Homogeneous Coordinates*

(Com S 477/577 Notes)

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1 Introduction

Geometry lies at the core of many application areas such as computer graphics, computer-aided design, computer vision, robotics, geographic information systems, etc. This course begins with projective geometry by describing how points and lines can be represented by Cartesian and homogeneous coordinates. We will introduce planar and spatial transformations to construct objects from 'geometric primitives', and to manipulate existing objects. Then we will study projections and look at how to render three-dimensional (3D) objects on a computer screen. This will be followed by an introduction to quaternions which constitute a very powerful tool in dealing with rotations. Later on in the course we will study the geometry of curves and surfaces.

In graphics applications, geometric objects are defined in terms of a number of building blocks called *graphical primitives*. These primitives may correspond to points, lines, curves, and surfaces. For example, a rectangle can be defined by its four sides (or four vertices). Each side is constructed from a line segment by applying a number of geometric operations, called transformations, which position, orientate, or scale the line primitives. Five types of transformation are particularly relevant in applications, namely, translations, scalings, reflections, rotations, and shears.

2 Planar Transformations

A *linear transformation* of the plane is a mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$ from the plane to itself such that

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + \boldsymbol{b}, \tag{1}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

A linear transformation is also called an *affine mapping* or *affine transformation*.

*Appendices are optional for reading unless specifically required.

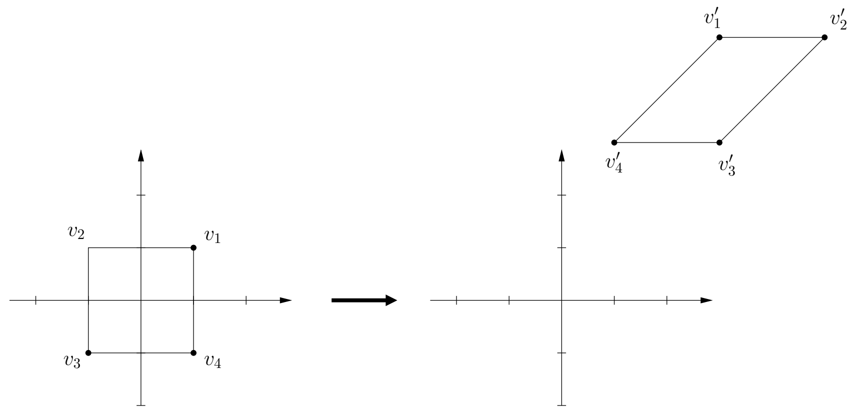

Figure 1: Affine transformation applied to a square of side 2.

Lemma 1 *The transformation given by (1) maps the line $cx + dy + e = 0$, where $c \neq 0$ or $d \neq 0$, to the line*

$$(a_{22}c - a_{21}d)x + (a_{11}d - a_{12}c)y + \Big((a_{12}b_2 - a_{22}b_1)c - (a_{11}b_2 - a_{21}b_1)d + (a_{11}a_{22} - a_{12}a_{21})e\Big) = 0.$$

If $a_{11}d - a_{12}c = 0$ and $a_{22}c - a_{21}d = 0$, then $a_{11}a_{22} - a_{12}a_{21} = 0$ and every point on the original line is mapped to the point $((b_1d - a_{12}e)/d, (b_2d - a_{22}e)/d)^\top$.[1]

Proof Use the line's parametric form to find the image of an arbitrary point on the original line. Then convert the obtained parametric coordinates of the image into an implicit equation. □

Besides collinearity, affine transformation also preserves ratios of distances [2, p. 36], for instance, the midpoint of a line segment remains the midpoint after the transformation.

Example 2. Consider the affine mapping (1) with

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

A square with vertices

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is mapped to a parallelogram with vertices

$$\boldsymbol{v}_1' = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad \boldsymbol{v}_2' = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \quad \boldsymbol{v}_3' = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_4' = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

As shown in Figure 1, the square becomes a parallelogram no longer centered at the origin. The vertices are ordered *clockwise* by index in the image (as opposed to counterclockwise in the square).

[1]The symbol $\top$ denotes the matrix transpose operation.

2.1 Translation

A *translation* is an affine transformation (1) with the matrix $A = I$. That is, a transformation maps every point $\boldsymbol{p}$ to a new point $\boldsymbol{p}'$ by adding a constant vector $\boldsymbol{b} = \binom{b_1}{b_2}$. It has the effect of moving the point in the direction of the x-axis by b_1 units, and in the direction of the y-axis by b_2 units. We denote the translation by $\mathrm{Trans}(b_1, b_2)$.

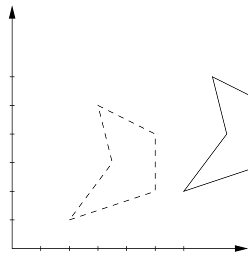

Figure 2: 5-gon before and after the translation $T(4, 1)$.

The transformation that maps $\boldsymbol{p}'$ back to $\boldsymbol{p}$ is the *inverse translation* $T^{-1} = \mathrm{Trans}(-b_1, -b_2)$.

2.2 Scaling

A *scaling* about the origin is an affine transformation (1) where the matrix $A = \mathrm{diag}(s_x, s_y)$ with $s_x \neq 0$ and $s_y \neq 0$, and $\boldsymbol{b} = 0$. This transformation, denoted by $\mathrm{Scale}(s_x, s_y)$, maps a point by multiplying its x and y coordinates by factors s_x and s_y, respectively. Here $s = \sqrt{s_x^2 + s_y^2}$ is the scaling factor. The scaling is said to be an *enlargement* if $s > 1$, and a *contraction* if $s < 1$. It is said to be *uniform* if $s_x = s_y$.

Scaling can be performed by a matrix multiplication

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We abuse the notation by letting

$$\mathrm{Scale}(s_x, s_y) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}.$$

This matrix is called the *scaling transformation matrix*.

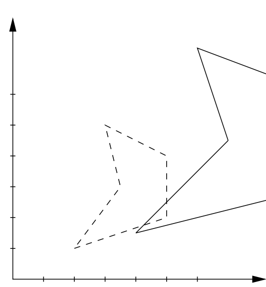

Figure 3: The same 5-gon in Figure 2 after scaling $S(2, 1.5)$.

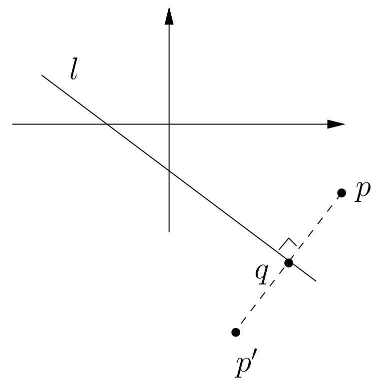

Figure 4: Reflection $\boldsymbol{p}'$ of a point $\boldsymbol{p}$ about a line l.

2.3 Reflection

Two common effects in Computer Aided Design (CAD) or computer drawing packages are the horizontal or vertical 'flip' or mirror effects. A flip of an object is obtained by applying a transformation known as *reflection*. Figure 4 shows a fixed line l in the plane and a point $\boldsymbol{p}$. To determine the reflected image of $\boldsymbol{p}$, move from $\boldsymbol{p}$ toward l in the direction normal to the line. Let $\boldsymbol{q}$ be the intersection of the movement with l. So $\boldsymbol{q}$ is the *projection* of $\boldsymbol{p}$ onto l and $d = \|\boldsymbol{p} - \boldsymbol{q}\|$ gives the shortest distance from $\boldsymbol{p}$ to l. A continuing movement from $\boldsymbol{q}$ for another distance of d will reach $\boldsymbol{p}'$, the reflection of $\boldsymbol{p}$.

It is easy to verify that the reflection Ref_x in the x-axis is the transformation $\binom{x}{y} \mapsto \binom{x}{-y}$, and the reflection Ref_y in the y-axis is the transformation $\binom{x}{y} \mapsto \binom{-x}{y}$. These two transformations can be denoted by matrices

$$\begin{aligned} \mathrm{Ref}_x &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \mathrm{Ref}_y &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Reflections in arbitrary lines can also be denoted by matrices. We will derive such matrices after the introduction of homogeneous coordinates.

2.4 Rotation about the Origin

A *rotation* of a point $\boldsymbol{p} = \binom{x}{y}$ about the origin through an angle θ maps it to another point $\boldsymbol{p}' = \binom{x'}{y'}$ such that $\boldsymbol{p}$ and $\boldsymbol{p}'$ are at the same distance from the origin and the angle from the vector $\boldsymbol{p}$ to the vector $\boldsymbol{p}'$ is θ. See Figure 5.

To determine the coordinates of the image point $\boldsymbol{p}'$, it is very convenient for us to use polar coordinates. Let $\binom{x}{y} = (r\cos\phi, r\sin\phi)^\top$, where r is the distance from $\boldsymbol{p}$ to the origin and ϕ the polar angle. Then we have

$$\begin{aligned} x' &= r\cos(\theta + \phi) \\ &= r\cos\theta\cos\phi - r\sin\theta\sin\phi \end{aligned}$$

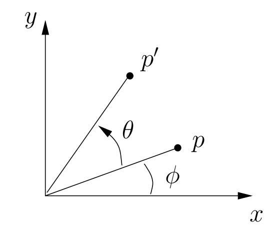

Figure 5: Rotation about the origin.

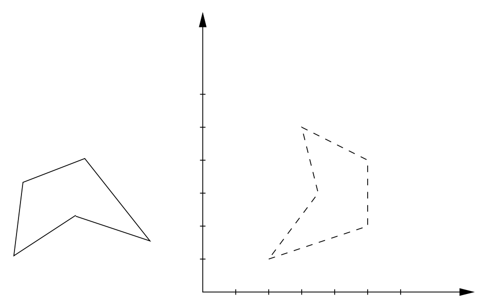

Figure 6: Rotation of a 5-gon about the origin by 110 degrees.

$$
\begin{aligned}
&= x\cos\theta - y\sin\theta, \\
y' &= r\sin(\theta+\phi) \\
&= r\sin\theta\cos\phi + r\cos\theta\sin\phi \\
&= x\sin\theta + y\cos\theta.
\end{aligned}
$$

More succinctly, the coordinates of $\boldsymbol{p}'$ can be obtained from those of $\boldsymbol{p}$ through a matrix multiplication:

$$\boldsymbol{p}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The orthogonal matrix[2]

$$\mathrm{Rot}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is called the *rotation matrix*. The inverse transform is the transpose $\mathrm{Rot}(\theta)^\top$ which rotates vectors back through $-\theta$.

Example 3. Consider a planar 2R robot manipulator (see Figure 7) consisting of two rigid links. The first link is attached to the base by a revolute joint J_1 which permits the link to rotate about the joint. The second link is attached to the first link by another revolute joint J_2. The robot's end effector is attached to the second link. The *pose* (i.e., position and orientation) of the end effector is controlled by exerting internal torques to turn the links about the two joints.

We set up a world (x, y)-coordinate system with the origin at J_1. The second link has its own (u, v)-coordinate system with J_2 as the origin. Let d be the distance between J_1 and J_2, θ_1 be the rotation angle from the x-axis to the axis of the first link, θ_2 be the rotation angle from the axis of the first link to that of the second link (this angle as shown in the figure is negative). The pose of the second link is obtained by

[2]A square matrix Q is *orthogonal* if $QQ^\top = Q^\top Q = I$.

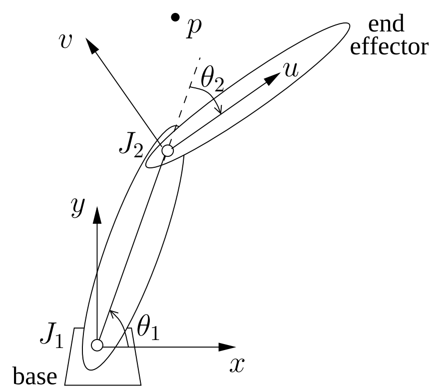

Figure 7: 2R robot manipulator.

applying a rotation $\mathrm{Rot}(\theta_1+\theta_2)$ followed by a translation $\mathrm{Trans}(d\cos\theta_1, d\sin\theta_1)$. Given the (u, v) coordinate of a point $\boldsymbol{p}$ with respect to the second link, the (x, y) coordinates of $\boldsymbol{p}$ in the world coordinate system is obtained by the transformation

$$
\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} d\cos\theta_1 \\ d\sin\theta_1 \end{pmatrix} \\
&= \begin{pmatrix} u\cos(\theta_1+\theta_2) - v\sin(\theta_1+\theta_2) + d\cos\theta_1 \\ u\sin(\theta_1+\theta_2) + v\cos(\theta_1+\theta_2) + d\sin\theta_1 \end{pmatrix}.
\end{aligned}
$$

When the joint angles are known, the world coordinates of an object can be determined from its local coordinates with respect to the robot's end effector. The calculation is referred to as the *forward kinematics* of the robot manipulator. We can generalize the above result to a robot manipulator with n revolute joints. The aim is to express the concatenations of all rotations and translations associated with the joints with one matrix multiplication. This will be possible with the assistance of homogeneous coordinates.

2.5 Shear

Let a fixed direction be represented by the unit vector $\boldsymbol{v} = \binom{v_x}{v_y}$. A *shear* about the origin of factor r in the direction $\boldsymbol{v}$ maps a point $\boldsymbol{p}$ to the point $\boldsymbol{p}' = \boldsymbol{p} + dr\boldsymbol{v}$, where d is the (signed) distance from the origin to the line through $\boldsymbol{p}$ in the direction $\boldsymbol{v}$.

Suppose $\boldsymbol{p} = \binom{x}{y}$. The unit vector normal to the line is $\boldsymbol{n} = \binom{-v_y}{v_x}$. Note that we choose the normal vector such that $\boldsymbol{v} \times \boldsymbol{n} = 1$. Therefore the distance d is given as

$$d = \boldsymbol{p} \cdot \boldsymbol{n} = y v_x - x v_y.$$

The shear transformation then maps $\boldsymbol{p}$ to

$$\boldsymbol{p}' = \boldsymbol{p} + dr\boldsymbol{v} = \begin{pmatrix} x + r(v_x y - v_y x) v_x \\ y + r(v_x y - v_y x) v_y \end{pmatrix}.$$

Thus the shear transformation matrix is

$$\mathrm{Shear}(\boldsymbol{v}, r) = \begin{pmatrix} 1 - r v_x v_y & r v_x^2 \\ -r v_y^2 & 1 + r v_x v_y \end{pmatrix}.$$

In particular, a shear along the x-axis has $\boldsymbol{v} = \binom{1}{0}$ and thus

$$\text{Shear}\left(\binom{1}{0}, r\right) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$$

.

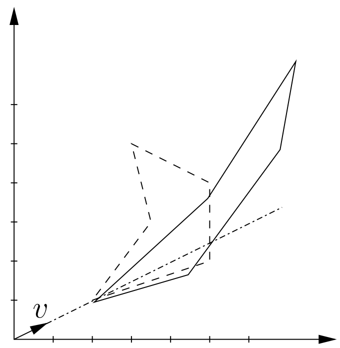

Figure 8: Shearing in $\boldsymbol{v} = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ by a factor $r = \frac{3}{2}$. The portion of the 5-gon to the left of $\boldsymbol{v}$ is extended along the direction $\boldsymbol{v}$ while the portion to the right of the vector is pulled back in the direction $-\boldsymbol{v}$.

Example 4. The shear in the direction $\boldsymbol{v} = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ with a factor $r = \frac{3}{2}$ has transformation matrix

$$\begin{aligned} \text{Shear}\left((\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}})^\top, \tfrac{3}{2}\right) &= \begin{pmatrix} 1 - \frac{3}{2}\frac{2}{\sqrt{5}}\frac{1}{\sqrt{5}} & \frac{3}{2}\left(\frac{2}{\sqrt{5}}\right)^2 \\ -\frac{3}{2}\left(\frac{1}{\sqrt{5}}\right)^2 & 1 + \frac{3}{2}\frac{2}{\sqrt{5}}\frac{1}{\sqrt{5}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{5} & \frac{6}{5} \\ -\frac{3}{10} & \frac{8}{5} \end{pmatrix}. \end{aligned}$$

Applying the shear to the 5-gon in Example 1, we have

$$\begin{pmatrix} \frac{2}{5} & \frac{6}{5} \\ -\frac{3}{10} & \frac{8}{5} \end{pmatrix} \begin{pmatrix} 2 & 5 & 5 & 3 & \frac{7}{2} \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4.4 & 6.8 & 7.2 & 5 \\ 1 & 1.7 & 4.9 & 7.1 & 3.75 \end{pmatrix}.$$

3 Consecutive Transformations

In many applications it is desirable to apply more than one transformations to an object. In robotics, for instance, an object has often undergone both translation and rotation after being manipulated. In vision, the image of an object results from a projection[3] of its model after some translation and rotation. It would be nice to concatenate all transformations into one equivalent transformation for the convenience of computation.

Example 1. A point $\boldsymbol{p} = (x, y)$ undergoes a rotation about the origin through an angle $\frac{\pi}{3}$. The resulting point $\boldsymbol{p}'$ has coordinates

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

[3]A projection is more general than affine transformation.

$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \binom{x}{y}.$$

Next, apply to $\boldsymbol{p}'$ a shear about the origin of factor 2 in the direction of the unit vector $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$. The new point $\boldsymbol{p}''$ has coordinates

$$\begin{aligned} \binom{x''}{y''} &= \begin{pmatrix} 1 - 2\frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right) & 2\left(\frac{\sqrt{3}}{2}\right)^2 \\ -2\cdot\left(-\frac{1}{2}\right)^2 & 1 + 2\frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right) \end{pmatrix} \binom{x'}{y'} \\ &= \begin{pmatrix} 1 + \frac{\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} \end{pmatrix} \binom{x'}{y'} \\ &= \begin{pmatrix} 1 + \frac{\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \binom{x}{y} \\ &= \begin{pmatrix} \frac{1}{2} + \sqrt{3} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} - 1 & \frac{1}{2} \end{pmatrix} \binom{x}{y}. \end{aligned}$$

The concatenated transformation from $\boldsymbol{p}$ to $\boldsymbol{p}''$ is thus represented by the matrix

$$\begin{pmatrix} \frac{1}{2} + \sqrt{3} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} - 1 & \frac{1}{2} \end{pmatrix}.$$

A problem is encountered when translations are involved. We would need to combine a matrix addition for the translation with a matrix multiplication for the other transformations. This is very awkward. There is a remedy, though, with the introduction of homogeneous coordinates. Then all transformations will be represented by matrices, and performed by matrix multiplications. And concatenation of transformations will be represented by the matrix product of the transformation matrices. Furthermore, an *inverse transformation*, which maps every image point back to its original position, will be obtained by taking a matrix inverse.

Recall that an affine transformation maps the point $\boldsymbol{p} = \binom{x}{y}$ to $\boldsymbol{p}' = \binom{x'}{y'} = A\boldsymbol{p} + \boldsymbol{b}$. To represent the mapping as a matrix multiplication, we introduce the homogeneous coordinates $(x, y, 1)$ such that

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} &= \binom{A\boldsymbol{p} + \boldsymbol{b}}{1} \\ &= \begin{pmatrix} A & \boldsymbol{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \end{aligned}$$

We can easily verify that the approach of homogeneous coordinates also works for the other linear transformations we have learned so far: scaling, rotation, reflection, and shear. All we need is to let $\boldsymbol{b} = 0$ in the new 3×3 transformation matrix. The problem is solved! But there is more about homogeneous coordinates we ought to know.

4 Definition of Homogeneous Coordinates

To formally introduce homogeneous coordinates, let us first recall that a *relation* $\sim$ on a set S is a subset of $S \times S$ such that u is related to v whenever (u, v) is in the subset. We also write $u \sim v$ when u and v are related. A relation $\sim$ is *reflexive* if $u \sim u$ for all $u \in S$; it is *symmetric* if $u \sim v$ whenever $v \sim u$; it is *transitive* if $u \sim w$ whenever $u \sim v$ and $v \sim w$. The relation $\sim$ is an *equivalence* if it is reflexive, symmetric, and transitive.

The relation $>$ on $\mathbb{R}$ is transitive, but not reflexive or symmetric. The relations $\leq$ and $\geq$ are both reflexive and transitive, but not symmetric. The most familiar equivalence relation on $\mathbb{R}$ is $=$. An equivalence relation on the set $\mathbb{Z}$ of integers is congruence $\equiv$ modulo integer $m > 0$.

Let $\sim$ be an equivalence relation on S. The subset of S consisting of all elements related to an element s is the *equivalent class* of s and denoted as $[s]$. For example, the congruence $\equiv$ (mod4) induces four equivalence classes $[0], [1], [2]$, and $[3]$, where $[i] = \{\, i + 4k \mid k \text{ integer} \,\}$ for $i = 0, 1, 2, 3$.

Let us now focus on the relation $\sim$ on the set $S = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ defined below:

$$(x, y, z) \sim (u, v, w) \qquad \text{iff} \qquad (x, y, z) = r(u, v, w) \qquad \text{for some } r \neq 0. \tag{2}$$

It is easy to show that the relation is reflexive, symmetric, and transitive. Hence it is an equivalence relation. The equivalence class of (x, y, z) is the set

$$[(x, y, z)] = \{\, r(x, y, z) \mid r \in \mathbb{R} \text{ and } r \neq 0 \,\}. \tag{3}$$

Homogeneous coordinates are equivalence classes of the relation $\sim$ defined by (2). The coordinates (x, y, z) is identified with $r(x, y, z)$ with $r \neq 0$. The *projective plane* $\mathbb{P}^2$ is defined to be the set of all equivalence classes, that is, $\{\, [(x, y, z)] \mid x \neq 0, y \neq 0, \text{ or } z \neq 0 \,\}$. An equivalence class is referred to as a *point* in the projective plane.

Operations of the projective plane are carried out by taking a representative from each equivalence class. For a class $[(u, v, w)]$ with $w \neq 0$, we use $(u/w, v/w, 1)$ as the representative. So there is a one-to-one correspondence between points (x, y) of the Cartesian plane and points $[(u, v, w)]$, $w \neq 0$ in the projective plane with $w \neq 0$.

5 Points at Infinity

Homogeneous coordinates of the form $(x, y, 0)$ do not correspond to a point in the Cartesian plane. Instead, they correspond to the unique point at infinity in the direction (x, y). To see this, consider the line through a point, say, (a, b), and with direction (x, y). It has the parametric form $(a+tx, b+ty)$. Every point on the line thus has homogeneous coordinates $(a + tx, b + ty, 1)$, and equivalently, $(\frac{a}{t} + x, \frac{b}{t} + y, \frac{1}{t})$. As t tends to infinity, that is, as a point moves along the line to infinity, the latter homogeneous coordinates become $(x, y, 0)$. Moving in the directions of (x, y) and $(-x, -y)$ will end up at the same infinity point represented by $(x, y, 0)$ as well as $(-x, -y, 0)$.

Hence the projective plane $\mathbb{P}^2$ can be seen as the plane $\mathbb{R}^2$ plus all the points at infinity, each of which along a different direction. The plane $\mathbb{P}^2$ also makes sense of the notion that two parallel lines intersect at infinity, as we will see in the example below.

Example 2. Consider the parallel lines $x + 2y = 1$ and $x + 2y = 2$. Let (u, v, w) be the homogeneous coordinates of a point (x, y) on the first line. Then $x = \frac{u}{w}$ and $y = \frac{v}{w}$. We have $\frac{u}{w} + \frac{2v}{w} = 1$ and thus the homogeneous equation of the line is

$$u + 2v = w. \tag{4}$$

Similarly, the second line has the homogeneous equation

$$u + 2v = 2w. \tag{5}$$

Equations (4) and (5) have solutions of the form $(-2r, r, 0)$. That is, the solutions are all homogeneous coordinates of the point $(-2, 1, 0)$, which is the unique intersection of the two parallel lines in the direction $(-2, 1)$. Similarly, we conclude that all lines parallel to $x + 2y = 1$ intersect in a unique point at infinity in the direction $(-2, 1)$.

Now we can visualize the relationship between the projective plane $\mathbb{P}^2$ and the three-dimensional space $\mathbb{R}^3$ induced from (3) as follows. The set $\mathbb{R}^3$ is made up of all the lines through the origin. Each line ℓ of this kind, excluding the origin, uniquely corresponds to a point in $\mathbb{P}^2$. If ℓ lies in the xy-plane, then it represents a point at infinity in $\mathbb{R}^2$. Otherwise, it represents a real point (i.e., one with finite coordinates) in $\mathbb{R}^2$, and the line is typically identified with its intersection point with the plane $z = 1$ (which has coordinates in the form of $(x, y, 1)$).

6 Point and Line Geometry in Homogeneous Coordinates

We have seen that a point (x, y) in the Cartesian plane has homogeneous coordinates $t(x, y, 1)$, $t \neq 0$. These coordinates would correspond to a line through the origin (excluded) if they were Cartesian coordinates in the 3-dimensional space. When homogeneous coordinates are "viewed" as Cartesian coordinates, the dimensions of the geometric object they describe "increase" by 1.

The geometry of a line in the Cartesian plane is reviewed in Appendix A. It has general equation $ax + by + c = 0$. Suppose (u, v, w) are the homogeneous coordinates of a point (x, y) on the line; hence $x = u/w$ and $y = v/w$. Substituting for x and y in the line equation and multiplying through by w, yields the conditions for (u, v, w) to be the homogeneous coordinates of a point on the line:

$$au + bv + cw = 0. \tag{6}$$

Equation (6) is known as the *homogeneous line equation*.[4] The line is uniquely specified by the coefficients a, b, and c, or any multiple ra, rb, and rc with $r \neq 0$. Therefore it is natural to specify the line by the homogeneous coordinates

$$\ell = (a, b, c).$$

Since any non-zero multiple of ℓ defines the same line, it is useful to consider ℓ as a vector of which only the direction matters. Let $\boldsymbol{p} = (u, v, w)$ be a point in homogeneous coordinates. Then in order for $\boldsymbol{p}$ to lie on the line, the dot product of $\boldsymbol{p}$ and ℓ must vanish, that is,

$$\boldsymbol{p} \cdot \ell = 0. \tag{7}$$

The identity (7) allows us to easily determine the line through two distinct points as well as the point of intersection of two lines.

Suppose ℓ is the vector that represents a line through two distinct points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, all in homogeneous coordinates. Then from (7) we have

$$\boldsymbol{p}_1 \cdot \ell = 0 \qquad \text{and} \qquad \boldsymbol{p}_2 \cdot \ell = 0.$$

[4] In Cartesian coordinates the same equation would describe a plane through the origin and with normal (a, b, c) .

Thus ℓ is perpendicular (or orthogonal) to both $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. To determine ℓ it suffices to find a vector perpendicular to $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ since only the direction matters. We choose the cross product by letting $\ell = \boldsymbol{p}_1 \times \boldsymbol{p}_2$ (or any multiple of $\boldsymbol{p}_1 \times \boldsymbol{p}_2$). *The equation of the line through two points can be determined by taking the 'cross product' of their homogeneous coordinates.*

EXAMPLE 4. The line ℓ passing through $(3,1)$ and $(-4,5)$ satisfies equations

$$\begin{aligned} \ell \cdot (3,1,1) &= 0, \\ \ell \cdot (-4,5,1) &= 0. \end{aligned}$$

Hence we have the line in homogeneous coordinates:

$$\ell = (3,1,1) \times (-4,5,1) = (-4,-7,19),$$

which give the line $4x + 7y - 19 = 0$. We can verify this equation using the original points $(3,1)$ and $(-4,5)$.

Next, suppose $\boldsymbol{p}$ is the intersection of two lines ℓ_1 and ℓ_2 (all in homogeneous coordinates). Then from (7) we have

$$\ell_1 \cdot \boldsymbol{p} = 0 \qquad \text{and} \qquad \ell_2 \cdot \boldsymbol{p} = 0.$$

In other words, $\boldsymbol{p}$ is orthogonal to both ℓ_1 and ℓ_2 when all are seen as vectors. Hence it is sufficient to take $\boldsymbol{p} = \ell_1 \times \ell_2$ (or any multiple of it) as the homogeneous coordinates of the point of intersection.

EXAMPLE 5. The intersection point $\boldsymbol{p}$ of the lines $x - 7y + 8 = 0$ and $3x - 4y + 1 = 0$ satisfies

$$(1,-7,8) \cdot \boldsymbol{p} = 0 \qquad \text{and} \qquad (3,-4,1) \cdot \boldsymbol{p} = 0.$$

Hence

$$\boldsymbol{p} = (1,-7,8) \times (3,-4,1) = (25,23,17).$$

And the two lines intersects at the point $(\frac{25}{17}, \frac{23}{17})$ in the Cartesian plane.

EXAMPLE 6. The two parallel lines $2x - 5y = 0$ and $2x - 5y = -3$ do not intersect in the Cartesian plane. In homogeneous coordinates, their intersection point $\boldsymbol{p}$ is $(2,-5,0) \times (2,-5,3) = (-15,-6,0)$, which is at infinity.

7 Projective Space

Homogeneous coordinates of the three-dimensional (3D) space $\mathbb{R}^3$ are derived in a similar manner as those of the plane. A point (x,y,z) in $\mathbb{R}^3$ is represented by the vector $(x,y,z,1)$, or by any multiple (rx,ry,rz,r) with $r \neq 0$. Conversely, the homogeneous coordinates (s,u,v,w) with $w \neq 0$ represent the point $(x,y,z) = (s/w, u/w, v/w)$ in the 3D space. A point of the form $(s,u,v,0)$ corresponds to the point at infinity in the 3D space in the direction of the vector (s,u,v). The set of all homogeneous coordinates (s,u,v,w) is called the (*three-dimensional*) *projective space* and denoted $\mathbb{P}^3$.

EXAMPLE 1. The homogeneous coordinates $(1,2,-1,5)$, $(\frac{1}{2},1,-\frac{1}{2},\frac{5}{2})$, $(-1,-2,1,-5)$ represent the same point $(\frac{1}{5},\frac{2}{5},-\frac{1}{5})$ in $\mathbb{R}^3$.

In Cartesian (i.e., Euclidean) coordinates a plane is described by an equation of the form $ax + by + cz + d = 0$. To obtain the corresponding equation in homogeneous coordinates, we substitute $x = s/w$, $y = u/w$, and $z = v/w$ in the equation and multiply both sides by w, yielding

$$as + bu + cv + dw = 0. \tag{8}$$

Recall that a line in the plane is represented by a line vector in homogeneous coordinates. Similarly, a plane in the space is specified by a *plane vector*

$$\boldsymbol{n} = (a, b, c, d).$$

Thus from (8) a point $\boldsymbol{p}$ in homogeneous coordinates (s, u, v, w) lies on a plane with plane vector $\boldsymbol{n}$ if and only if $\boldsymbol{p} \cdot \boldsymbol{n} = 0$.

7.1 Plane through Three Distinct Points

We have learned that the line through two points in the plane can be obtained by carrying out a cross product. The analogous problem in space is to determine the unique plane, represented by $\boldsymbol{n}$, that passes through three distinct points $\boldsymbol{p}_i = (s_i, u_i, v_i, w_i)$, $i = 1, 2, 3$. The plane vector $\boldsymbol{n}$ satisfies

$$\boldsymbol{n} \cdot \boldsymbol{p}_1 = 0, \qquad \boldsymbol{n} \cdot \boldsymbol{p}_2 = 0, \qquad \boldsymbol{n} \cdot \boldsymbol{p}_3 = 0.$$

Namely, $\boldsymbol{n}$ is perpendicular to three vectors $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{p}_3$. The condition for this to occur is that

$$\boldsymbol{n} = k \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ s_1 & u_1 & v_1 & w_1 \\ s_2 & u_2 & v_2 & w_2 \\ s_3 & u_3 & v_3 & w_3 \end{vmatrix}, \tag{9}$$

where $k \neq 0$, $\boldsymbol{e}_1 = (1, 0, 0, 0)$, $\boldsymbol{e}_2 = (0, 1, 0, 0)$, $\boldsymbol{e}_3 = (0, 0, 1, 0)$, and $\boldsymbol{e}_4 = (0, 0, 0, 1)$. For convenience, we choose $k = 1$. The determinant above is a vector determinant in the sense that all $\boldsymbol{e}_i$s are treated as scalars in the expansion into a summation. Accordingly, we can verify that

$$\begin{aligned} \boldsymbol{n} \cdot \boldsymbol{p}_i &= \begin{vmatrix} \boldsymbol{e}_1 \cdot \boldsymbol{p}_i & \boldsymbol{e}_2 \cdot \boldsymbol{p}_i & \boldsymbol{e}_3 \cdot \boldsymbol{p}_i & \boldsymbol{e}_4 \cdot \boldsymbol{p}_i \\ s_1 & u_1 & v_1 & w_1 \\ s_2 & u_2 & v_2 & w_2 \\ s_3 & u_3 & v_3 & w_3 \end{vmatrix} \\ &= \begin{vmatrix} s_i & u_i & v_i & w_i \\ s_1 & u_1 & v_1 & w_1 \\ s_2 & u_2 & v_2 & w_2 \\ s_3 & u_3 & v_3 & w_3 \end{vmatrix} \\ &= 0. \end{aligned}$$

Example 2. The plane through the points $(5, 4, 2)$, $(-1, 7, 3)$, and $(2, -2, 9)$ is described by the line vector

$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ 5 & 4 & 2 & 1 \\ -1 & 7 & 3 & 1 \\ 2 & -2 & 9 & 1 \end{vmatrix} = \boldsymbol{e}_1 \begin{vmatrix} 4 & 2 & 1 \\ 7 & 3 & 1 \\ -2 & 9 & 1 \end{vmatrix} - \boldsymbol{e}_2 \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 1 \\ 2 & 9 & 1 \end{vmatrix} + \boldsymbol{e}_3 \begin{vmatrix} 5 & 4 & 1 \\ -1 & 7 & 1 \\ 2 & -2 & 1 \end{vmatrix} - \boldsymbol{e}_4 \begin{vmatrix} 5 & 4 & 2 \\ -1 & 7 & 3 \\ 2 & -2 & 9 \end{vmatrix}$$

$$
\begin{aligned}
&= 27\boldsymbol{e}_1 + 39\boldsymbol{e}_2 + 45\boldsymbol{e}_3 - 381\boldsymbol{e}_4 \\
&= \begin{pmatrix} 27 \\ 39 \\ 45 \\ -381 \end{pmatrix}.
\end{aligned}
$$

If $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{p}_3$ are three collinear points, it can be shown that

$$(s_3, u_3, v_3, w_3) = a(s_1, u_1, v_1, w_1) + b(s_2, u_2, v_2, w_2)$$

for some a and b. The determinant in (9) can have the entries of its fourth row all made zero, yielding $\boldsymbol{n} = \boldsymbol{0}$.

7.2 Intersection of Three Planes

Analogously, the point of intersection $\boldsymbol{p}$ of three planes respectively determined by vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ satisfies

$$\boldsymbol{n}_1 \cdot \boldsymbol{p} = 0, \qquad \boldsymbol{n}_2 \cdot \boldsymbol{p} = 0, \qquad \boldsymbol{n}_3 \cdot \boldsymbol{p} = 0.$$

Denote $\boldsymbol{n}_i = (s_i, u_i, v_i, w_i)$. Hence the homogeneous coordinates of $\boldsymbol{p}$ are given as

$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ s_1 & u_1 & v_1 & w_1 \\ s_2 & u_2 & v_2 & w_2 \\ s_3 & u_3 & v_3 & w_3 \end{vmatrix}$$

or any multiple of the above vector determinant.

Example 3. The point of intersection of the three planes

$$3x + 5y + z = 2, \qquad 7x - 4z = -1, \qquad 2y + 5z + 8 = 0$$

is obtained by computing the determinant

$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ 3 & 5 & 1 & -2 \\ 7 & 0 & -4 & 1 \\ 0 & 2 & 5 & 8 \end{vmatrix} = -199\boldsymbol{e}_1 + 237\boldsymbol{e}_2 - 314\boldsymbol{e}_3 + 137\boldsymbol{e}_4.$$

Namely, the intersection has homogeneous coordinates $(-199, 237, -314, 137)$ and Cartesian coordinates $(-\frac{199}{137}, \frac{237}{137}, -\frac{314}{137})$.

A Line in the Plane

Recall the general equation of a line:

$$ax + by + c = 0, \qquad \text{where } a \neq 0 \text{ or } b \neq 0. \tag{10}$$

The equation above is the *implicit form* of the line. We normalize the coefficients and obtain

$$\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y + \frac{c}{\sqrt{a^2 + b^2}} = 0. \tag{11}$$

For convenience, let us introduce the unit vector $\boldsymbol{n} = (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})^\top$. Now, move the term involving c to the right hand side of equation (11) and rewrite the remaining terms on the left hand side as a dot product:

$$\boldsymbol{n} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{c}{\sqrt{a^2+b^2}}. \tag{12}$$

For any two points $\binom{x_1}{y_1}$ and $\binom{x_2}{y_2}$ on the line, we have that

$$\boldsymbol{n} \cdot \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) = 0.$$

Thus, the unit vector $\boldsymbol{n}$ is perpendicular to the line. Equation (12) states that the distance to the line from the origin is $\frac{|c|}{\sqrt{a^2+b^2}}$. The vector $\boldsymbol{n}$ points toward the line when $c < 0$ and away from the line when $c > 0$. We easily see that the vector $\binom{a}{b}$, just like $\boldsymbol{n}$, is perpendicular to the line while

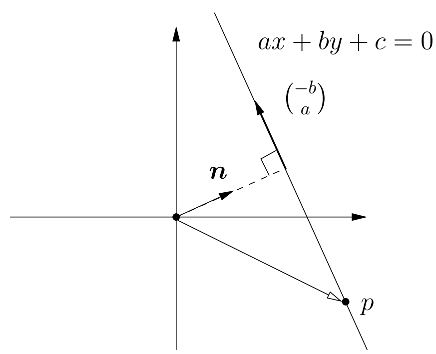

Figure 9: A line.

its orthogonal vector $\binom{-b}{a}$ is parallel to the line.

The line through a point $\boldsymbol{p} = \binom{p_x}{p_y}$ in the direction of the vector $\boldsymbol{v} = \binom{v_x}{v_y}$ can also be defined parametrically as

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \boldsymbol{p} + t\boldsymbol{v} \\ &= \begin{pmatrix} p_x + tv_x \\ p_y + tv_y \end{pmatrix}. \end{aligned} \tag{13}$$

From this parametric (or explicit) form, we immediately derive the implicit form of the line by eliminating t from $x = p_x + v_x t$ and $y = p_y + v_y t$:

$$v_y x - v_x y + (p_y v_x - p_x v_y) = 0.$$

Conversely, given the implicit form (10), we may set $\boldsymbol{v} = \binom{-b}{a}$ parallel to the line in deriving the parametric form (13). A point on the line can be chosen by setting $x = 0$ in case $b \neq 0$ (or $y = 0$ otherwise) and determining y (or x, respectively).

Example 1. Consider two lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$. Their directions are $\boldsymbol{v} = (-b_1, a_1)$ and $\boldsymbol{w} = (-b_2, a_2)$, respectively. Let θ be the angle between the two lines, more specifically, from $\boldsymbol{v}$ to $\boldsymbol{w}$.

Then, the identities $\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \cos\theta$ and $\boldsymbol{v} \times \boldsymbol{w} = \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \sin\theta$ give rise to

$$\begin{aligned}\cos\theta &= \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2+b_1^2} \cdot \sqrt{a_2^2+b_2^2}},\\ \sin\theta &= \frac{a_1b_2 - a_2b_1}{\sqrt{a_1^2+b_1^2} \cdot \sqrt{a_2^2+b_2^2}}.\end{aligned}$$

Hence

$$\tan\theta = \frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}.$$

The two lines are parallel if and only if $\theta = 0$, that is, if and only if $a_1b_2 = a_2b_1$.

B Visualization of the Projective Plane

Two models exist for us to visualize the projective plane and understand homogeneous coordinates geometrically. They are the line model and the spherical model.

The line model represents the point with homogeneous coordinates $\lambda(u, v, w)$, $\lambda \neq 0$ by the line through the origin with direction (u, v, w). A one-to-one correspondence exists between the points (x, y) in the plane $\mathbb{R}^2$ and the lines parametrized as $t(x, y, 1)$, $t \in \mathbb{R}$. Another one-to-one correspondence exists between the point (x, y) and the point $(x, y, 1)$ in the plane $w = 1$.

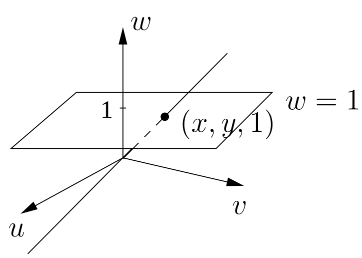

Figure 10: The line model of the projective plane.

But points at infinity, which have homogeneous coordinates of the form $(x, y, 0)$, are not on the $w = 1$ plane. Instead, they correspond to lines in the $w = 0$ plane. Also, lines in the Cartesian plane $\mathbb{R}^2$ correspond to planes in the u-v-w space. This is the difficulty with the line model.

Example 3. The two parallel lines $x + 2y = 1$ and $x + 2y = 2$ in the previous example correspond to the planes in the u-v-w space given by equations

$$\begin{aligned}u + 2v &= w,\\ u + 2v &= 2w.\end{aligned}$$

The planes intersect in a line $(-2t, t, 0)$ through the origin in the $w = 0$ plane. This line corresponds to the point at infinity $(-2, 1, 0)$, which is intersection of the two parallel lines.

In the spherical model of the projective plane, a point with homogeneous coordinates (x, y, z) maps to the point of the intersection of the corresponding line $t(x, y, z)$ and the unit sphere centered

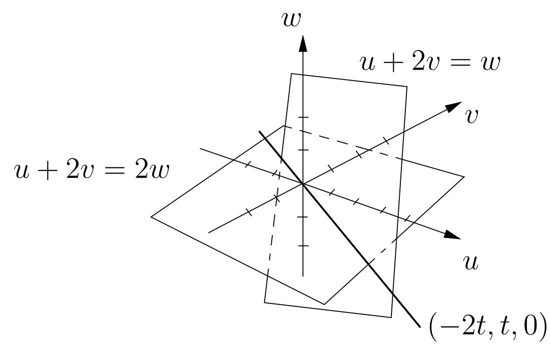

Figure 11: Intersection of plane images of parallel lines in the Cartesian plane.

at the origin $x^2 + y^2 + z^2 = 1$. In other words,

$$(x, y, z) \mapsto \pm \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}. \tag{14}$$

The two image points are antipodal. Since the two antipodal points correspond to the same point $[(x, y, z)]$ in the projective plane, it suffices to consider the upper half sphere together with (half of) the equator.

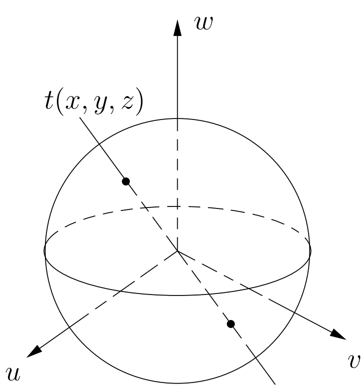

Figure 12: Spherical model of the projective plane. Antipodal points represent the same homogeneous point.

Every point at infinity in the Cartesian plane $\mathbb{R}^2$ has homogeneous coordinates $(x, y, 0)$. By (14) it corresponds to two antipodal points on the equator. Thus the sphere (or half sphere) provides a way of visualizing all homogeneous coordinates. A line in the plane $\mathbb{R}^2$ corresponds to a great circle (which is the intersection of the sphere with the plane containing the origin and the line elevated to the $z = 1$ plane). The intersection of two parallel lines correspond to the intersection points of the two great circles on the sphere, namely, two antipodal points on the equator (which represent two points at infinity in the direction of these lines).

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