{x}_{ld}$ in decreasing priority.

$$\boldsymbol{\tau} = M\boldsymbol{\alpha} + C(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) - \boldsymbol{\tau}_{\text{ext}}$$

Idea: Design $\boldsymbol{\alpha}$ in terms of $\boldsymbol{x}_{1d}, \dots, \boldsymbol{x}_{ld}$ and actual trajectories $\boldsymbol{x}_1, \dots, \boldsymbol{x}_l$.

a) Let $\boldsymbol{\alpha}$ assume the same structure of $\ddot{\boldsymbol{\theta}}$:

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_l$$

Let $\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_l$ (to be designed) correspond to $\ddot{\boldsymbol{x}}_1, \cdots, \ddot{\boldsymbol{x}}_l$:

$$\boldsymbol{\alpha}_1 = J_1^{\dagger}\left(\boldsymbol{\beta}_1 - \dot{J}_1\dot{\boldsymbol{\theta}}\right)$$

$$\boldsymbol{\alpha}_k = \bar{J}_k^{\dagger}\left(\boldsymbol{\beta}_k - \dot{J}_k\dot{\boldsymbol{\theta}} - J_k \sum_{i=1}^{k-1} \boldsymbol{\alpha}_i\right), \quad 2 \le k \le l$$

Note that $\dot{\boldsymbol{\theta}}$ can be measured so J_k and N_k are evaluated using its value.

Tracking the Top-Level Trajectory

b) Subtract the dynamics equation from the controller:

$$\left.\begin{aligned}\boldsymbol{\tau} &= M\ddot{\boldsymbol{\theta}} + C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) - \boldsymbol{\tau}_{\text{ext}} \\ \boldsymbol{\tau} &= M\boldsymbol{\alpha} + C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) - \boldsymbol{\tau}_{\text{ext}}\end{aligned}\right\} \Longrightarrow M\boldsymbol{\alpha} = M\ddot{\boldsymbol{\theta}} \Longrightarrow \boldsymbol{\alpha} = \ddot{\boldsymbol{\theta}}$$

Expand $\boldsymbol{\alpha}$ and $\ddot{\boldsymbol{\theta}}$:

$$\sum_{k=1}^{l} \bar{J}_k^{\dagger}\left(\boldsymbol{\beta}_k - \dot{J}_k\dot{\boldsymbol{\theta}} - J_k \sum_{i=1}^{k-1} \boldsymbol{\alpha}_i\right) = \sum_{k=1}^{l} \bar{J}_k^{\dagger}\left(\ddot{\boldsymbol{x}}_k - \dot{J}_k\dot{\boldsymbol{\theta}} - J_k \sum_{i=1}^{k-1} \ddot{\boldsymbol{\theta}}_i\right) \qquad \left(\textstyle\sum_{i=1}^{0} \boldsymbol{\alpha}_i = \sum_{i=1}^{0} \ddot{\boldsymbol{\theta}}_i = 0\right)$$

c) Multiply both sides of the above equation with J_1. Applying Corollary 5, this will zero out all summands with indices $k > 1$ on both sides, yielding

$$J_1 J_1^{\dagger}(\boldsymbol{\beta}_1 - \dot{J}_1\dot{\boldsymbol{\theta}}) = J_1 J_1^{\dagger}(\ddot{\boldsymbol{x}}_1 - \dot{J}_1\dot{\boldsymbol{\theta}})$$

Top-Level Trajectory

$$J_1 J_1^{\dagger}(\boldsymbol{\beta}_1 - \dot{J}_1 \dot{\boldsymbol{\theta}}) = J_1 J_1^{\dagger}(\ddot{\boldsymbol{x}}_1 - \dot{J}_1 \dot{\boldsymbol{\theta}})$$

$$\Downarrow$$

$$J_1 J_1^{\dagger} \boldsymbol{\beta}_1 = J_1 J_1^{\dagger} \ddot{\boldsymbol{x}}_1$$

$\Downarrow$ $J_1 J_1^{\dagger} = I_{m_1}$ since J_1 has full rank.

$$\boldsymbol{\beta}_1 = \ddot{\boldsymbol{x}}_1$$

d) We can track $\boldsymbol{x}_{1d}$ asymptotically by setting

$$\boldsymbol{\beta}_1 = \ddot{\boldsymbol{x}}_{1d} + K_{1v}(\dot{\boldsymbol{x}}_{1d} - \dot{\boldsymbol{x}}_1) + K_{1p}(\boldsymbol{x}_{1d} - \boldsymbol{x}_1)$$

Tracking Trajectories at All Levels

e) Repeat the steps c) and d) as k increases from 2 to l. For each k, make use of $\boldsymbol{\beta}_i = \ddot{\boldsymbol{x}}_i$ and $\boldsymbol{\alpha}_i = \ddot{\boldsymbol{\theta}}_i$, for $1 \le i \le k-1$ to establish $\boldsymbol{\beta}_k = \ddot{\boldsymbol{x}}_k$.

$$\bar{J}_k \bar{J}_k^{\dagger}(\boldsymbol{\beta}_k - \ddot{\boldsymbol{x}}_k) = 0$$

- If the kth task is independent of all $k-1$ tasks with higher priorities, then $\bar{J}_k$ has full rank and $\boldsymbol{x}_{kd}$ is tracked by setting

$$\boldsymbol{\beta}_k = \ddot{\boldsymbol{x}}_{kd} + K_{kv}(\dot{\boldsymbol{x}}_{kd} - \dot{\boldsymbol{x}}_k) + K_{kp}(\boldsymbol{x}_{kd} - \boldsymbol{x}_k)$$

- If the kth task depends on some the $k-1$ tasks with higher priorities, then $\bar{J}_k$ does not have full rank. The trajectory $\boldsymbol{x}_{kd}$ cannot be tracked.

$$\bar{J}_k \bar{J}_k^{\dagger}(\ddot{\boldsymbol{x}}_{kd} - \ddot{\boldsymbol{x}}_k) = 0$$

Instead, the norm $\|\ddot{\boldsymbol{x}}_{kd} - \ddot{\boldsymbol{x}}_k\|$ is minimized in the null space of all higher priority tasks.

Application 4: Null Space Impedance Control

Follow a task trajectory $\boldsymbol{x}_{1d}(t)$ (top priority) while exhibiting some impedance behavior (secondary priority).

Let

$$\boldsymbol{x}_1 = \boldsymbol{x}_{1d}$$

$$\boldsymbol{x}_2 = \boldsymbol{\theta} \quad \Longrightarrow \quad J_2 = I_n \quad \Longrightarrow \quad \bar{J}_2 = J_2 N_1 = N_1$$

$$\Longrightarrow \quad \ddot{\boldsymbol{\theta}} = J_1^{\dagger}\left(\ddot{\boldsymbol{x}}_1 - \dot{J}_1\dot{\boldsymbol{\theta}}\right) + N_1\ddot{\boldsymbol{x}}_2$$

Set impedance as

$$\boldsymbol{\beta}_2 = \ddot{\boldsymbol{\theta}}_d + M_d^{-1}\left(\boldsymbol{\tau}_{\text{ext}} + B_d\left(\dot{\boldsymbol{\theta}}_d - \dot{\boldsymbol{\theta}}\right) + K_d(\boldsymbol{\theta}_d - \boldsymbol{\theta})\right)$$

- $\boldsymbol{\tau}_{\text{ext}}$ is compensated (via sensing). Set $\boldsymbol{\tau} = M\boldsymbol{\alpha} + C\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) - \boldsymbol{\tau}_{\text{ext}}$ where

$$\boldsymbol{\alpha} = J_1^{\dagger}\left(\ddot{\boldsymbol{x}}_1 - \dot{J}_1\dot{\boldsymbol{\theta}}\right) + N_1\left(\ddot{\boldsymbol{\theta}}_d + M_d^{-1}\left(\boldsymbol{\tau}_{\text{ext}} + B_d\left(\dot{\boldsymbol{\theta}}_d - \dot{\boldsymbol{\theta}}\right) + K_d(\boldsymbol{\theta}_d - \boldsymbol{\theta})\right)\right)$$

$$\Downarrow \quad \boldsymbol{\theta}_e = \boldsymbol{\theta}_d - \boldsymbol{\theta}$$

$$\ddot{\boldsymbol{x}}_1 = \ddot{\boldsymbol{x}}_{1d} \qquad N_1\left(\ddot{\boldsymbol{\theta}}_e + M_d^{-1}\left(B_d\dot{\boldsymbol{\theta}}_e + K_d\boldsymbol{\theta}_e\right) + M_d^{-1}\boldsymbol{\tau}_{\text{ext}}\right) = 0$$

Application 4 (cont'd)

- $\boldsymbol{\tau}_{\text{ext}}$ is not available. Set $\boldsymbol{\tau} = M\boldsymbol{\alpha} + C(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta})$ where

$$\boldsymbol{\alpha} = J_1^{\dagger}(\ddot{\boldsymbol{x}}_1 - \dot{J}_1\dot{\boldsymbol{\theta}}) + N_1\left(\ddot{\boldsymbol{\theta}}_d + M_d^{-1}(B_d\dot{\boldsymbol{\theta}}_e + K_d\boldsymbol{\theta}_e)\right)$$

$$\Downarrow$$

$$\ddot{\boldsymbol{x}}_1 - \ddot{\boldsymbol{x}}_{1d} = J_1 M^{-1}\boldsymbol{\tau}_{\text{ext}} \qquad N_1\left(\ddot{\boldsymbol{\theta}}_e + M_d^{-1}(B_d\dot{\boldsymbol{\theta}}_e + K_d\boldsymbol{\theta}_e) - M^{-1}\boldsymbol{\tau}_{\text{ext}}\right) = 0$$

In $\boldsymbol{\alpha}_1$, if we choose $M_d = M$, and use the dynamically consistent generalized inverse $J_1{}^{*} \equiv M^{-1}\, J_1{}^{T}\left(J_1 M^{-1} J_1{}^{T}\right)^{-1}$ instead of $J_1^{\dagger}$ and $I_n - J_1{}^{*}J_1$ instead of N_1, then we can show

$$(I_n - J_1{}^{*}J_1)^T\left(M\ddot{\boldsymbol{\theta}}_e + B_d\dot{\boldsymbol{\theta}}_e + K_d\,\boldsymbol{\theta}_e + \boldsymbol{\tau}_{\text{ext}}\right) = 0$$

V. Augmented Torque Projection

$\boldsymbol{\tau}_1 \in \mathbb{R}^{m_1}$: torque on the first priority level

$\vdots$

$\boldsymbol{\tau}_l \in \mathbb{R}^{m_l}$: torque on the lth priority level

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \sum_{k=2}^{r} \widetilde{\boldsymbol{N}}_k(\boldsymbol{\theta})\boldsymbol{\tau}_k$$

where

$$\widetilde{N}_k = I_n - L_k^T \left(L_k^{\#}\right)^T \text{ with } L_k = \begin{pmatrix} J_1 \\ \vdots \\ J_k \end{pmatrix}$$

Iterative computation [3], [6]:

$$\widetilde{N}_1 = I_n$$
$$\tilde{J}_k(\boldsymbol{\theta}) = J_k(\boldsymbol{\theta})\widetilde{N}_k(\boldsymbol{\theta})^T$$
$$\widetilde{N}_k = \widetilde{N}_{k-1}\left(I_n - \tilde{J}_{k-1}(\boldsymbol{\theta})^T \left(\tilde{J}_{k-1}(\boldsymbol{\theta})^{\#}\right)^T\right)$$

♦ Enforces orthogonality of all involved tasks.

♠ Singularities in L_k needs some special treatment.

Application 5: Secondary Task

We can set $\boldsymbol{\tau}_1$ to achieve, for instance, impedance control in the workspace.

Assign a secondary task using the control law [7]:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \left(I_n - J_1^T J_1^{T\#}\right)\left(\boldsymbol{u} \underbrace{- K_d \dot{\boldsymbol{\theta}}}_{\text{Damping torque}}\right)$$

- Maximize the manipulability index.

$$\boldsymbol{u} = K_m \left(\frac{\partial}{\partial \boldsymbol{\theta}}\left(\sqrt{\det(JJ^T)}\right)\right)^T$$

- Or, minimize the dynamic conditioning (DC) index ω, which is the least-squares difference between the generalized inertia matrix and some diagonal matrix.

$$\boldsymbol{u} = -K_c \left(\frac{\partial \omega(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)^T$$

Successive Torque Projection

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \sum_{k=2}^{r} \breve{\boldsymbol{N}}_k(\boldsymbol{\theta})\boldsymbol{\tau}_k$$

where

$$\breve{N}_2 = I_n - J_2^T \left(J_2^{\#}\right)^T$$

$$\breve{N}_k = \breve{N}_{k-1} \left(I_n - J_{k-1}^T (J_{k-1}^{\#})^T\right), \qquad k > 2$$

Iterative computation [6].

- ♠ Task hierarch is not strict. Multiplication ensures orthogonality with the task at the current level k but corrupts all preceding projections.
- ♦ Computationally efficient.
- ♦ Singularities are avoided.

References

1. John M. Hollerbach and Ki C. Suh. Redundancy resolution of manipulators through torque optimization. *IEEE J. Robot. Automat.*, vol. 3, no. 4, pp. 308-316, 1987.

2. Oussama Khatib. "**Motion/Force Redundancy of Manipulators**" , *Proc. of the Japan-USA Symposium on Flexible Automation*, 1, 337--342, Kyoto, Japan, July, 1990.

3. B. Siciliano and J. J. Slotine. Synthesis of whole-body behaviors through hierarchical control of behavioral primitives. *Proc. Int. Conf. Advanced Robots,* pp. 1211-1216, 1991.

4. Oussama Khatib. Inertial properties in robotic manipulation: An object-level framework. *Int. J. Robot. Res*, vol. 14, no. 4, pp. 19-36, 1995.

5. Hamid Sadeghian, Luigi Villani, Mehdi Keshmiri, and Bruno Siciliano. Dynamic multi-priority control in redundant robotic systems. *Robotica*, vol. 31, pp. 1155-1167, 2013.

6. Alexander Dietrich, Christian Ott, and Alin Albu-Schäffer. An overview of null-space projections for redundant, torque-controlled robots. *Int. J. Robot. Res*, vol. 34, no. 11, pp. 1385-1400, 2015.

7. Fanny Ficuciello, Luigi Villani, and Bruno Siciliano. Variable Impedance control of redundant manipulators for intuitive human-robot physical interaction. *IEEE Trans. Robot.*, vol. 31, no. 4, pp. 850-863, 2015.