Tutorial on Rigid Body Dynamics

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Outline

I. Particle dynamics

II. Angular velocity

III. Angular momentum

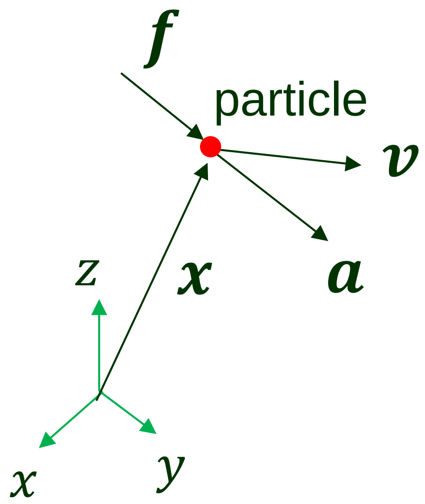
IV. Inertia tensor

V. Euler’s equation

VI. Body frame

VII. Dynamic simulation

I. Particle Dynamics



$\{w\}$: world frame, stationary
(inertial frame)

m: mass of the particle

$\boldsymbol{f}$: force on the particle

$\boldsymbol{x}$: position

$\boldsymbol{v}$: velocity ($\boldsymbol{v} = \dot{\boldsymbol{x}}$)

$\boldsymbol{a}$: acceleration ($\boldsymbol{a} = \dot{\boldsymbol{v}}$)

Momentum

$$\boldsymbol{p} = m\boldsymbol{v}$$

Moment of force (torque)

$$\boldsymbol{\tau} = \boldsymbol{x} \times \boldsymbol{f}$$

Moment of momentum (angular momentum)

$$\boldsymbol{l} = \boldsymbol{x} \times \boldsymbol{p}$$

Newton's 2nd Law

$$\boldsymbol{f} = m\boldsymbol{a}$$

$$= \dot{\boldsymbol{p}}$$

$$\dot{\boldsymbol{l}} = \frac{d\boldsymbol{l}}{dt} = \frac{d}{dt}(\boldsymbol{x} \times \boldsymbol{p})$$

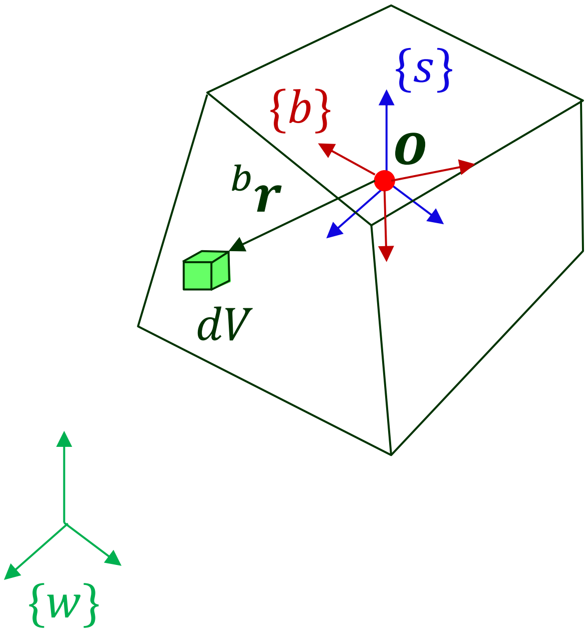
$$= \frac{d\boldsymbol{x}}{dt} \times (m\boldsymbol{v}) + \boldsymbol{x} \times \frac{d\boldsymbol{p}}{dt}$$

$$= \boldsymbol{v} \times (m\boldsymbol{v}) + \boldsymbol{x} \times (m\boldsymbol{a})$$

$$= \boldsymbol{x} \times \boldsymbol{f}$$

$$= \boldsymbol{\tau} \qquad \text{(torque)}$$

II. Rigid Body



$\boldsymbol{o}$: position of center of mass in $\{w\}$

$\boldsymbol{v}$: velocity of center of mass ($\boldsymbol{v} = \dot{\boldsymbol{o}}$)

ρ: mass density

$\{b\}$: body frame located at $\boldsymbol{o}$

$\{s\}$: stationary frame instantaneously located at $\boldsymbol{o}$ and oriented the same as $\{w\}$.

$$\boldsymbol{v} = {}^{s}\boldsymbol{v}$$

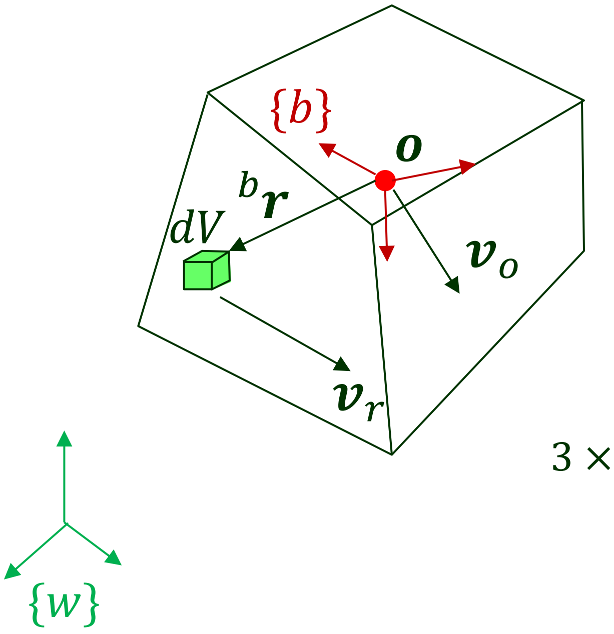
R: matrix describing the rotation from $\{s\}$ to $\{b\}$, i.e., from $\{w\}$ to $\{b\}$.

${}^{b}\boldsymbol{r}$: location of a mass element $\rho\, dV$ in $\{b\}$.

In $\{w\}$, the mass element has the position

$$\boldsymbol{o} + R\,{}^{b}\boldsymbol{r}$$

Velocity of a Mass Element



In $\{w\}$, the mass element has the velocity

$$\boldsymbol{v}_r = \frac{d}{dt}(\boldsymbol{o} + R\,{}^{b}\boldsymbol{r})$$

$$= \boldsymbol{v} + \dot{R}\,{}^{b}\boldsymbol{r} \qquad ({}^{b}\boldsymbol{r} \text{ is a constant vector})$$

To obtain $\dot{R}$, we make use of R being an orthogonal matrix:

$$RR^T = I_3 \quad \Longrightarrow \quad \dot{R}R^T + R\dot{R}^T = 0$$

$$\uparrow$$

3 × 3 identity matrix

$$\Longrightarrow \quad \dot{R}R^T + (\dot{R}R^T)^T = 0$$

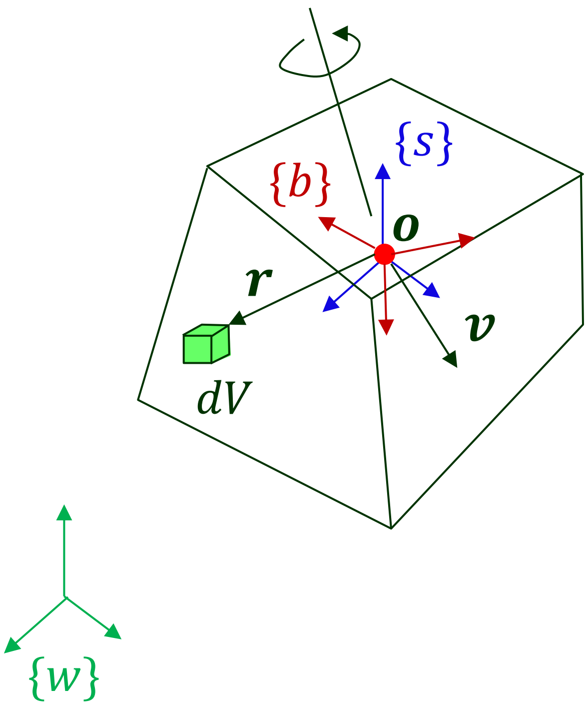
$$\Longrightarrow \quad \dot{R}R^T = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \equiv [\boldsymbol{\omega}]_\times$$

for some $\omega_x, \omega_y, \omega_z$.

Hence, we have

$$\dot{R} = [\boldsymbol{\omega}]_\times R$$

Angular Velocity



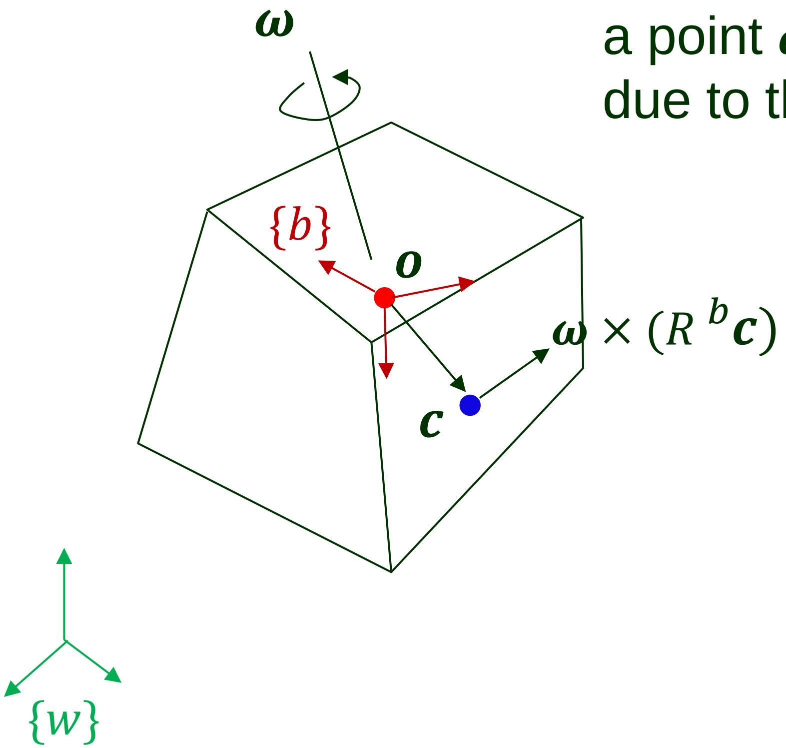
$$\boldsymbol{\omega} = {}^{s}\boldsymbol{\omega} \equiv \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad [\boldsymbol{\omega}]_\times = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

- ♦ The angular velocity $\boldsymbol{\omega}$ has its axis through the center of mass $\boldsymbol{o}$.
- ♦ It is described in the world frame $\{w\}$, or in the instantaneous frame $\{s\}$.
- ♦ The mass element dV has velocity:

$$\begin{aligned} \boldsymbol{v}_r &= \boldsymbol{v} + \dot{R}\,{}^{b}\boldsymbol{r} \\ &= \boldsymbol{v} + (\dot{R}R^T)R\,{}^{b}\boldsymbol{r} \\ &= \boldsymbol{v} + \boldsymbol{\omega} \times (R\,{}^{b}\boldsymbol{r}) \end{aligned}$$

- $\boldsymbol{v}$, due to its movement with the center of mass $\boldsymbol{o}$
- $\boldsymbol{\omega} \times (R\,{}^{b}\boldsymbol{r})$, due to its rotation about $\boldsymbol{o}$

Differentiating a Constant Vector in the Rotating Body



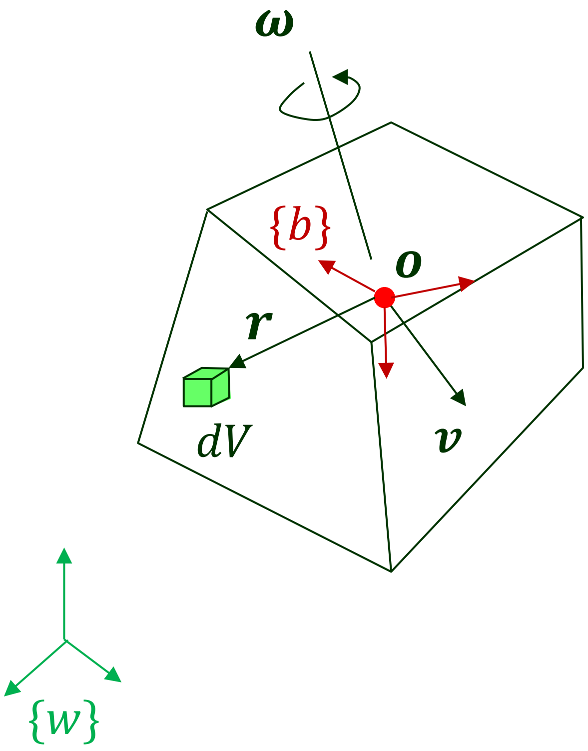
Even when the body's center of mass is not moving, a point $\boldsymbol{c}$ in $\{b\}$ is observed to have a velocity in $\{w\}$, due to the angular velocity.

$$\frac{d}{dt}(R\,{}^{b}\boldsymbol{c}) = \dot{R}\,{}^{b}\boldsymbol{c}$$

$$= \dot{R}R^{T}\left(R\,{}^{b}\boldsymbol{c}\right)$$

$$= \boldsymbol{\omega} \times (R\,{}^{b}\boldsymbol{c})$$

III. Angular Momentum



Location of the mass element in $\{w\}$:

$$\boldsymbol{r} = R\,{}^{b}\boldsymbol{r}$$

It has angular momentum:

$$d\boldsymbol{l} = \boldsymbol{r} \times (\rho dV \boldsymbol{v}_r)$$

$$= \rho dV(\boldsymbol{r} \times \boldsymbol{v}_r)$$

$$= \rho \boldsymbol{r} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r}) dV$$

Angular Momentum (cont’d)

$$\boldsymbol{l} = \int d\boldsymbol{l}$$

$$= \int \rho \boldsymbol{r} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r})\, dV$$

$$= \int \rho \boldsymbol{r} \times \boldsymbol{v}\, dV + \int \rho \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\ dV$$

$$= \int \rho \boldsymbol{r}\, dV \times \boldsymbol{v} + \int \rho\big((\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{\omega} - \boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{\omega})\big)\ dV$$

Angular Momentum (simplified)

We can pick $\{s\}$ as the reference frame so the center of mass $\boldsymbol{o}$ is the origin. This implies

$$\int \rho \; {}^{s}\boldsymbol{r} \, dV = \mathbf{0}$$

$${}^{s}\boldsymbol{l} = \int \rho \; {}^{s}\boldsymbol{r} \, dV \times \; {}^{s}\boldsymbol{v} + \int \rho \Big(\big(\; {}^{s}\boldsymbol{r} \cdot \; {}^{s}\boldsymbol{r} \big) \; {}^{s}\boldsymbol{\omega} - \; {}^{s}\boldsymbol{r} \big(\; {}^{s}\boldsymbol{r} \cdot \; {}^{s}\boldsymbol{\omega} \big) \Big) \; dV$$

$$\Big\Downarrow$$

${}^{s}\boldsymbol{v} = \boldsymbol{v}$ and ${}^{s}\boldsymbol{\omega} = \boldsymbol{\omega}$ because $\{w\}$ and $\{s\}$ are stationary and oriented the same.

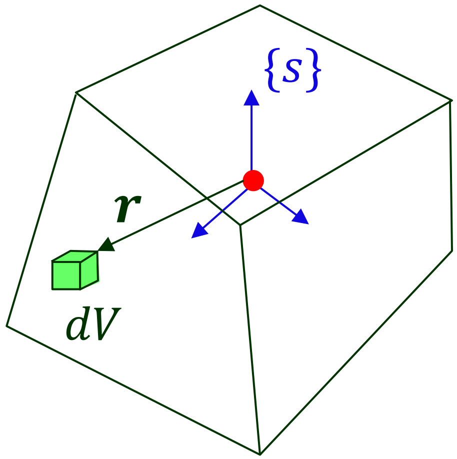
$$= \int \rho \left(\big(\; {}^{s}\boldsymbol{r} \cdot \; {}^{s}\boldsymbol{r} \big) I_3 - \; {}^{s}\boldsymbol{r} \; {}^{s}\boldsymbol{r}^T \right) \boldsymbol{\omega} \; dV$$

$$= \left(\int \rho \left(\big(\; {}^{s}\boldsymbol{r} \cdot \; {}^{s}\boldsymbol{r} \big) I_3 - \; {}^{s}\boldsymbol{r} \; {}^{s}\boldsymbol{r}^T \right) dV \right) \boldsymbol{\omega}$$

- The above formula is correct in the frame $\{s\}$ not in the frame $\{w\}$.
- As the object is moving, at another time instant the above form of ${}^{s}\boldsymbol{l}$ holds for a different stationary frame situated at its center of mass at that instant.

IV. Inertia Tensor

Angular inertia matrix (*inertia tensor*)



$$ {}^{s}I \equiv \int \rho \left(({}^{s}\boldsymbol{r} \cdot {}^{s}\boldsymbol{r}) I_3 - {}^{s}\boldsymbol{r}\, {}^{s}\boldsymbol{r}^T \right) dV $$

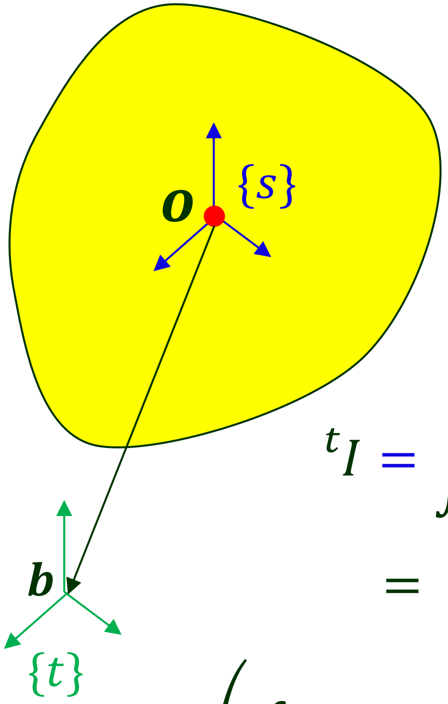
- The density ρ is a function of ${}^{s}\boldsymbol{r}$, but is often constant since most objects have uniform density.

$$ {}^{s}\boldsymbol{l} = \left(\int \rho \left(({}^{s}\boldsymbol{r} \cdot {}^{s}\boldsymbol{r}) I_3 - {}^{s}\boldsymbol{r}\, {}^{s}\boldsymbol{r}^T \right) dV \right) {}^{s}\boldsymbol{\omega} $$

$$ = {}^{s}I\ {}^{s}\boldsymbol{\omega} $$

Inertia Tensor Under Translation

The inertial tensor is defined in the frame $\{s\}$ located at $\boldsymbol{o}$.



$$^{s}I = \int \rho\left((\boldsymbol{x}\cdot\boldsymbol{x})I_3 - \boldsymbol{x}\boldsymbol{x}^{\boldsymbol{T}}\right)\, dV$$

Sometimes we need to calculate it with respect to a frame $\{t\}$ that results from translating $\{s\}$ by $\boldsymbol{b}$. The new inertia tensor is

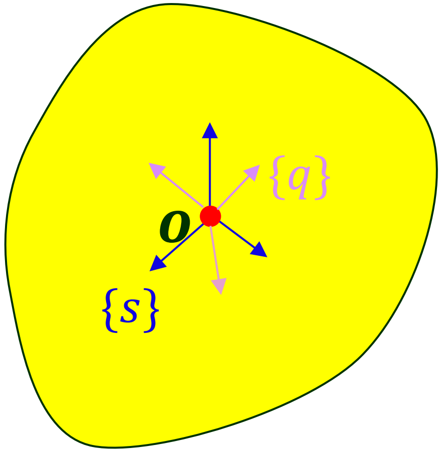
$$^{t}I = \int \rho\left(\left((\boldsymbol{x}-\boldsymbol{b})\cdot(\boldsymbol{x}-\boldsymbol{b})\right)I_3 - (\boldsymbol{x}-\boldsymbol{b})(\boldsymbol{x}-\boldsymbol{b})^{\boldsymbol{T}}\right)\, dV$$

$$= {}^{s}I + \int \rho\left((-2\boldsymbol{x}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{b})I_3 + \boldsymbol{x}\boldsymbol{b}^{\boldsymbol{T}} + \boldsymbol{b}\boldsymbol{x}^{\boldsymbol{T}} - \boldsymbol{b}\boldsymbol{b}^{\boldsymbol{T}}\right) dV$$

$$= {}^{s}I - 2\left(\underbrace{\int \rho\boldsymbol{x}\, dV}_{=\,\mathbf{0}} \cdot \boldsymbol{b}\right) I_3 + \underbrace{\int \rho\boldsymbol{x}\, dV}_{=\,\mathbf{0}}\, \boldsymbol{b}^{\boldsymbol{T}} + \boldsymbol{b}\underbrace{\int \rho\boldsymbol{x}^{\boldsymbol{T}} dV}_{=\,\mathbf{0}^{T}} + \int \rho\left((\boldsymbol{b}\cdot\boldsymbol{b})I_3 - \boldsymbol{b}\boldsymbol{b}^{\boldsymbol{T}}\right) dV$$

$$= {}^{s}I + \int \rho\left((\boldsymbol{b}\cdot\boldsymbol{b})I_3 - \boldsymbol{b}\boldsymbol{b}^{\boldsymbol{T}}\right) dV = {}^{s}I + m\left((\boldsymbol{b}\cdot\boldsymbol{b})I_3 - \boldsymbol{b}\boldsymbol{b}^{\boldsymbol{T}}\right)$$

Inertia Tensor Under Rotation



Consider a frame $\{q\}$ also located at $\boldsymbol{o}$ that rotates from the frame $\{s\}$ as described by a matrix R.

$$\boldsymbol{x} = \underbrace{{}^s\boldsymbol{r}} \longmapsto {}^q\boldsymbol{r} = R^T\boldsymbol{x}$$

Inertia tensor expressed in $\{q\}$:

$$\begin{aligned}
{}^qI &= \int \rho \left((R^T\boldsymbol{x}) \cdot (R^T\boldsymbol{x}) I_3 - R^T\boldsymbol{x}(R^T\boldsymbol{x})^{\boldsymbol{T}} \right) \, dV \\
&= \int \rho \left((\boldsymbol{x}^T R R^T \boldsymbol{x}) I_3 - R^T \boldsymbol{x}\boldsymbol{x}^T R \right) dV \\
&= \int \rho \left((\boldsymbol{x}^T \boldsymbol{x}) I_3 - R^T \boldsymbol{x}\boldsymbol{x}^T R \right) dV \\
&= \int \rho \left((\boldsymbol{x}^T \boldsymbol{x}) R^T I_3 R - R^T \boldsymbol{x}\boldsymbol{x}^T R \right) dV \\
&= R^T \int \rho \left((\boldsymbol{x}^T \boldsymbol{x}) I_3 - \boldsymbol{x}\boldsymbol{x}^T \right) dV \; R \\
&= R^T \, {}^sI R
\end{aligned}$$

Principal Axes

The inertia tensor I in a frame at $\boldsymbol{o}$ is symmetric and positive definite.

$$I = U\Lambda U^T$$ (spectral decomposition)

rotational matrix   diagonal matrix with positive eigenvalues

We now define the frame $\{b\}$ to be one whose orientation relative to $\{s\}$ is given by U. The inertia tensor is diagonalized in this frame:

$${}^bI = U^T U\Lambda U^T U = \Lambda$$

The three eigenvector of I in $\{s\}$, now the three axes of $\{b\}$, are the body's principal axes.

From now on, the *body frame* $\{b\}$ will refer to the frame defined by the principal axes.

Moment of Inertia

We can use the inertia tensor I to obtain the *moment of inertia* about an axis in the direction $\hat{\boldsymbol{\imath}}$ through the origin of the frame $\{s\}$.

$$
\begin{aligned}
I_l &\equiv \rho \int (\boldsymbol{x} \times \hat{\boldsymbol{\imath}}) \cdot (\boldsymbol{x} \times \hat{\boldsymbol{\imath}}) \; dV \\
&= \rho \int \boldsymbol{x} \cdot \left(\hat{\boldsymbol{\imath}} \times (\boldsymbol{x} \times \hat{\boldsymbol{\imath}})\right) \; dV \\
&= \rho \int \boldsymbol{x} \cdot \left((\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}})\boldsymbol{x} - (\hat{\boldsymbol{\imath}} \cdot \boldsymbol{x})\hat{\boldsymbol{\imath}}\right) \; dV \\
&= \rho \int \boldsymbol{x} \cdot \boldsymbol{x} - \hat{\boldsymbol{\imath}}^T \, \boldsymbol{x}\boldsymbol{x}^T \hat{\boldsymbol{\imath}} \, dV \\
&= \rho \int \hat{\boldsymbol{\imath}}^T (\boldsymbol{x} \cdot \boldsymbol{x} I_3) \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\imath}}^T \, \boldsymbol{x}\boldsymbol{x}^T \hat{\boldsymbol{\imath}} \, dV \\
&= \hat{\boldsymbol{\imath}}^T \left(\rho \int (\boldsymbol{x} \cdot \boldsymbol{x}) I_3 - \boldsymbol{x}\boldsymbol{x}^T dV \right) \hat{\boldsymbol{\imath}} \\
&= \hat{\boldsymbol{\imath}}^T I \hat{\boldsymbol{\imath}}
\end{aligned}
$$

// measures resistance to angular
// acceleration about the axis

Velocity Notation

$\boldsymbol{u}$: a vector/point in the world frame $\{w\}$

${}^{a}\boldsymbol{u}$: vector $\boldsymbol{u}$ expressed in the frame $\{a\}$

$\boldsymbol{u} = R\,{}^{a}\boldsymbol{u}$ where R is the rotation matrix from $\{w\}$ to $\{a\}$.

$\dot{\boldsymbol{u}}$: derivative of $\boldsymbol{u}$ in $\{w\}$

${}^{a}\dot{\boldsymbol{u}}$: derivative of $\boldsymbol{u}$ in $\{w\}$ as expressed in $\{a\}$

$$\dot{\boldsymbol{u}} = R\,{}^{a}\dot{\boldsymbol{u}}$$

$\frac{d}{dt}({}^{a}\boldsymbol{u})$: derivative of ${}^{a}\boldsymbol{u}$

$$\boldsymbol{u} = R\,{}^{a}\boldsymbol{u} \quad \Longrightarrow \quad \dot{\boldsymbol{u}} = \dot{R}\,{}^{a}\boldsymbol{u} + R\frac{d}{dt}({}^{a}\boldsymbol{u}) = \boldsymbol{\omega} \times (R\,{}^{a}\boldsymbol{u}) + R\frac{d}{dt}({}^{a}\boldsymbol{u})$$

$$= \boldsymbol{\omega} \times \boldsymbol{u} + R\frac{d}{dt}({}^{a}\boldsymbol{u})$$

$\dot{\boldsymbol{u}} \neq R\frac{d}{dt}({}^{a}\boldsymbol{u})$, i.e., ${}^{a}\dot{\boldsymbol{u}} \neq \frac{d}{dt}({}^{a}\boldsymbol{u})$, unless $\boldsymbol{\omega} \times \boldsymbol{u} = \boldsymbol{0}$

V. Euler's Equation

The torque $\boldsymbol{\tau}$, momentum $\boldsymbol{l}$, and inertia tensor I are all in terms of the frame $\{s\}$ from now on, so the superscript s is omitted.

$$
\begin{aligned}
\boldsymbol{\tau} &= \dot{\boldsymbol{l}} \\
&= \frac{d}{dt}(I\boldsymbol{\omega}) \\
&= \dot{I}\boldsymbol{\omega} + I\dot{\boldsymbol{\omega}}
\end{aligned}
$$

$\{s\}$ undergoes a rotation of R^T from $\{b\}$.

$$
I = R\,{}^bI R^T \xRightarrow[{}^bI = \Lambda]{} I = R\Lambda R^T \Rightarrow \dot{I} = \dot{R}\Lambda R^T + R\Lambda\dot{R}^T
$$

$$
\begin{aligned}
&= [\boldsymbol{\omega}]_\times R\Lambda R^T + R\Lambda([\boldsymbol{\omega}]_\times R)^T \\
&= [\boldsymbol{\omega}]_\times I + R\Lambda R^T[\boldsymbol{\omega}]_\times{}^T \\
&= [\boldsymbol{\omega}]_\times I + I[\boldsymbol{\omega}]_\times{}^T \\
&= [\boldsymbol{\omega}]_\times I - I[\boldsymbol{\omega}]_\times
\end{aligned}
$$

Euler’s Equation (cont’d)

$$\boldsymbol{\tau} = I\dot{\boldsymbol{\omega}} + [\boldsymbol{\omega}]_\times I\boldsymbol{\omega} - I[\boldsymbol{\omega}]_\times \boldsymbol{\omega}$$

$$= I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega})$$

⇓ when $\boldsymbol{\tau} = \mathbf{0}$

$$\dot{\boldsymbol{\omega}} = I^{-1}(\boldsymbol{\omega} \times (I\boldsymbol{\omega}))$$

- In the absence of external torque, angular acceleration is non-zero unless it is an eigenvector of I.

- A body tumbling in space has constant angular momentum ($\boldsymbol{\tau} = \mathbf{0}$) but continuously changing angular velocity.

Newton's 2nd Law for a Rigid Body

$$\boldsymbol{f} = m\dot{\boldsymbol{v}}$$

// in both {w} and {s}
// superscripts omitted

(Newton's equation)

$$\boldsymbol{\tau} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega})$$

(Euler's equation)

- Newton's equation governs how the body's center of mass moves.
- Newton's equation is described in the world frame $\{w\}$, which does not change over time.
- It also holds in the instantaneous frame $\{s\}$, but it would be meaningless for us to consider the body's trajectory in this frame in which it stays for zero time.
- Euler's equation governs how the orientation of the rigid body changes.
- It is described in the instantaneous, stationary frame $\{s\}$, whose origin changes with the object's center of mass.

Trajectory of a Rigid Body

$$\boldsymbol{f} = m\dot{\boldsymbol{v}}$$ (Newton's equation)

$$\boldsymbol{\tau} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega})$$ (Euler's equation)

- Integration of Newton's equation in the world frame $\{w\}$ yields the trajectory of the center of mass.

♠ Can we integrate Euler's equation in $\{s\}$ which changes as the object moves?

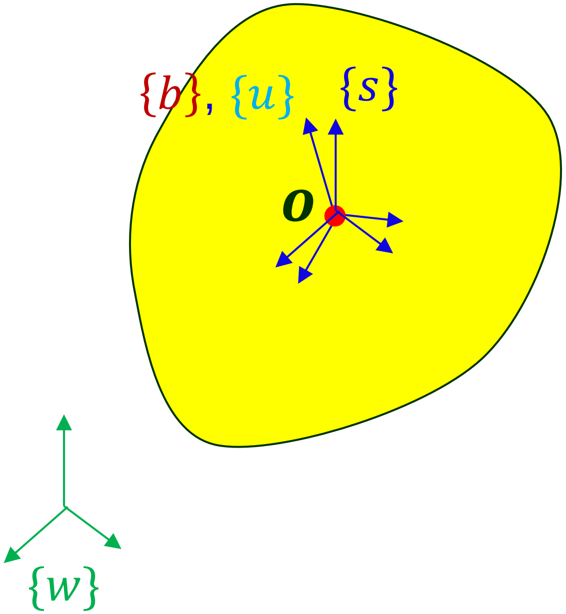
♠ Shouldn't integration be done within a single fixed frame?

Well, let us picture another scenario in which the same object has its center of mass fixed but its angular velocity is governed by the same Euler's equation. In this scenario, the frame $\{s\}$ does not change.

The object's orientation will be changing exactly the same way in the above two scenarios.

- Euler's equation can also be integrated directly – except that the inertia tensor I is changing!

Four Frames



$\{b\}$: body frame attached to the body at its center of mass $\boldsymbol{o}$ and defined by the three principal axes

The inertia tensor is diagonalized in $\{b\}$:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$\{u\}$: instantaneous frame (stationary) that coincides with $\{b\}$

$\{w\}$: world frame (stationary)

$\{s\}$: instantaneous frame (stationary) oriented the same as $\{w\}$

Solution of Euler’s Equation

- In the reference state, the body frame $\{b\}$ has the same orientation as the world frame $\{w\}$.
- During the motion, the changing orientation of $\{b\}$ relative to $\{w\}$ is described by the rotation matrix R (which is determined by three rotation angles $\boldsymbol{\theta}$).
- I is described in $\{s\}$, which undergoes a rotation of R^T from $\{b\}$.

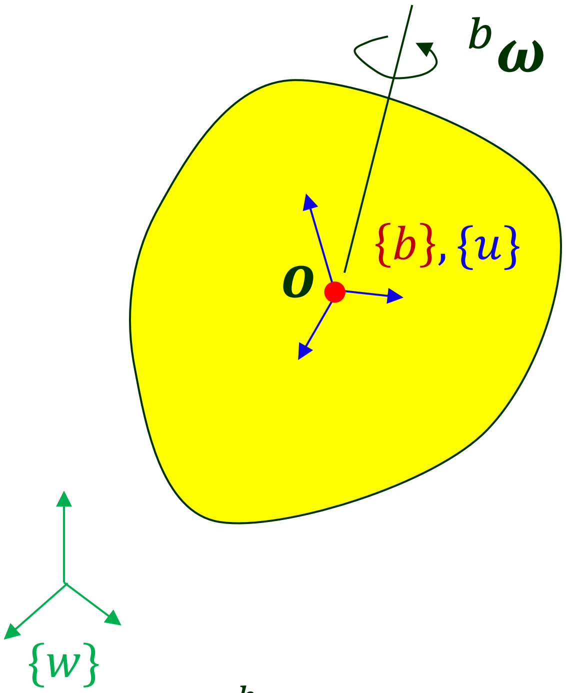
$$I = \left(R^T\right)^T \Lambda R^T = R\Lambda R^T$$

- Euler’s equation is transformed into

$$\boldsymbol{\tau} = \left(R\Lambda R^T\right)\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \left(\left(R\Lambda R^T\right)\boldsymbol{\omega}\right)$$

$$\dot{\boldsymbol{\omega}} \xrightarrow{\int} \boldsymbol{\omega} \xrightarrow{\int} \boldsymbol{\theta} \longrightarrow R$$

VI. Angular Velocity "in the Body Frame"



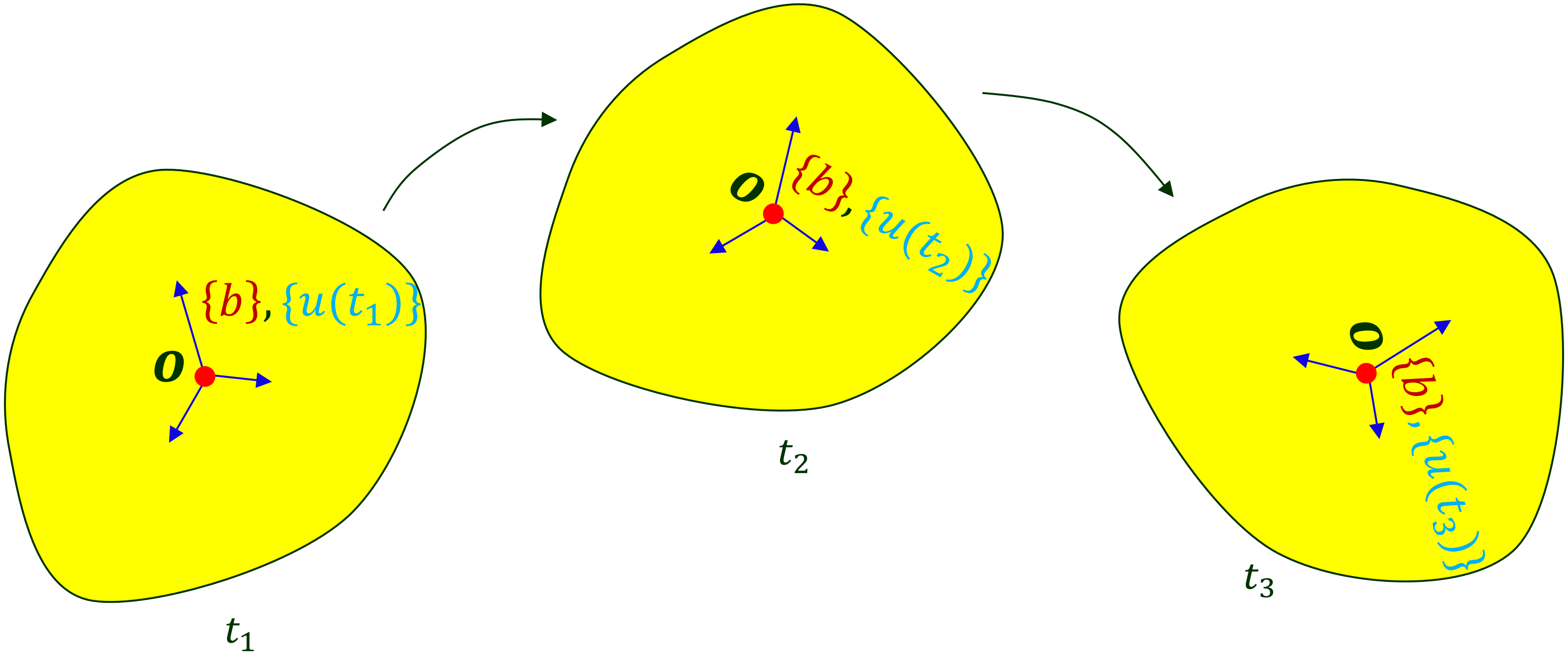
Often the angular velocity is conveniently described relative to $\{u\}$, or equivalently, described in $\{b\}$.

$${}^b\boldsymbol{\omega} = R^T\boldsymbol{\omega}$$

- ${}^b\boldsymbol{\omega}$ is observed from a stationary frame that instantaneously coincides with the body frame $\{b\}$.

- It is given in a different stationary frame at a different time instant.

A Family of Instantaneous Frames



◆ During the object's motion, there exists a continuum of instantaneous frames $\{u(t)\}$.

◆ At time t, the frame $\{u(t)\}$ coincides with the moving body frame $\{b\}$.

◆ The object has zero velocity and angular velocity relative to $\{b\}$.

◆ Its velocity ${}^b\boldsymbol{v}$ and angular velocity ${}^b\boldsymbol{\omega}$ expressed in $\{b\}$ are relative to $\{u(t)\}$.

At One Time Instant

Let's consider a fixed time instant, say, $t = t_0$.

${}^b\boldsymbol{\tau}$: moment of force in terms of $\{b\}$

$$\boldsymbol{\tau} = R\,{}^b\boldsymbol{\tau}$$

$$\boldsymbol{\omega} = R\,{}^b\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\omega}} = R\frac{d}{dt}\,{}^b\boldsymbol{\omega}|_{\{u(t_0)\}}$$ (R is constant as we fix time at t_0)

derivative w.r.t. t within the frame $\{u(t_0)\}$

Euler's equation:

$$ {}^b\boldsymbol{n} = \Lambda\frac{d}{dt}\,{}^b\boldsymbol{\omega}|_{\{u(t_0)\}} + {}^b\boldsymbol{\omega} \times (\Lambda\,{}^b\boldsymbol{\omega})$$ (expressed in $\{u(t_0)\}$)

Issue: The equation can only be integrated if ${}^b\boldsymbol{\omega}(t)$, for all t, is measured in the single frame $\{b(t_0)\}$, which is only convenient for t_0.

Angular Acceleration “in the Body Frame”

- Euler’s equation is not automatically applicable to ${}^{b}\boldsymbol{\omega}$, which is not associated with a single fixed frame. In this case, we are concerned with a continuum of frames with changing orientations.

- Nevertheless, we have

$$\boldsymbol{\omega} = R\,{}^{b}\boldsymbol{\omega} \quad \Longrightarrow \quad \dot{\boldsymbol{\omega}} = \frac{d}{dt}\left(R\,{}^{b}\boldsymbol{\omega}\right) = \dot{R}\,{}^{b}\boldsymbol{\omega} + R\frac{d}{dt}\left({}^{b}\boldsymbol{\omega}\right)$$

$$= \left(\dot{R}R^{T}\right)\left(R\,{}^{b}\boldsymbol{\omega}\right) + R\frac{d}{dt}\left({}^{b}\boldsymbol{\omega}\right) = [\,{}^{s}\boldsymbol{\omega}]_{\times}\,{}^{s}\boldsymbol{\omega} + R\frac{d}{dt}\left({}^{b}\boldsymbol{\omega}\right)$$

$$= \boldsymbol{\omega} \times \boldsymbol{\omega} + R\frac{d}{dt}\left({}^{b}\boldsymbol{\omega}\right) = R\frac{d}{dt}\left({}^{b}\boldsymbol{\omega}\right)$$

The derivative $\frac{d}{dt}\left({}^{b}\boldsymbol{\omega}\right)$ is also related to the real angular acceleration $\dot{\boldsymbol{\omega}}$ via a rotation.

$$\frac{d}{dt}\left({}^{b}\boldsymbol{\omega}\right) = {}^{b}\dot{\boldsymbol{\omega}} \equiv {}^{b}(\dot{\boldsymbol{\omega}})$$

- We can treat $\frac{d}{dt}\left({}^{b}\boldsymbol{\omega}\right)$ as the “angular acceleration”.

Rewriting Euler’s Equation

Recall that $\{s\}$ is the stationary frame instantaneously located at $\boldsymbol{o}$ and oriented the same as $\{w\}$.

$$\boldsymbol{\tau} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega})$$

$$\Downarrow$$

$$\underbrace{I = R\Lambda R^T \qquad\qquad R\,{}^b\boldsymbol{\tau} = I\left(R\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right)\right) + (R\,{}^b\boldsymbol{\omega}) \times (I(R\,{}^b\boldsymbol{\omega}))}$$

$$\Downarrow$$

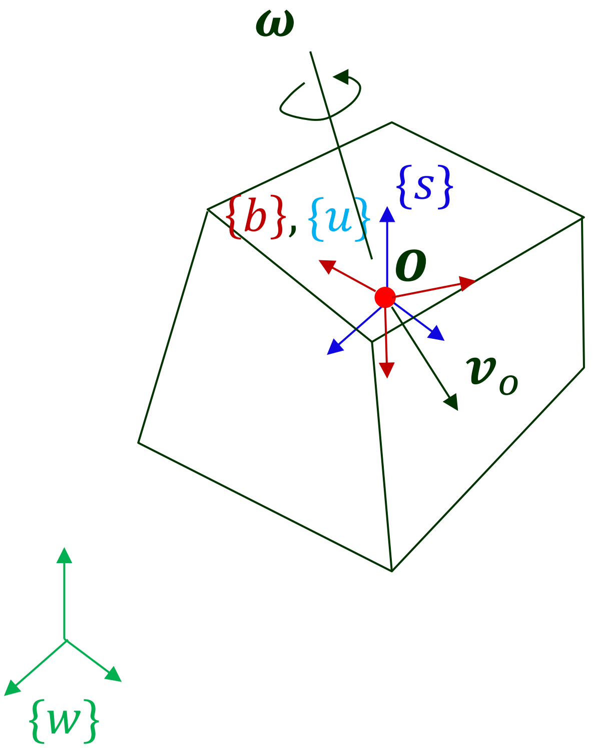
$$R\,{}^b\boldsymbol{\tau} = (R\Lambda R^T)\left(R\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right)\right) + (R\,{}^b\boldsymbol{\omega}) \times ((R\Lambda R^T)(R\,{}^b\boldsymbol{\omega}))$$

$$\Rightarrow \qquad R\,{}^b\boldsymbol{\tau} = R\Lambda\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right) + (R\,{}^b\boldsymbol{\omega}) \times (R\Lambda\,{}^b\boldsymbol{\omega})$$

$$\Rightarrow \qquad R\,{}^b\boldsymbol{\tau} = R\Lambda\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right) + R\left({}^b\boldsymbol{\omega} \times \Lambda\boldsymbol{\omega_b}\right)$$

$$\Rightarrow \qquad {}^b\boldsymbol{\tau} = \Lambda\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right) + {}^b\boldsymbol{\omega} \times \left(\Lambda\,{}^b\boldsymbol{\omega}\right)$$

VII. Simulating Dynamics

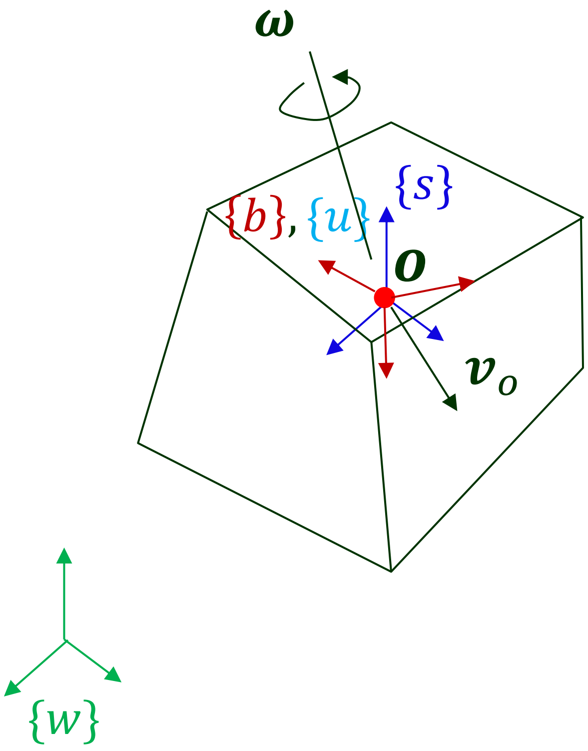


We (numerically) integrate the following system with $\boldsymbol{v} = {}^{w}\boldsymbol{v}$ and $\boldsymbol{\omega} = {}^{w}\boldsymbol{\omega} = {}^{s}\boldsymbol{\omega}$ as variables:

$$\boldsymbol{f} = m\dot{\boldsymbol{v}}$$

$$\boldsymbol{\tau} = (R\Lambda R^T)\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \left((R\Lambda R^T)\boldsymbol{\omega}\right)$$

Dynamics in Mixed Frames



We could also integrate the velocity in the world frame while angular velocity in the body frame..

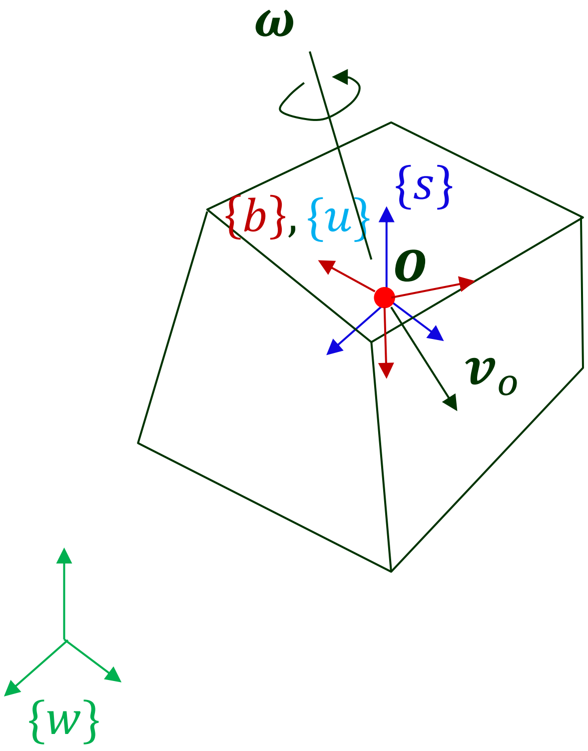
$$\boldsymbol{f} = m\dot{\boldsymbol{v}}$$

✓

$${}^{b}\boldsymbol{\tau} = \Lambda \frac{d}{dt}\left({}^{b}\boldsymbol{\omega}\right) + {}^{b}\boldsymbol{\omega} \times \left(\Lambda\, {}^{b}\boldsymbol{\omega}\right)$$

More efficient!

Dynamics in the Body Frame



$$\boldsymbol{v} = R\,{}^b\boldsymbol{v} \implies \dot{\boldsymbol{v}} = \dot{R}\,{}^b\boldsymbol{v} + R\frac{d}{dt}\left({}^b\boldsymbol{v}\right)$$

$$\implies \dot{\boldsymbol{v}} = \boldsymbol{\omega} \times \left(R\,{}^b\boldsymbol{v}\right) + R\frac{d}{dt}\left({}^b\boldsymbol{v}\right)$$

$$= \left(R\,{}^b\boldsymbol{\omega}\right) \times \left(R\,{}^b\boldsymbol{v}\right) + R\frac{d}{dt}\left({}^b\boldsymbol{v}\right)$$

$$= R\left({}^b\boldsymbol{\omega} \times {}^b\boldsymbol{v}\right) + R\frac{d}{dt}\left({}^b\boldsymbol{v}\right)$$

$$\implies \boldsymbol{f} = m\dot{\boldsymbol{v}} = Rm\left(\frac{d}{dt}\left({}^b\boldsymbol{v}\right) + {}^b\boldsymbol{\omega} \times {}^b\boldsymbol{v}\right)$$

$$\Big\Downarrow$$

Multiply both sides with R^T

$${}^b\boldsymbol{f} = m\left(\frac{d}{dt}\left({}^b\boldsymbol{v}\right) + {}^b\boldsymbol{\omega} \times {}^b\boldsymbol{v}\right)$$

$${}^b\boldsymbol{\tau} = \Lambda\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right) + {}^b\boldsymbol{\omega} \times \left(\Lambda\,{}^b\boldsymbol{\omega}\right)$$