Fourier Series

(Com S 477/577 Notes)

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1 Introduction

Many functions in nature are periodic, that is,

$$f(x+\tau) = f(x),$$

for some fixed τ, which is called the *period* of f. Though function approximation using orthogonal polynomials is very convenient, there is only one kind of periodic polynomial, that is, a constant. So, polynomials are not good for approximating periodic functions. In this case, trigonometric functions are quite useful.

A large class of important computational problems falls under the category of "Fourier transform methods" or "spectral methods". For some of these problems, the Fourier transform is simply an efficient computational tool for data manipulation. For other problems, the Fourier transform is itself of intrinsic interest.

Fourier methods have revolutionized fields of science and engineering, from radio astronomy to medical imaging, from seismology to spectroscopy. The wide application of Fourier methods is credited principally to the existence of the fast Fourier transform (FFT). The most direct applications of the FFT are to the convolution or deconvolution of data, correlation and autocorrelation, optimal filtering, power spectrum estimation, and the computational of Fourier integrals.

A physical process can be described either in the *time domain*, by the values of some quantity h as a function of time t, or else in the *frequency domain*, where the process is specified by giving its amplitude H as a function of frequency ξ with $-\infty < \xi < \infty$. For many purposes it is useful to think of $h(t)$ and $H(\xi)$ as being two different representations of the same function. These two representations are related to each other by the Fourier transform equations,

$$\begin{aligned} H(\xi) &= \int_{-\infty}^{\infty} h(t)e^{2\pi i\xi t}\,dt, \\ h(t) &= \int_{-\infty}^{\infty} H(\xi)e^{-2\pi i\xi t}\,d\xi, \qquad i^2 = 1. \end{aligned}$$

2 Fourier Series

A *trigonometric polynomial* of order n is any function of the form

$$p(x) = \frac{a_0}{2} + \sum_{j=1}^{n}(a_j \cos jx + b_j \sin jx), \tag{1}$$

where $a_0, \ldots, a_n$ and $b_1, \ldots, b_n$ are real or complex numbers. Such a trigonometric polynomial has period 2π.

When approximating a function $f(x)$ with period $\tau \neq 2\pi$, we have to make some adjustment by considering instead the function

$$g(x) = f\Big(\tau x/(2\pi)\Big),$$

which has period 2π. Having constructed a trigonometric polynomial approximation $p(x)$ to $g(x)$, we obtain a τ-periodic polynomial approximation $p(2\pi x/\tau)$ to $f(x)$. For this reason we will from now on assume that the function $f(x)$ to be approximated is already 2π-periodic.

The trigonometric polynomial (1) has an equivalent complex form

$$p(x) = \sum_{j=-n}^{n} c_j e^{ijx}, \qquad \text{where } i^2 = -1, \tag{2}$$

under *Euler's formula*

$$e^{ix} = \cos x + i \sin x.$$

We then expand (2):

$$p(x) = c_0 + \sum_{j=1}^{n} (c_j + c_{-j}) \cos jx + i \sum_{j=1}^{n} (c_j - c_{-j}) \sin jx.$$

A comparison between the above and (1) yields

$$\begin{aligned} a_j &= c_j + c_{-j}, \\ b_j &= i(c_j - c_{-j}), \end{aligned}$$

$j = 0, \ldots, n$. Solution of each pair of such equations gives us, for $j = 0, \ldots, n$,

$$\begin{aligned} c_j &= \frac{a_j - ib_j}{2}, \\ c_{-j} &= \frac{a_j + ib_j}{2}. \end{aligned}$$

The functions $1, e^{\pm ix}, e^{\pm 2ix}, \ldots,$ form an orthonormal basis with respect to the inner product

$$\langle g, h \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(x) \overline{h(x)} \, dx, \tag{3}$$

where $\overline{h(x)}$ is the complex conjugate of $h(x)$. More specifically,

$$\langle e^{ijx}, e^{ikx} \rangle = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

The *Fourier series* for a function $f(x)$ is given by

$$f(x) \approx \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{ijx}, \tag{4}$$

where

$$\begin{aligned}\hat{f}(j) &= \langle f(x), e^{ijx} \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ijx} dx.\end{aligned}$$

are the *Fourier coefficients*. From the above definition we easily see that $\hat{f}(-j) = \overline{\hat{f}(j)}$. In (4), "$\approx$" means that the Fourier series converges to $f(x)$ under rather mild conditions. For example, the series converges uniformly if $f(x)$ is continuous and $f'(x)$ is piecewise continuous.

Theorem 1 *The partial sum*

$$\sum_{j=-n}^{n} \hat{f}(j) e^{ijx}$$

of the Fourier series for $f(x)$ is the best approximation to $f(x)$ by trigonometric polynomials of order n under the inner product defined in (3); that is, with respect to the norm

$$\|g\|_2 = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |g(x)|^2 \, dx}.$$

Furthermore, it can be shown that *Parseval's relation*

$$\sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx$$

holds.

The Fourier coefficients $\hat{f}(j)$ can help us "understand" the function $f(x)$. Suppose $f(x)$ is a *real* function with period 2π. It can be viewed as the motion of a point at time x on a line. Substitute the *polar form*

$$\hat{f}(j) = |\hat{f}(j)| e^{i\theta_j}$$

into the Fourier series (4) and use the fact that $\hat{f}(j)$ and $\hat{f}(-j)$ are complex conjugates:

$$f(x) \approx |\hat{f}(0)| \cos\theta_0 + 2 \sum_{j=1}^{\infty} |\hat{f}(j)| \cos(\theta_j + jx).$$

Thus we have obtained a representation of the periodic motion as a superposition of simple harmonic oscillations. The jth such motion (with $j > 0$), $2|\hat{f}(j)| \cos(\theta_j + jx)$, has

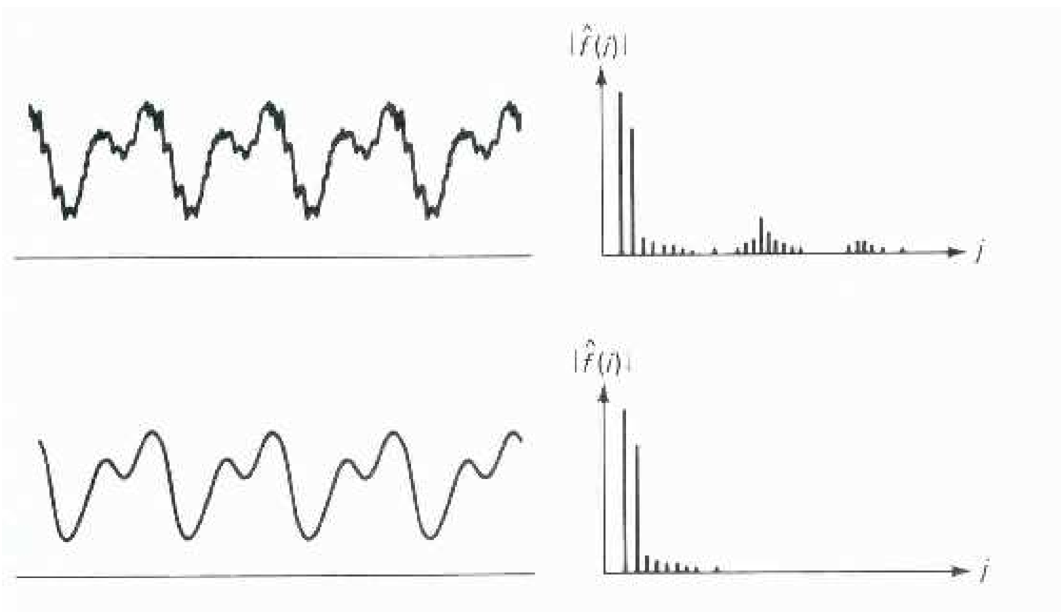
amplitude:	$2\|\hat{f}(j)\|$,
frequency:	$\frac{j}{2\pi}$,
angular frequency:	j,
period or *wavelength*:	$\frac{2\pi}{j}$,
phase angle:	θ_j.

The number $|\hat{f}(j)|$ measures the strength of the presence of a simple harmonic motion of frequency $\frac{j}{2\pi}$ in the total motion. It can be shown that

$$\hat{f}(j) = O\left(|j|^{-l-1}\right), \tag{5}$$

when the lth derivative of $f(x)$ exists and is piecewise continuous. The sequence $|\hat{f}(0)|, |\hat{f}(1)|, \ldots$ is called the *spectrum* of $f(x)$ over which the "total energy" $\|f\|_2^2$ is distributed. A "noisy" function will have sizeable $|\hat{f}(j)|$ for large j. For a "smooth" function, the spectrum will decrease rapidly as j increases.

The method of *smoothing* often consists in generating the Fourier coefficients of $f(x)$ from data, *filtering* these coefficients to suppress high frequencies (which usually correspond to noise), and then reconstructing the function as a Fourier series with "purified" or "filtered" coefficients. The figure[1] below shows two 2π-periodic functions and their power spetrums. The second function is obtained from the first by filtering out its higher frequencies.



Since it is generally difficult or impossible to compute the Fourier coefficients $\{\hat{f}(j)\}$ exactly, we use their discrete approximations that result from *sampling* f at the points $x_k = \frac{2\pi k}{N}$ for $k = 0, \ldots, N-1$. They are

$$\begin{aligned} \hat{f}_N(j) &= \langle f, e^{ijx} \rangle_N, \qquad j = 0, \ldots, N-1 & (6) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} f(x_k) e^{-ijx_k}. & (7) \end{aligned}$$

Here the discrete inner product $\langle\ ,\ \rangle_N$ is defined as

$$\langle g, h \rangle_N = \frac{1}{N} \sum_{i=0}^{N-1} g(x_i) \overline{h(x_i)},$$

[1] From [1, p. 271].

Under this inner product, the functions $1, e^{\pm ix}, e^{\pm i2x}, \dots$ are still orthogonal, namely,

$$\langle e^{ikx}, e^{ijx}\rangle_N = \begin{cases} 1, & \text{if } k = j \pmod N, \\ 0, & \text{otherwise.} \end{cases}$$

But now we have

$$\begin{aligned} \hat{f}_N(j) &= \langle f, e^{ijx}\rangle_N \\ &= \langle \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}, e^{ijx}\rangle_N \qquad \text{by (4)} \\ &= \sum_{k=-\infty}^{\infty} \hat{f}(k) \cdot \langle e^{ikx}, e^{ijx}\rangle_N \\ &= \sum_{k=j \pmod N} \hat{f}(k). \end{aligned} \tag{8}$$

Equation (8) immediately implies that

$$\cdots = \hat{f}_N(j-N) = \hat{f}_N(j) = \hat{f}_N(j+N) = \cdots.$$

The points $x_0, \dots, x_{N-1}$ are called the *sampling points*, $f(x_0), \dots, f(x_{N-1})$ the *sampling values*, $\frac{2\pi}{N}$ the *sampling interval*, and $\frac{N}{2\pi}$ the *sampling frequency*.[2] From equation (8) we see that all the Fourier coefficients $\hat{f}(k)$, $k = j \bmod N$, get mashed together and show up indistinguishably in the discrete Fourier series. This is referred to as *aliasing*. We cannot tell the difference between two basis functions $e^{ik\frac{2\pi}{N}}$ and $e^{ij\frac{2\pi}{N}}$, $k = j \pmod N$, because they agree at all sampling point $x_0, \dots, x_{N-1}$.

Aliasing is illustrated[3] on the next page on a continuous function in (a) which is nonzero only for a finite time interval T. The Fourier transform of the function, shown in (b), has no limited bandwidth but rather finite amplitude for all frequencies. Suppose the original function is sampled with a sampling interval Δ, then the resulting Fourier transform in (c) is defined between frequencies $-\frac{1}{2\Delta}$ and $\frac{1}{2\Delta}$. Power outside that frequency range is folded over or "aliased" into the range.[4] To eliminate this effect, the original function should go through low-pass filtering before sampling.

Plugging (5) into (8) yields

$$\begin{aligned} \hat{f}_N(j) &= \sum_{k=j \pmod N} O\Big(|k|^{-l-1}\Big) \\ &= \hat{f}_N\big(\hat{j}\big) + O\left(\left(\frac{N}{2}\right)^{-l-1}\right), \end{aligned}$$

where

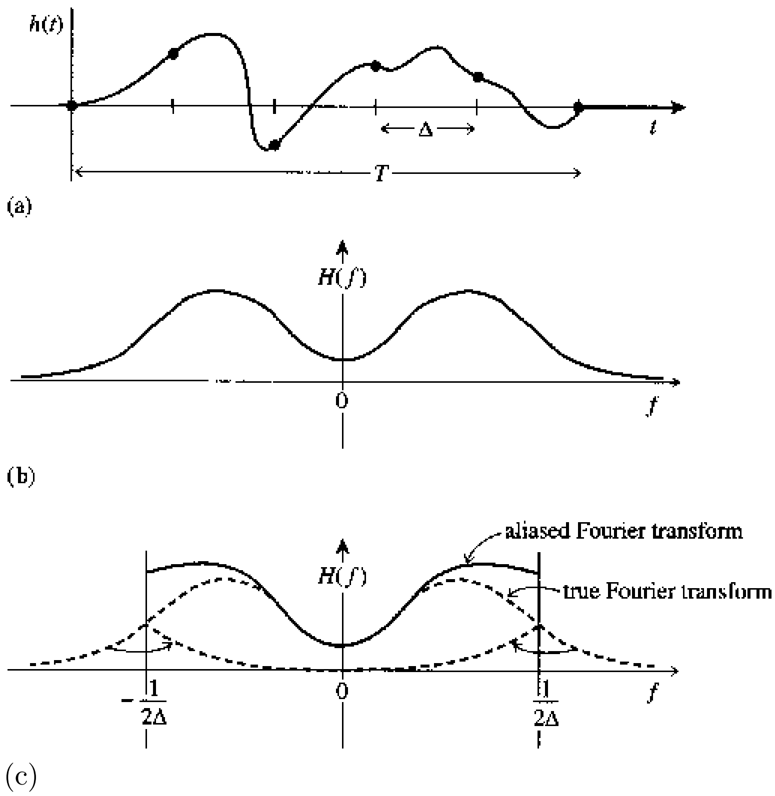
$$\hat{j} = \begin{cases} j \bmod N, & \text{if } 0 \le j \bmod N \le \frac{N}{2}; \\ (j \bmod N) - N, & \text{if } j \bmod N > \frac{N}{2}. \end{cases}$$

[2] To generalize, for a function with period τ, the sampling frequency is $\frac{N}{\tau}$.

[3] The figure is from [3, p. 507].

[4] This folding effect is in part created by $H(-f) = \overline{H(f)}$.

So we see that the Fourier coefficients $\hat{f}(j)$ with $|j| \leq \frac{N}{2}$ dominate other coefficients. For this reason, $\hat{f}_N(j)$ is usually taken only as an approximation to $\hat{f}(j)$ with $|j| \leq \frac{N}{2}$. Thus when we sample a real function at N equally spaced points, in the interval $[0, 2\pi)$, the aliasing effect prevents the observation of periodic phenomena in $f(x)$ with frequencies higher than $(N/2)/2\pi$. Phrased differently, we have the following result.



Theorem 2 (Sampling Theorem) *If we wish to observe a certain periodic phenomenon of frequency v, then we must sample at a frequency at least as large as $2v$.*

Observe that $\hat{f}_N(-j)$ is a conjugate of $\hat{f}_N(j)$, for all j, since

$$\begin{aligned}
\hat{f}_N(-j) &= \langle f, e^{i(-j)x}\rangle_N && \text{by (6)} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} f(x_k)e^{-i(-j)x_k} && \text{by (7)} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} f(x_k)e^{ijx_k} \\
&= \overline{\hat{f}_N(j)}. && \text{by (7)}
\end{aligned}$$

The corresponding trigonometric polynomial approximant of $f(x)$ has the form:

$$p(x) = \sum_{|j|<N/2} \hat{f}_N(j)e^{ijx} + \mathrm{Re}\left(\hat{f}_N(N/2)e^{i(N/2)x}\right)$$

$$= \hat{f}_N(0) + 2\,\mathrm{Re}\left(\sum_{j=1}^{N/2-1} \hat{f}_N(j)e^{ijx}\right) + \mathrm{Re}\left(\hat{f}_N(N/2)e^{i(N/2)x}\right).$$

The last term is present only when N is even. Having mentioned it for completeness's sake, we will now discuss the case when N is odd, that, is $N = 2n + 1$ for some integer n. In this case, the N functions $1, e^{\pm ix}, \ldots, e^{\pm inx}$ are orthonormal with respect to the discrete inner product $\langle\ ,\ \rangle_N$. By virtually the same reasoning of least-squares approximation by orthogonal polynomials, we have the following theorem.

Theorem 3 *For any $m \le n = \frac{N-1}{2}$, the mth order trigonometric polynomial*

$$p_m(x) = \sum_{j=-m}^{m} \hat{f}_N(j)e^{ijx}$$

is the best approximation to $f(x)$ by trigonometric polynomials of order m with respect to the discrete mean-square norm

$$\|g\|_2 = \Big(\langle g, g\rangle_N\Big)^{\frac{1}{2}} = \frac{1}{N}\left(\sum_{k=0}^{N-1}\left|g\Big(\frac{2\pi k}{N}\Big)\right|^2\right)^{\frac{1}{2}}.$$

3 Fast Computation — FFT

We are interested in the frequencies present in $f(x)$ and their strength (or magnitude). But due to aliasing, $\hat{f}_N(j)$, defined in (7), is good as an approximation to $f(j)$ only for $-\frac{N}{2} < j \le \frac{N}{2}$. Thus, for very large N we want to be able to calculate $\hat{f}_N(j)$, for $0 \le |j| \le \frac{N}{2}$, or equivalently, for $0 \le j \le N-1$, from $f(x_0), \ldots, f(x_{N-1})$, where $x_k = \frac{2\pi k}{N}$. A straightforward calculation would take time $O(N^2)$.

A significant improvement can be achieved by reducing the above problem to a discrete Fourier transform (DFT). DFT is the mapping

$$\boldsymbol{z} = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{pmatrix} \quad \longmapsto \quad \hat{\boldsymbol{z}} = \begin{pmatrix} \hat{z}_0 \\ \hat{z}_1 \\ \vdots \\ \hat{z}_{N-1} \end{pmatrix}$$

such that

$$\hat{z}_j = \sum_{k=0}^{N-1} z_k \omega_N^{jk}, \qquad j = 0, \ldots, N-1,$$

where ω_N is an Nth root of 1, that is, $\omega_N = e^{-i\frac{2\pi}{N}}$. The mapping can also be written as a matrix equation:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)^2} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{pmatrix} = \begin{pmatrix} \hat{z}_0 \\ \hat{z}_1 \\ \vdots \\ \hat{z}_{N-1} \end{pmatrix}.$$

If we take $z_j = f(x_j)$, $0 \le j \le N-1$, then

$$\hat{f}_N(j) = \frac{1}{N}\hat{z}_j, \qquad j = 0, 1, \ldots, N-1.$$

To verify, by definition (7) we have

$$\begin{aligned}
\hat{f}_N(j) &= \frac{1}{N}\sum_{k=0}^{N-1} f(x_k)e^{-ijx_k} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} f(x_k)e^{-ij\frac{2\pi k}{N}} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} f(x_k)\left(\omega_N^j\right)^k \\
&= \frac{1}{N}\hat{z}_j.
\end{aligned}$$

Fast Fourier transform allows us to compute the discrete Fourier coefficients in time $O(N \log N)$. With $N = 10^6$, for example, the improvement is from roughly two weeks of CPU time to 30 seconds!

References

[1] S. D. Conte and C de Boor. *Elementary Numerical Analysis: An Algorithmic Approach.* McGraw-Hill, Inc., 3rd edition, 1980.

[2] M. Erdmann. Lecture notes for *16-811 Mathematical Fundamentals for Robotics.* The Robotics Institute, Carnegie Mellon University, 1998.

[3] W. H. Press, *et al.* *Numerical Recipes in C++: The Art of Scientific Computing.* Cambridge University Press, 2nd edition, 2002.