Supplementary Material for **Multiple Impacts: A State Transition Diagram Approach**

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1 Maximum Possible Impulse Accumulation in State S_3

In state S_3, only the ball-ball impact is active. We derive the maximum possible in-state accumulation $\Delta I_{1\max}$ of I_1 given in (34). The value was used there to decide whether a transition $S_3 \to S_1$ or $S_3 \to S_4$ would take place. During the state, the change in the strain energy E_1 stored at the ball-ball contact is simplified from (21) to

$$\Delta E_1 = \left(v_2^{(0)} - v_1^{(0)}\right) \Delta I_1 - \frac{1}{2M} \Delta I_1^2 + \alpha_1 (e_1^2 - 1) E_{1\max}, \tag{1}$$

where $M = 1/(\frac{1}{m_1} + \frac{1}{m_2})$. There are two cases depending on the phase of the ball-ball impact during which S_3 starts.

- Compression. Before compression ends, $\alpha_1 = 0$; it follows from (70) that

$$\Delta E_1 = \left(v_2^{(0)} - v_1^{(0)}\right) \Delta I_1 - \frac{1}{2M} \Delta I_1^2. \tag{2}$$

  By (7), (18), and (33), compression ends when

$$\dot{x}_1 = v_1 - v_2 = v_1^{(0)} - v_2^{(0)} + \frac{1}{M} \Delta I_1 = 0,$$

  from which we obtain

$$\Delta I_1 = M(v_2^{(0)} - v_1^{(0)}).$$

  Note that $v_2^{(0)} > v_1^{(0)}$ since the state starts during compression of the ball-ball impact. Combining the above equation with (71), we obtain the maximum strain energy that would occur at the end of compression

$$E_{1\max} = E_1^{(0)} + \Delta E_1 = E_1^{(0)} + \frac{1}{2} M \left(v_2^{(0)} - v_1^{(0)}\right)^2. \tag{3}$$

Immediately afterward, a loss of $(1-e_1^2)$ of the energy would occur. Substitute (72) along with $\alpha = 1$ into (70) to obtain the value of ΔE_1 (from the start of the state) at a point during restitution if the phase would ever start:

$$\Delta E_1 = \left(v_2^{(0)} - v_1^{(0)}\right)\Delta I_1 - \frac{1}{2M}\Delta I_1^2 + (e_1^2 - 1)\left(E_1^{(0)} + \frac{1}{2}M\left(v_2^{(0)} - v_1^{(0)}\right)^2\right). \tag{4}$$

The maximum possible impulse accumulation $\Delta I_{1\max}$ is attained when the strain energy becomes zero, that is,

$$E_1^{(0)} + \Delta E_1 = 0. \tag{5}$$

Now we substitute (73) into (74) and, after rearrangement and merging of terms, obtain a quadratic equation in ΔI_1:

$$-\frac{1}{2M}\Delta I_1^2 + \left(v_2^{(0)} - v_1^{(0)}\right)\Delta I_1 + e_1^2 E_1^{(0)} + \frac{1}{2}(e_1^2 - 1)M\left(v_2^{(0)} - v_1^{(0)}\right)^2 = 0.$$

Solution of the equation yields

$$\begin{aligned} \Delta I_{1\max} &= M\left(v_2^{(0)} - v_1^{(0)} + \sqrt{\left(v_2^{(0)} - v_1^{(0)}\right)^2 + \frac{2}{M}\left(e_1^2 E_1^{(0)} + \frac{1}{2}(e_1^2 - 1)M\left(v_2^{(0)} - v_1^{(0)}\right)^2\right)}\right) \\ &= M\left(v_2^{(0)} - v_1^{(0)} + e_1\sqrt{\left(v_2^{(0)} - v_1^{(0)}\right)^2 + \frac{2}{M}E_1^{(0)}}\right). \end{aligned}$$

Only one root is chosen because the impulse value must be greater than $M(v_2^{(0)} - v_1^{(0)})$ at which compression ends.

- Restitution. Then the energy change (70) under $\alpha_1 = 0$ becomes

$$\Delta E_1 = \left(v_2^{(0)} - v_1^{(0)}\right)\Delta I_1 - \frac{1}{2M}\Delta I_1^2.$$

We solve equation (74) with a substitution of the above, obtaining

$$\Delta I_{1\max} = M\left(v_2^{(0)} - v_1^{(0)} + \sqrt{\left(v_2^{(0)} - v_1^{(0)}\right)^2 + \frac{2}{M}E_1^{(0)}}\right).$$

Note that $v_2^{(0)} < v_1^{(0)}$ since the state starts during restitution.

2 Proof of Theorem 5 on Stiffness and Mass Ratios, and Input/Output Scalability

In this section, we prove Theorem 5 which consists of three parts. First, we look at part i) about the stiffness ratio of the ball-ball and ball-table contacts. From Section 4.2, state S_1 is characterized by the differential equations (13)–(15) which have only one occurrence of the stiffness ratio k_2/k_1 but none of k_1 or k_2 separately. Meanwhile, the outcomes of the states S_2 and S_3 are independent of k_1 or k_2, from the analyses in Sections 4.4 and 4.3. The scaled values of I_i and E_i constitute a solution to the new collision problem, and the unique one by Theorem 4.

Moving on, we look at part ii) regarding the ratio of the masses of the two balls. Let us scale the masses m_i, impulses I_i, and energies E_i, along with increments ΔI_i and ΔE_i, $i = 1, 2$, all by some $s > 0$, while keeping v_i, $i = 1, 2$, unchanged. The system (13)–(15) governing state S_1 is still satisfied. A solution to the system exists and is unique. All energy-based conditions for impact phases in Sections 3.5 and 3.2 hold if and only if they did before the scaling. Also, there is no change to the truths of the conditions in Sections 4.3 and 4.4 that determine the outcomes of states S_2 and S_3. Since the initial velocity v_0 of $\mathcal{B}_1$ does not change, the impulses will accumulate exactly the same way as before, up to the scale factor s.

To prove part iii) of Theorem 5 on the invariance of the output-input velocity ratios to the initial upper ball velocity, we first establish the invariance of the state sequence to velocity scaling. Let us refer to the collision with the initial velocity v_0 as instance A, while the collision with the initial velocity sv_0, $s > 0$, as instance B. A collision instance can be viewed as the two impulses I_1 and I_2 growing from zeros to their final values. So we can identify a point during the collision with some intermediate value of (I_1, I_2).

Lemma 1 *Collision instances A and B yield exactly the same sequence of states. A one-to-one correspondence exists between a point (I_1, I_2) during A and a point (sI_1, sI_2) during B such that the following hold for the two points:*

i) The values of v_1 and v_2 in instance B are s times those in instance A, respectively.

ii) The values of E_1 and E_2 in instance B are s^2 times those in instance A, respectively

Proof By induction on the number of states. We need only establish correspondences for impulses and strain energies between the two instances, since those for velocities then follow from (11) and (12).

We first look at the starting state (S_1) of the two collision instances. In instance A, the impulses I_1, I_2 and the strain energies E_1, E_2 evolve according to the system of (13)–(15) which specify the derivatives of I_2, E_1, and E_2 in terms of I_1. In other words, I_2, E_1, E_2 constitute the solution to the system as functions of I_1.

Now, we scale the solution terms as $sI_1, sI_2, s^2E_1, s^2E_2$, for any $s > 0$. The result is the solution to a second system consisting of the same three differential equations except with v_0 in (13) replaced by sv_0. To verify (15), we have

$$\frac{d(sI_2)}{d(sI_1)} = \frac{dI_2}{dI_1} = \sqrt{\frac{k_2}{k_1}} \cdot \sqrt{\frac{E_2}{E_1}} = \sqrt{\frac{k_2}{k_1}} \cdot \sqrt{\frac{s^2E_2}{s^2E_1}}.$$

We can also multiply both sides of (13) by s and rewrite it as

$$\frac{d(s^2E_1)}{d(sI_1)} = s\frac{dE_1}{dI_1} = -\left(sv_0 + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)(sI_1) - \frac{1}{m_2}(sI_2)\right).$$

This shows that the scaled impulses and initial velocity satisfy (13). Similarly, equation (14) for the derivative dE_2/dI_2 is also satisfied after the scaling. The initial correspondence between the two instances then follows from the uniqueness of a solution to the system of (13)–(15).

Before compression of either impact ends in S_1, at any instant of the collision instance A with the ball-ball impulse value I_1, there exists a corresponding instant of the collision instance B with the impulse value sI_1. The values of I_2, E_1, E_2 in A correspond to s or s^2 times these values of their counterparts in B.

It is easy to see that the same conditions for restitution (and restart of compression) introduced in Sections 3.5 and 3.2 hold for B except with the replacement of v_0 by sv_0. The correspondences last through the impact phases of the two impact instances. As a result, at the end of the first state, the impulse values in instance B will be s times those in instance A, while the values of the strain energies in B will be s^2 times.

Similarly, the relationships maintain through the two S_1 states in instances A and B that follow S_2 or S_3, if they hold at the beginnings of the S_1 states.

To avoid the boredom of induction, we overview the rest of the proof rather than proceed with rigor and detail. It is more or less straightforward to verify that the same scaling effects carry over to two corresponding S_2 states or S_3 states in the two instances. Note that the conditions for restitution and end of collision are invariant to the prescribed scaling.

The velocity-based conditions for a transition $S_2 \to S_1$ or $S_3 \to S_1$ shown according to the diagram in Figure 4 remain unchanged by scaling. The conditions for transitions $S_1 \to S_2$, $S_1 \to S_3$, $S_2 \to S_4$ and $S_3 \to S_4$ are based on restitution and not affected. We have just seen such correspondence over the first state of both instances A and B. □

Part iii) of Theorem 5 now follows from an application of Lemma 13 to the last states of both collision instances, and from the velocity equations (11) and (12) determined by the impulses. The ratios v_1/v_0 and v_2/v_0 are invariant to v_0. They depend on the stiffness ratio k_1/k_2, the mass ratio m_2/m_1, and the coefficients of restitution e_1 and e_2 only.

3 Degenerate Simultaneous Collisions

The original paper has assumed that the stiffness ratio k_2/k_1 is non-zero and finite and neither of the two coefficients of restitution e_1 and e_2 is zero. We will now analyze the cases $k_2/k_1 = 0$, $k_2/k_1 = \infty$, $e_1 = 0$, and $e_2 = 0$ by coupling the multiple impact model with the following hypothesis about the ball-ball-table impact problem:

> *If the contact between a pair of objects is infinitely stiffer than that between another pair, then the first pair of objects will act as one object once they reach the same velocity, and the rest of the collision will reduce to single impact with the third object.*

Rather than proving that the hypothesis follows from the multiple impact model in the limiting case, we will reason differently over solution of the governing differential equations (13)–(15).

As assumed in the paper, the upper ball has unit mass ($m_1 = 1$) and unit initial velocity ($v_0 = -1$).

3.1 Infinitely Stiffer Ball-Table Contact Than Ball-Ball Contact

Let us first consider the stiffness ratio $k_2/k_1 = \infty$. In other words, the ball-table contact is infinitely stiffer than the ball-ball contact. Under the hypothesis, the lower ball can be treated as an integral part of the table so $v_2 = 0$ and the collision reduces to single impact. This immediately gives us the post-impact ball velocities:

$$v_1^* = e_1 \qquad \text{and} \qquad v_2^* = 0. \tag{6}$$

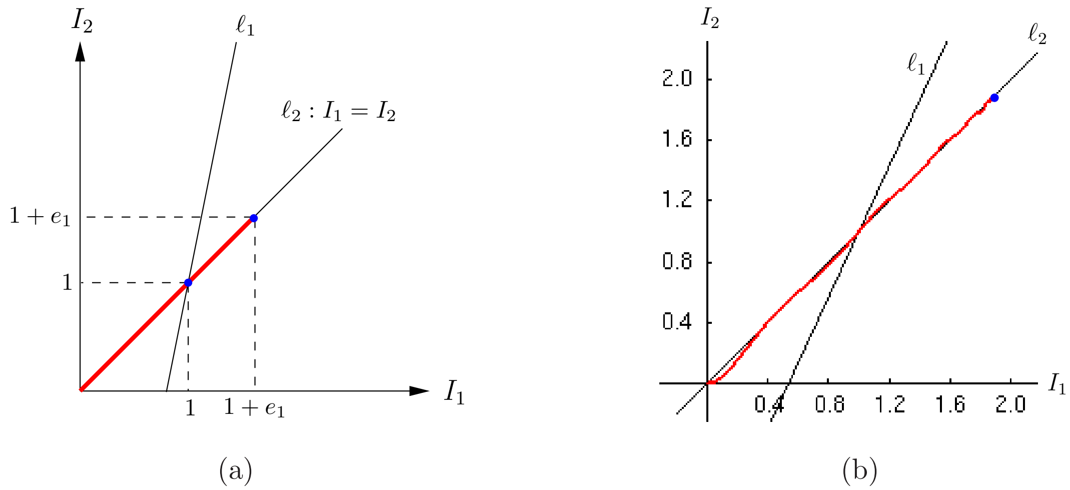

Figure 1: (a) Impulse curve (red) under $k_2/k_1 = \infty$ grows along the compression line ℓ_2; (b) impulse curve of a collision instance with $k_2/k_1 = 40$ shows tendency of convergence to ℓ_2.

Since $v_2 = 0$, the velocity equation (12) implies that $I_1 = I_2$ during the impulse accumulation. Figure 17(a) shows that the impulses accumulate along the compression line ℓ_2 and stop at the point $(1 + e_1, 1 + e_1)$. Compression ends at the point $(1, 1)$. Figure 17(b) shows the impulse curve of a collision instance computed by our model with the same physical parameter values as given in (44) except $k_2/k_1 = 40$. The state sequence $\langle S_1, S_3, ..., S_1, S_3, S_1, S_2, S_4 \rangle$ begins with six repeats of the pair of S_1 and S_3. The impulse curve quickly grows around the compression line ℓ_1. The post-collision ball velocities $v_1 = 0.88922$ and $v_2 = 0.00022$ approach the limit values 0.9 and 0 determined from (75) for $k_2/k_1 = \infty$.

To prove that the outcome (75) follows from the model in Figure 4 in the limiting case, we would have to show that as $k_2/k_1 \to \infty$ the impulse curve will converge to the compression line ℓ_2 and the final impulse (I_1^*, I_2^*) will converge to $(1 + e_1, 1 + e_1)$. Instead of pursuing such a proof, let us reason as follows over solution of the governing differential equations (13)–(15).

First, the impulse derivative dI_2/dI_1 has value 1 at the start of collision. To see this, we take the limit of (27) as $k_2/k_1 \to \infty$ and obtain 1. As $\delta \to 0$, the derivative has the limit 1. Under Theorem 4, equations (13)–(15) governing the first state S_1 are uniquely satisfied by

$$I_1 = I_2, \qquad E_1 = I_1 - \frac{1}{2} I_1^2, \qquad \text{and} \qquad E_2 = \frac{k_1 E_1}{k_2}. \tag{7}$$

The state sequence is $\langle S_1, S_4 \rangle$. Here is why. The first equation above establishes $\dot{x}_2 = 0$ under (14). The third one implies that $E_2 = 0$ since $k_2/k_1 = \infty$. The ball-table impact is inactive. This is consistent with the second equation, which also holds for single impact. According to the state transition diagram in Figure 4, the transition $S_1 \to S_4$ takes place once the ball-ball impact ends (and the ball-table impact trivially ends because it never started).

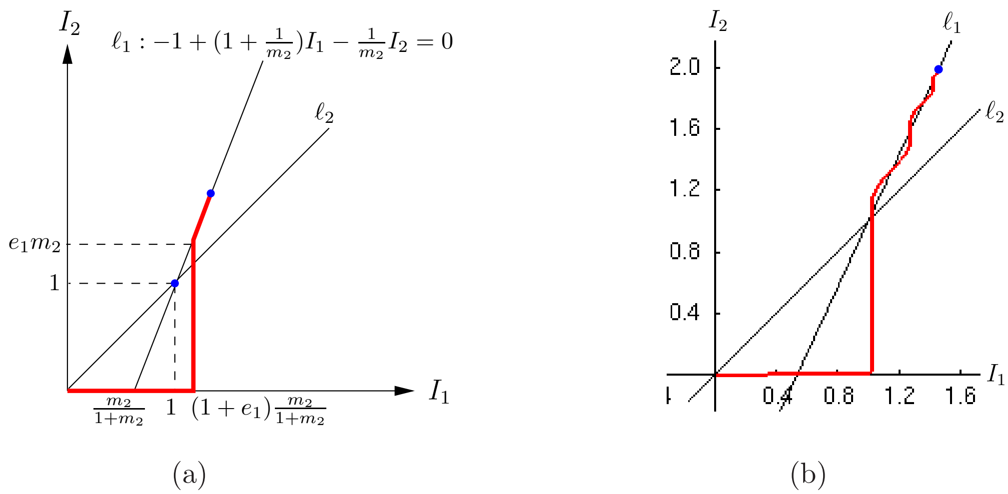

Figure 2: (a) Impulse curve under $k_2/k_1 = 0$ accumulates first horizontally, then vertically, and if there are more than three states, finally along the compression line ℓ_1; (b) impulse curve from a collision with $k_2/k_1 = 1/40$ shows this tendency.

3.2 Infinitely Stiffer Ball-Ball Contact Than Ball-Table Contact

With $k_2/k_1 = 0$ we need to determine when the two ball's velocities become equal and then apply the hypothesis at the beginning of Section E. Section 4.2 establishes the initial derivative values:

$$\frac{dI_2}{dI_1} = 0, \qquad \frac{dE_1}{dI_1} = -v_0 = 1, \qquad \text{and} \qquad \frac{dE_2}{dI_1} = 0.$$

Thus, E_1 grows much faster than E_2 in at least the beginning period of the collision. Since $k_2/k_1 = 0$ and both E_1 and E_2 are bounded by the initial total kinetic energy 1/2 under energy conservation according to Theorem 2, by (15) we infer that $dI_2/dI_1 = 0$ and thus $dE_2/dI_1 = (dE_2/dI_2) \cdot (dI_2/dI_1) = 0$ unless E_1 is zero within the first state S_1. But $E_1 \neq 0$ before the state ends. So $dI_2/dI_1 \equiv 0$ implies $I_2 \equiv 0$ and $E_2 \equiv 0$ in the state.

In summary, in state S_1 only the ball-ball impact is active. Consequently, the impulse curve grows along the I_1-axis, as illustrated in Figure 18(a). Compression ends when the curve crosses the point $(\frac{m_2}{1+m_2}, 0)$ where the compression line ℓ_1 intersects the I_1-axis. Restitution ends when the curve reaches the point $(1+e_1)\frac{m_2}{1+m_2}$.

The next state is S_2 during which I_2 accumulates an amount of $(1+e_1)(1+e_2)\frac{m_2}{1+m_2}$ until the ball-table impact finishes restitution, or of $e_1 m_2$ until the impulse curve reaches the compression line ℓ_1. The derivation of these two amounts can be found in Appendix E.2. In the former case, collision ends with a transition to state S_4, yielding the state sequence $\langle S_1, S_2, S_4 \rangle$.

In the latter case, the two balls reach the same velocity, and the rest of the collision is essentially a problem of single impact between the two balls (treated as one rigid object) and the table. Note that the spring at the ball-table contact has stored some strain energy which may grow or decrease depending on the current phase of the ball-table impact. The impulse curve will be growing upward along ℓ_1 because $v_1 = v_2$. To explain such growth, let E_{22} be the value of the strain energy E_2 at the end of the second state S_2. Consider impulses and strain energies determined from I_1 as follows:

$$
\begin{aligned}
-1+\left(1+\frac{1}{m_2}\right)I_1-\frac{1}{m_2}I_2 &= 0, && (8)\\
E_1 &= \frac{k_2}{k_1}\cdot\frac{E_2}{(1+m_2)^2}, && (9)\\
E_2 &= E_{22}+\frac{1}{1+m_2}\left(I_2-\frac{1}{2}I_2^2\right). && (10)
\end{aligned}
$$

It is straightforward to verify that the third state S_1's governing differential equations are all satisfied: (13) under (77), (14) under (77) and (79), and (15) under (77) and (78). Equation (77) states that the impulses grow along ℓ_1 in the third state S_1, while $dE_1/dI_1 = 0$ (by (13) and (77)) implies that the ball-ball impact is essentially inactive given $k_2/k_1 = 0$. The collision will end with the ball-table impact, yielding the state sequence $\langle S_1, S_2, S_1, S_4\rangle$.

Adopt the notation I_{ij} for the value of the impulse I_i at the end of the jth state in the state sequence. Also, denote by v_{ij} the velocity of the ith ball (the upper ball if $i = 1$ and the lower ball if $i = 2$) at the end of the jth state. Recall that from the start of the collision the impulse curve accumulates along the I_1-axis with inactive ball-table impact, ending state S_1 with the value

$$
I_{11} = (1+e_1)\frac{m_2}{1+m_2}.
$$

Since $I_2 = 0$ during S_1, the state ends with the ball velocities according to (11) and (12):

$$
\begin{aligned}
v_{11} &= -1+I_{11} = -1+(1+e_1)\frac{m_2}{1+m_2} = \frac{e_1m_2-1}{1+m_2}; && (11)\\
v_{21} &= -\frac{I_{11}}{m_2} = -\frac{1+e_1}{1+m_2}. && (12)
\end{aligned}
$$

In the second state (S_2), if the ball-table impact ends restitution, the next state will be S_4 to end the collision. This happens if and only if $v_{11} \geq 0$ and $-e_2v_{21} \leq v_{11}$; or equivalently, by (80) and (81), if and only if $e_1m_2 \geq 1$ and $e_2(1+e_1) \leq e_1m_2 - 1$. The impulse I_2 will accumulate an amount of

$$
I_{22} = (1+e_2)m\cdot(-v_{21}) = (1+e_1)(1+e_2)\frac{m_2}{1+m_2}
$$

in the state. The final velocities are $v_1^* = v_{11}$ and v_2^* determined from (12) with $I_1 = I_{11}$ and $I_2 = I_{22}$.

If the ball-table impact does not end restitution in S_2, the impulse I_2 will accumulate enough to reach the compression line ℓ_1, triggering a transition to S_1 with the two balls having the same velocity v_{11}. Thus, $v_{22} = v_{11}$. Solve for I_{22} from equation (16) of ℓ_1 which now becomes

$$
-1+\left(1+\frac{1}{m_2}\right)I_{12}-\frac{1}{m_2}I_{22} = 0.
$$

Here $I_{12} = I_{11}$ since the impulse I_1 does not increase in S_2. So we obtain the impulse value

$$
\begin{aligned}
I_{22} &= -m_2+(1+m_2)I_{11}\\
&= e_1m_2.
\end{aligned}
$$

After state S_2 ends, the two balls are essentially treated as one object so the problem becomes a single impact one. The virtual spring between the lower ball and the table still has some strain energy either to increase or to release. Calculation of the final ball velocities depends on whether the ball-table impact is during compression or restitution when S_2 ends.

- $e_1 m_2 < 1$. In this case, $v_{11} = v_{12} < 0$. The ball-table impact has not finished compression in S_2 when the state ends. In Section E.2, we showed that the strain energy E_2 was zero in the first state. As S_2 ends, the strain energy stored at the ball-table contact is

$$\begin{aligned} E_{22} &= \frac{1}{2} m_2 (v_{21}^2 - v_{22}^2) \\ &= \frac{1}{2} m_2 (v_{21}^2 - v_{11}^2) \\ &= \frac{1}{2} m_2 \cdot \frac{1}{(1+m_2)^2} \Big((1+e_1)^2 - (e_1 m_2 - 1)^2 \Big) \qquad \text{from (80) and (81)} \\ &= \frac{m_2 e_1}{2(1+m_2)} \Big(2 + e_1(1-m_2) \Big). \end{aligned}$$

  Both balls have downward velocity v_{12} as compression of the ball-table spring continues. The two balls start acting as one rigid body with combined mass $1 + m_2$. Compression at the ball-table contact then ends with the maximum strain energy:

$$\begin{aligned} E_{2\max} &= E_{22} + \frac{1}{2}(1+m_2) v_{12}^2 \\ &= E_{22} + \frac{1}{2}(1+m_2) v_{11}^2 \\ &= \frac{1 + m_2 e_1^2}{2(1+m_2)}. \end{aligned}$$

  During restitution, the amount $e_2^2 E_{2\max}$ is released. The final ball velocities are

$$\begin{aligned} v_1^* = v_2^* &= \sqrt{\frac{2 e_2^2 E_{2\max}}{1+m_2}} \\ &= \frac{e_2 \sqrt{1 + m_2 e_1^2}}{1+m_2}. \end{aligned}$$

- $e_1 m_2 \geq 1$ and $-e_2 v_{21} > v_{11}$. The second condition is equivalent to $e_2(1+e_1) > e_1 m_2 - 1$. In this case, $v_{11} = v_{12} > 0$. Compression first took place in $\mathcal{S}_2$ since $v_{21} < 0$. The ball-table impact is during restitution as state S_2 finishes with $v_2 = v_{12} > 0$. The virtual spring stored energy $\frac{1}{2} m_2 e_2^2 v_{21}^2$ when restitution started in the state. When S_2 ends, the virtual spring at the ball-table contact still stores energy

$$\begin{aligned} E_{22} &= \frac{1}{2} m_2 \Big(e_2^2 v_{21}^2 - v_{22}^2 \Big) \\ &= \frac{1}{2} m_2 \Big(e_2^2 v_{21}^2 - v_{11}^2 \Big) \\ &= \frac{m_2}{2(1+m_2)^2} \Big(e_2^2 (1+e_1)^2 - (e_1 m_2 - 1)^2 \Big). \end{aligned}$$

After S_2 the two balls act as one object with mass $1+m_2$. Since the ball-table impact is now in restitution, the strain energy E_{22} will be released without loss to reach state S_4. The final velocity v_1 of the two balls satisfies the equation

$$\frac{1}{2}(1+m_2)v_1^2 = \frac{1}{2}(1+m_2)v_{11}^2 + E_{22}.$$

We obtain the solution

$$v_1^* = v_2^* = \frac{\sqrt{(e_2 m_2 - 1)^2 + m_2 e_2^2 (1+e_1)^2}}{(1+m_2)^{3/2}}. \tag{13}$$

Figure 18(b) shows the impulse curve of a collision instance with the same physical parameter values as given in (44) except $k_2/k_1 = 1/40$. There are ten states with the first eight alternating between S_1 and S_2. In the first state, the impulse curve grows almost horizontally. After the second state, it starts snaking around the compression line ℓ_1. At the end of the collision, the two balls have velocities $v_1^* = 0.4573$ and $v_2^* = 0.44281$ which are very close to the velocity (≈ 0.45203) calculated according to (82.

3.3 Perfectly Plastic Collisions

A perfectly plastic collision happens at the ith contact if the coefficient of restitution $e_i = 0$. The strain energy is completely lost as compression ends. No restitution follows. This is a limiting case of inelastic collision in which the two bodies stick together, after compression ends at their contact.

Under the impact assumption i) in Section 3.3, after compression ends the stiffness becomes k_i/e_i^2, which is infinity. So the stiffness ratio k_2/k_1 becomes either infinite or zero. From this point we can apply the analysis similar to that used in Section E.1 or E.2.

First, let us consider perfectly plastic ball-ball collision ($e_1 = 0$). The ball-ball impact ends with compression. Afterward, the two balls will have the same velocity $v_1 = v_2$. We continue the collision analysis depending on the state (S_1 or S_3) within which the ball-ball impact ends.

- State S_1. A transition to state S_2 takes place according to the diagram in Figure 4. At the moment, the strain energy stored at the ball-table contact is E_2. The two balls, treated now as one object, have combined mass of $1+m_2$. It is rather straightforward to derive the final ball velocities as follows:

$$v_1^* = v_2^* = \begin{cases} e_2\sqrt{\frac{2E_2}{1+m_2} + v_1^2}, & \text{if the ball-table impact during compression at } S_1 \to S_2; \\ \sqrt{\frac{2E_2}{1+m_2} + v_1^2}, & \text{if the ball-table impact during restitution at } S_1 \to S_2. \end{cases}$$

- State S_3. Since restitution ends in the meantime, a transition to S_4 takes place. In S_4, look at whether the two balls are moving downward or upward, deriving the final ball velocities as

$$v_1^* = v_1^* = \begin{cases} -e_2 v_1, & \text{if } v_1 \le 0; \\ v_1, & \text{otherwise.} \end{cases}$$

This is the only situation where computation is needed in the terminal state S_4.

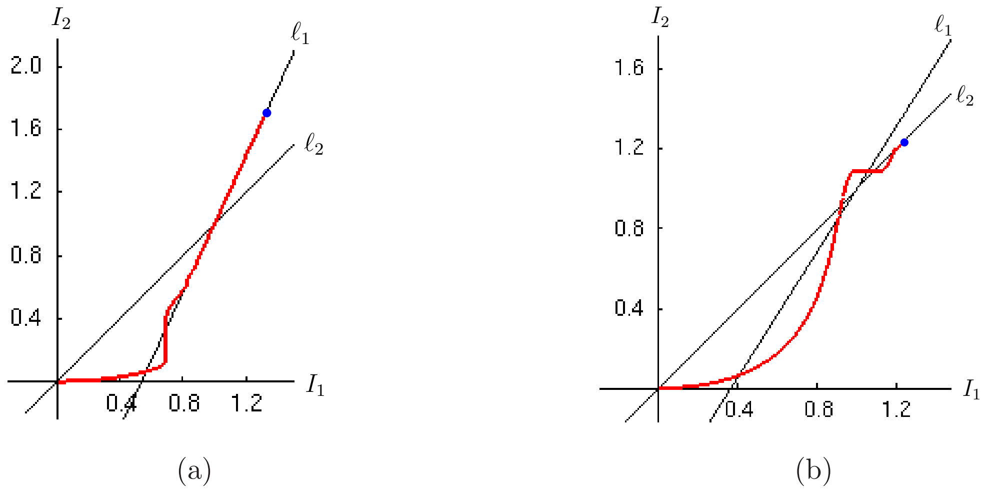

Figure 3: Impulse curves could converge to the compression lines ℓ_1 in (a) and ℓ_2 in (b) even though the stiffness ratio k_2/k_1 assumes a finite and non-zero initial value ($\frac{7}{2}$ in (a) and $\frac{2}{9}$ in (b)).

Next, we consider perfectly plastic ball-table collision ($e_2 = 0$). Compression of the ball-table impact has no restitution to follow. The lower ball gets final velocity $v_2^* = 0$. The upper ball has velocity v_1 when the ball-table impact ends compression. The spring at the ball-ball contact stores strain energy E_1 at the moment. We look at the state s during which the end of ball-table compression happens. If $s = S_1$, state S_3 is next. A transition $S_3 \rightarrow S_4$ then follows. If $s = S_2$, We infer that $v_1 \geq 0$, because otherwise compression would not have ended in S_2, and a transition to S_1 would have taken place before that point. A transition to $S_2 \rightarrow S_4$ takes place immediately to end the collision. The final velocity of the upper ball depends on the status of the collision at the moment the ball-table impact ends:

$$v_1^* = \begin{cases} e_1\sqrt{2E_1 + v_1^2}, & \text{if } s = S_1 \text{ and the ball-ball impact during compression;} \\ \sqrt{2E_1 + v_1^2}, & \text{if } s = S_1 \text{ and the ball-ball impact during restitution;} \\ v_1, & \text{if } s = S_2. \end{cases}$$

3.4 Exponentially Increasing or Decreasing Stiffness Ratios

Under impact law i), the contact stiffness $k_i = \bar{k}_i/e_i^{2n}$ for the ith impact increases exponentially in the number n of times this impact finishes compression. As a result, the stiffness ratio k_2/k_1 may increase or decrease exponentially unless the number of times the other impact finishes compression differs from n by a constant.

In Figure 19, the impulse curves of two collision instances are plotted. The values of the physical parameters are the same as in (44) except $e_1 = 0.2$ and $k_2/k_1 = 7/2$ in instance (a) and $e_2 = 0.2$, $m_2 = 1/\sqrt{3}$ and $k_2/k_1 = 2/9$ in instance (b). In (a), the states alternate between S_1 and S_2. In every state S_1, the ball-ball impact finishes compression once, increasing the stiffness at the contact by a factor of $1/e_1^2 = 25$. Meanwhile, the ball-table impact never finishes compression. So the ratio k_2/k_1 decreases by a factor of $1/25$ every two states. The impulse curve converges to the compression line ℓ_1 quickly. To avoid instability due to floating point arithmetic, we deem

$k_2/k_1 < 10^{-5}$ as zero and then use the close-form solution for the limiting case $k_2/k_1 = 0$ in Section E.2.

In (b), the states alternate between S_1 and S_3. The ball-table impact ends compression in every S_1 state while the ball-ball impact never ends compression. The stiffness ratio k_2/k_1 increases by a factor of 25 every two states with the impulse curve converging to the compression line ℓ_2. For numerical stability, we let the state sequence terminate once $k_2/k_1 > 10^5$, and use the closed-form solution for the corresponding case in Section E.1.