Displacement and Motion of a Rigid Body*

(COMS 4770/5770 Notes)

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Sep 26, 2024

A rigid body consists of particles whose relative distances and orientations do not change as they move. A *rigid body transformation* is any change which preserves the relative distance and orientation[1] of all the points inside the body. Every rigid body transformation can be described as a rotation about some reference point on or inside the body and a translation by the point. It is therefore an affine transformation.

A robot can often be viewed as a single rigid body or a mechanical linkage as multiple rigid bodies connected at joints. Its body or end-effector (e.g., a gripper or hand) has a local coordinate frame, in terms of which the robot's commands, controls, even motions are most conveniently specified. On the other hand, the environment or objects interacting with the robot are often described in a stationary (world) frame. Constantly, transformations between the two frames and among the frames of multiple robots are needed for task executions.

In computer vision, a camera is mounted either at a fixed location in the environment or on a robot — typically at its end-effector or hand (the camera is said to be in "eye-in-hand" configuration). The camera's position and orientation are not accurately known in advance and often need to be calibrated before the usage. Calibration sets up a transformation from the camera's local frame to the world frame in the environment or the robot's tool frame.

1 Transformation in the Plane

A rigid body transformation T in the plane is also called a *planar displacement*. For convenience, a body frame $\{b\}$ is set up at a chosen reference point $\boldsymbol{c}$ inside the body (described in world coordinates), as illustrated in Figure 1. The body's geometry is conveniently described in $\{b\}$. In the world frame $\{w\}$ located at $\boldsymbol{o}$, its *pose* (position and orientation) is completely determined by its translation and rotation from $\{w\}$. The translation is represented by the position $\boldsymbol{c}$ of the reference point, expressed already in $\{w\}$. The rotation, characterized by a matrix $R(\theta)$, is the same as that of $\{b\}$ from a frame $\{s\}$ that is located at $\boldsymbol{c}$ and oriented the same as $\{w\}$. Here, θ is the angle of rotation which can specify the rotation alone. A point inside the body with coordinates ${}^b\boldsymbol{p}$ in $\{b\}$ has the following coordinates in $\{w\}$:

$$\boldsymbol{p} = R(\theta)\,{}^b\boldsymbol{p} + \boldsymbol{c} \tag{1}$$

*The appendix is optional for reading unless specifically required.

[1]Reflection is thus excluded.

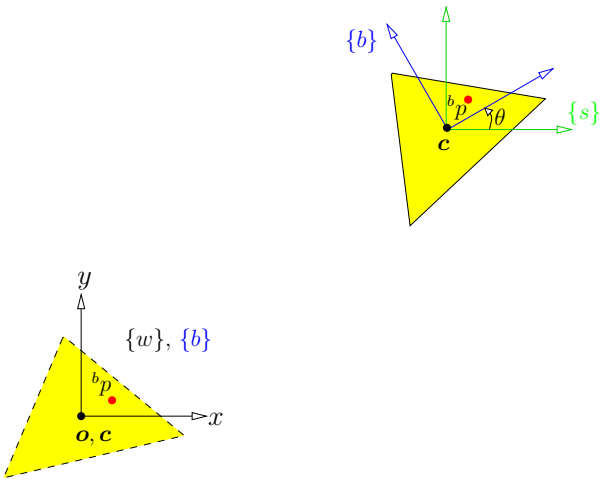

Figure 1: Plane displacement.

where

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The rotation axis always points outward. We denote the transformation as $T = (R, \boldsymbol{c})$. From (1) we see that the rotation comonent does not depend on the choice of the world frame, as long as its orientation stays the same.

Theorem 1 *If $R(\theta) = I_2$, the 2×2 identity matrix, the transformation $T = (R, \boldsymbol{c})$ is a pure translation. If $R \neq I_2$, T is equivalent to a rotation about the center*

$$\boldsymbol{s} = (I_2 - R)^{-1}\boldsymbol{c}. \tag{2}$$

through θ.

Proof In the specical case $R = I_2$, equation (1) reduces to $\boldsymbol{p} = {}^b\boldsymbol{p} + \boldsymbol{c}$, describing a pure translation by $\boldsymbol{c}$.

In the general case, $R \neq I_2$. Let us first verify that $\boldsymbol{s}$ is invariant under T. Multiplying both sides of (2) with $(I_2 - R)$, we immediately have

$$(I_2 - R)\boldsymbol{s} = \boldsymbol{c},$$

i.e., $\boldsymbol{s} = R\boldsymbol{s} + \boldsymbol{c}$. Next, we rewrite the image point of ${}^b\boldsymbol{p}$ under T as follows:

$$\begin{aligned} R\,{}^b\boldsymbol{p} + \boldsymbol{c} &= R\,{}^b\boldsymbol{p} + (I_2 - R)(I_2 - R)^{-1}\boldsymbol{c} \\ &= R\,{}^b\boldsymbol{p} + (I_2 - R)\boldsymbol{s} \\ &= R\,{}^b\boldsymbol{p} - R\boldsymbol{s} + \boldsymbol{s} \\ &= R\left({}^b\boldsymbol{p} - \boldsymbol{s}\right) + \boldsymbol{s}. \end{aligned}$$

The last equation above describes a rotation of ${}^b\boldsymbol{p}$ about $\boldsymbol{s}$. □

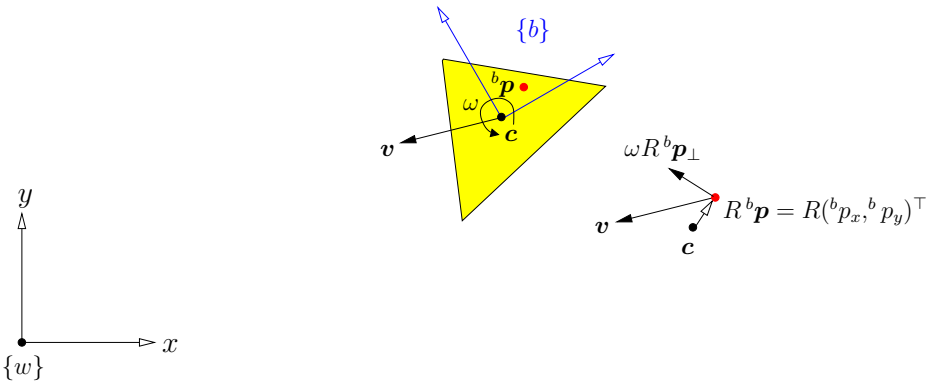

Figure 2: Motion of an object in the plane is characterized by its velocity $\boldsymbol{v}$ and angular velocity ω. Here, ${}^b\boldsymbol{p}$ is a point of the body expressed in the body frame $\{b\}$, and ${}^b\boldsymbol{p}_\perp$ is the result of rotating the point through $\frac{\pi}{2}$ in the frame.

The point $\boldsymbol{s}$ in (2) is called the *pole* of the transformation T. It does not exist if T is a pure translation. The matrix $I_2 - R$ has determinant $2 - 2\cos\theta$, which is non-zero as long as $\theta \neq 0$. Note that the rotation center $\boldsymbol{s}$ may be outside of the (bounded) rigid body. We may also treat the transformation T as applied to the entire plane, that is, $T : \boldsymbol{q} \mapsto R\boldsymbol{q} + \boldsymbol{c}$, $\boldsymbol{q} \in \mathbb{R}^2$.

2 Motion in the Plane

The world frame $\{w\}$ is stationary. Suppose that the rigid body is in a motion, so its position $\boldsymbol{c}(t)$ and orientation $\theta(t)$ are both functions of time t. The body has a translational velocity, or simply, velocity. This velocity, denoted $\boldsymbol{v}$, is of its reference point $\boldsymbol{c}$; namely, $\boldsymbol{v}(t) = \dot{\boldsymbol{c}} \equiv d\boldsymbol{c}(t)/dt$. Here, '$\cdot$' denotes differentiation with respect to time.

We differentiate equation (1) to inspect the velocity of a point $\boldsymbol{p}$ inside the body:

$$\begin{aligned} \boldsymbol{v}_p \equiv \dot{\boldsymbol{p}} &= \frac{dR}{dt}\,{}^b\boldsymbol{p} + R\frac{d\,{}^b\boldsymbol{p}}{dt} + \dot{\boldsymbol{c}} \\ &= \dot{\theta}\begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}{}^b\boldsymbol{p} + \boldsymbol{v} \end{aligned}$$

where the derivative $d\,{}^b\boldsymbol{p}/dt$ vanishes because the point is fixed in the rigid body. Now we introduce the angular velocity $\omega = \dot{\theta}$. Let ${}^b\boldsymbol{p} = ({}^bp_x,\, {}^bp_y)^\top$. The above velocity can be written as

$$\boldsymbol{v}_p = \boldsymbol{v} + \omega R\,{}^b\boldsymbol{p}_\perp, \tag{3}$$

where ${}^b\boldsymbol{p}_\perp = (-{}^bp_y,\, {}^bp_x)^\top$ is the vector from rotating ${}^b\boldsymbol{p}$ through $\frac{\pi}{2}$ within the body frame $\{b\}$.[2] The first velocity component $\boldsymbol{v}$ in (3) is due to the linear motion of the rigid body. The second velocity component $\omega R\,{}^b\boldsymbol{p}_\perp$ is due to the rate of rotation ω of the rigid body, i.e., of $\{b\}$ relative to the world frame $\{w\}$. More intuitively, we can picture $\omega R\,{}^b\boldsymbol{p}_\perp$ as the rate of rotation of $\{b\}$ with respect to a stationary frame $\{s\}$ which shares the origin with $\{b\}$ but has the orientation of $\{w\}$.

[2] It is easy to check that ${}^b\boldsymbol{p} \cdot {}^b\boldsymbol{p}_\perp = 0$ and ${}^b\boldsymbol{p} \times {}^b\boldsymbol{p}_\perp > 0$.

Assume $w \neq 0$. Denote $\boldsymbol{p}_\perp = R^b \boldsymbol{p}_\perp$ as the result of rotating $\boldsymbol{p} = (p_x, p_y)^\top$ through $\pi/2$ in $\{w\}$. Namely, $\boldsymbol{p}_\perp = (-p_y, p_x)^\top$. Letting $\boldsymbol{v}_p = 0$ in (3), we solve the equation to obtain

$$p_y = \frac{v_x}{\omega} \qquad \text{and} \qquad p_x = -\frac{v_y}{\omega}.$$

The point

$$\boldsymbol{r} = \begin{pmatrix} -v_y \\ v_x \end{pmatrix} \Big/ \omega$$

has zero velocity at the instant. It is called the *instantaneous rotation center* and evolves with time.

3 Rigid Body Transformation in the Space

We move on to three dimensions. A point $\boldsymbol{p}$ under a rotation about some axis ℓ through a point $\boldsymbol{q}$ is transformed to $R(\boldsymbol{p} - \boldsymbol{q}) + \boldsymbol{q}$, where R is the rotation matrix. Rewriting the image point as $R\boldsymbol{p} + (I_3 - R)\boldsymbol{q}$, we see that the same rotation can be represented as one through the same angle about an axis through the origin with the orientation of ℓ, followed by a translation. Similarly, a transformation in which a translation $\boldsymbol{b}$ precedes a rotation R is equivalent to one in which the same rotation precedes a new translation $R\boldsymbol{b}$, because

$$R(\boldsymbol{p} + \boldsymbol{b}) = R\boldsymbol{p} + R\boldsymbol{b}.$$

In fact, any sequence of translations and rotations can be rewritten as a sequence of rotations followed by a sequence of translations. Further, the rotation sequence is equivalent to a single rotation, while the translation sequence is equivalent to a single translation. Every rigid body transformation T in three dimensions is thus equivalent to a rotation followed by a translation. Namely, it is described as a mapping $T : \boldsymbol{p} \mapsto R\boldsymbol{p} + \boldsymbol{b}$, where $R = (r_{ij})$ is a rotation matrix, and $\boldsymbol{b}$ is a vector. The transformation T is often referred to as a *spatial displacement*.

Theorem 2 (Chasles' Theorem) *Every spatial displacement $T(\boldsymbol{p}) = R\boldsymbol{p} + \boldsymbol{b}$ is the composition of a rotation about some axis ℓ and a translation along the same axis.*

As illustrated in Figure 3, a spatial displacement mimics the movement of a screw. The rotation axis ℓ is called the *screw axis*. Any point on ℓ is constrained to move along the line under the transformation. Instead of proving the theorem, below we describe how to determine the screw axis ℓ and the translation [1, pp. 19–20].

Corollary 3 *The screw axis for a spatial displacement $T = (R, \boldsymbol{b})$, with $R = (r_{ij})$, is described by Plücker coordinates $(\boldsymbol{l}, -(R^T + I_3)\boldsymbol{b}_\perp)$, where $\boldsymbol{l} = (r_{23} - r_{32}, r_{31} - r_{13}, r_{12} - r_{21})^\top$ is the direction of the axis of rotation specified by R and $\boldsymbol{b}_\perp = \boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{l})\boldsymbol{l}/\|\boldsymbol{l}\|^2$ is the component of $\boldsymbol{b}$ orthogonal to $\boldsymbol{l}$.*

Proof Given the rotation matrix $R = (r_{ij})$, its corresponding axis of rotation satisfies $R\boldsymbol{l} = \boldsymbol{l}$ and $R^\top \boldsymbol{l} = R^\top R\boldsymbol{l} = \boldsymbol{l}$. It follows that

$$\begin{aligned} \mathbf{0} &= (R - R^\top)\boldsymbol{l} \\ &= \begin{pmatrix} r_{23} - r_{32} \\ r_{31} - r_{13} \\ r_{12} - r_{21} \end{pmatrix} \times \boldsymbol{l}. \end{aligned}$$

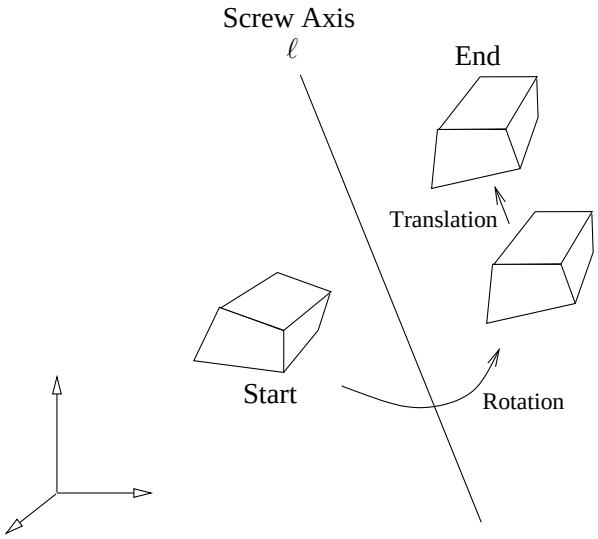

Figure 3: Equivalence of a rigid body transformation to a rotation about the screw axis followed by a translation along the same axis.

We can set

$$\boldsymbol{l} = \begin{pmatrix} r_{23} - r_{32} \\ r_{31} - r_{13} \\ r_{12} - r_{21} \end{pmatrix}. \tag{4}$$

This is the direction of the screw axis ℓ.

Next, the translation vector $\boldsymbol{b}$ is decomposed into $\boldsymbol{b}_{\parallel}$ along the axis ℓ and $\boldsymbol{b}_{\perp}$ orthogonal to it:

$$\boldsymbol{b} = \boldsymbol{b}_{\parallel} + \boldsymbol{b}_{\perp},$$

where

$$\begin{aligned} \boldsymbol{b}_{\parallel} &= \left(\boldsymbol{b} \cdot \frac{\boldsymbol{l}}{\|\boldsymbol{l}\|}\right) \frac{\boldsymbol{l}}{\|\boldsymbol{l}\|} \\ &= \frac{1}{\|\boldsymbol{l}\|^2}(\boldsymbol{b} \cdot \boldsymbol{l})\boldsymbol{l}. \end{aligned}$$

Let $\boldsymbol{a}$ be the foot of the perpendicular from the origin to ℓ. See Figure 4. The foot is transformed into

$$R\boldsymbol{a} + \boldsymbol{b} \;=\; R\boldsymbol{a} + \boldsymbol{b}_{\perp} + \boldsymbol{b}_{\parallel},$$

which, by Theorem 2, must lie on ℓ. The vector $R\boldsymbol{a}$ is generated from rotating $\boldsymbol{a}$ about an axis through the origin and parallel to ℓ. Since $\boldsymbol{a} \perp \ell$, $R\boldsymbol{a} \perp \ell$ must hold as well. The point $R\boldsymbol{a} + \boldsymbol{b}_{\perp}$ must be in the plane Π that is perpendicular to ℓ and contains both $\boldsymbol{a}$ and $R\boldsymbol{a}$. Meanwhile, $R\boldsymbol{a}+\boldsymbol{b}_{\perp}$ must also lie on the screw axis ℓ in order for $R\boldsymbol{a} + \boldsymbol{b}$ to lie on ℓ, since $\boldsymbol{b}_{\parallel}$ is parallel to $\boldsymbol{l}$. We thus infer, as the unique intersection of ℓ with Π,

$$\boldsymbol{a} = R\boldsymbol{a} + \boldsymbol{b}_{\perp},$$

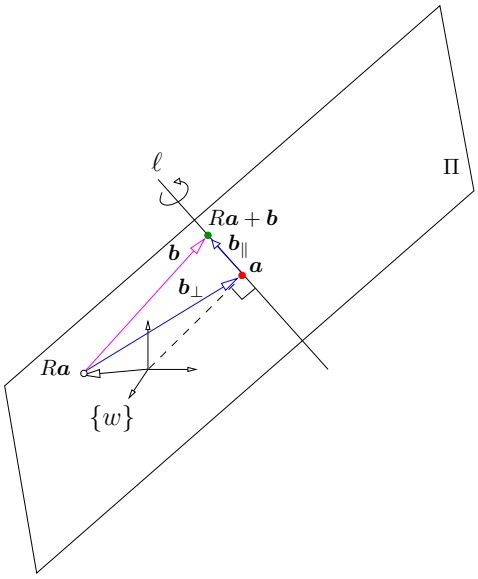

Figure 4: Foot of the perpendicular $\boldsymbol{a}$ from the origin to the screw axis ℓ.

Immediately, we obtain

$$\boldsymbol{a} = (I_3 - R)^{-1}\boldsymbol{b}_\perp,$$

where the inverse $(I_3 - R)^{-1}$ exists due to the uniqueness of $\boldsymbol{a}$.

The cross product of $\boldsymbol{l}$ with any vector equals the product of $R - R^\top$ with that vector. The moment $\boldsymbol{m}$ of ℓ is

$$\begin{aligned} \boldsymbol{l} \times \boldsymbol{a} &= (R - R^\top)\boldsymbol{a} \\ &= (R - R^\top)(I_3 - R)^{-1}\boldsymbol{b}_\perp. \end{aligned} \tag{5}$$

Since

$$\begin{aligned} (R^\top + I_3)(I_3 - R) &= R^\top + I_3 - I_3 - R \\ &= R^\top - R, \end{aligned}$$

we have

$$R^\top + I_3 = (R^\top - R)(I_3 - R)^{-1},$$

namely,

$$(R - R^\top)(I_3 - R)^{-1} = -(R^\top + I_3)$$

Substitute the above into (5):

$$\boldsymbol{m} = \boldsymbol{l} \times \boldsymbol{a} = -(R^\top + I_3)\boldsymbol{b}_\perp. \tag{6}$$

The screw axis ℓ is described by the Plücker coordinates $(\boldsymbol{l}, \boldsymbol{m})$. Clearly, the translation along the axis is given by $\boldsymbol{b}_\perp$. □

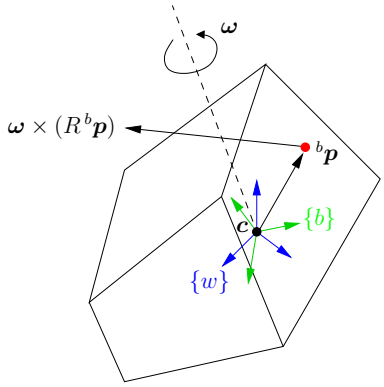

Figure 5: Velocity of a point ${}^b\boldsymbol{p}$ inside the rotating rigid body due to its angular velocity $\boldsymbol{\omega}$.

4 Angular Velocity in the Space

Before examining the motion of an unconstrained rigid body in the space, we study the special case where the origin $\boldsymbol{c}$ of the body frame $\{b\}$ is fixed at the origin of the world frame $\{w\}$. The rigid body is thus constrained and can change only its orientation, described by the rotation matrix R, which is a function of time t.

The coordinates of a point ${}^b\boldsymbol{p}$ inside the body as expressed in $\{w\}$ is

$$\boldsymbol{p} = R(t)\,{}^b\boldsymbol{p}. \tag{7}$$

The body coordinates ${}^b\boldsymbol{p}$ are constant. Unless ${}^b\boldsymbol{p} = \boldsymbol{0}$ (i.e., ${}^b\boldsymbol{p}$ coincides with $\boldsymbol{c}$), $\boldsymbol{p}$ changes with time. Differentiating (7), we obtain the velocity of this point:

$$\begin{aligned} \dot{\boldsymbol{p}} &= \dot{R}\,{}^b\boldsymbol{p} + R\frac{d\,{}^b\boldsymbol{p}}{dt} \\ &= \dot{R}\,{}^b\boldsymbol{p}. \end{aligned} \tag{8}$$

The derivative $\dot{R}$ measures how R changes with time. To obtain it, we make use of R being an orthogonal matrix, that is,

$$I_3 = RR^\top.$$

Differentiate both sides of the above equation:

$$\begin{aligned} 0 &= \dot{R}R^\top + R\dot{R}^\top \\ &= \dot{R}R^\top + \left(\dot{R}R^\top\right)^\top. \end{aligned}$$

Thus, $\dot{R}R^\top$ is an antisymmetric matrix; it can be represented as

$$\dot{R}R^\top = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}, \tag{9}$$

for some ω_x, ω_y, and ω_z.

Let us gather the non-zero entries in the matrix $\dot{R}R^\top$ into a vector:

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \tag{10}$$

For any vector $\boldsymbol{u} \in \mathbb{R}^3$, we have $\dot{R}R^\top \boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{u}$. This helps us rewrite equation (8) as following:

$$\begin{aligned} \dot{\boldsymbol{p}} &= \dot{R}R^\top \left(R\,{}^b\boldsymbol{p} \right) \\ &= \boldsymbol{\omega} \times \left(R\,{}^b\boldsymbol{p} \right) \\ &= \boldsymbol{\omega} \times \boldsymbol{p}. \end{aligned} \tag{11}$$

In (11), both $\boldsymbol{\omega}$ and $\boldsymbol{p}$ are expressed in the world frame $\{w\}$, just as $\dot{\boldsymbol{p}}$. The point ${}^b\boldsymbol{p}$ in the rigid body thus acquires a velocity due to the body's ongoing rotation. The vector $\boldsymbol{\omega}$, which measures the rate of rotation, is the *angular velocity* of the rigid body in three dimensions.

Let us introduce an operator $[\cdot]_\times$ that turns a vector $\boldsymbol{a} = (a_x, a_y, a_z)^\top \in \mathbb{R}^3$ into the following antisymmetric matrix:

$$[\boldsymbol{a}]_\times = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix},$$

so that the cross product of $\boldsymbol{a}$ with another vector $\boldsymbol{b}$ can be written as $[\boldsymbol{a}]_\times \boldsymbol{b}$. This notation eliminates the need for parentheses within a cross product involving operands that are matrix products themselves. Multiplying both sides of (9) with R gives us the derivative of the rotation matrix:

$$\dot{R} = [\boldsymbol{\omega}]_\times R. \tag{12}$$

Since the rotation matrix R is always multiplied with some vector, the derivative $\dot{R}$ alone is not meaningful, for we can always turns such multiplication into a cross product:

$$\dot{R}\boldsymbol{a} = \dot{R}R^\top R\boldsymbol{a} = \boldsymbol{\omega} \times (R\boldsymbol{a}).$$

5 Motion of a Rigid Body

Finally, we consider that the rigid rody is translating and rotating simultaneously. The body frame $\{b\}$ and the world frame $\{w\}$ generally do not share the same origin as shown in Figure 6. The reference point $\boldsymbol{c}(t)$ of $\{b\}$ has velocity

$$\boldsymbol{v} = \dot{\boldsymbol{c}}.$$

A point ${}^b\boldsymbol{p}$ in the body has the coordinates

$$\boldsymbol{p} = \boldsymbol{c} + R\,{}^b\boldsymbol{p}$$

in $\{w\}$. Differentiating the above equation, we have the point's velocity expressed in the world frame $\{w\}$:

$$\begin{aligned} \boldsymbol{v}_p \equiv \dot{\boldsymbol{v}} &= \dot{\boldsymbol{c}} + \dot{R}\,{}^b\boldsymbol{p} \\ &= \boldsymbol{v} + \boldsymbol{\omega} \times (R\,{}^b\boldsymbol{p}). \end{aligned} \tag{13}$$

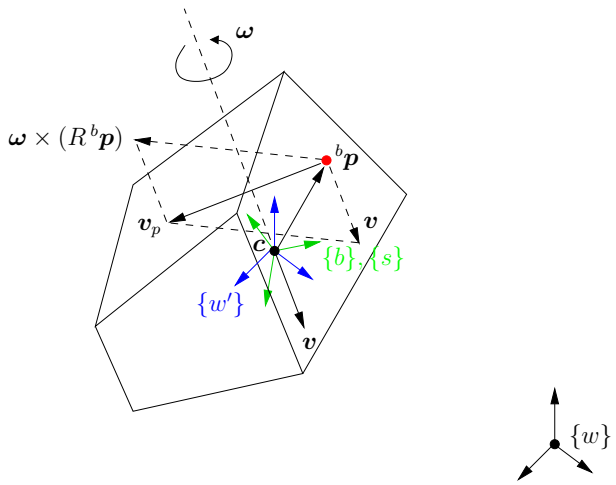

Figure 6: Velocity of a point ${}^b\boldsymbol{p}$ inside a translating and rotating rigid body.

The velocity $\boldsymbol{v}_p$ has two components: the first due to the velocity of the body to which the point ${}^b\boldsymbol{p}$ is rigidly attached and the second due to the rotation of the body as an observer in the world frame sees it.

Consider a frame $\{w'\}$ that is stationary, shares its origin with the body frame $\{b\}$ at the time instant, and is oriented the same as the world frame $\{w\}$. See Figure 6. Because $\{w'\}$ is stationary, the velocity $\boldsymbol{v}$ and angular velocity $\boldsymbol{\omega}$ of the rigid body, expressed in this frame, have respectively the same values as in $\{w\}$. The frame $\{w'\}$ helps us visualize $\boldsymbol{\omega}$ since it and $\{b\}$ share the same origin. The frame $\{w'\}$ is *instantaneous* in the sense that, at a different time instant, a different stationary frame will be considered.

A more useful instantaneous frame in rigid body dynamics, robotic manipulation, and localization is a stationary frame $\{s\}$ that is completely aligned with the body frame $\{b\}$ at the time instant t. A big reason is that, in a spacecraft or robot, motion commands and torques are conveniently expressed in terms of $\{b\}$. Suppose the body frame $\{b\}$ has its rotation matrix R, velocity $\boldsymbol{v}$, and angular velocity $\boldsymbol{\omega}$ relative to the world frame $\{w\}$. Then the velocities are $R^T\boldsymbol{v}$ and $R^T\boldsymbol{\omega}$ when expressed in $\{s\}$. We often blur the difference and refer to $R^T\boldsymbol{v}$ and $R^T\boldsymbol{\omega}$ as the velocities of the body "expressed in the body frame", and denote them as ${}^b\boldsymbol{v}$ and ${}^b\boldsymbol{\omega}$, respectively. The left superscript ^{b} here has the meaning "in terms of" besides "relative to" as in ${}^b\boldsymbol{p}$. Note that differentiations are still taken with respect to the world frame $\{w\}$ or any stationary frame for that matter.

Let us try to express the velocity $\boldsymbol{v}_p$ from (13) in the body frame $\{b\}$. We use the fact that $(R\boldsymbol{a}) \times (R\boldsymbol{b}) = R(\boldsymbol{a} \times \boldsymbol{b})$, that is, taking the cross product of two vetors $\boldsymbol{a}$ and $\boldsymbol{b}$ after a rotation is equivalent to rotating their cross product obtained before the rotation. The following steps are made:

$$\begin{aligned} R^\top \boldsymbol{v}_p &= R^\top \boldsymbol{v} + R^\top \left(\boldsymbol{\omega} \times \left(R\,{}^b\boldsymbol{p}\right)\right) \\ &= R^\top \boldsymbol{v} + R^\top \left(\left(RR^\top \boldsymbol{\omega}\right) \times \left(R\,{}^b\boldsymbol{p}\right)\right) \\ &= R^\top \boldsymbol{v} + R^\top R \left(\left(R^\top \boldsymbol{\omega}\right) \times {}^b\boldsymbol{p}\right) \end{aligned}$$

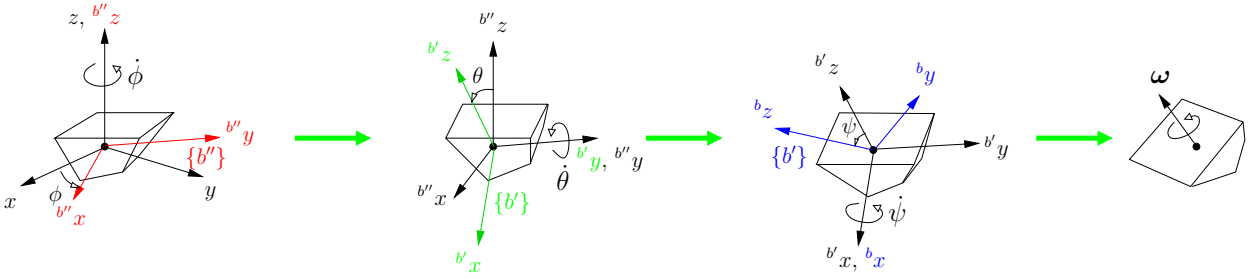

Figure 7: From the rates of the Z-Y-X Euler angle rates $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$ to the angular velocity $\boldsymbol{\omega}$.

$$= \quad R^\top \boldsymbol{v} + \left(R^\top \boldsymbol{\omega}\right) \times {}^b\boldsymbol{p}.$$

In terms of the body frame, the above equation becomes

$${}^b\boldsymbol{v}_p = {}^b\boldsymbol{v} + {}^b\boldsymbol{\omega} \times {}^b\boldsymbol{p}.$$

A Conversion from Euler Angle Rates to Angular Velocity

Euler angles are commonly used for describing the orientation of a rigid body because they are generally non-redundant. For instance, the attitude of aircraft is expressed with yaw, pitch, and roll angles in order, namely, the Z-Y-X Euler angles. In applications, there is often a need to convert the rates of Euler angles into the angular velocity $\boldsymbol{\omega}$.

We consider the representation of a rotation using the Z-Y-X Euler angles: yaw ϕ, pitch θ, and roll ψ, which describe the angles of three consecutive rotations about the body z-, y-, and x-axes, respectively. See Figure 7. The corresponding rotation matrices are

$$\begin{aligned}
R_z(\phi) &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
R_y(\theta) &= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \\
R_x(\psi) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}.
\end{aligned}$$

The rotation of the body frame from the world frame is described by the matrix

$$R = R_z(\phi) R_y(\theta) R_x(\psi),$$

which transforms the body coordinates of a point to its world coordinates via a left multiplication.

Let $\boldsymbol{\omega}$ be the angular velocity of the body frame as expressed in the world frame. It has three (non-orthogonal) components. The first component is due to the rate $\dot{\phi}$ of rotation about the z-axis of the world frame $\{w\}$. The second component is due to the rate $\dot{\theta}$ of rotation about the y-axis of the intermediate body frame $\{b'\}$. This axis is expressed as $R_z \hat{\boldsymbol{y}}$ in $\{w\}$, from which $\{b'\}$ rotates

through ϕ about its z-axis. The third component is due to the rate $\dot{\psi}$ of rotation about the x-axis of the final body frame $\{b\}$. This axis is expressed as $R_z R_y \hat{\boldsymbol{x}}$ in $\{w\}$, since $\{b\}$ further rotates through θ about the y-axis of $\{b'\}$. Adding up the three component, all in $\{w\}$, gives us the angular velocity

$$\boldsymbol{\omega} = \dot{\phi}\hat{\boldsymbol{z}} + \dot{\theta}\left(R_z(\phi)\hat{\boldsymbol{y}}\right) + \dot{\psi}\left(R_z(\phi)R_y(\theta)\hat{\boldsymbol{x}}\right). \tag{14}$$

The Euler angle rates $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$ are merely viewed as speeds of three rotations about the (instantaneously) fixed axes in the directions $\hat{\boldsymbol{z}}$, $R_z\hat{\boldsymbol{y}}$, and $R_yR_z\hat{\boldsymbol{x}}$, respectively.

We multiply out the matrix products in the above equation:

$$\begin{aligned}
R_z\hat{\boldsymbol{y}} &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \\
R_zR_y\hat{\boldsymbol{x}} &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix}.
\end{aligned}$$

Substitute the above expressions for $R_z\hat{\boldsymbol{y}}$ and $R_zR_y\hat{\boldsymbol{x}}$ into equation (14):

$$\begin{aligned}
\boldsymbol{\omega} &= \dot{\phi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \dot{\theta}\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} + \dot{\psi}\begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix} \\
&= \begin{pmatrix} -\dot{\theta}\sin\phi + \dot{\psi}\cos\phi\cos\theta \\ \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta \\ \dot{\phi} - \dot{\psi}\sin\theta \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\sin\phi & \cos\phi\cos\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 1 & 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}
\end{aligned} \tag{15}$$

We can easily write out the angular velocity $\boldsymbol{\omega}$ in terms of the rates of another type of Euler angles, for instance, the Z-Y-Z Euler angles α, β, γ. In this case, the three rotation axes are expressed in the world frame as $\hat{\boldsymbol{z}}$, $R_z(\alpha)\hat{\boldsymbol{y}}$, and $R_z(\alpha)R_y(\beta)\hat{\boldsymbol{z}}$, leading to

$$\boldsymbol{\omega} = \dot{\alpha}\hat{\boldsymbol{z}} + \dot{\beta}\left(R_z(\alpha)\hat{\boldsymbol{y}}\right) + \dot{\gamma}\left(R_z(\alpha)R_y(\beta)\hat{\boldsymbol{z}}\right).$$

References

[1] J. Michael McCarthy. *An Introduction to Theoretical Kinematics.* The MIT Press, 1990.