Tutorial on Particle Kinematics

Yan-Bin Jia

Department of Computer Science

Iowa State University

Ames, IA 50011

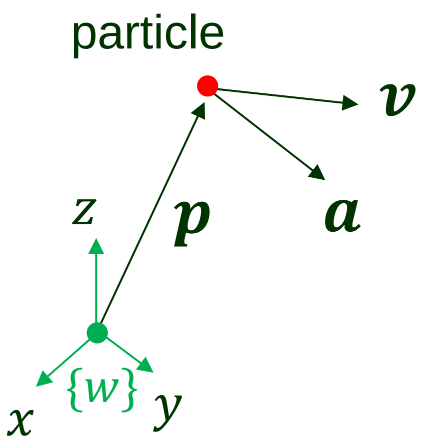
Outline

I. Particle fixed in a rotating frame

II. Translating and rotating body frame

III. Particle moving in a translating and rotating body frame

Particle Kinematics



$\{w\}$: world frame, stationary
(inertial frame)

$\boldsymbol{p}$: position

$\boldsymbol{v}$: velocity

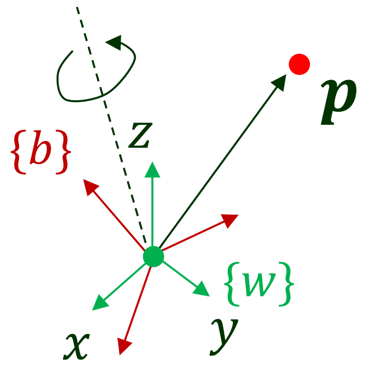
$$\boldsymbol{v} = \dot{\boldsymbol{p}}$$

$\boldsymbol{a}$: acceleration

$$\boldsymbol{a} = \dot{\boldsymbol{v}}$$

I. Particle Fixed in a Rotating Frame

$\{b\}$ can be seen as attached to a rigid body spinning about its center of mass while $\boldsymbol{p}$ as a point inside the body.



$\{w\}$: world frame, stationary (inertial frame)

$\{b\}$: a body frame sharing its origin with $\{w\}$ and rotating about the origin.

R: matrix describing the rotation from from $\{w\}$ to $\{b\}$.

The particle (point) $\boldsymbol{p}$ is fixed within $\{b\}$ with its coordinates in the frame denoted as ${}^{b}\boldsymbol{p}$.

Its coordinates in the world frame $\{w\}$ is

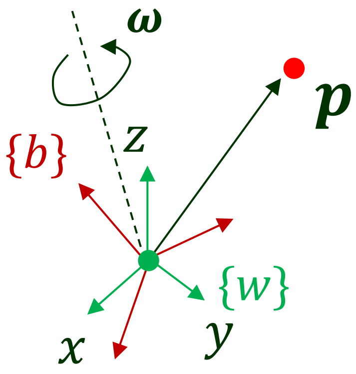
$$\boldsymbol{p} = R\,{}^{b}\boldsymbol{p}$$

Its velocity is

$$\dot{\boldsymbol{p}} = \dot{R}\,{}^{b}\boldsymbol{p}$$

Angular Velocity

To obtain $\dot{R}$, we make use of R being an orthogonal matrix:



$$RR^T = I_3 \quad \Longrightarrow \quad \dot{R}R^T + R\dot{R}^T = 0$$

↑ 3×3 identity matrix

$$\Longrightarrow \quad \dot{R}R^T + (\dot{R}R^T)^T = 0$$

$$\Longrightarrow \quad \dot{R}R^T = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

for some $\omega_x, \omega_y, \omega_z$.

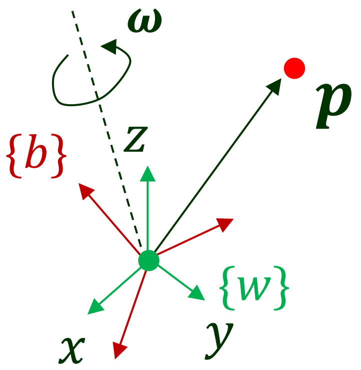
Angular velocity: $\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$

$$\dot{\boldsymbol{p}} = \dot{R}\,{}^b\boldsymbol{p} = \dot{R}R^T\left(R\,{}^b\boldsymbol{p}\right) = \underbrace{\boldsymbol{\omega} \times (R\,{}^b\boldsymbol{p})}$$

Velocity of $\boldsymbol{p}$ due to the rotation of $\{b\}$.

$\dot{R}\boldsymbol{a} = \boldsymbol{\omega} \times (R\boldsymbol{a})$ for any 3-vector $\boldsymbol{a}$

Acceleration



$$\dot{\boldsymbol{p}} = \dot{R}\,{}^{b}\boldsymbol{p} \quad = \boldsymbol{\omega} \times (R\,{}^{b}\boldsymbol{p})$$

$$\Downarrow$$

$$\ddot{\boldsymbol{p}} = \dot{\boldsymbol{\omega}} \times \left(R\,{}^{b}\boldsymbol{p}\right) + \boldsymbol{\omega} \times \left(\dot{R}\,{}^{b}\boldsymbol{p}\right)$$

$$= \dot{\boldsymbol{\omega}} \times \left(R\,{}^{b}\boldsymbol{p}\right) + \boldsymbol{\omega} \times \left(\dot{R}R^{T}R\,{}^{b}\boldsymbol{p}\right)$$

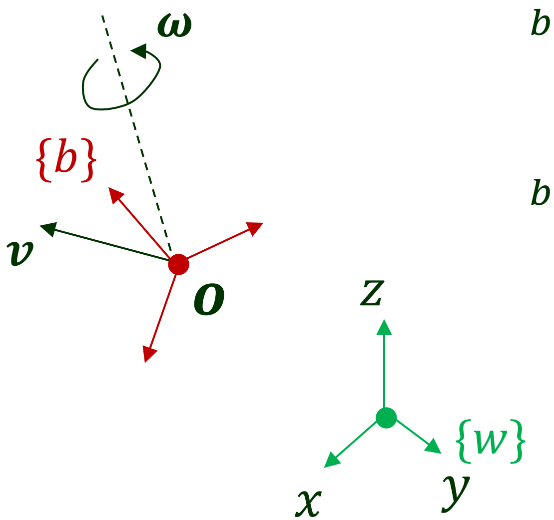
$$= \dot{\boldsymbol{\omega}} \times \left(R\,{}^{b}\boldsymbol{p}\right) + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times (R\,{}^{b}\boldsymbol{p})\right)$$

Rate of change in the velocity due to that in $\boldsymbol{\omega}$.

Rate of change in the velocity $\dot{\boldsymbol{p}}$ due to the rotation.

II. Translating & Rotating Body Frame

$\boldsymbol{v}, \boldsymbol{\omega}$: velocity and angular velocity of the body frame $\{b\}$ relative to the world frame $\{w\}$ as expressed in $\{w\}$



${}^b\boldsymbol{v},\ {}^b\boldsymbol{\omega}$: velocity and angular velocity of $\{b\}$ relative to $\{w\}$ as expressed in the moving frame $\{b\}$.

${}^b\dot{\boldsymbol{v}},\ {}^b\dot{\boldsymbol{\omega}}$: acceleration and angular acceleration of $\{b\}$ relative to $\{w\}$ as expressed in $\{b\}$ (differentiation carried out in $\{w\}$) :

$$\boldsymbol{v} = R\,{}^b\boldsymbol{v} \qquad \boldsymbol{\omega} = R\,{}^b\boldsymbol{\omega}$$

$$\dot{\boldsymbol{v}} = R\,{}^b\dot{\boldsymbol{v}} \qquad \dot{\boldsymbol{\omega}} = R\,{}^b\dot{\boldsymbol{\omega}}$$

$$\boldsymbol{v} = R\,{}^b\boldsymbol{v} \Longrightarrow \dot{\boldsymbol{v}} = \dot{R}\,{}^b\boldsymbol{v} + R\frac{d}{dt}\left({}^b\boldsymbol{v}\right) = \boldsymbol{\omega} \times \left(R\,{}^b\boldsymbol{v}\right) + R\frac{d}{dt}\left({}^b\boldsymbol{v}\right)$$

$$= \boldsymbol{\omega} \times \boldsymbol{v} + R\frac{d}{dt}\left({}^b\boldsymbol{v}\right)$$

Different Meanings of ${}^{b}\dot{\boldsymbol{v}}$ and $\frac{d}{dt}\left({}^{b}\boldsymbol{v}\right)$

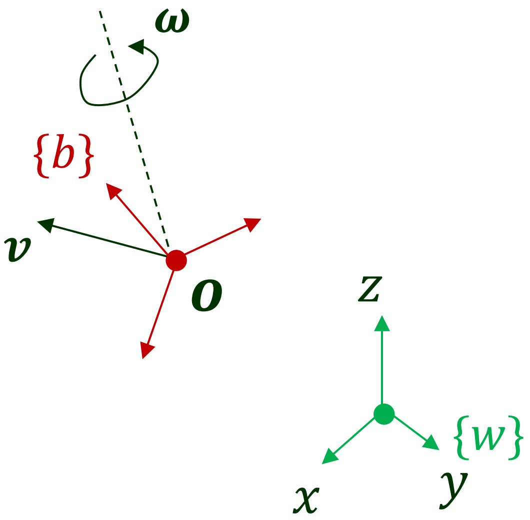
$$\boldsymbol{v} = R\,{}^{b}\dot{\boldsymbol{v}} = \left(R\,{}^{b}\boldsymbol{\omega}\right) \times \left(R\,{}^{b}\boldsymbol{v}\right) + R\frac{d}{dt}\left({}^{b}\boldsymbol{v}\right)$$

$$\Downarrow \quad (R\boldsymbol{a}) \times (R\boldsymbol{b}) = R(\boldsymbol{a} \times \boldsymbol{b})$$

$$R\,{}^{b}\dot{\boldsymbol{v}} = R\left({}^{b}\boldsymbol{\omega} \times {}^{b}\boldsymbol{v}\right) + R\frac{d}{dt}\left({}^{b}\boldsymbol{v}\right)$$

$$\Downarrow$$

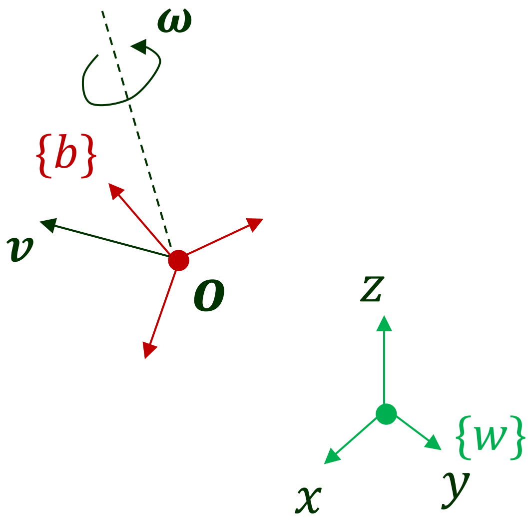
$$\underbrace{{}^{b}\dot{\boldsymbol{v}}} = {}^{b}\boldsymbol{\omega} \times {}^{b}\boldsymbol{v} + \underbrace{\frac{d}{dt}\left({}^{b}\boldsymbol{v}\right)}$$



Acceleration of $\{b\}$ expressed in itself

Rate of change in ${}^{b}\boldsymbol{v}$ (not a real acceleration)

Different Meanings But the Same Value of ${}^b\dot{\boldsymbol{\omega}}$ and $\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right)$



$$\boldsymbol{\omega} = R\,{}^b\boldsymbol{\omega}$$

$$\Downarrow$$

$$\begin{aligned}\dot{\boldsymbol{\omega}} &= \boldsymbol{\omega} \times \left(R\,{}^b\boldsymbol{\omega}\right) + R\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right)\\ &= \boldsymbol{\omega} \times \boldsymbol{\omega} + R\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right)\\ &= R\frac{d}{dt}\left({}^b\boldsymbol{\omega}\right)\end{aligned}$$

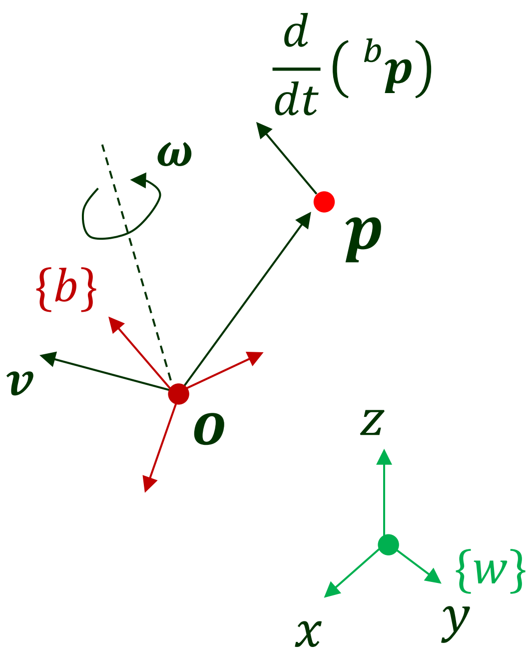
$$\Downarrow \quad \dot{\boldsymbol{\omega}} = R\,{}^b\dot{\boldsymbol{\omega}}$$

$${}^b\dot{\boldsymbol{\omega}} = \frac{d}{dt}\left({}^b\boldsymbol{\omega}\right)$$

Angular acceleration of $\{b\}$ expressed in itself

Rate of change in ${}^b\boldsymbol{\omega}$

III. Particle Moving Relative to a Moving Body Frame



$\boldsymbol{v}, \boldsymbol{\omega}$: velocity and angular velocity of $\{b\}$ relative to $\{w\}$ as expressed in $\{w\}$.

$\boldsymbol{p}$: position of the particle in $\{w\}$

$\dot{\boldsymbol{p}}$: velocity of the particle in $\{w\}$

$\ddot{\boldsymbol{p}}$: acceleration of the particle in $\{w\}$

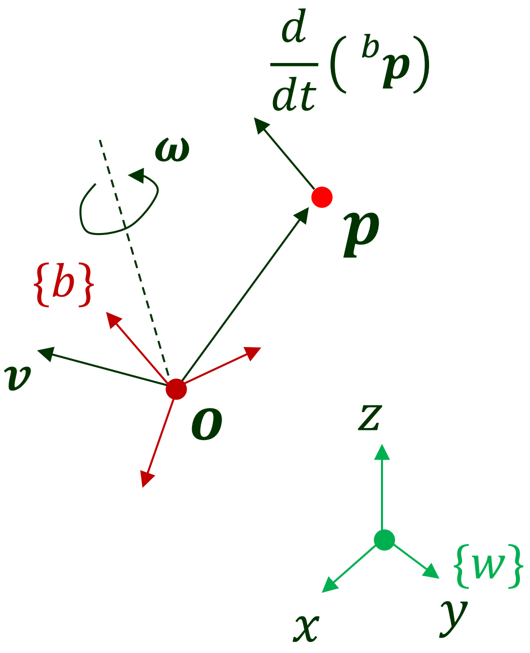
${}^b\boldsymbol{p}$: position of the particle relative to $\{b\}$ (and expressed in it)

$$\boldsymbol{p} = \boldsymbol{o} + R\,{}^b\boldsymbol{p}$$

$\frac{d}{dt}({}^b\boldsymbol{p})$: velocity of the particle relative to $\{b\}$

$\frac{d^2}{dt^2}({}^b\boldsymbol{p})$: acceleration of the particle relative to $\{b\}$

Velocity and Acceleration

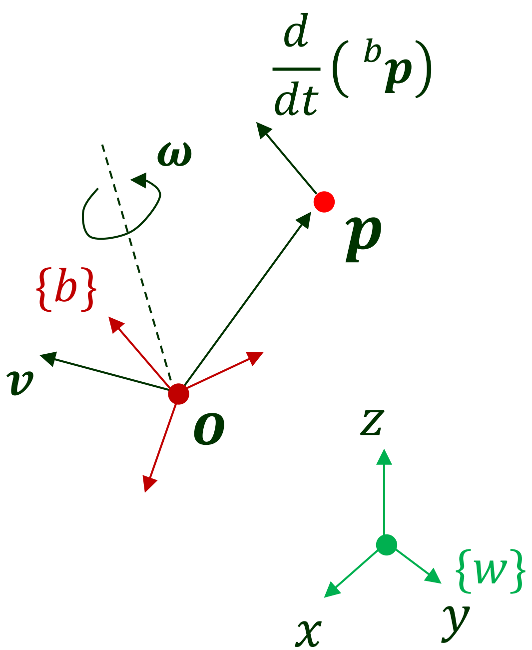


$$\boldsymbol{p} = \boldsymbol{o} + R\,{}^b\boldsymbol{p}$$

$$\dot{\boldsymbol{p}} = \dot{\boldsymbol{o}} + \dot{R}\,{}^b\boldsymbol{p} + R\,\frac{d}{dt}({}^b\boldsymbol{p})$$
$$= \boldsymbol{v} + \boldsymbol{\omega} \times (R\,{}^b\boldsymbol{p}) + R\frac{d}{dt}({}^b\boldsymbol{p})$$

$$\ddot{\boldsymbol{p}} = \dot{\boldsymbol{v}} + \dot{\boldsymbol{\omega}} \times (R\,{}^b\boldsymbol{p}) + \boldsymbol{\omega} \times \left(\dot{R}\,{}^b\boldsymbol{p} + R\frac{d}{dt}({}^b\boldsymbol{p})\right)$$
$$+ \dot{R}\frac{d}{dt}({}^b\boldsymbol{p}) + R\frac{d^2}{dt^2}({}^b\boldsymbol{p})$$
$$= \dot{\boldsymbol{v}} + \dot{\boldsymbol{\omega}} \times (R\,{}^b\boldsymbol{p}) + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times (R\,{}^b\boldsymbol{p})\right)$$
$$+ 2\boldsymbol{\omega} \times \left(R\frac{d}{dt}({}^b\boldsymbol{p})\right) + R\frac{d^2}{dt^2}({}^b\boldsymbol{p})$$

Velocity Expressed in the Body Frame



${}^b\dot{\boldsymbol{p}}$: velocity of the particle relative to $\{w\}$ and expressed in $\{b\}$ (differentiation carried out in $\{w\}$)

${}^b\ddot{\boldsymbol{p}}$: acceleration of the particle relative to $\{w\}$ and expressed in $\{b\}$ (differentiation carried out in $\{w\}$)

$$\dot{\boldsymbol{p}} = R\,{}^b\dot{\boldsymbol{p}} \qquad \ddot{\boldsymbol{p}} = R\,{}^b\ddot{\boldsymbol{p}}$$

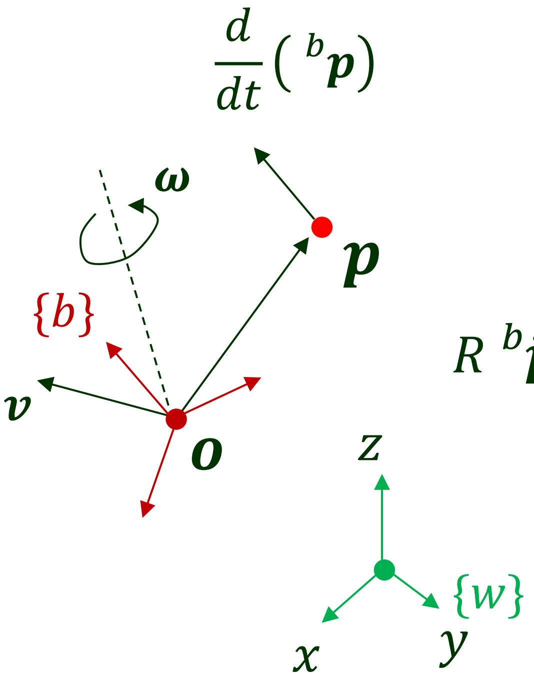
$${}^b\dot{\boldsymbol{p}} \neq \frac{d}{dt}({}^b\boldsymbol{p}) \qquad {}^b\ddot{\boldsymbol{p}} \neq \frac{d^2}{dt^2}({}^b\boldsymbol{p})$$

$$\dot{\boldsymbol{p}} = \boldsymbol{v} + \boldsymbol{\omega} \times (R\,{}^b\boldsymbol{p}) + R\frac{d}{dt}({}^b\boldsymbol{p})$$

$$\Rightarrow R\,{}^b\dot{\boldsymbol{p}} = R\,{}^b\boldsymbol{v} + (R\,{}^b\boldsymbol{\omega}) \times (R\,{}^b\boldsymbol{p}) + R\frac{d}{dt}({}^b\boldsymbol{p})$$

$$\Rightarrow {}^b\dot{\boldsymbol{p}} = \underbrace{{}^b\boldsymbol{v}}_{\text{Velocity of the body frame}} + \underbrace{{}^b\boldsymbol{\omega} \times {}^b\boldsymbol{p}}_{\text{Velocity due to the body frame's angular velocity}} + \underbrace{\frac{d}{dt}({}^b\boldsymbol{p})}_{\text{Velocity within the body frame}}$$

Acceleration Expressed in the Body Frame



$$\ddot{\mathbf{p}} = \dot{\boldsymbol{v}} + \dot{\boldsymbol{\omega}} \times (R\,{}^b\mathbf{p}) + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times (R\,{}^b\mathbf{p})\right) + 2\boldsymbol{\omega} \times \left(R\frac{d}{dt}({}^b\mathbf{p})\right) + R\frac{d^2}{dt^2}({}^b\mathbf{p})$$

$$\Downarrow \quad \ddot{\mathbf{p}} = R\,{}^b\ddot{\mathbf{p}}, \boldsymbol{\omega} = R\,{}^b\boldsymbol{\omega}, \dot{\boldsymbol{\omega}} = R\,{}^b\dot{\boldsymbol{\omega}}$$

$$R\,{}^b\ddot{\mathbf{p}} = R\,{}^b\dot{\boldsymbol{v}} + (R\,{}^b\dot{\boldsymbol{\omega}}) \times (R\,{}^b\mathbf{p}) + (R\,{}^b\boldsymbol{\omega}) \times \left((R\,{}^b\boldsymbol{\omega}) \times (R\,{}^b\mathbf{p})\right) + 2(R\,{}^b\boldsymbol{\omega}) \times \left(R\frac{d}{dt}({}^b\mathbf{p})\right) + R\frac{d^2}{dt^2}({}^b\mathbf{p})$$

$$\Downarrow$$

$$R\,{}^b\ddot{\mathbf{p}} = R\,{}^b\dot{\boldsymbol{v}} + R({}^b\dot{\boldsymbol{\omega}} \times {}^b\mathbf{p}) + R\left({}^b\boldsymbol{\omega} \times ({}^b\boldsymbol{\omega} \times {}^b\mathbf{p})\right) + 2R\left({}^b\boldsymbol{\omega} \times \frac{d}{dt}({}^b\mathbf{p})\right) + R\frac{d^2}{dt^2}({}^b\mathbf{p})$$

$$\Downarrow$$

$${}^b\ddot{\mathbf{p}} = {}^b\dot{\boldsymbol{v}} + {}^b\dot{\boldsymbol{\omega}} \times {}^b\mathbf{p} + {}^b\boldsymbol{\omega} \times ({}^b\boldsymbol{\omega} \times {}^b\mathbf{p}) + 2\,{}^b\boldsymbol{\omega} \times \frac{d}{dt}({}^b\mathbf{p}) + \frac{d^2}{dt^2}({}^b\mathbf{p})$$