Plücker Coordinates for Lines in the Space

(COMS 4770/5770 Notes)

Yan-Bin Jia

Sep 3, 2024

In the space, a plane can be conveniently represented by four homogeneous coordinates. Representation of a line, however, turns out to be more complicated and requires the use of some special coordinates. Before presenting the material, it helps to go over some identities involving cross and dot products. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be three vectors in $\mathbb{R}^3$. The following identities hold:

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= \boldsymbol{b} \cdot \boldsymbol{a}, & (1)\\ \boldsymbol{a} \times \boldsymbol{b} &= -\boldsymbol{b} \times \boldsymbol{a}, & (2)\\ \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) &= (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}, & (3)\\ \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}. & (4) \end{aligned}$$

Applying the above identities, we can remove concatenation of cross product operators. For example, letting $\boldsymbol{d}$ be a fourth vector, we apply (4) as follows:

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \Big((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{d}\Big)\boldsymbol{c} - \Big((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}\Big)\boldsymbol{d}.$$

1 Plücker Coordinates

A rigid body in the space is known to have six degrees of freedom: three translational and three rotational. It can translate along either one of x-, y-, and z-axes, or rotate about either one of them or any of other three orthogonal directions. A line ℓ in the space, however, has only four degrees of freedom. This is because it will remain the same line when undergoing a rotation about itself as the axis or a translation in its own direction.

As shown in Figure 1, a line ℓ can be represented by its direction $\boldsymbol{l}$ and a point $\boldsymbol{p}$ that it passes through. The *Plücker coordinates* [1, 2] of the line ℓ is defined to be $(\boldsymbol{l}, \boldsymbol{m})$, where $\boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{l}$ is referred to as the *moment vector*. These coordinates were introduced by Julius Plücker in the 19th century. They are also known as the *Grassmann coordinates*, or referred to as the *homogeneous line coordinates* of the line. The second coordinate $\boldsymbol{m}$ is independent of the choice of $\boldsymbol{p}$ because a different point $\boldsymbol{p}'$ on the line satisfies $\boldsymbol{p}' - \boldsymbol{p} = \lambda \boldsymbol{l}$, for some λ, thereby inducing

$$\begin{aligned} \boldsymbol{p}' \times \boldsymbol{l} &= \Big(\boldsymbol{p} + (\boldsymbol{p}' - \boldsymbol{p})\Big) \times \boldsymbol{l} \\ &= \boldsymbol{p} \times \boldsymbol{l} + (\boldsymbol{p}' - \boldsymbol{p}) \times \boldsymbol{l} \\ &= \boldsymbol{m} + \lambda \boldsymbol{l} \times \boldsymbol{l} \\ &= \boldsymbol{m}. \end{aligned}$$

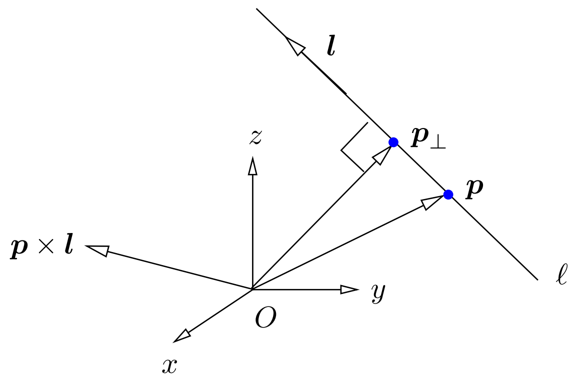

Figure 1: Plücker coordinates $(\boldsymbol{l}, \boldsymbol{p}\times\boldsymbol{l})$ of a line. Here $\boldsymbol{p}_\perp$ is the closest point on the line to the origin O.

A point $\boldsymbol{q}$ lies on the line ℓ if and only if $\boldsymbol{q}\times\boldsymbol{l} = \boldsymbol{m}$. The necessity follows from the independence of $\boldsymbol{m}$ from the point location on the line as just reasoned above. For the sufficiency, we see that $\boldsymbol{q}\times\boldsymbol{l} = \boldsymbol{m}$ implies $(\boldsymbol{q}-\boldsymbol{r})\times\boldsymbol{l} = \boldsymbol{m}-\boldsymbol{m} = \boldsymbol{0}$, where $\boldsymbol{r}$ is a point on the line. Thus, the vector $\boldsymbol{q}-\boldsymbol{r}$ is collinear with the line direction $\boldsymbol{l}$. So $\boldsymbol{q}$ must be on the line as well.

When $\boldsymbol{l}$ is a unit vector, the moment $\|\boldsymbol{m}\|$ gives the distance from the origin to the line. Generally, such distance is given by $\|\boldsymbol{m}\|/\|\boldsymbol{l}\|$. If $\boldsymbol{m}$ is zero, the line is through the origin.

The six coordinates in $(\boldsymbol{l}, \boldsymbol{m})$ have two redundancies. Since $\boldsymbol{m}$ scales with $\boldsymbol{l}$, the Plücker coordinates $(\boldsymbol{l}, \boldsymbol{m})$ and $(c\boldsymbol{l}, c\boldsymbol{m})$, $c \neq 0$, describe the same line. First, we can choose $\boldsymbol{l}$ to be a unit vector. Second, $\boldsymbol{l}\cdot\boldsymbol{m} = \boldsymbol{l}\cdot(\boldsymbol{p}\times\boldsymbol{l}) = 0$. The constraints $\|\boldsymbol{l}\| = 1$ and $\boldsymbol{l}\cdot\boldsymbol{m} = 0$ essentially remove two variables from $(\boldsymbol{l}, \boldsymbol{m})$, reflecting the line's four degrees of freedom in the space.

For the Plücker coordinates as $(\hat{\boldsymbol{l}}, \boldsymbol{m})$, where $\hat{\boldsymbol{l}}$ is a unit vector, $\boldsymbol{m} = \boldsymbol{p}\times\hat{\boldsymbol{l}}$ is called the *moment of the line*. It is the moment of a unit force acting at $\boldsymbol{p}$ in the direction $\hat{\boldsymbol{l}}$ with respect to the origin. The norm $\|\boldsymbol{m}\|$ gives the distance from the origin to the line, achieved at the foot $\boldsymbol{p}_\perp$ of the line's perpendicular through the origin, where

$$\begin{aligned}
\boldsymbol{p}_\perp &= \boldsymbol{p} - (\hat{\boldsymbol{l}}\cdot\boldsymbol{p})\hat{\boldsymbol{l}} \\
&= (\hat{\boldsymbol{l}}\cdot\hat{\boldsymbol{l}})\boldsymbol{p} - (\hat{\boldsymbol{l}}\cdot\boldsymbol{p})\hat{\boldsymbol{l}} \\
&= \hat{\boldsymbol{l}}\times(\boldsymbol{p}\times\hat{\boldsymbol{l}}) \\
&= \hat{\boldsymbol{l}}\times\boldsymbol{m}.
\end{aligned}$$

For a line with general Plücker coordinates $(\boldsymbol{l}, \boldsymbol{m})$, its distance to the origin is $\|\boldsymbol{m}\|/\|\boldsymbol{l}\|$ achieved at $\boldsymbol{p}_\perp = \boldsymbol{l}\times\boldsymbol{m}/\|\boldsymbol{l}\|^2$. Hence, the line passes through the origin if $\boldsymbol{m} = \boldsymbol{0}$ (and $\boldsymbol{l} \neq \boldsymbol{0}$), as illustrated in Figure 2(a).

What happens if $\boldsymbol{l} = \boldsymbol{0}$ and $\boldsymbol{m} \neq \boldsymbol{0}$? Let us first consider a line ℓ with Plücker coordinates $(\boldsymbol{l}, \boldsymbol{m})$ such that both $\boldsymbol{l}$ and $\boldsymbol{m}$ are nonzero. Let $\boldsymbol{p}$ be a point on the line, hence $\boldsymbol{m} = \boldsymbol{p}\times\boldsymbol{l}$. Also, let Π be the plane containing the line and origin. The vector $\boldsymbol{m}$ is normal to Π. Now we translate ℓ in the direction of $\boldsymbol{p}$ within Π. The line's new Plücker coordinates becomes $(\boldsymbol{l}, t\boldsymbol{p}\times\boldsymbol{l}) = (\boldsymbol{l}, t\boldsymbol{m})$, where t represents some scalar multiple of the line's original distance from the origin. Equivalently, these coordinates are $(\boldsymbol{l}/t, \boldsymbol{m})$ since they are independent of scaling. As the line is translated to infinity, we have

$$\lim_{t\to\infty}\left(\frac{\boldsymbol{l}}{t}, \boldsymbol{m}\right) = (\boldsymbol{0}, \boldsymbol{m}).$$

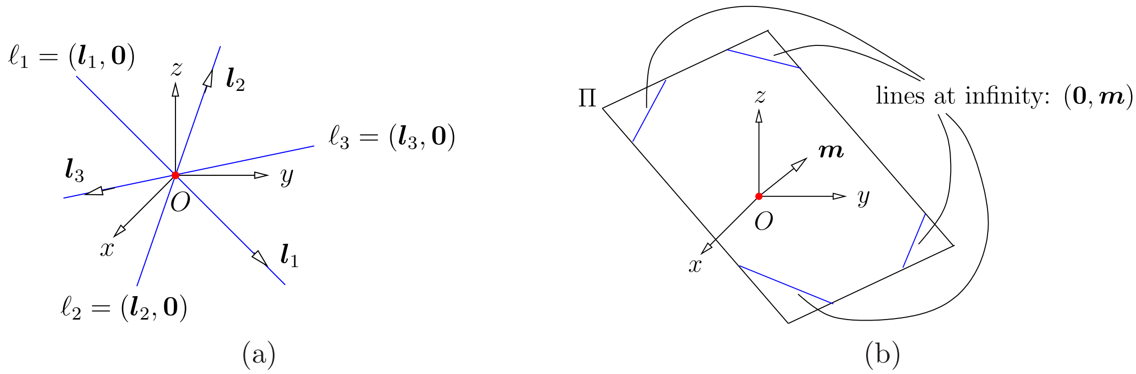

Figure 2: (a) Lines through the origin have Plücker coordinates in the form $(\_, \mathbf{0})$. (b) All the lines at infinity in the plane Π through the origin and with the normal $\boldsymbol{m}$ have the same Plücker coordinates $(\mathbf{0}, \boldsymbol{m})$.

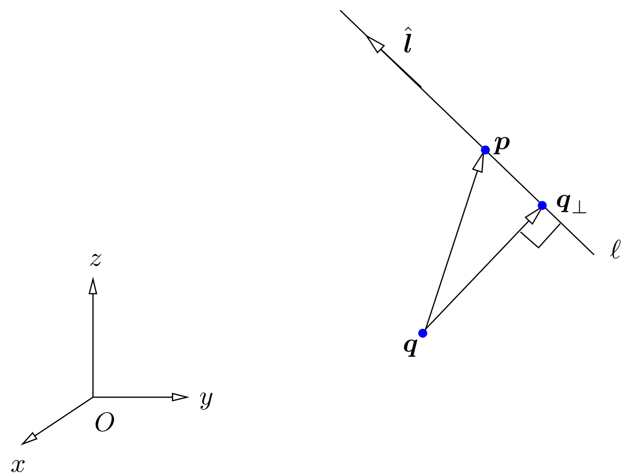

Figure 3: Moment of a line ℓ about a point $\boldsymbol{q}$.

Note that $\boldsymbol{m}$ is normal to the plane Π, in which every line may have its last three Plücker coordinates described by $\boldsymbol{m}$. Hence, all the lines at infinity in Π, regardless of their directions, are described by the same Plücker coordinates $(\mathbf{0}, \boldsymbol{m})$. This is illustrated in Figure 2(b). Parallel lines at infinity, however, lie in different planes through the origin, and thus have different values of $\boldsymbol{m}$.

The Plücker coordinates do not include $(\mathbf{0}, \mathbf{0})$. They are considered as homogeneous coordinates (in a five-dimensional projective space) which uniquely represent lines in the three-dimensional space.

2 Reciprocal Product

Now we look at the moment of the line ℓ about a point $\boldsymbol{q}$ in the space. See Figure 3. The moment corresponds to the torque generated by the unit force $\hat{\boldsymbol{l}}$ acting at any point $\boldsymbol{p}$ on the line with respect to $\boldsymbol{q}$. Denoted by $\boldsymbol{m}_q$, it is derived below:

$$\boldsymbol{m}_q \quad = \quad (\boldsymbol{p} - \boldsymbol{q}) \times \hat{\boldsymbol{l}}$$

$$
\begin{aligned}
&= \boldsymbol{p} \times \hat{\boldsymbol{l}} - \boldsymbol{q} \times \hat{\boldsymbol{l}} \\
&= \boldsymbol{m} - \boldsymbol{q} \times \hat{\boldsymbol{l}}. \qquad (5)
\end{aligned}
$$

Zero dot product of $\hat{\boldsymbol{l}}$ with (5) shows that the moment is orthogonal to $\hat{\boldsymbol{l}}$. Its norm $\|\boldsymbol{m} - \boldsymbol{q} \times \hat{\boldsymbol{l}}\|$ is the distance from $\boldsymbol{q}$ to the line.

Let $\boldsymbol{q}_\perp$ be the foot of the perpendicular from $\boldsymbol{q}$ to the line ℓ. Hence, $\boldsymbol{q}_\perp - \boldsymbol{q}$ is orthogonal to $\hat{\boldsymbol{l}}$, implying

$$
\begin{aligned}
\hat{\boldsymbol{l}} \times \boldsymbol{m}_q &= \hat{\boldsymbol{l}} \times \Big((\boldsymbol{q}_\perp - \boldsymbol{q}) \times \hat{\boldsymbol{l}} \Big) \\
&= (\hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{l}})(\boldsymbol{q}_\perp - \boldsymbol{q}) - \Big(\hat{\boldsymbol{l}} \cdot (\boldsymbol{q}_\perp - \boldsymbol{q}) \Big) \hat{\boldsymbol{l}} \\
&= \boldsymbol{q}_\perp - \boldsymbol{q}.
\end{aligned}
$$

We obtain

$$
\boldsymbol{q}_\perp = \boldsymbol{q} + \hat{\boldsymbol{l}} \times \boldsymbol{m}_q. \qquad (6)
$$

Suppose there are two lines ℓ_1 and ℓ_2 described by Plücker coordinates $(\hat{\boldsymbol{l}}_1, \boldsymbol{m}_1)$ and $(\hat{\boldsymbol{l}}_2, \boldsymbol{m}_2)$, respectively. The moment of ℓ_2 about a point $\boldsymbol{p}_1$ on ℓ_1 is $\boldsymbol{m}_2 - \boldsymbol{p}_1 \times \hat{\boldsymbol{l}}_2$ according to (5). Its

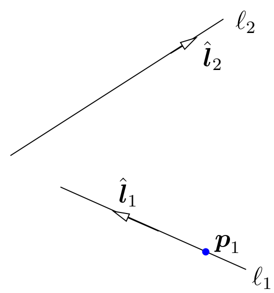

Figure 4: Two lines.

projection along the latter line's direction $\hat{\boldsymbol{l}}_1$ is then calculated as follows using (5):

$$
\begin{aligned}
\hat{\boldsymbol{l}}_1 \cdot (\boldsymbol{m}_2 - \boldsymbol{p}_1 \times \hat{\boldsymbol{l}}_2) &= \hat{\boldsymbol{l}}_1 \cdot \boldsymbol{m}_2 - \hat{\boldsymbol{l}}_1 \cdot (\boldsymbol{p}_1 \times \hat{\boldsymbol{l}}_2) \\
&= \boldsymbol{l}_1 \cdot \boldsymbol{m}_2 - (\hat{\boldsymbol{l}}_1 \times \boldsymbol{p}_1) \cdot \hat{\boldsymbol{l}}_2 \\
&= \hat{\boldsymbol{l}}_1 \cdot \boldsymbol{m}_2 + \hat{\boldsymbol{l}}_2 \cdot \boldsymbol{m}_1. \qquad (7)
\end{aligned}
$$

The result, independent of the location of $\boldsymbol{p}_1$ on ℓ_1, is called the *moment* of ℓ_2 *about* ℓ_1. The sum in (7) is called be the *reciprocal product* of the Plücker coordinates of the two lines:

$$
(\hat{\boldsymbol{l}}_1, \boldsymbol{m}_1) * (\hat{\boldsymbol{l}}_2, \boldsymbol{m}_2) \stackrel{\text{def}}{=} \hat{\boldsymbol{l}}_1 \cdot \boldsymbol{m}_2 + \hat{\boldsymbol{l}}_2 \cdot \boldsymbol{m}_1. \qquad (8)
$$

This sum stays the same even if we switch the subscripts 1 and 2, which implies that the moment of ℓ_2 about ℓ_1 is equal to that of ℓ_1 about ℓ_2. Nevertheless, the reciprocal product depends on the directions $\hat{\boldsymbol{l}}_1$ and $\hat{\boldsymbol{l}}_2$ chosen for the two lines.

3 Distance Between Two Lines

Two lines in the space may be parallel to each other. They may intersect. Or they may not simultaneously lie in any single plane, that is, they may not be co-planar. In the last situation, the two lines are called *skew lines*. They have a *common perpendicular* which intersects both at right angles. In Figure 5, the points $\boldsymbol{p}_1^*$ and $\boldsymbol{p}_2^*$ are respectively the intersections of ℓ_1 and ℓ_2 with their

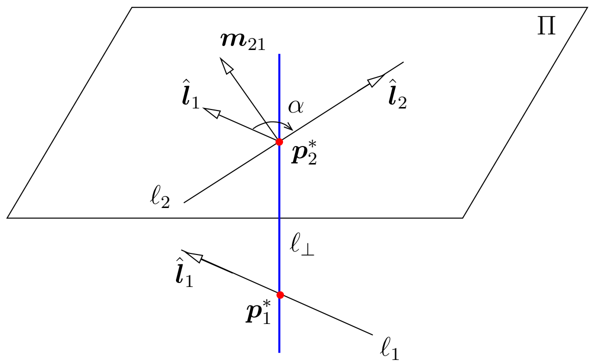

Figure 5: Distance between lines ℓ_1 and ℓ_2. The common perpendicular intersects the two lines at $\boldsymbol{p}_1^*$ and $\boldsymbol{p}_2^*$, respectively.

common perpendicular $\ell_\perp$. These two points are called the *feet* of the common perpendicular. The distance between the two lines equals that between $\boldsymbol{p}_1^*$ and $\boldsymbol{p}_2^*$, which are the closest pair of points from different lines. The distance is equal to that from $\boldsymbol{p}_1^*$ to ℓ_2, or from $\boldsymbol{p}_2^*$ to ℓ_1. That from $\boldsymbol{p}_1^*$ to ℓ_2 is equal to the norm of the moment $\boldsymbol{m}_{21}$ of the line ℓ_2 about $\boldsymbol{p}_1^*$ which, by (5), is

$$\begin{aligned} \boldsymbol{m}_{21} &= \boldsymbol{m}_2 - \boldsymbol{p}_1^* \times \hat{\boldsymbol{l}}_2 \\ &= (\boldsymbol{p}_2^* - \boldsymbol{p}_1^*) \times \hat{\boldsymbol{l}}_2. \end{aligned} \tag{9}$$

The last equation implies that $\boldsymbol{m}_{21} \perp \hat{\boldsymbol{l}}_2$.

We can make use of reciprocal products to determine the relative location of the two lines. If they intersect, we can find their intersection. If they are skew, we can compute their distance, construct their *common perpendicular*, and its intersections with the two lines.

Theorem 1 *Two lines ℓ_1 and ℓ_2, in Plücker coordinates $(\boldsymbol{l}_1, \boldsymbol{m}_1)$ and $(\boldsymbol{l}_2, \boldsymbol{m}_2)$, have the distance*

$$d = \begin{cases} \dfrac{|(\boldsymbol{l}_1, \boldsymbol{m}_1) * (\boldsymbol{l}_2, \boldsymbol{m}_2)|}{\|\boldsymbol{l}_1 \times \boldsymbol{l}_2\|}, & \textit{if } \boldsymbol{l}_1 \times \boldsymbol{l}_2 \neq 0; \\ \dfrac{\|\boldsymbol{l}_1 \times (\boldsymbol{m}_1 - \boldsymbol{m}_2/s)\|}{\|\boldsymbol{l}_1\|^2}, & \textit{otherwise, where } \boldsymbol{l}_2 = s\boldsymbol{l}_1 \textit{ for some } s \neq 0. \end{cases} \tag{10}$$

Proof Denote by $\hat{\boldsymbol{l}}_1$ and $\hat{\boldsymbol{l}}_2$ the unit vectors in the directions of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, respectively. First, we assume that the two lines ℓ_1 and ℓ_2 are not parallel to each other; in other words, $\boldsymbol{l}_1 \times \boldsymbol{l}_2 \neq 0$. We will consider the case of them being parallel later. Refer to Figure 5, because $\boldsymbol{p}_1^*$ is the closest point on ℓ_1 to ℓ_2, $(\boldsymbol{p}_2^* - \boldsymbol{p}_1^*) \perp \hat{\boldsymbol{l}}_2$, i.e., $\ell_\perp \perp \ell_2$. The plane Π containing ℓ_2^* and $\boldsymbol{m}_{21} = \boldsymbol{m}_2/\|\boldsymbol{l}_2\| - \boldsymbol{p}_1^* \times \hat{\boldsymbol{l}}_2$ is orthogonal to $\ell_\perp$ and thus parallel to ℓ_1. It consequently contains $\hat{\boldsymbol{l}}_1$. When the latter gets translated to $\boldsymbol{p}_2^*$, let α be the angle of rotation from $\hat{\boldsymbol{l}}_1$ to $\hat{\boldsymbol{l}}_2$. Since both are unit vectors, it trivially holds that $|\sin\alpha| = \|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|$. Since $\boldsymbol{m}_{21} \perp \hat{\boldsymbol{l}}_2$, we derive the moment of ℓ_2 about ℓ_1 below:

$$
\begin{aligned}
\hat{\boldsymbol{l}}_1 \cdot \boldsymbol{m}_{21} &= \|\boldsymbol{m}_{21}\| \cos\left(\alpha + \frac{\pi}{2}\right) && (11)\\
&= \|\boldsymbol{m}_{21}\|(-\sin\alpha). && (12)
\end{aligned}
$$

The above leads to derivation of the distance between ℓ_1 and ℓ_2:

$$
\begin{aligned}
d &= \|\boldsymbol{m}_{21}\| \\
&= \frac{|\hat{\boldsymbol{l}}_1 \cdot \boldsymbol{m}_{21}|}{|\sin\alpha|} && \text{(by (12))}\\
&= \frac{|\hat{\boldsymbol{l}}_1 \cdot (\boldsymbol{m}_2/\|\boldsymbol{l}_2\| - \boldsymbol{p}_1^* \times \hat{\boldsymbol{l}}_2)|}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} && \text{(by (9))}\\
&= \frac{|\hat{\boldsymbol{l}}_1 \cdot (\boldsymbol{m}_2/\|\boldsymbol{l}_1\|) - (\hat{\boldsymbol{l}}_1 \times \boldsymbol{p}_1^*) \cdot \hat{\boldsymbol{l}}_2)|}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \\
&= \frac{|(\hat{\boldsymbol{l}}_1, \boldsymbol{m}_1/\|\boldsymbol{l}_1\|) * (\hat{\boldsymbol{l}}_2, \boldsymbol{m}_2/\|\boldsymbol{l}_2\|)|}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|}. && \text{(by (7) and (8))} \quad (13)
\end{aligned}
$$

If the two lines intersect, then $d = 0$ and equation (13) implies that the reciprocal product is zero, that is,

$$
(\boldsymbol{l}_1, \boldsymbol{m}_1) * (\boldsymbol{l}_2, \boldsymbol{m}_2) = 0. \tag{14}
$$

When the two lines ℓ_1 and ℓ_2 are parallel to each other. That $\hat{\boldsymbol{l}}_1 = \pm\hat{\boldsymbol{l}}_2$ immediately implies the following:

$$
\begin{aligned}
\hat{\boldsymbol{l}}_1 \cdot \boldsymbol{m}_2 &= \pm\hat{\boldsymbol{l}}_2 \cdot \boldsymbol{m}_2 = 0;\\
\hat{\boldsymbol{l}}_2 \cdot \boldsymbol{m}_1 &= \pm\hat{\boldsymbol{l}}_1 \cdot \boldsymbol{m}_1 = 0,
\end{aligned}
$$

By definition (8) the reciprocal product vanishes. Consequently, (13) yields $d = 0$, which is incorrect. In this case, we let $\hat{\boldsymbol{l}} = \hat{\boldsymbol{l}}_1$. With $\boldsymbol{l}_2 = s\boldsymbol{l}_1$ for some s, we rewrite the Plücker coordinates as $(\hat{\boldsymbol{l}}, \boldsymbol{m}_1')$ and $(\hat{\boldsymbol{l}}, \boldsymbol{m}_2')$, where $\boldsymbol{m}_1' = \boldsymbol{m}_1/\|\boldsymbol{l}_1\|$ and $\boldsymbol{m}_2' = \boldsymbol{m}_2/(s\|\boldsymbol{l}_1\|)$. Let $\boldsymbol{p}_1^*$ and $\boldsymbol{p}_2^*$ be two points on ℓ_1 and ℓ_2, respectively, such that $\boldsymbol{p}_1^* - \boldsymbol{p}_2^*$ is orthogonal to $\hat{\boldsymbol{l}}$. The distance between the two lines is thus $\|\boldsymbol{p}_1^* - \boldsymbol{p}_2^*\|$. We have that

$$
\boldsymbol{m}_1' - \boldsymbol{m}_2' = (\boldsymbol{p}_1^* - \boldsymbol{p}_2^*) \times \hat{\boldsymbol{l}},
$$

from which we obtain

$$
\begin{aligned}
\boldsymbol{p}_1^* - \boldsymbol{p}_2^* &= (\hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{l}})(\boldsymbol{p}_1^* - \boldsymbol{p}_2^*) - \Big(\hat{\boldsymbol{l}} \cdot (\boldsymbol{p}_1^* - \boldsymbol{p}_2^*)\Big)\hat{\boldsymbol{l}} \\
&= \hat{\boldsymbol{l}} \times \Big((\boldsymbol{p}_1^* - \boldsymbol{p}_2^*) \times \hat{\boldsymbol{l}}\Big) \\
&= \hat{\boldsymbol{l}} \times (\boldsymbol{m}_1' - \boldsymbol{m}_2') \\
&= \frac{\boldsymbol{l}_1}{\|\boldsymbol{l}_1\|} \times \left(\frac{\boldsymbol{m}_1}{\|\boldsymbol{l}_1\|} - \frac{\boldsymbol{m}_2/s}{\|\boldsymbol{l}_1\|}\right) \\
&= \frac{\boldsymbol{l}_1 \times (\boldsymbol{m}_1 - \boldsymbol{m}_2/s)}{\|\boldsymbol{l}_1\|^2}. && (15)
\end{aligned}
$$

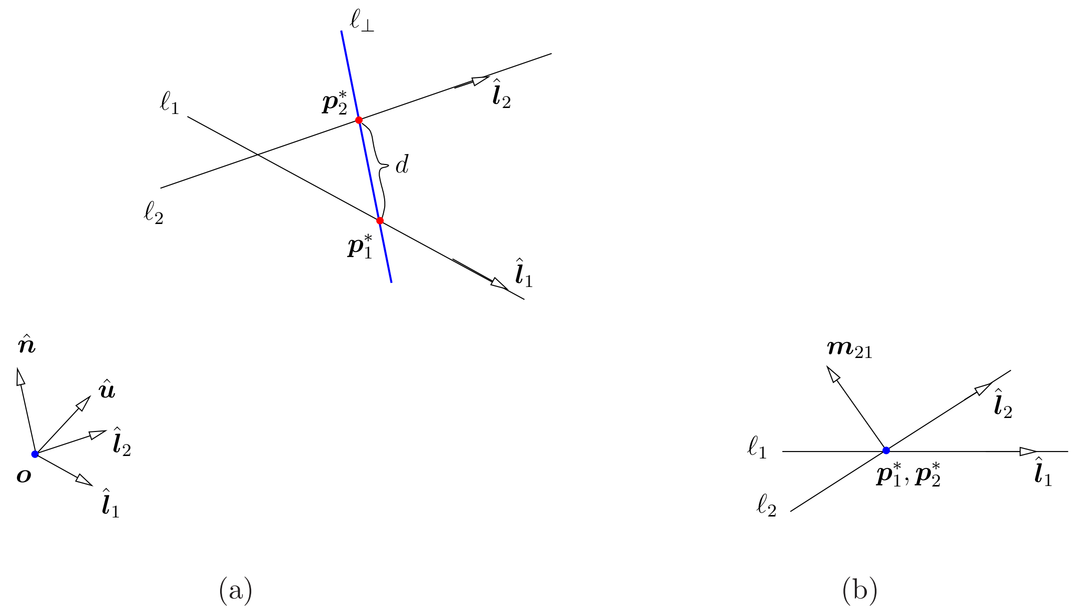

Figure 6: Two skew lines ℓ_1 and ℓ_2 at distance d: (a) with their common perpendicular $\ell_\perp$, and (b) in a top-down view. The local frame in (a) is formed by $\hat{\boldsymbol{l}}_1$, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2 / \|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|$, and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{l}}_1$.

Now, we combine the above result with (13) into the formula (10) for the line distance, taking into account that the direction vectors are not unit in general. $\square$

The proof of Theorem 1 also establishes a condition for checking if two lines are co-planar.

Corollary 2 *Two lines ℓ_1 and ℓ_2 are co-planar if and only if the reciprocal product of their Plücker coordinates is zero.*

4 Common Perpendicular of Two Skew Lines

We quickly look at the simpler case where two lines ℓ_1 and ℓ_2 are parallel. There are infinitely many common perpendiculars in the direction given in (15) from the analysis in Section 3. The moments of these lines about the origin vary in both direction and magnitude.

In the rest of this section, the two lines ℓ_1 and ℓ_2 are considered to be skew. Their common perpendicular $\ell_\perp$ is orthogonal to their directions $\hat{\boldsymbol{l}}_1$ and $\hat{\boldsymbol{l}}_2$. It is therefore in the direction of the cross product $\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2$. See Figure 6(a). The sign of the reciprocal product can tell us the relative position between the two lines, as stated in the following lemma.

Lemma 3 *Suppose the reciprocal product $(\hat{\boldsymbol{l}}_1, \boldsymbol{m}_1) * (\hat{\boldsymbol{l}}_2, \boldsymbol{m}_2) \neq 0$. It is positive if and only if the cross product $\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2$ is in the direction of $\boldsymbol{p}_1^* - \boldsymbol{p}_2^*$, where $\boldsymbol{p}_1^*$ and $\boldsymbol{p}_2^*$ are the feet of the common perpendicular on ℓ_1 and ℓ_2, respectively.*

Proof To facilitate our reasoning, we set up a frame at the origin $\boldsymbol{o}$ as shown in Figure 6(a) that is formed by $\hat{\boldsymbol{l}}_1$, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2 / \|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|$, and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{l}}_1$. Part (b) of the figure gives a top-down view of the two lines with $\hat{\boldsymbol{n}}$ pointing outward. Clearly, $\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2$ and $\boldsymbol{p}_1^* - \boldsymbol{p}_2^*$ are in either the same or opposite directions.

Denote by $\boldsymbol{m}_{21}$ the moment of ℓ_2 about $\boldsymbol{p}_1^*$. We have

$$\begin{aligned}\boldsymbol{m}_{21} \cdot \hat{\boldsymbol{l}}_1 &= \boldsymbol{m}_2 \cdot \hat{\boldsymbol{l}}_1 - (\boldsymbol{p}_1^* \times \hat{\boldsymbol{l}}_2) \cdot \hat{\boldsymbol{l}}_1 \\ &= \boldsymbol{m}_2 \cdot \hat{\boldsymbol{l}}_1 + (\boldsymbol{p}_1^* \times \hat{\boldsymbol{l}}_1) \cdot \hat{\boldsymbol{l}}_2 \\ &= \boldsymbol{m}_2 \cdot \hat{\boldsymbol{l}}_1 + \boldsymbol{m}_1 \cdot \hat{\boldsymbol{l}}_2 \\ &= (\hat{\boldsymbol{l}}_1, \boldsymbol{m}_1) * (\hat{\boldsymbol{l}}_2, \boldsymbol{m}_2),\end{aligned}$$

by the definition of the reciprocal product. In Figure 6(b), we see that $(\boldsymbol{p}_2^* - \boldsymbol{p}_1^*) \cdot \hat{\boldsymbol{n}}$ and $\boldsymbol{m}_{21} \cdot \hat{\boldsymbol{l}}_1$ always have opposite signs. Namely, $(\boldsymbol{p}_1^* - \boldsymbol{p}_2^*) \cdot \hat{\boldsymbol{n}}$ and the reciprocal product always have the same sign. □

4.1 Plücker Coordinates of the Common Perpendicular

Lemma 3 implies that $\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2$ multiplied with the reciprocal product is always in the direction of $\boldsymbol{p}_1^* - \boldsymbol{p}_2^*$. We thus obtain the vector $\boldsymbol{p}_1^* - \boldsymbol{p}_2^*$ as follows:

$$\begin{aligned}\boldsymbol{p}_1^* - \boldsymbol{p}_2^* &= d\left(\pm\frac{\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|}\right) \\ &= \frac{(\hat{\boldsymbol{l}}_1, \boldsymbol{m}_1) * (\hat{\boldsymbol{l}}_2, \boldsymbol{m}_2)}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \cdot \frac{\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \qquad \text{(by (13) and Lemma 3)} \\ &= \frac{(\hat{\boldsymbol{l}}_1, \boldsymbol{m}_1) * (\hat{\boldsymbol{l}}_2, \boldsymbol{m}_2)}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|^2}\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2. \qquad (16)\end{aligned}$$

Theorem 4 *Let ℓ_1 and ℓ_2 be two skew lines in the Plücker coordinates $(\boldsymbol{l}_1, \boldsymbol{m}_1)$ and $(\boldsymbol{l}_2, \boldsymbol{m}_2)$. Their common perpendicular $\ell_\perp$ has the Plücker coordinates $(\boldsymbol{l}_\perp, \boldsymbol{m}_\perp)$, where*

$$\boldsymbol{l}_\perp = \boldsymbol{l}_1 \times \boldsymbol{l}_2, \qquad (17)$$

$$\boldsymbol{m}_\perp = \boldsymbol{m}_1 \times \boldsymbol{l}_2 - \boldsymbol{m}_2 \times \boldsymbol{l}_1 + \frac{\Big((\boldsymbol{l}_1, \boldsymbol{m}_1) * (\boldsymbol{l}_2, \boldsymbol{m}_2)\Big)(\boldsymbol{l}_1 \cdot \boldsymbol{l}_2)}{\|\boldsymbol{l}_1 \times \boldsymbol{l}_2\|^2}\boldsymbol{l}_1 \times \boldsymbol{l}_2. \qquad (18)$$

Proof The line $\ell_\perp$ is already known to be in the direction of $\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2$. We need only determine the cross product of a point on $\ell_\perp$ with $\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2$. Choosing this point to be $\boldsymbol{p}_2^*$, we have

$$\begin{aligned}\boldsymbol{p}_2^* \times (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2) &= (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_2)\hat{\boldsymbol{l}}_1 - (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_1)\hat{\boldsymbol{l}}_2 \\ &= \Big((\boldsymbol{p}_1^* + \boldsymbol{p}_2^* - \boldsymbol{p}_1^*) \cdot \hat{\boldsymbol{l}}_2\Big)\hat{\boldsymbol{l}}_1 - (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_1)\hat{\boldsymbol{l}}_2 \\ &= (\boldsymbol{p}_1^* \cdot \hat{\boldsymbol{l}}_2)\hat{\boldsymbol{l}}_1 - (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_1)\hat{\boldsymbol{l}}_2 \qquad \text{(since } \boldsymbol{p}_2^* - \boldsymbol{p}_1^* \text{ is orthogonal to } \hat{\boldsymbol{l}}_2\text{)} \\ &= \Big((\hat{\boldsymbol{l}}_2 \cdot \hat{\boldsymbol{l}}_1)\boldsymbol{p}_1^* - \hat{\boldsymbol{l}}_2 \times (\boldsymbol{p}_1^* \times \hat{\boldsymbol{l}}_1)\Big) - \Big((\hat{\boldsymbol{l}}_2 \cdot \hat{\boldsymbol{l}}_1)\boldsymbol{p}_2^* - \hat{\boldsymbol{l}}_1 \times (\boldsymbol{p}_2^* \times \hat{\boldsymbol{l}}_2)\Big) \qquad \text{(by (4))} \\ &= \Big((\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\boldsymbol{p}_1^* - \hat{\boldsymbol{l}}_2 \times \boldsymbol{m}_1\Big) - \Big((\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\boldsymbol{p}_2^* - \hat{\boldsymbol{l}}_1 \times \boldsymbol{m}_2\Big) \\ &= \boldsymbol{m}_1 \times \hat{\boldsymbol{l}}_2 - \boldsymbol{m}_2 \times \hat{\boldsymbol{l}}_1 + (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)(\boldsymbol{p}_1^* - \boldsymbol{p}_2^*) \\ &= \boldsymbol{m}_1 \times \hat{\boldsymbol{l}}_2 - \boldsymbol{m}_2 \times \hat{\boldsymbol{l}}_1 + \frac{\Big((\hat{\boldsymbol{l}}_1, \boldsymbol{m}_1) * (\hat{\boldsymbol{l}}_2, \boldsymbol{m}_2)\Big)(\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|^2}\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2, \qquad (19)\end{aligned}$$

after substitution of (16).

If the directions of ℓ_1 and ℓ_2 are given by two vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ that are not unit, the cross products $\boldsymbol{l}_\perp = \boldsymbol{l}_1 \times \boldsymbol{l}_2$ and $\boldsymbol{m}_\perp = \boldsymbol{p}_2^* \times (\boldsymbol{l}_1 \times \boldsymbol{l}_2)$ will scale by the same factor. The resulting Plücker coordinates will describe the same line. $\square$

4.2 Feet of the Common Perpendicular

Moving on, let us find the feet $\boldsymbol{p}_1^*$ and $\boldsymbol{p}_2^*$ of the common perpendicular $\ell_\perp$, where it intersects the lines ℓ_1 and ℓ_2, respectively.

Theorem 5 *Let ℓ_1 and ℓ_2 be two skew lines in Plücker coordinates $(\boldsymbol{l}_1, \boldsymbol{m}_1)$ and $(\boldsymbol{l}_2, \boldsymbol{m}_2)$. Their common perpendicular $\boldsymbol{l}_\perp$ intersects with ℓ_1 and ℓ_2 respectively at the following points:*

$$\boldsymbol{p}_1^* = \frac{-\boldsymbol{m}_1 \times \Big(\boldsymbol{l}_2 \times (\boldsymbol{l}_1 \times \boldsymbol{l}_2)\Big) + \Big(\boldsymbol{m}_2 \cdot (\boldsymbol{l}_1 \times \boldsymbol{l}_2)\Big)\boldsymbol{l}_1}{\|\boldsymbol{l}_1 \times \boldsymbol{l}_2\|^2}, \tag{20}$$

$$\boldsymbol{p}_2^* = \frac{\boldsymbol{m}_2 \times \Big(\boldsymbol{l}_1 \times (\boldsymbol{l}_1 \times \boldsymbol{l}_2)\Big) - \Big(\boldsymbol{m}_1 \cdot (\boldsymbol{l}_1 \times \boldsymbol{l}_2)\Big)\boldsymbol{l}_2}{\|\boldsymbol{l}_1 \times \boldsymbol{l}_2\|^2}. \tag{21}$$

Proof We need only derive an expression for $\boldsymbol{p}_2^*$ and then obtain that for $\boldsymbol{p}_1^*$ by switching the subscripts 1 and 2. The idea is to project $\boldsymbol{p}_2^*$ onto the three orthogonal axes $\hat{\boldsymbol{l}}_1$, $\hat{\boldsymbol{u}}$, and $\hat{\boldsymbol{n}}$ in Figure 6(a). First, we observe

$$\begin{aligned} \boldsymbol{m}_2 \cdot \hat{\boldsymbol{l}}_1 &= (\boldsymbol{p}_2^* \times \hat{\boldsymbol{l}}_2) \cdot \hat{\boldsymbol{l}}_1 \\ &= -\boldsymbol{p}_2^* \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2) \\ &= -(\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{n}})\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|, \end{aligned}$$

from which it immediately follows that

$$\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{n}} = -\frac{\boldsymbol{m}_2 \cdot \hat{\boldsymbol{l}}_1}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|}. \tag{22}$$

Next, we derive the projection onto $\hat{\boldsymbol{n}}$ as follows:

$$\begin{aligned} \boldsymbol{p}_2^* \cdot \hat{\boldsymbol{u}} &= \boldsymbol{p}_2^* \cdot (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{l}}_1) \\ &= (\boldsymbol{p}_2^* \times \hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{l}}_1 \\ &= \frac{\Big(\boldsymbol{p}_2^* \times (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)\Big) \cdot \hat{\boldsymbol{l}}_1}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \\ &= \frac{(\boldsymbol{m}_1 \times \hat{\boldsymbol{l}}_2) \cdot \hat{\boldsymbol{l}}_1}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \qquad \text{(substituting (19))} \\ &= -\frac{\boldsymbol{m}_1 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \qquad (23) \\ &= -\boldsymbol{m}_1 \cdot \hat{\boldsymbol{n}}. \qquad (24) \end{aligned}$$

Last, we determine the projection of $\boldsymbol{p}_2^*$ onto $\hat{\boldsymbol{l}}_1$. The vector $\hat{\boldsymbol{l}}_2$ lies in the plane spanned by $\hat{\boldsymbol{l}}_1$ and $\hat{\boldsymbol{u}}$ (cf. Figure 6(a)) such that

$$\hat{\boldsymbol{l}}_2 = (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\hat{\boldsymbol{l}}_1 + \|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|\hat{\boldsymbol{u}}.$$

It follows that

$$\begin{aligned} \boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_2 &= (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)(\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_1) + \|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|(\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{u}}) \\ &= (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)(\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_1) - \boldsymbol{m}_1 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2), \end{aligned} \tag{25}$$

after substitution of (24). We now make use of the cross product

$$\begin{aligned} \hat{\boldsymbol{l}}_2 \times \boldsymbol{m}_2 &= \hat{\boldsymbol{l}}_2 \times (\boldsymbol{p}_2^* \times \hat{\boldsymbol{l}}_2) \\ &= \boldsymbol{p}_2^* - (\hat{\boldsymbol{l}}_2 \cdot \boldsymbol{p}_2^*)\hat{\boldsymbol{l}}_2 \end{aligned}$$

to obtain

$$\boldsymbol{p}_2^* = (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_2)\hat{\boldsymbol{l}}_2 + \hat{\boldsymbol{l}}_2 \times \boldsymbol{m}_2.$$

Take the dot products of the both sides of the above equation with $\hat{\boldsymbol{l}}_1$:

$$\begin{aligned} \boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_1 &= (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_2)(\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2) + (\hat{\boldsymbol{l}}_2 \times \boldsymbol{m}_2) \cdot \hat{\boldsymbol{l}}_1 \\ &= (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_2)(\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2) + (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2) \cdot \boldsymbol{m}_2 \\ &= (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)^2(\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_1) - (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\Big(\boldsymbol{m}_1 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)\Big) + \boldsymbol{m}_2 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2), \end{aligned}$$

after substitution of (25). Solve the above equation:

$$\begin{aligned} \boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_1 &= \frac{\boldsymbol{m}_2 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2) - (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\Big(\boldsymbol{m}_1 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)\Big)}{1 - (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)^2} \\ &= \frac{\boldsymbol{m}_2 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2) - (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\Big(\boldsymbol{m}_1 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)\Big)}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|^2}. \end{aligned} \tag{26}$$

Now we are ready to assemble the projections (22), (24), and (26) to form the intersection point between $\boldsymbol{l}_2$ and $\boldsymbol{l}_\perp$ as follows:

$$\begin{aligned} \boldsymbol{p}_2^* &= (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} + (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}} + (\boldsymbol{p}_2^* \cdot \hat{\boldsymbol{l}}_1)\hat{\boldsymbol{l}}_1 \\ &= -\frac{\boldsymbol{m}_2 \cdot \hat{\boldsymbol{l}}_1}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|}\hat{\boldsymbol{n}} - (\boldsymbol{m}_1 \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \times \hat{\boldsymbol{l}}_1 + \left(\frac{\boldsymbol{m}_2 - (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\boldsymbol{m}_1}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \cdot \hat{\boldsymbol{n}}\right)\hat{\boldsymbol{l}}_1 \\ &= \frac{-(\boldsymbol{m}_2 \cdot \hat{\boldsymbol{l}}_1)\hat{\boldsymbol{n}} + (\boldsymbol{m}_2 \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{l}}_1}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} - (\boldsymbol{m}_1 \cdot \hat{\boldsymbol{n}})\left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{l}}_1 + \frac{\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|}\hat{\boldsymbol{l}}_1\right) \\ &= \frac{\boldsymbol{m}_2 \times (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{n}})}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} - (\boldsymbol{m}_1 \cdot \hat{\boldsymbol{n}})\frac{(\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2) \times \hat{\boldsymbol{l}}_1 + (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\hat{\boldsymbol{l}}_1}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \\ &= \frac{\boldsymbol{m}_2 \times (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{n}})}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} - (\boldsymbol{m}_1 \cdot \hat{\boldsymbol{n}})\frac{(\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_1)\hat{\boldsymbol{l}}_2 - (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\hat{\boldsymbol{l}}_1 + (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2)\hat{\boldsymbol{l}}_1}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \\ &= \frac{\boldsymbol{m}_2 \times (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{n}}) - (\boldsymbol{m}_1 \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{l}}_2}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|} \\ &= \frac{\boldsymbol{m}_2 \times \Big(\hat{\boldsymbol{l}}_1 \times (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)\Big) - \Big(\boldsymbol{m}_1 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)\Big)\hat{\boldsymbol{l}}_2}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|^2}. \end{aligned} \tag{27}$$

The expression for the intersection point $\boldsymbol{p}_1^*$ of $\boldsymbol{l}_1$ and $\boldsymbol{l}_\perp$ is obtained by simply switching the subscripts 1 and 2 in the above expression:

$$\boldsymbol{p}_1^* = \frac{-\boldsymbol{m}_1 \times \left(\hat{\boldsymbol{l}}_2 \times (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)\right) + \left(\boldsymbol{m}_2 \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)\right)\hat{\boldsymbol{l}}_1}{\|\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2\|^2}. \tag{28}$$

It is straightforward to verify (16) using (27) and (28).

The expressions (27) and (28) will still be valid even if the vectors $\hat{\boldsymbol{l}}_1$ and $\hat{\boldsymbol{l}}_2$ are replaced with general direction vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, respectively. This is because the moment vectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ will scale by the magnitudes of the direction vectors. □

The derivation of (27) and (28) was independent of the distance value d. Hence these formulae (and subsequently (20) and (21)) hold when $d = 0$, that is, when ℓ_1 and ℓ_2 intersect. In this case, the reciprocal product $\hat{\boldsymbol{l}}_1 \cdot \boldsymbol{m}_2 + \hat{\boldsymbol{l}}_2 \cdot \boldsymbol{m}_1 = 0$, and it is easy to verify that $\boldsymbol{p}_1^* = \boldsymbol{p}_2^*$. We can extract $\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2$ out from the numerator and rewrite the expression as a matrix product.

Corollary 6 *Suppose two lines ℓ_1 and ℓ_2 in Plücker coordinates $(\boldsymbol{l}_1, \boldsymbol{m}_1)$ and $(\boldsymbol{l}_2, \boldsymbol{m}_2)$ intersect each other. Therefore, $\boldsymbol{l}_1 \cdot \boldsymbol{m}_2 + \boldsymbol{l}_2 \cdot \boldsymbol{m}_1 = 0$ and $\boldsymbol{l}_1 \times \boldsymbol{l}_2 \neq \boldsymbol{0}$ hold. The intersection point is given as*

$$\boldsymbol{p} = \left((\boldsymbol{m}_1 \cdot \boldsymbol{l}_2) I_3 + \boldsymbol{l}_1 \boldsymbol{m}_2^\top - \boldsymbol{l}_2 \boldsymbol{m}_1^\top\right) \frac{\boldsymbol{l}_1 \times \boldsymbol{l}_2}{\|\boldsymbol{l}_1 \times \boldsymbol{l}_2\|^2}, \tag{29}$$

where I_3 is the 3×3 identity matrix.

5 Computing the Spatial Relationship of Two Lines

Given two lines ℓ_1 and ℓ_2 in Plücker coordinates $(\boldsymbol{l}_1, \boldsymbol{m}_1)$ and $(\boldsymbol{l}_2, \boldsymbol{m}_2)$, respectively, we use the following procedure to determine how they are positioned relative to each other in the space.

1. Check if $\boldsymbol{l}_1 \times \boldsymbol{l}_2 = 0$ holds. If so, they are parallel with distance given in (10).
2. Otherwise, evaluate the reciprocal product $(\boldsymbol{l}_1, \boldsymbol{m}_1) * (\boldsymbol{l}_2, \boldsymbol{m}_2)$, and check if it is zero.
   (a) If zero, then the two lines intersect at the point $\boldsymbol{p}^*$ given by (29).
   (b) If not zero, then the two lines are not co-planar. We can evaluate their distance according to (10), common perpendicular in the Plüker coordinates $(\boldsymbol{l}_\perp, \boldsymbol{m}_\perp)$ according to (17) and (18), and its two feet $\boldsymbol{p}_1^*$ and $\boldsymbol{p}_2^*$ according to (20) and (21).

References

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