

Supplemental Material for **Pose and Motion Estimation of Free-Flying Objects: Aerodynamics, Constrained Filtering, and Graph-Based Feature Tracking**

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1 Solving the Laplace Equation

The finite differencing method (FDM) is used to compute an approximate solution to the Laplace equation and its boundary conditions of Section III. A linear system is constructed from grid points distributed throughout the domain of fluid, that is, relative to a stationary frame instantaneously coinciding with \mathcal{F}_b .

1.1 Generating the Non-uniform Grid

Suppose the object’s surface is defined implicitly as $f(x, y, z) = 0$ in the body frame \mathcal{F}_b . The object is encompassed in fluid flowing at a uniform rate and satisfying the Laplace equation in (6). The fluid is bounded on the interior by the function f , and on the exterior by a 3D box. The domain encompassing the flow is discretized to produce grid points. Since the number of grid points in a volume can quickly exceed computational requirements, finer discretization is used near the surface where higher accuracy is desired. More specifically, grid points near the surface (within some distance threshold of f) are projected onto it along the normal direction of the closest surface point. Details of this projection are given in Appendix 1.3. Fig. 1 shows the non-uniform grid for the two objects used in our second and third experiments presented in Sections VII-B and VII-C: an ellipsoidal rugby ball in (a) and polyhedral object in (b). The ellipsoid has the surface function $f(x, y, z) = x^2/0.06^2 + y^2/0.03^2 + z^2/0.03^2 - 1 = 0$. The surface function of the polyhedron is a piecewise one defined as follows. The distance of a grid point $\mathbf{p} = (x, y, z)^\top$ to the polyhedron is defined by finding the first intersection of the surface with a ray from \mathbf{p} to the center of mass of the object. The intersection point \mathbf{q} induces a surface function locally defined as $f(\mathbf{p}) = \|\mathbf{p} - \mathbf{q}\| = 0$. The function f becomes piecewise smooth when extended over the polyhedron’s surface.

1.2 Approximation by Finite Differences

The Laplace equation (6) and its boundary conditions (7) and (8) are approximated with finite differences at grid points and solved together. Let \mathbf{p} be a grid point and $\phi_{i,j,k}$ its velocity potential.

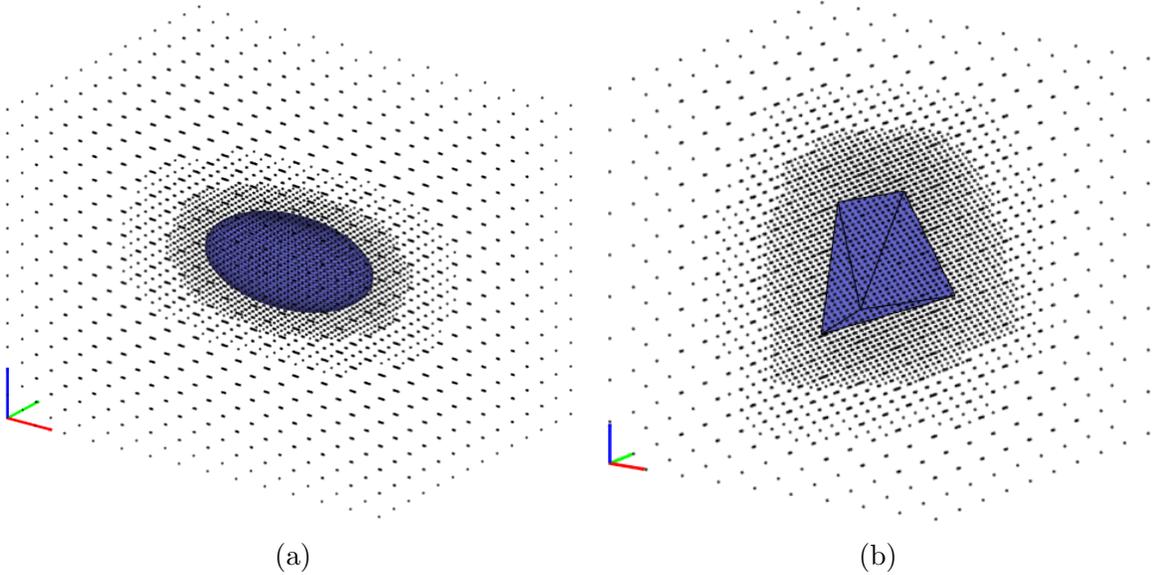


Figure 1: Non-uniform grid for the (a) ellipsoidal rugby ball and (b) polyhedral object used in the second and third experiments of Section VII.

Denote the velocity potential of grid points neighboring \mathbf{p} as $\phi_{i\pm 1,j,k}$, $\phi_{i,j\pm 1,k}$, $\phi_{i,j,k\pm 1}$ in the x , y , and z directions, respectively. The ‘+’ or ‘-’ in each subscript is chosen based on which direction has a neighboring grid point. We let Δx^+ , Δy^+ , Δz^+ denote the distance between \mathbf{p} and its respective neighbor in the positive x , y , and z direction, and similarly Δx^- , Δy^- , and Δz^- in the negative directions.

The partial derivatives of the velocity potential ϕ are approximated by finite differences derived from the Taylor series expansion of ϕ with respect to x , y , and z [1, pp. 186–187]. Using forward differences, the boundary condition in (7) with respect to x is approximated as

$$\frac{\phi_{i+1,j,k} - \phi_{i,j,k}}{\Delta x^+} = -b v_x, \quad (1)$$

where in a slight abuse of notation, $\phi_{i,j,k}$ corresponds to grid points only along the exterior boundary. The partial derivative with respect to y is formed by replacing occurrences of Δx^+ with Δy^+ , and $\phi_{i+1,j,k}$ with $\phi_{i,j+1,k}$, and partial derivatives with respect to z follow in the same fashion. Backward and central differences make use of grid points in the negative direction, i.e., Δx^- and $\phi_{i-1,j,k}$ for partial derivatives with respect to x . Letting $\hat{\mathbf{n}} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)$, the boundary condition of (8) is approximated similarly as

$$\frac{\phi_{i+1,j,k} - \phi_{i,j,k}}{\Delta x^+} \hat{n}_x + \frac{\phi_{i,j+1,k} - \phi_{i,j,k}}{\Delta y^+} \hat{n}_y + \frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta z^+} \hat{n}_z = 0, \quad (2)$$

where $\phi_{i,j,k}$ corresponds with grid points at the interior boundary.

Last, the Laplace equation (6) expands into

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (3)$$

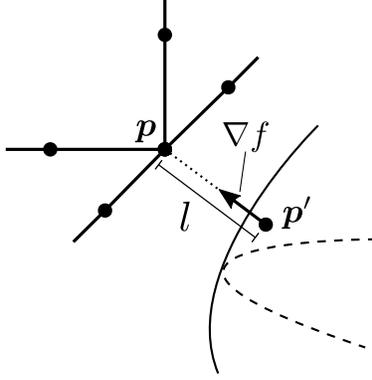


Figure 2: Projection of a grid point onto the interior surface.

From the second order Taylor series approximations of ϕ , the central finite difference is written for non-uniform grid points as

$$\frac{\partial^2 \phi}{\partial x^2} = 2 \frac{\Delta x^- (\phi_{i+1,j,k} - \phi_{i,j,k}) + \Delta x^+ (\phi_{i-1,j,k} - \phi_{i,j,k})}{\Delta x^- \Delta x^+ (\Delta x^+ + \Delta x^-)}$$

At the boundaries, forward and backward differences are used under the assumption of uniformly spaced points (i.e. $\Delta x = \Delta x^- = \Delta x^+$)¹. Partial derivatives with respect to y and z follow similarly. Substitution of the appropriate finite differences into equation (3) yields a linear equation in velocity potentials.

With the Laplace equation and its two boundary conditions approximated, we gather the velocity potentials at all n_L grid points into a vector ϕ . Equations (1)–(3) are subsequently combined to form a linear system in ϕ :

$$C\phi = \mathbf{b}, \quad (4)$$

where the coefficient matrix C has dimensions $m_L \times n_L$. We have $m_L > n_L$ since the Laplace equation over all grid points contributes n_L rows to C , and the boundary conditions contribute $m_L - n_L$ rows from points lying on the boundaries. The system (4) can be solved via singular value decomposition. Since the matrices C and \mathbf{b} are sparse, factorization methods such as sparse LU decomposition can be used.

1.3 Finite Differences of Surface Points

While the system (4) can be employed to solve for the velocity field, we note that the partial derivatives for grid points near the surface could be better approximated. These grid points, previously considered on the surface, are actually a small distance away. Let a point \mathbf{p} near the surface be projected along the direction opposite to the surface gradient $\nabla f(\mathbf{p})$. That is,

$$\begin{aligned} \mathbf{p}' &= \mathbf{p} - l \nabla f(\mathbf{p})^\top \\ &= \mathbf{p} + (h_x, h_y, h_z)^\top \end{aligned} \quad (5)$$

¹The assumption is met as long as the non-uniform grid is generated with three evenly spaced grid points moving out from each boundary in any direction.

for some constant $l > 0$. Fig. 2 shows the projection of \mathbf{p} onto the surface.

Let ϕ' denote the velocity potential at \mathbf{p}' and $\hat{\mathbf{n}}'$ its surface normal. Taking the first order Taylor series approximation of ϕ' about ϕ gives

$$\phi' = \phi + h_x \frac{\partial \phi}{\partial x} + h_y \frac{\partial \phi}{\partial y} + h_z \frac{\partial \phi}{\partial z}. \quad (6)$$

The Laplace equation and boundary condition at the surface are then respectively

$$\nabla^2 \phi' = 0 \quad \text{and} \quad \nabla \phi' \cdot \hat{\mathbf{n}}' = 0, \quad (7)$$

where the first and second order partial derivatives are obtained from differentiating equation (6). The resulting equations yield mixed partial derivatives of ϕ up to the third order. The subsequent first and second order partial derivatives with respect to single variables are obtained from finite differences of the previous section, while the second and third order mixed partial derivatives are approximated below.

Consider the second order partial derivative $\partial^2 \phi / \partial x \partial y$. Differentiate the first order forward finite difference of ϕ along the x -direction with respect to y , and plug in two more forward finite differences in the y -direction to yield

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\phi_{i+1,j+1,k} - \phi_{i+1,j,k} - \phi_{i,j+1,k} + \phi_{i,j,k}}{h_x h_y}.$$

For the third order mixed partial derivative $\partial^3 \phi / \partial x^2 \partial y$, start with the second order central finite difference in the x -direction and differentiate with respect to y . Substituting in three forward finite differences in the y -direction yields

$$\frac{\partial^3 \phi}{\partial x^2 \partial y} = \frac{1}{h_x^2 h_y} (\phi_{i+1,j+1,k} - \phi_{i+1,j,k} - 2\phi_{i,j+1,k} + 2\phi_{i,j,k} + \phi_{i-1,j+1,k} - \phi_{i-1,j,k}).$$

Other second and third order partial derivatives are determined by choosing the appropriate forward, backward, or central differences based on available neighboring grid points. Substituting the finite differences into equations (7) yields linear equations in ϕ . For points lying close to the interior surface, the equation's coefficients are written as rows in C , replacing the corresponding rows from (2) and (3) for these points. The resulting system is solved in the same manner as before.

References

- [1] W. H. Press, B. P. Flannery, S. A. Teukolsky, W. T. Vetterling, *et al.*, *Numerical Recipes in C*. Cambridge University Press, second ed., 2002.