Tutorial on Robotic Manipulator Control

(Part II: Impedance & Hybrid Controls)

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Outline

I. Force Control

II. Impedance control

III. Hybrid control

I. Force Control (Simple)

Assume $\ddot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}} = \mathbf{0}$.

$$M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + C\big(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\big)\dot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}) + J^T\boldsymbol{f} = \boldsymbol{\tau}$$ (dynamics)

$$\Big\Downarrow \quad \ddot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}} = \mathbf{0}$$

$$V(\boldsymbol{\theta}) + J^T\boldsymbol{f} = \boldsymbol{\tau}$$ (reduced dynamics)

Introduce

$\boldsymbol{f}_d$: desired force

$\boldsymbol{f}_e = \boldsymbol{f}_d - \boldsymbol{f}$: force error vector

Controller (from replacing $\boldsymbol{f}$ in the reduced dynamics with a PID servo):

$$\boldsymbol{\tau} = V(\boldsymbol{\theta}) + J^T\left(\boldsymbol{f}_d + K_p\boldsymbol{f}_e + K_v\dot{\boldsymbol{f}}_e + K_i\int \boldsymbol{f}_e\,dt\right)$$

$\Big\Downarrow$ 1. equate with $V(\boldsymbol{\theta}) + J^T\boldsymbol{f} = \boldsymbol{\tau}$
2. left multiply with J

$$\boldsymbol{f}_e + K_p\boldsymbol{f}_e + K_v\dot{\boldsymbol{f}}_e + K_i\int \boldsymbol{f}_e\,dt = 0$$ (error dynamics)

Force Control (General)

- We can control any physical entity $\boldsymbol{p}$ that is linearly related to the joint angles $\boldsymbol{\theta}$ or the end-effector position $\boldsymbol{x}$.
  - Express $\ddot{\boldsymbol{\theta}}$ (or $\ddot{\boldsymbol{x}}$) in terms of $\ddot{\boldsymbol{p}}$, $\dot{\boldsymbol{\theta}}$, and $\boldsymbol{\theta}$ (or $\ddot{\boldsymbol{p}}$, $\dot{\boldsymbol{x}}$, and $\boldsymbol{x}$).
  - Substitute a servo expression for $\ddot{\boldsymbol{p}}$ that is in terms of some desired value $\boldsymbol{p}_d$ and its first two derivatives.
- To realize force control, linear stiffness provides such a relation in the task space:

stiffness matrix

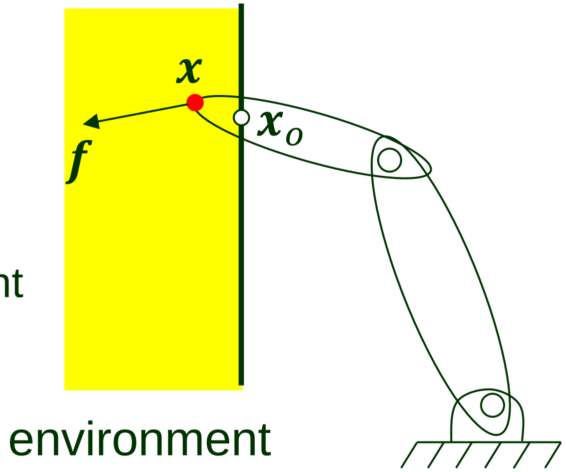
force exerted on environment

$$\boldsymbol{f} = K_o(\boldsymbol{x} - \boldsymbol{x}_o)$$

static location of environment

$$\Longrightarrow \quad \ddot{\boldsymbol{f}} = K_o \ddot{\boldsymbol{x}}$$

$$\Longrightarrow \quad \ddot{\boldsymbol{x}} = K_o^{-1} \ddot{\boldsymbol{f}}$$



Force Control (cont'd)

$\boldsymbol{f}_d$: desired force

$\boldsymbol{f}_e = \boldsymbol{f}_d - \boldsymbol{f}$: force error vector

Conducted in the task space:

$$\ddot{\boldsymbol{\theta}} = J^{\dagger}\left(\ddot{\boldsymbol{x}} - \dot{J}\dot{\boldsymbol{\theta}}\right)$$

$$\ddot{\boldsymbol{x}} = K_o^{-1}\ddot{\boldsymbol{f}} \quad \Downarrow$$

$$\ddot{\boldsymbol{\theta}} = J^{\dagger}\left(K_o^{-1}\ddot{\boldsymbol{f}} - \dot{J}\dot{\boldsymbol{\theta}}\right)$$

Construct a servo using PD control $\Downarrow$

$$\boldsymbol{a} = J^{\dagger}\left(K_o^{-1}(\ddot{\boldsymbol{f}}_d + K_v\dot{\boldsymbol{f}}_e + K_p\boldsymbol{f}_e) - \dot{J}\dot{\boldsymbol{\theta}}\right)$$

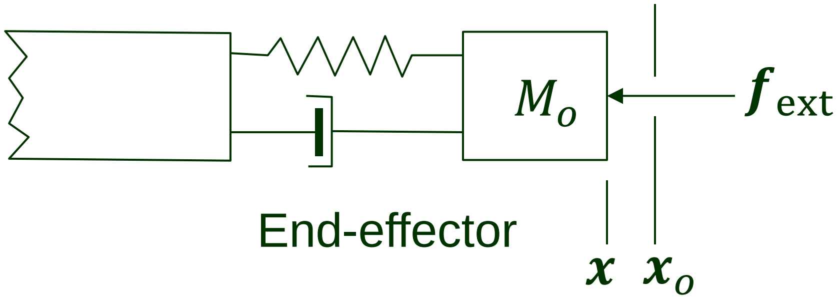
Equate & left multiply with J $\Longrightarrow$

$$\ddot{\boldsymbol{f}}_e + K_v\dot{\boldsymbol{f}}_e + K_p\boldsymbol{f}_e = \boldsymbol{0}$$

(error dynamics)

II. Impedance Control

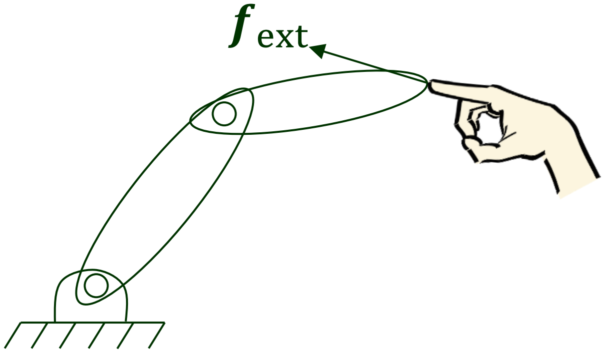
Simulate the dynamic behavior of a mass-spring-damper.



Impedance: ratio of force output to displacement (motion) input

M_o: effective mass

$\boldsymbol{f}_{\text{ext}}$: external force applied by the environment (e.g., human) and measured by a force/torque (F/T) sensor if exists.

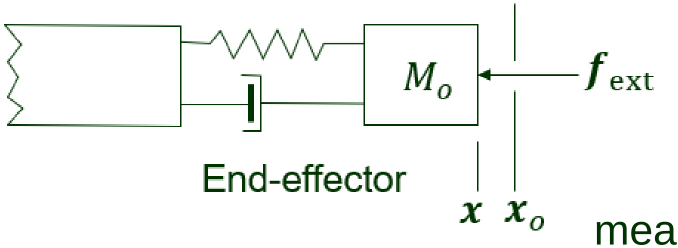


Haptics: Create desired sensation via force interaction

$\boldsymbol{f} = -\boldsymbol{f}_{\text{ext}}$: force applied by the end-effector on the environment.

$\boldsymbol{x}_o$: end-effector equilibrium position in the absence of $\boldsymbol{f}_{\text{ext}}$

Targeted Compliant Behavior



Goal: Control the robot such that, from the perspective of the party applying $\boldsymbol{f}_{\text{ext}}$, the robot's behavior mimics that of a system with mass M_o, stiffness K_v, and damping K_v.

measurable

$$M_o(\ddot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}_o) + K_v(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_o) + K_p(\boldsymbol{x} - \boldsymbol{x}_o) = \boldsymbol{f}_{\text{ext}}$$

effective mass matrix

preset

damping

preset

stiffness

preset (known)

- Construct the torque $\boldsymbol{\tau}$ by determining either $\boldsymbol{f}_{\text{ext}}$ from $\ddot{\boldsymbol{x}}$ or $\ddot{\boldsymbol{x}}$ from $\boldsymbol{f}_{\text{ext}}$.
- Treat the environment as an admittance, and the manipulator as an impedance.
- When interacting with a surface, set $\boldsymbol{x}_o$ to "slightly penetrate" inside the surface.

II.1 With Position Sensing (Force-Based)

Generate torque to control impedance:

$$\boldsymbol{x} \quad \longrightarrow \quad \boldsymbol{f} = -\boldsymbol{f}_{\text{ext}} \quad \longrightarrow \quad \boldsymbol{\tau}$$

(displacement input)

(force output)

(torque to account for $\boldsymbol{f}$ and realize the impedance)

- Compute $\boldsymbol{f}$ as

$$-\boldsymbol{f} = M_o(\ddot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}_{\boldsymbol{o}}) + K_v(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_{\boldsymbol{o}}) + K_p(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{o}})$$

The trajectory of the nominal interaction point $\boldsymbol{x}_{\boldsymbol{o}}$ is available. (Often the point is not moving.)

- Next, compute $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = M(\boldsymbol{\theta})J^{\dagger}\left(\ddot{\boldsymbol{x}} - \dot{J}\dot{\boldsymbol{\theta}}\right) + C\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}) + J^T\boldsymbol{f}$$

♣ Force indirectly assigned by controlling position $\boldsymbol{x}$.

♣ Small K_p and large M_o in directions expecting contact .

♣ Large K_p and small M_o in directions that are free.

II.2 With F/T Sensing (Position-Based)

F/T sensor $\longrightarrow$ $\boxed{\boldsymbol{f}_{\text{ext}} \longrightarrow \ddot{\boldsymbol{x}}} \longrightarrow \boldsymbol{\tau}$

The objective now is to implement the targeted force-displacement behavior.

- $M_o(\ddot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}_{\boldsymbol{o}}) + K_v(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_{\boldsymbol{o}}) + K_p(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{o}}) = \boldsymbol{f}_{\text{ext}}$

$$\Downarrow$$

$$\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}_{\boldsymbol{o}} + M_o^{-1}\left(\boldsymbol{f}_{\text{ext}} - K_v(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_{\boldsymbol{o}}) - K_p(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{o}})\right)$$

- Servo $\boldsymbol{a} = J^{\dagger}(\ddot{\boldsymbol{x}} - \dot{J}\dot{\boldsymbol{\theta}})$ // desired joint acceleration

$$= J^{\dagger}\left(\ddot{\boldsymbol{x}}_{\boldsymbol{o}} + M_o^{-1}\left(\boldsymbol{f}_{\text{ext}} - K_v(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_{\boldsymbol{o}}) - K_p(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{o}})\right) - \dot{J}\dot{\boldsymbol{\theta}}\right)$$

- Torque $\boldsymbol{\tau} = M(\boldsymbol{\theta})\boldsymbol{a} + C\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}) - J^T\boldsymbol{f}_{\text{ext}}$

// needed torque to generate a

♣ If no contact, $\boldsymbol{f}_{\text{ext}} = \boldsymbol{0}$.

$$\boldsymbol{a} = J^{\dagger}\left(\ddot{\boldsymbol{x}}_{\boldsymbol{o}} + M_o^{-1}\left(-K_v(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_{\boldsymbol{o}}) - K_p(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{o}})\right) - \dot{J}\dot{\boldsymbol{\theta}}\right)$$

Position control with velocity gain $-M_o^{-1}\,K_v$ and position gain $-M_o^{-1}\,K_p$!

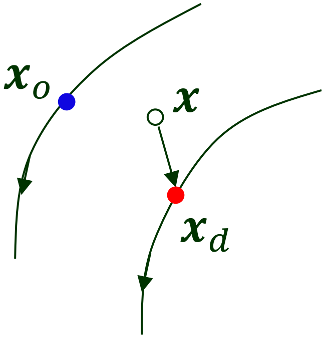
II.3 Admittance Control as a Variation

F/T sensor $\longrightarrow$ $\boldsymbol{f}_{\text{ext}} \longrightarrow \boldsymbol{x}_d \longrightarrow \boldsymbol{\tau}$

Position control

Instead of evaluating $\ddot{\boldsymbol{x}}$ to construct $\boldsymbol{\tau}$ directly, we treat $\ddot{\boldsymbol{x}}$ as $\ddot{\boldsymbol{x}}_d$ to realize the same targeted compliant behavior:

$$M_o(\ddot{\boldsymbol{x}}_d - \ddot{\boldsymbol{x}}_o) + K_v(\dot{\boldsymbol{x}}_d - \dot{\boldsymbol{x}}_o) + K_p(\boldsymbol{x}_d - \boldsymbol{x}_o) = \boldsymbol{f}_{\text{ext}} = -\boldsymbol{f}$$



1) Generate the desired trajectories $\boldsymbol{x}_d(t)$ and $\boldsymbol{\theta}_d(t)$.

$$\ddot{\boldsymbol{x}}_d = \ddot{\boldsymbol{x}}_o + M_o^{-1}\left(\boldsymbol{f}_{\text{ext}} - K_v(\dot{\boldsymbol{x}}_d - \dot{\boldsymbol{x}}_o) - K_p(\boldsymbol{x}_d - \boldsymbol{x}_o)\right)$$

$$\begin{tikzcd}
\ddot{\boldsymbol{x}}_d \arrow[d, Rightarrow, "\text{integrate}"] & \\
\dot{\boldsymbol{x}}_d \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\text{integrate}"'] & \dot{\boldsymbol{\theta}}_d = J^{\dagger}\dot{\boldsymbol{x}}_d \arrow[d, Rightarrow, "\text{integrate}"] \\
\boldsymbol{x}_d & \boldsymbol{\theta}_d
\end{tikzcd}$$

$\boldsymbol{x}_o$: desired equilibrium trajectory in the absence of the environmental force

$\boldsymbol{x}_d$: desired trajectory

Position Control in the Inner Loop

2) Use $\boldsymbol{x}_d(t)$ and $\boldsymbol{\theta}_d(t)$ for position control in the task space.

$$\boldsymbol{\tau} = M(\boldsymbol{\theta})\boldsymbol{a} + C(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}) - J^T \boldsymbol{f}_{\text{ext}}$$

$$\boldsymbol{a} = J^{\dagger}\left(\ddot{\boldsymbol{x}}_d + K_v \dot{\boldsymbol{x}}_e + K_p \boldsymbol{x}_e + K_i \int \boldsymbol{x}_e \, dt - \dot{J}\dot{\boldsymbol{\theta}}\right)$$

where $\boldsymbol{x}_e = \boldsymbol{x}_d - \boldsymbol{x}$ and $\boldsymbol{\theta}_e = \boldsymbol{\theta}_d - \boldsymbol{\theta}$.

$$\boldsymbol{f}_{\text{ext}} \longrightarrow \boldsymbol{x}_d \xrightarrow{\text{Position control}} \boldsymbol{\tau}$$

(force input) (desired displacement) (torque to track $\boldsymbol{x}_d$)

Admittance defines the displacement that results from a force input.

Goal: Let the robot control its motion in response to a (sensed) external force (i.e., let the robot yield to the force with a movement).

II.4 Impedance Control (Joint Space)

Dynamics:

$$M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}) = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{ext}} \qquad (1)$$

Torque applied by the environment

Desired impedance:

$$M_d(\ddot{\boldsymbol{\theta}} - \ddot{\boldsymbol{\theta}}_{\boldsymbol{o}}) + B_d(\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{\theta}}_{\boldsymbol{o}}) + K_d(\boldsymbol{\theta} - \boldsymbol{\theta}_{\boldsymbol{o}}) = \boldsymbol{\tau}_{\text{ext}} \qquad (2)$$

- $\boldsymbol{\tau}_{\text{ext}}$ can be measured. Apply the control

$$\boldsymbol{\tau} = M(\boldsymbol{\theta})\boldsymbol{\alpha} + C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}) - \boldsymbol{\tau}_{\text{ext}} \qquad (3)$$

where

$$\boldsymbol{\alpha} = \ddot{\boldsymbol{\theta}}_o + M_d^{-1}\left(\boldsymbol{\tau}_{\text{ext}} - B_d(\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{\theta}}_{\boldsymbol{o}}) - K_d(\boldsymbol{\theta} - \boldsymbol{\theta}_{\boldsymbol{o}})\right)$$

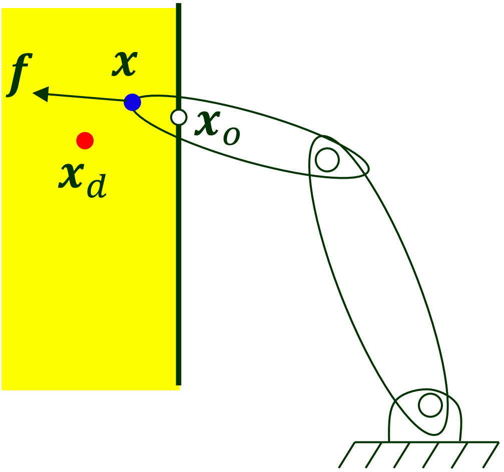
(3) + (1) ⇒ (2)

- $\boldsymbol{\tau}_{\text{ext}}$ is not available. (2) can be achieved only for $M_d = M(\boldsymbol{\theta})$ using the control law:

$$\boldsymbol{\tau} = M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_o - B_d(\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{\theta}}_{\boldsymbol{o}}) - K_d(\boldsymbol{\theta} - \boldsymbol{\theta}_{\boldsymbol{o}}) + C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}) \qquad (4)$$

(4) + (1) ⇒ (2)

II.5 Stiffness Control



Special case of impedance control

$\boldsymbol{x}_o$: static location of the environment (end-effector equilibrium position in the absence of environmental force)

$\boldsymbol{x}_d$: desired constant end-effector position

$\boldsymbol{x}$: actual position

$\boldsymbol{f} = K_o(\boldsymbol{x} - \boldsymbol{x}_o)$: force acting on the environment

Dynamics: $M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + C\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}) + J^{\boldsymbol{T}}\boldsymbol{f} = \boldsymbol{\tau}$

Control: $\boldsymbol{\tau} = J^T(\boldsymbol{\theta})\left(-K_v\dot{\boldsymbol{x}} + K_p(\boldsymbol{x}_d - \boldsymbol{x})\right) + V(\boldsymbol{\theta})$

$\dot{\boldsymbol{x}}_d = 0$

Closed-loop dynamics:

$$M\ddot{\boldsymbol{\theta}} + C\dot{\boldsymbol{\theta}} = J^T\left(-K_v\dot{\boldsymbol{x}} + K_p(\boldsymbol{x}_d - \boldsymbol{x}) + K_o(\boldsymbol{x}_o - \boldsymbol{x})\right)$$

Regulating Stiffness

$$M\ddot{\boldsymbol{\theta}} + C\dot{\boldsymbol{\theta}} = J^T\left(-K_v\dot{\boldsymbol{x}} + K_p(\boldsymbol{x}_d - \boldsymbol{x}) + K_o(\boldsymbol{x}_o - \boldsymbol{x})\right)$$

$$\Downarrow$$

$\dot{\boldsymbol{\theta}} = \mathbf{0}$ and $\ddot{\boldsymbol{\theta}} = \mathbf{0}$ in the steady state ($t \to \infty$)

$\dot{\boldsymbol{x}} = J\ \dot{\boldsymbol{\theta}}$

$$\lim_{t\to\infty}\left(K_p(\boldsymbol{x}_d - \boldsymbol{x}) + K_o(\boldsymbol{x}_o - \boldsymbol{x})\right) = \mathbf{0}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \quad \boldsymbol{x_d} = \begin{pmatrix} x_{d1} \\ \vdots \\ x_{dm} \end{pmatrix} \quad \boldsymbol{x_o} = \begin{pmatrix} x_{o1} \\ \vdots \\ x_{om} \end{pmatrix} \quad \Downarrow \quad K_p = \begin{pmatrix} k_{p1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & k_{pm} \end{pmatrix} \qquad K_o = \begin{pmatrix} k_{o1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & k_{om} \end{pmatrix}$$

$$\lim_{t\to\infty} x_i = \frac{k_{pi}x_{di} + k_{oi}x_{oi}}{k_{pi} + k_{oi}} \qquad 1 \leq i \leq m$$

$$\Downarrow$$

$$\lim_{t\to\infty} f_i = k_{oi}\,(x_i - x_{oi}) = \frac{k_{pi}k_{oi}}{k_{pi} + k_{oi}}(x_{di} - x_{oi})$$

$$\approx k_{pi}\,(x_{di} - x_{oi})$$

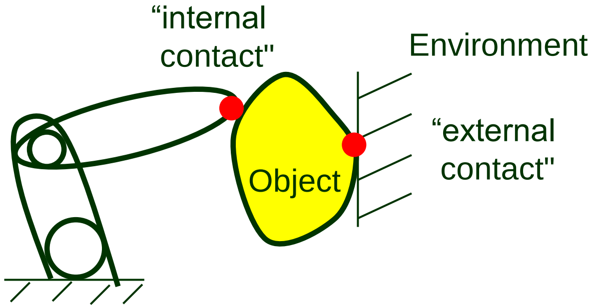
since $k_{oi} \gg k_{pi}$ (the environment is a lot stiffer than the end-effector)

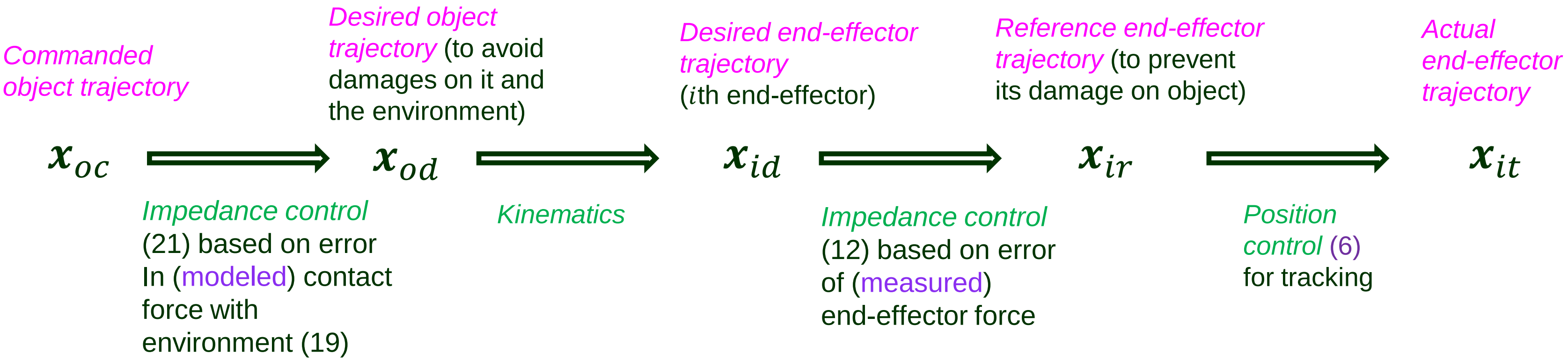
II.6 Impedance Control Example

D. Heck, D. Kostic, A. Densai, and H. Nijmeijer. Internal and external force-based impedance control for cooperative manipulation. *Proceedings of the European Control Conference*, pp. 2299-2304, 2013.

n manipulators manipulating a rigid object

- External force (object contact with the environment) is modeled..
- Interaction forces at the manipulator tips with the object are sensed and decomposed into: .
  - motion inducing forces.
  - “internal” forces in the null space that do not contribute to the motion
- Both “internal” and “external” forces are under impedance control.
- Desired values of “internal” and “external” forces are assumed available.





$x_{oc} \Rightarrow x_{od} \Rightarrow x_{id} \Rightarrow x_{ir} \Rightarrow x_{it}$

x_{oc}: *Commanded object trajectory*

$x_{oc} \Rightarrow x_{od}$: *Impedance control* (21) based on error In (*modeled*) contact force with environment (19)

x_{od}: *Desired object trajectory* (to avoid damages on it and the environment)

$x_{od} \Rightarrow x_{id}$: *Kinematics*

x_{id}: *Desired end-effector trajectory* (ith end-effector)

$x_{id} \Rightarrow x_{ir}$: *Impedance control* (12) based on error of (*measured*) end-effector force

x_{ir}: *Reference end-effector trajectory* (to prevent its damage on object)

$x_{ir} \Rightarrow x_{it}$: *Position control* (6) for tracking

x_{it}: *Actual end-effector trajectory*

III. Simple Hybrid Control

Goal: Control motion and force along orthogonal directions in the task space.

- ♠ Decouple the position and force control into subtasks via a task space formulation. .
- ♠ A direction can be specified with either a desired force or a desired position, but not both.

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \qquad \ddot{\boldsymbol{\theta}} = J^{\dagger}(\ddot{\boldsymbol{x}} - \dot{J}\dot{\boldsymbol{\theta}})$$

$$\Downarrow$$

Servo $$\boldsymbol{a} = J^{\dagger}\left(\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} - \dot{J}\dot{\boldsymbol{\theta}}\right)$$

- ♦ Position control along x_i with desired trajectory $x_{di}(t)$.

$$\alpha_i = \ddot{x}_{di} + k_{vi}(\dot{x}_{di} - \dot{x}_i) + k_{pi}(x_{di} - x_i)$$

Simple Hybrid Control (cont’d)

- Force control along x_j needs a stiffness model to relate force to position. Assume the environment can be modeled as a spring in the direction.

$$f_j = k_j\left(x_j - x_{oj}\right) \quad \Longrightarrow \quad \ddot{x}_j = k_j^{-1}\ddot{f}_j$$

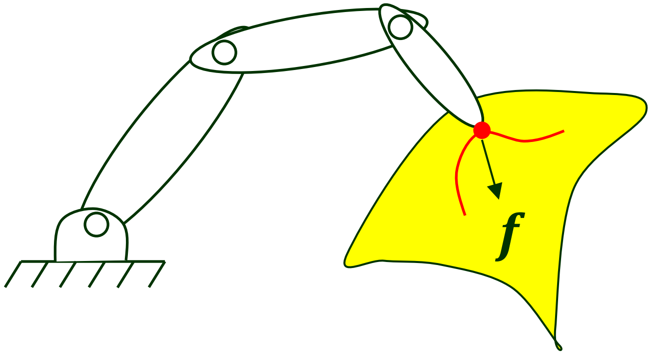
Stiffness in the
jth direction

Static location in
the jth direction

$$\alpha_j = \frac{1}{k_j}\left(\ddot{f}_{dj} + k_{vj}\left(\dot{f}_{dj} - \dot{f}_j\right) + k_{pj}\left(f_{dj} - f_j\right)\right)$$

General Hybrid Control

Goal Move the end-effector along a desired position trajectory on a (frictionless) surface while tracking a desired force trajectory exerted on the same surface.



Velocity and angular velocity: $\boldsymbol{v} = \dot{\boldsymbol{x}}$

m constraints: $\boldsymbol{\phi}(\boldsymbol{\theta}) = \mathbf{0}$

Manipulator torque: $\boldsymbol{\tau}$

Manipulator Jacobian: J

End-effector force: $\boldsymbol{f} = J^{-T}\boldsymbol{\tau}$

Applications: writing, painting, cleaning, etc.

Dynamics in the task space $\mathbb{R}^6$:

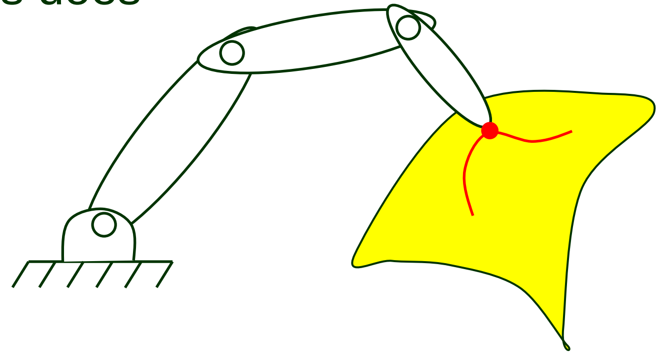
$$\Lambda(\boldsymbol{\theta})\dot{\boldsymbol{v}} + \underbrace{\boldsymbol{\Gamma}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{\eta}(\boldsymbol{\theta})}_{\boldsymbol{\mu}(\boldsymbol{\theta})} = \boldsymbol{f} - \boldsymbol{f}_{\mathrm{ee}}$$

Force and torque applied by the end-effector along the constraint surface normal

Constrained Surface

$$\boldsymbol{\phi}(\boldsymbol{\theta}) = \mathbf{0} \quad \Longrightarrow \quad J_\phi \dot{\boldsymbol{\theta}} = \mathbf{0} \qquad \text{where} \quad J_\phi = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\theta}}$$

Constraint force are normal to $\boldsymbol{\phi}(\boldsymbol{\theta}) = \mathbf{0}$ and thus does zero work:



$$\underbrace{\boldsymbol{\tau}_\phi^T \dot{\boldsymbol{\theta}} = \mathbf{0} \qquad\qquad \left(J_\phi^T\right)^T \dot{\boldsymbol{\theta}} = \mathbf{0}}_{\Downarrow}$$

$$\boldsymbol{\tau}_\phi = J_\phi^T \boldsymbol{\lambda}$$

It is exerted by the end-effector :

$$\boldsymbol{\tau}_\phi = J^T \boldsymbol{f}_{\text{ee}}$$

$$\Downarrow$$

$$\boldsymbol{f}_{\text{ee}} = J^{-T} \boldsymbol{\tau}_\phi = J^{-T} J_\phi^T \boldsymbol{\lambda} = S_f \boldsymbol{\lambda}$$

where $S_f \equiv J^{-T} J_\phi^T : \quad \lambda \mapsto \boldsymbol{f}_{\text{ee}}$ (maps constraint force to that on the end-effector)

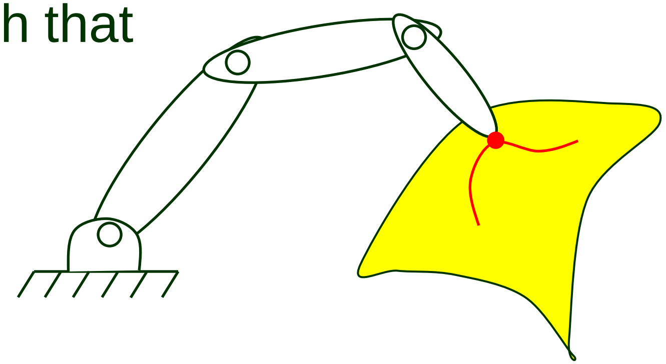
$$J_\phi \dot{\boldsymbol{\theta}} = \mathbf{0} \quad \Longrightarrow \quad J_\phi J^{-1} J \dot{\boldsymbol{\theta}} = \mathbf{0} \quad \Longrightarrow \quad S_f^T \boldsymbol{v} = \mathbf{0}$$

Reparameterization

$$\boldsymbol{\phi}(\boldsymbol{\theta}) = \mathbf{0}$$

$$\Downarrow$$

$\boldsymbol{r}$: $6 - m$ independent coordinates such that



$$\boldsymbol{\theta} = \boldsymbol{\rho}(\boldsymbol{r})$$

$$\Downarrow$$

$$\left.\begin{array}{l} \dot{\boldsymbol{\theta}} = J_\rho \dot{\boldsymbol{r}} \\ J_\phi \dot{\boldsymbol{\theta}} = \mathbf{0} \end{array}\right\} \Longrightarrow J_\phi J_\rho \dot{\boldsymbol{r}} = \mathbf{0}$$

$\Downarrow$ $\dot{\boldsymbol{r}}$ is arbitrary

$$J_\phi J_\rho = \mathbf{0}$$

$$\boldsymbol{v} = J\dot{\boldsymbol{\theta}} = JJ_\rho \dot{\boldsymbol{r}} = S_v \dot{\boldsymbol{r}}$$

where $S_v \equiv JJ_\rho$: $\dot{\boldsymbol{r}} \mapsto \boldsymbol{v}$ (maps change rates of independent coordinates to velocity in the task space)

Selection Matrices

S_f, S_v are called selection matrices.

$$S_f^T \boldsymbol{v} = \mathbf{0}$$

$$\Big\Downarrow \quad \boldsymbol{v} = S_v \dot{\boldsymbol{r}}$$

$$S_f^T S_v \dot{\boldsymbol{r}} = \mathbf{0}$$

$$\Big\Downarrow \quad \dot{\boldsymbol{r}} \text{ is arbitrary}$$

$$S_f^T S_v = 0$$

- The column spaces of S_f and S_v are orthogonal.

Solution of the Constraint Force

from dynamics:

$$\Lambda(\boldsymbol{\theta})\dot{\boldsymbol{v}} + \boldsymbol{\mu}(\boldsymbol{\theta}) = \boldsymbol{f} - \boldsymbol{f}_{\mathrm{ee}}$$

$$\Big\Downarrow \quad \boldsymbol{f}_{\mathrm{ee}} = S_f\boldsymbol{\lambda}$$

$$\dot{\boldsymbol{v}} = \Lambda^{-\mathbf{1}}\left(\boldsymbol{f} - S_f\boldsymbol{\lambda} - \ \boldsymbol{\mu}\right) \qquad\qquad S_f^T\boldsymbol{v} = \mathbf{0}$$

$$\Big\Downarrow \qquad\qquad\qquad\qquad\qquad \Big\Downarrow$$

$$\underbrace{S_f^T\dot{\boldsymbol{v}} = S_f^T\Lambda^{-\mathbf{1}}\left(\boldsymbol{f} - S_f\boldsymbol{\lambda} - \ \boldsymbol{\mu}\right) \qquad\qquad \dot{S}_f^T\boldsymbol{v} + S_f^T\dot{\boldsymbol{v}} = \mathbf{0}}$$

$$\Big\Downarrow$$

$$-\dot{S}_f^T\boldsymbol{v} = S_f^T\Lambda^{-\mathbf{1}}\left(\boldsymbol{f} - S_f\boldsymbol{\lambda} - \ \boldsymbol{\mu}\right)$$

Constraint Force

$$-\dot{S}_f^T \boldsymbol{v} = S_f^T \Lambda^{-\mathbf{1}}\left(\boldsymbol{f} - S_f \boldsymbol{\lambda} - \boldsymbol{\mu}\right)$$

$$\Downarrow$$

$$S_f^T \Lambda^{-\mathbf{1}} S_f \boldsymbol{\lambda} = S_f^T \Lambda^{-\mathbf{1}}(\boldsymbol{f} - \boldsymbol{\mu}) + \dot{S}_{\boldsymbol{f}}^{\boldsymbol{T}} \boldsymbol{v}$$

$$\Downarrow$$

$$\lambda = S_f^+(\boldsymbol{f} - \boldsymbol{\mu}) + \Lambda_f \dot{S}_{\boldsymbol{f}}^{\boldsymbol{T}} \boldsymbol{v}$$

where

$S_f^+ \equiv \left(S_f^T \Lambda^{-\mathbf{1}} S_f\right)^{-\mathbf{1}} S_f^T \Lambda^{-\mathbf{1}}$ (generalized inverse of S_f)

$\Lambda_f \equiv \left(S_f^T \Lambda^{-\mathbf{1}} S_f\right)^{-\mathbf{1}}$

Dynamics in Independent Coordinates $\boldsymbol{r}$

$$\Lambda\dot{\boldsymbol{v}} + \boldsymbol{\mu} = \boldsymbol{f} - S_f\boldsymbol{\lambda}$$ (dynamics)

$$\Big\Downarrow \quad \lambda = S_f^{+}(\boldsymbol{f} - \boldsymbol{\mu}) + \Lambda_f \dot{S}_{\boldsymbol{f}}^{\boldsymbol{T}}\boldsymbol{v}$$

$$\Lambda\dot{\boldsymbol{v}} + S_f\Lambda_f\dot{S}_f^T\boldsymbol{v} = (I_6 - S_f S_f^{+})(\boldsymbol{f} - \boldsymbol{\mu})$$

$\Big\Downarrow$ Multiply both sides with S_v^T

$$S_v^T\Lambda\dot{\boldsymbol{v}} + S_v^T S_f\Lambda_f\dot{S}_f^T\boldsymbol{v} = S_v^T(I_6 - S_f S_f^{+})(\boldsymbol{f} - \boldsymbol{\mu})$$

$\Big\Downarrow$ $S_v^T S_f = 0$ since $S_f^T S_v = 0$

$$S_v^T\Lambda\dot{\boldsymbol{v}} = S_v^T(\boldsymbol{f} - \boldsymbol{\mu})$$

$\Big\Downarrow$ $\dot{\boldsymbol{v}} = S_v\ddot{\boldsymbol{r}} + \dot{S}_v\dot{\boldsymbol{r}}$ from $\boldsymbol{v} = S_v\dot{\boldsymbol{r}}$

$$\Lambda_v\ddot{\boldsymbol{r}} = S_v^T\left(\boldsymbol{f} - \boldsymbol{\mu} - \Lambda\dot{S}_v\dot{\boldsymbol{r}}\right)$$

where $\Lambda_v \equiv S_v^T\Lambda S_v$

Hybrid Controller

We apply control:

$$\boldsymbol{f} = \Lambda S_v \underbrace{\boldsymbol{a}}_{\text{position servo}} + S_f \underbrace{\boldsymbol{b}}_{\text{force servo}} + \boldsymbol{\mu} + \Lambda \dot{S}_v \dot{\boldsymbol{r}}$$

$$\boldsymbol{\tau} = J^T \boldsymbol{f} \qquad \text{(control law)}$$

- Derive the position servo.

Substitute control into dynamics $\Lambda_v \ddot{\boldsymbol{r}} = S_v^T(\boldsymbol{f} - \boldsymbol{\mu} - \Lambda \dot{S}_v \dot{\boldsymbol{r}})$:

$$\begin{aligned}\Lambda_v \ddot{\boldsymbol{r}} &= S_v^T(\Lambda S_v \boldsymbol{a} + S_f \boldsymbol{b}) \\ &= S_v^T \Lambda S_v \boldsymbol{a} && (S_v^T S_f = 0) \\ &= \Lambda_v \boldsymbol{a}\end{aligned}$$

$$\Downarrow$$

$$\ddot{\boldsymbol{r}} = \boldsymbol{a}$$

Desired trajectory in independent coordinates

$$\boldsymbol{a} = \ddot{\boldsymbol{r}}_d + K_v(\dot{\boldsymbol{r}}_d - \dot{\boldsymbol{r}}) + K_p(\boldsymbol{r}_d - \boldsymbol{r})$$

Force Control Component

$$\lambda = S_f^{+}(\boldsymbol{f} - \boldsymbol{\mu}) + \Lambda_f \dot{S}_{\boldsymbol{f}}^{\boldsymbol{T}} \boldsymbol{v}$$

$$\Downarrow \quad \boldsymbol{f} = \Lambda S_v \boldsymbol{a} + S_f \boldsymbol{b} + \boldsymbol{\mu} + \Lambda \dot{S}_v \dot{\boldsymbol{r}}$$

$$\lambda = S_f^{+}\left(\Lambda S_v \boldsymbol{a} + S_f \boldsymbol{b} + \Lambda \dot{S}_v \dot{\boldsymbol{r}}\right) + \Lambda_f \dot{S}_{\boldsymbol{f}}^{\boldsymbol{T}} \boldsymbol{v}$$

$$\Downarrow \quad S_f^{+} \equiv \left(S_f^T \Lambda^{-\mathbf{1}} S_f\right)^{-\mathbf{1}} S_f^T \Lambda^{-\mathbf{1}}$$

$$\lambda = \left(S_f^T \Lambda^{-\mathbf{1}} S_f\right)^{-\mathbf{1}} \underbrace{S_f^T \Lambda^{-\mathbf{1}} \Lambda S_v}_{=0} \boldsymbol{a} + \boldsymbol{b} + \underbrace{\left(S_f^T \Lambda^{-\mathbf{1}} S_f\right)^{-\mathbf{1}}}_{=\Lambda_f} S_f^T \Lambda^{-\mathbf{1}} \Lambda \dot{S}_v \dot{\boldsymbol{r}} + \Lambda_f \dot{S}_{\boldsymbol{f}}^{\boldsymbol{T}} \boldsymbol{v}$$

$$\Downarrow$$

$$\begin{aligned} \boldsymbol{\lambda} &= \boldsymbol{b} + \Lambda_f\left(S_f^T \dot{S}_v \dot{\boldsymbol{r}} + \dot{S}_{\boldsymbol{f}}^{\boldsymbol{T}} \boldsymbol{v}\right) \\ &= \boldsymbol{b} + \Lambda_f\left(S_f^T (\dot{\boldsymbol{v}} - S_v \ddot{\boldsymbol{r}}) + \dot{S}_{\boldsymbol{f}}^{\boldsymbol{T}} \boldsymbol{v}\right) \\ &= \boldsymbol{b} + \Lambda_f\left(\frac{d}{dt}\underbrace{\left(S_f^T \boldsymbol{v}\right)}_{=0} - \underbrace{S_f^T S_v}_{=0} \ddot{\boldsymbol{r}}\right) \\ &= \boldsymbol{b} \end{aligned}$$

Desired force trajectory

$$\boldsymbol{b} = \boldsymbol{\lambda}_d + K_{fp}(\boldsymbol{\lambda}_d - \boldsymbol{\lambda}) + K_{fi}\int (\boldsymbol{\lambda}_d - \boldsymbol{\lambda})\ dt$$

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