Dual Quaternions

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1 Introduction

As we know, quaternions are very efficient for representing rotations with clear geometric meaning (rotation axis and angle) and only one redundancy. Unfortunately, they do not handle translations, which meanwhile can be made multiplicative along with rotations via the use of homogeneous coordinates. Despite also being 4-tuples, homogeneous coordinates are algebraically incompatible with quaternions. In 1873, dual quaternions were introduced by William Kingdom Clifford [1] in an effort to combine rotations and translations while retaining the benefits of the quaternion representation of rotations.

2 Quaternion

A *quaternion* q is defined to be the sum of a scalar q_0 and a vector $\boldsymbol{q} = (q_1, q_2, q_3)$; namely,

$$q = q_0 + \boldsymbol{q} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k},$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors along the x-, y-, z-axes, respectively. The quaternion can also be viewed as a 4-tuple (q_0, q_1, q_2, q_3). A vector $\boldsymbol{v}$ is called a *pure quaternion* in the form of $0 + \boldsymbol{v}$.

Addition of the quaternion q to another quaternion $p = p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}$ acts in a component-wise way:

$$p + q = (p_0 + q_0) + (p_1 + q_1)\boldsymbol{i} + (p_2 + q_2)\boldsymbol{j} + (p_3 + q_3)\boldsymbol{k}. \quad (1)$$

Multiplication of the two quaternions is carried out as follows:

$$pq = p_0q_0 - \boldsymbol{p} \cdot \boldsymbol{q} + p_0\boldsymbol{q} + q_0\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{q}. \quad (2)$$

The *conjugate* of q is the quaternion

$$q^* = q_0 - \boldsymbol{q} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}$$

such that

$$qq^* = q^*q = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

It is easy to verify that the conjugate of a product quaternion is the product of the conjugates of its factor quaternions:

$$(pq)^* = q^*p^*.$$

The *norm* of q is $|q| = \sqrt{qq^*}$, and its *inverse* is

$$q^{-1} = \frac{q^*}{|q|^2}.$$

The quaternion q is a *unit quaternion* if $|q| = 1$. In this case, the inverse of q is simply its conjugate, i.e., $q^{-1} = q^*$. Furthermore, q can be written as

$$q = \cos\frac{\theta}{2} + \hat{\boldsymbol{u}}\sin\frac{\theta}{2}, \tag{3}$$

where $\theta \in [0, \pi]$ and $\hat{\boldsymbol{u}}$ is a unit vector.

Given the unit quaternion q above and a vector $\boldsymbol{v}$, the quaternion product $q\boldsymbol{v}q^*$ is the vector that results from rotating $\boldsymbol{v}$ through the angle θ about an axis though the origin in the direction $\hat{\boldsymbol{u}}$. Thus, a unit quaternion encodes the axis and angle of some rotation, which can be trivially recovered from the quaternion itself.

Suppose p is another unit quaternion. To the vector $\boldsymbol{v}$ if we first apply the rotation represented by q and then the rotation represented by p, the resulting vector is

$$\boldsymbol{v}' = p(q\boldsymbol{v}q^*)p^* = (pq)\boldsymbol{v}(pq)^*.$$

Thus, the effect is essentially the same as that of applying the rotation represented by the product quaternion pq.

The power of a unit quaternion $q = \cos\theta + \hat{\boldsymbol{u}}\sin\theta$ is defined in a way similar to that of a complex number:

$$q^\rho = \cos(\rho\theta) + \hat{\boldsymbol{u}}\sin(\rho\theta), \qquad \rho \in \mathbb{R}. \tag{4}$$

Intuitively, it represents the rotation about the same axis through the angle $\rho\theta$.

3 Dual Numbers

Dual numbers were first proposed by Clifford [1], and further developed by Study [14]. A dual number is defined to be

$$\bar{d} = a + \epsilon b,$$

where a and b are real numbers, or more generally, elements of a (algebraic) field, and ϵ is a *dual unit* with $\epsilon^2 = 0$. In the above, a and b are referred to as the *real* and *dual* parts of $\bar{d}$.

Addition and multiplication act on two dual numbers $\bar{d}_i = a_i + \epsilon b_i$, $i = 1, 2$, as follows:

$$\begin{aligned} \bar{d}_1 + \bar{d}_2 &= (a_1 + a_2) + \epsilon(b_1 + b_2), & (5)\\ \bar{d}_1 \otimes \bar{d}_2 &= a_1a_2 + \epsilon a_1b_2 + \epsilon a_2b_1 + \epsilon^2 a_2b_2 \\ &= a_1a_2 + \epsilon(a_1b_2 + a_2b_1). & (6) \end{aligned}$$

Under the above arithmetic rules, a dual number $\bar{d} = a + \epsilon b$ with $a \neq 0$ has an inverse

$$\bar{d}^{-1} = a^{-1}\left(1 - \epsilon b a^{-1}\right). \tag{7}$$

If $a = 0$, then $\bar{d} = \epsilon b$ has no inverse. Thus, algebraically dual numbers form a ring but not a field.

A dual vector $\bar{\boldsymbol{d}} = (\bar{d}_1, \bar{d}_2, \bar{d}_3)$ has its every entry being a dual number. The product of a dual number $\bar{d}$ with $\bar{\boldsymbol{d}}$ is a dual vector obtained under the multiplication rule (14):

$$\bar{d}\bar{\boldsymbol{d}} = (\bar{d} \otimes \bar{d}_1, \bar{d} \otimes \bar{d}_2, \bar{d} \otimes \bar{d}_3).$$

The inner and cross products of $\bar{\boldsymbol{d}}$ with another dual vector $\bar{\boldsymbol{e}} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ can be defined similarly as those of two vectors:

$$\bar{\boldsymbol{d}} \cdot \bar{\boldsymbol{e}} = \bar{d}_1 \otimes \bar{e}_1 + \bar{d}_2 \otimes \bar{e}_2 + \bar{d}_3 \otimes \bar{e}_3, \tag{8}$$

$$\bar{\boldsymbol{d}} \times \bar{\boldsymbol{e}} = \begin{pmatrix} \bar{d}_2 \otimes \bar{e}_3 - \bar{d}_3 \otimes \bar{e}_2 \\ \bar{d}_3 \otimes \bar{e}_1 - \bar{d}_1 \otimes \bar{e}_3 \\ \bar{d}_1 \otimes \bar{e}_2 - \bar{d}_2 \otimes \bar{e}_1 \end{pmatrix}^T . \tag{9}$$

A function of the dual number $a + \epsilon b$ can be defined using the Taylor expansion at its real part:

$$\begin{aligned} f(a + \epsilon b) &= f(a) + \epsilon b f'(a) + \frac{1}{2}\epsilon^2 b^2 f''(a) + \cdots \\ &= f(a) + \epsilon b f'(a), \end{aligned} \tag{10}$$

since all powers of ϵ greater than one vanish.

4 Dual Quaternion Arithmetics

Dual quaternion algebra is an extension of the dual-number theory by Clifford [1] in an effort to combine with Hamilton's quaternion algebra [7]. It provides an elegant way of solving a range of problems that are otherwise complex. Rigid transformations, in particular, can be each represented with eight scalar variables, and combined through concatenation. Smooth constant interpolation between two rigid transformations can also be achieved effortlessly.

A dual quaternion σ has the form

$$\sigma = p + \epsilon q, \tag{11}$$

where p and q are quaternions given below:

$$\begin{aligned} p &= p_0 + \boldsymbol{p} = p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}, \\ q &= q_0 + \boldsymbol{q} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}. \end{aligned}$$

The quaternions p and q are the real and dual part of σ, respectively. We can also view σ as an 8-tuple:

$$\sigma = (p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3). \tag{12}$$

Addition (5) and multiplication (6) of two dual numbers generalize to two dual quaternions $\sigma_1 = p_1 + \epsilon q_1$ and $\sigma_2 = p_2 + \epsilon q_2$ as follows:

$$\sigma_1 + \sigma_2 = (p_1 + p_2) + \epsilon(q_1 + q_2), \tag{13}$$

$$\sigma_1 \otimes \sigma_2 = p_1 p_2 + \epsilon(p_1 q_2 + q_1 p_2). \tag{14}$$

A dual quaternion $\sigma = p + \epsilon q$ with $p \neq 0$ has an inverse $p^{-1}(1 - \epsilon q p^{-1})$ with the form inherited from that of the inverse of a dual number.

The dual quaternion σ can also be viewed as the sum of a dual number $\bar{d} = p_0 + \epsilon q_0$ and a dual vector $\bar{\boldsymbol{d}} = \boldsymbol{p} + \epsilon \boldsymbol{q}$. Here, $\bar{d}$ and $\bar{\boldsymbol{d}}$ are referred to as the dual scalar part and dual vector part of σ, respectively:

$$\sigma = \bar{d} + \bar{\boldsymbol{d}}. \tag{15}$$

This representation interprets σ as a "quaternion" whose four terms are dual numbers. Rules of the quaternion algebra carry over under the condition that $\epsilon^2 = 0$. For example, it is not difficult (though somewhat tedious) to verify that the multiplication rule (14), applied to two dual quaternions $\sigma_i = (\bar{d}_i, \bar{\boldsymbol{d}}_i)$, $i = 1, 2$, can be written equivalently as follows:

$$
\begin{aligned}
\sigma_1 \otimes \sigma_2 &= (\bar{d}_1 + \bar{\boldsymbol{d}}_1)(\bar{d}_2 + \bar{\boldsymbol{d}}_2) \\
&= (\bar{d}_1 \otimes \bar{d}_2 - \bar{\boldsymbol{d}}_1 \cdot \bar{\boldsymbol{d}}_2) + (\bar{d}_1 \bar{\boldsymbol{d}}_2 + \bar{d}_2 \bar{\boldsymbol{d}}_1 + \bar{\boldsymbol{d}}_1 \times \bar{\boldsymbol{d}}_2),
\end{aligned}
$$

where the inner and cross products of two dual vectors are given in (8) and (9). There is an apparent parallel between the above and multiplication of two quaternions given in (2).

Just as numbers and vectors are special cases of quaternions, numbers, vectors, dual numbers, dual numbers, dual vectors, and quaternions are all special cases of dual quaternions.

5 Three Conjugates

If we view a dual quaternion $\sigma = p + \epsilon q$ as just an 8-tuple, its conjugate could be a vector generated from σ by flipping the signs of some of its elements. The dual quaternion indeed has three conjugates, the first of which is from the conjugate of a dual number:

$$\sigma^{\bullet} = p - \epsilon q. \tag{16}$$

Or, viewed as an 8-tuple, it is

$$\sigma^{\bullet} = (p_0, p_1, p_2, p_3, -q_0, -q_1, -q_2, -q_3).$$

In this sense, the product of σ with $\sigma^{\bullet}$ ought to be treated as an inner product to yield a scalar. This is not the case, however, if under the multiplication rule (14), for we have

$$
\begin{aligned}
\sigma \otimes \sigma^{\bullet} &= (p + \epsilon q)(p - \epsilon q) \\
&= pp + \epsilon(qp - pq).
\end{aligned}
$$

Generally, the quaternion product pp is not a scalar, and $qp \neq pq$ following the non-commutativity of quaternion multiplication. The first conjugate is seldom used other than for deriving the third conjugate to be introduced shortly.

The second conjugate of σ follows from the classical quaternion conjugation:

$$\sigma^* = p^* + \epsilon q^*, \tag{17}$$

where p^* and q^* are the conjugates of the quaternions p and q, respectively. The conjugate σ^* can also be viewed as an 8-tuple:

$$\sigma^* = (p_0, -p_1, -p_2, -p_3, q_0, -q_1, -q_2, -q_3).$$

Let us calculate the product of σ with σ^* below:

$$
\begin{aligned}
\sigma \otimes \sigma^* &= (p + \epsilon q)(p^* + \epsilon q^*) \\
&= pp^* + \epsilon(pq^* + qp^*) \\
&= (p_0^2 + p_1^2 + p_2^2 + p_3^2) + 2\epsilon(p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3). \\
&= \|p\|^2 + 2\epsilon(p_0 q_0 + \boldsymbol{p} \cdot \boldsymbol{q}).
\end{aligned} \tag{18}
$$

It is a dual scalar but not a scalar unless $p_0 q_0 + \boldsymbol{p} \cdot \boldsymbol{q} = 0$, that is, unless p and q are orthogonal 4-tuples.

The third conjugate of σ combines the first two conjugation operators:

$$\sigma^{\diamond} = p^* - \epsilon q^*. \tag{19}$$

As an 8-tuple, it takes the form

$$\sigma^{\diamond} = (p_0, -p_1, -p_2, -p_3, -q_0, q_1, q_2, q_3).$$

Similarly, we calculate the product of σ with the above conjugate:

$$\begin{aligned} \sigma \otimes \sigma^{\diamond} &= (p + \epsilon q)(p^* - \epsilon q^*) \\ &= pp^* + \epsilon(qp^* - pq^*). \end{aligned}$$

The product is a dual quaternion whose real part is a scalar and dual part is a vector.

For each of the three introduced conjugates, the conjugate of the conjugate of σ is σ itself. The conjugate of the product of dual quaternions equals the product of the individual conjugates of these dual quaternions in the reverse order. This is established below for all three conjugates:

$$\begin{aligned}
(\sigma_1 \otimes \sigma_2)^{\bullet} &= \Big(p_1 p_2 + \epsilon(p_1 q_2 + q_1 p_2)\Big)^{\bullet} \qquad \text{(by (14))} \\
&= p_1 p_2 - \epsilon(p_1 q_2 + q_1 p_2) \\
&= (p_1 - \epsilon q_1)(p_2 - \epsilon q_2) \\
&= \sigma_1^{\bullet} \otimes \sigma_2^{\bullet}; \\
(\sigma_1 \otimes \sigma_2)^{*} &= \Big(p_1 p_2 + \epsilon(p_1 q_2 + q_1 p_2)\Big)^{*} \\
&= (p_1 p_2)^* + \epsilon(p_1 q_2 + q_1 p_2)^* \\
&= (p_1 p_2)^* + \epsilon\Big((p_1 q_2)^* + (q_1 p_2)^*\Big) \\
&= p_2^* p_1^* + \epsilon(q_2^* p_1^* + p_2^* q_1^*) \\
&= (p_2^* + \epsilon q_2^*)(p_1^* + \epsilon q_1^*) \\
&= \sigma_2^* \otimes \sigma_1^*; \\
(\sigma_1 \otimes \sigma_2)^{\diamond} &= \Big(p_1 p_2 + \epsilon(p_1 q_2 + q_1 p_2)\Big)^{\diamond} \\
&= (p_1 p_2)^* - \epsilon(p_1 q_2 + q_1 p_2)^* \\
&= (p_1 p_2)^* - \epsilon\Big((p_1 q_2)^* + (q_1 p_2)^*\Big) \\
&= p_2^* p_1^* - \epsilon(q_2^* p_1^* + p_2^* q_1^*) \\
&= (p_2^* - \epsilon q_2^*)(p_1^* - \epsilon q_1^*) \\
&= \sigma_2^{\diamond} \otimes \sigma_1^{\diamond}.
\end{aligned}$$

6 Unit Dual Quaternion

A dual quaternion $\sigma = p + \epsilon q$ is *unit* if $\sigma \otimes \sigma^* = 1$. By (18), it satisfies the following pair of conditions:

$$p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1, \tag{20}$$

$$p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3 = 0. \tag{21}$$

Namely, the real part p must be a unit quaternion, and it must be orthogonal to the dual part q as 4-tuples. The inverse of σ is thus σ^*.

The conditions (20) and (21) imposed on σ reduce the number of degrees of freedom (DOFs) from eight to six, exactly the number of DOFs of a rigid body in 3D.

Unit dual quaternions constitute a six-dimensional submanifold of $\mathbb{R}^8$, called the *image space of spatial displacements* [11, p. 65]. We can visualize this manifold in $\mathbb{R}^4$ as follows. Equation (20) defines a unit hypersphere of three dimensions, and equation (21) defines the three-dimensional hyperplane perpendicular to the normal at the point (p_0, p_1, p_2, p_3) on the hypersphere. Thus, the image space consists of the hypersphere and all of its tangent spaces (which have been translated to contain the origin).

7 Rigid Displacement

Let $r = \cos\frac{\theta}{2} + \hat{\boldsymbol{u}}\sin\frac{\theta}{2}$ be a quaternion that represents a rotation about the unit vector $\hat{\boldsymbol{u}}$ through θ. Its conjugate is $r^* = \cos\frac{\theta}{2} - \hat{\boldsymbol{u}}\sin\frac{\theta}{2}$ such that $rr^* = r^*r = 1$. Denote by R the corresponding rotation matrix. Let $\boldsymbol{t} = (t_1, t_2, t_3)$ be a translation. A point $\boldsymbol{v}$ under the rotation R followed by the translation $\boldsymbol{t}$ becomes the point $R\boldsymbol{v} + \boldsymbol{t}$.

The transformation sequence $R, \boldsymbol{t}$ can be compactly represented by a dual quaternion. The translation vector $\boldsymbol{t}$ is a pure quaternion. We combine it with the rotation quaternion r into the following dual quaternion:

$$\begin{aligned}\sigma &= r + \frac{\epsilon}{2}\boldsymbol{t}r \qquad (22)\\ &= \cos\frac{\theta}{2} + \hat{\boldsymbol{u}}\sin\frac{\theta}{2} + \frac{\epsilon}{2}\left(-\sin\frac{\theta}{2}(\boldsymbol{t}\cdot\hat{\boldsymbol{u}}) + \cos\frac{\theta}{2}\boldsymbol{t} + \sin\frac{\theta}{2}\boldsymbol{t}\times\hat{\boldsymbol{u}}\right).\end{aligned}$$

If the transformation is a pure rotation, i.e., $\boldsymbol{t} = 0$, we end up with $\sigma = r$. If the transformation is a pure translation, that is, $\theta = 0$, we end up with $\sigma = 1 + \frac{\epsilon}{2}\boldsymbol{t}$.

It is easy to verify that the dual quaternion (22) is a unit one:

$$\begin{aligned}\sigma\otimes\sigma^* &= \left(r + \frac{\epsilon}{2}\boldsymbol{t}r\right)\left(r^* + \frac{\epsilon}{2}(\boldsymbol{t}r)^*\right)\\ &= \left(r + \frac{\epsilon}{2}\boldsymbol{t}r\right)\left(r^* + \frac{\epsilon}{2}r^*\boldsymbol{t}^*\right)\\ &= rr^* + \frac{\epsilon}{2}(rr^*\boldsymbol{t}^* + \boldsymbol{t}rr^*)\\ &= 1 + \frac{\epsilon}{2}(\boldsymbol{t}^* + \boldsymbol{t})\\ &= 1,\end{aligned}$$

where the last step made use of the fact that $\boldsymbol{t}$ is a pure quaternion with $\boldsymbol{t}^* = -\boldsymbol{t}$.

From a unit dual quaternion $\sigma = p + \epsilon q$, we can easily extract the encapsulated rotation r and translation $\boldsymbol{t}$ in quaternion form:

$$\begin{aligned}r &= p, \qquad (23)\\ \boldsymbol{t} &= 2qp^*. \qquad (24)\end{aligned}$$

Equation (24) is derived from multiplying $q = \frac{1}{2}\boldsymbol{t}r$ by r^* and then applying (23).

To be operated on by the dual quaternion σ, a point $\boldsymbol{v}$ as a pure quaternion $0+\boldsymbol{v}$ needs to be converted into a dual quaternion $1+\epsilon\boldsymbol{v}$. The transformation formula is defined as a sequence of two multiplications much like the one for the rotation of a vector specified by a quaternion:

$$\begin{aligned}
\sigma \otimes (1+\epsilon\boldsymbol{v}) \otimes \sigma^{\diamond} &= \left(r + \frac{\epsilon}{2}\boldsymbol{t}r\right) \otimes (1+\epsilon\boldsymbol{v}) \otimes \left(r^* - \frac{\epsilon}{2}(\boldsymbol{t}r)^*\right) \\
&= \left(r + \frac{\epsilon}{2}\boldsymbol{t}r + \epsilon r\boldsymbol{v}\right) \otimes \left(r^* - \frac{\epsilon}{2}r^*\boldsymbol{t}^*\right) \\
&= rr^* + \epsilon\left(\frac{1}{2}(\boldsymbol{t}rr^* - rr^*\boldsymbol{t}^*) + r\boldsymbol{v}r^*\right) \\
&= 1 + \epsilon\left(\frac{1}{2}(\boldsymbol{t} - \boldsymbol{t}^*) + r\boldsymbol{v}r^*\right) \\
&= 1 + \epsilon(r\boldsymbol{v}r^* + \boldsymbol{t}). \qquad (25)
\end{aligned}$$

Note that $r\boldsymbol{v}r^*$ is a pure quaternion representing the vector $R\boldsymbol{v}$ after the rotation of $\boldsymbol{v}$. The resulting dual quaternion $1+\epsilon(r\boldsymbol{v}r^*+\boldsymbol{t})$ thus encapsulates the vector $R\boldsymbol{v}+\boldsymbol{t}$.

EXAMPLE 1. Consider a point $\boldsymbol{v} = (a,0,0)$ on the x-axis. Suppose we rotate $\boldsymbol{v}$ about an axis defined by $(1,1,1)$ through an angle of $2\pi/3$, and then translate the resulting point by $(0,0,b)$. The rotation is defined by the unit quaternion

$$\begin{aligned}
r &= \cos\frac{\pi}{3} + \frac{1}{\sqrt{3}}(1,1,1)\sin\frac{\pi}{3} \\
&= \frac{1}{2} + \frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}.
\end{aligned}$$

The point is represented by the quaternion $a\boldsymbol{i}$. The result of rotation is the quaternion product

$$\begin{aligned}
r\boldsymbol{v}r^* &= \left(\frac{1}{2} + \frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}\right) a\boldsymbol{i} \left(\frac{1}{2} - \frac{1}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} - \frac{1}{2}\boldsymbol{k}\right) \\
&= \frac{a}{4}(-1+\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k})(1-\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}) \\
&= \frac{a}{4}(4\boldsymbol{j}) \\
&= a\boldsymbol{j}.
\end{aligned}$$

After the rotation, the point becomes $(0,a,0)$, which will then be translated to $(0,a,b)$.

It is easy to verify that dual quaternion will yield the same result. The quaternion $\boldsymbol{t} = b\boldsymbol{k}$ represents the intended translation. The dual quaternion $\sigma = r + \frac{\epsilon}{2}\boldsymbol{t}r$ represents the transformation that starts with the rotation and follows with the translation. We make use of (25):

$$\begin{aligned}
\sigma \otimes (1+\epsilon\boldsymbol{v}) \otimes \sigma^{\diamond} &= 1 + \epsilon(r\boldsymbol{v}r^* + \boldsymbol{t}) \\
&= 1 + \epsilon(a\boldsymbol{j} + b\boldsymbol{k}).
\end{aligned}$$

If the translation $\boldsymbol{t}$ is applied before the rotation, the combined transformation is represented by the dual quaternion

$$\zeta = r + \frac{\epsilon}{2}r\boldsymbol{t}. \qquad (26)$$

The resulting dual quaternion can be derived similarly:

$$\zeta \otimes (1+\epsilon\boldsymbol{v}) \otimes \zeta^{\diamond} = \left(r + \frac{\epsilon}{2}r\boldsymbol{t}\right) \otimes (1+\epsilon\boldsymbol{v}) \otimes \left(r^* - \frac{\epsilon}{2}\epsilon(r\boldsymbol{t})^*\right)$$

$$
\begin{aligned}
&= \left(r + \frac{\epsilon}{2} r\boldsymbol{t} + \epsilon r\boldsymbol{v}\right) \otimes \left(r^* - \frac{\epsilon}{2}\boldsymbol{t}^* r^*\right) \\
&= rr^* + \epsilon\left(\frac{1}{2}(r\boldsymbol{t}r^* - r\boldsymbol{t}^* r^*) + r\boldsymbol{v}r^*\right) \\
&= 1 + \epsilon\,(r\boldsymbol{t}r^* + r\boldsymbol{v}r^*) \\
&= 1 + \epsilon r(\boldsymbol{v} + \boldsymbol{t})r^*. \qquad (27)
\end{aligned}
$$

Clearly, the dual quaternion (27) encapsulates the new location $R(\boldsymbol{v} + \boldsymbol{t})$.

Since the dual quaternion $\sigma = r + \frac{\epsilon}{2}\boldsymbol{t}r$ is unit, is inverse is

$$
\begin{aligned}
\sigma^* &= r^* + \frac{\epsilon}{2}(\boldsymbol{t}r)^* \\
&= r^* + \frac{\epsilon}{2} r^* \boldsymbol{t}^* \\
&= r^* - \frac{\epsilon}{2} r^* \boldsymbol{t} \qquad (28) \\
&= r^* + \frac{\epsilon}{2} r^*(-\boldsymbol{t}).
\end{aligned}
$$

Comparing it with ζ in (26), we see that σ^* represents a translation $-\boldsymbol{t}$ followed by a rotation R^{-1} represented by r^*. This is indeed the inverse of the rigid body transformation represented by σ. Similarly, the inverse dual quaternion $\zeta^* = r^* + \frac{\epsilon}{2}(-\boldsymbol{t})r^*$ represents a rotation R^{-1} followed by a translation $-\boldsymbol{t}$; that is, the inverse transformation of that represented by ζ.

Suppose that the vector $\boldsymbol{v}$ in its dual quaternion form $1 + \epsilon\boldsymbol{v}$ is under a sequence of rigid transformations represented by the dual quaternions $\sigma_1, \sigma_2, \ldots, \sigma_n$. The resulting vector is encapsulated in the dual quaternion:

$$
\begin{aligned}
1 + \epsilon\boldsymbol{v}' &= \sigma_n \otimes (\sigma_{n-1} \otimes \cdots \otimes (\sigma_1 \otimes (1 + \epsilon\boldsymbol{v}) \otimes \sigma_1^\diamond) \otimes \cdots \otimes \sigma_{n-1}^\diamond) \otimes \sigma_n^\diamond \\
&= (\sigma_n \otimes \cdots \otimes \sigma_1) \otimes (1 + \epsilon\boldsymbol{v}) \otimes (\sigma_n \otimes \cdots \otimes \sigma_1)^\diamond.
\end{aligned}
$$

We denote the product dual quaternion as $\sigma = \sigma_n \otimes \cdots \otimes \sigma_1$. The effect is equivalent to a single rigid transformation represented by σ; namely,

$$
1 + \epsilon\boldsymbol{v}' = \sigma \otimes (1 + \epsilon\boldsymbol{v}) \otimes \sigma^\diamond.
$$

The dual quaternion $\sigma = r + \frac{\epsilon}{2}\boldsymbol{t}r$ itself can be rewritten as the product of two dual quaternions:

$$
\sigma = \left(1 + \frac{\epsilon}{2}\boldsymbol{t}\right) r.
$$

The above form states exactly that σ is a composition of a rotation, represented by the quaternion r (a dual quaternion without the dual part), and a translation, represented by $1 + \frac{\epsilon}{2}\boldsymbol{t}$.

8 Screw Coordinates and Dual Quaternions

In this section we will first look at how to carry out rigid transformations of lines using dual quaternions. We will show that screw displacements can be elegantly represented by dual quaternions. It is known that any rigid displacement is equivalent to a screw motion.

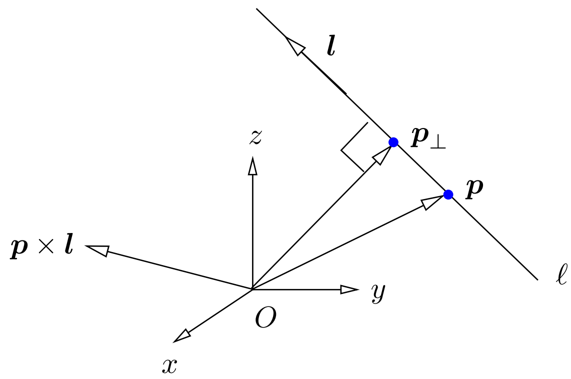

Figure 1: Plücker coordinates $(\boldsymbol{l}, \boldsymbol{p} \times \boldsymbol{l})$ of a line.

8.1 Line Transformation

As shown in Figure 1, a line ℓ in the space is determined by its direction $\boldsymbol{l}$ and a point $\boldsymbol{p}$ that it passes through. The *Plücker coordinates* of the line is defined to be $(\boldsymbol{l}, \boldsymbol{m})$, where $\boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{l}$. These coordinates are independent of the choice of $\boldsymbol{p}$, since a different point $\boldsymbol{p}'$ on the line has

$$\begin{aligned} \boldsymbol{p}' \times \boldsymbol{l} &= (\boldsymbol{p} + (\boldsymbol{p}' - \boldsymbol{p})) \times \boldsymbol{l} \\ &= \boldsymbol{m} + \lambda \boldsymbol{l} \times \boldsymbol{l} \qquad \text{(for some } \lambda) \\ &= \boldsymbol{m}. \end{aligned}$$

The six coordinates in $(\boldsymbol{l}, \boldsymbol{m})$ have two redundancies. Since $\boldsymbol{m}$ scales with $\boldsymbol{l}$, $(\boldsymbol{l}, \boldsymbol{m})$ and $(c\boldsymbol{l}, c\boldsymbol{m})$, $c \neq 0$, describe the same line. We can choose $\boldsymbol{l}$ to be a unit vector. Second, $\boldsymbol{l} \cdot \boldsymbol{m} = \boldsymbol{l} \cdot (\boldsymbol{p} \times \boldsymbol{l}) = 0$. The constraints $\|\boldsymbol{l}\| = 1$ and $\boldsymbol{l} \cdot \boldsymbol{m} = 0$ essentially remove two variables from $(\boldsymbol{l}, \boldsymbol{m})$, reflecting the four degrees of freedom of a line in the space.

From now on, we will write the Plücker coordinates as $(\hat{\boldsymbol{l}}, \boldsymbol{m})$ with $\hat{\boldsymbol{l}}$ being a unit vector. Under such choice, $\boldsymbol{m} = \boldsymbol{p} \times \hat{\boldsymbol{l}}$ is called the *moment of the line*. It is the moment of a unit force acting at $\boldsymbol{p}$ in the direction $\hat{\boldsymbol{l}}$ with respect to the origin. The norm $\|\boldsymbol{m}\|$ gives the distance from the origin to the line, achieved at its closest point $\boldsymbol{p}_\perp$ on the line:

$$\begin{aligned} \boldsymbol{p}_\perp &= (\hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{l}})\boldsymbol{p} - (\hat{\boldsymbol{l}} \cdot \boldsymbol{p})\hat{\boldsymbol{l}} \\ &= \hat{\boldsymbol{l}} \times (\boldsymbol{p} \times \hat{\boldsymbol{l}}) \\ &= \hat{\boldsymbol{l}} \times \boldsymbol{m}. \end{aligned}$$

A line ℓ in the Plücker coordinates $(\hat{\boldsymbol{l}}, \boldsymbol{m})$ is represented by the dual quaternion $l = \hat{\boldsymbol{l}} + \epsilon \boldsymbol{m}$.

Theorem 1 *Suppose a line ℓ_1 undergoes a rotation represented by the quaternion r followed by a translation $\boldsymbol{t}$. It is transformed into a line ℓ_2. Let $l_i = \hat{\boldsymbol{l}}_i + \epsilon \boldsymbol{m}_i$, $i = 1, 2$, be the dual quaternion describing the line ℓ_i. Then*

$$l_2 = \sigma \otimes l_1 \otimes \sigma^*, \tag{29}$$

where $\sigma = r + \frac{\epsilon}{2} r \boldsymbol{t}$.

Proof The proof is adapted from that given by Daniilidis [2]. A point $\boldsymbol{p}$ on l_1 under the overall transformation becomes the point $\boldsymbol{p}' = r\boldsymbol{p}r^* + \boldsymbol{t}$. We have

$$\begin{aligned}
\hat{\boldsymbol{l}}_2 &= r\hat{\boldsymbol{l}}_1 r^*, && (30)\\
\boldsymbol{m}_2 &= \boldsymbol{p}' \times \hat{\boldsymbol{l}}_2 \\
&= (r\boldsymbol{p}r^* + \boldsymbol{t}) \times (r\hat{\boldsymbol{l}}_1 r^*) \\
&= (r\boldsymbol{p}r^*) \times (r\hat{\boldsymbol{l}}_1 r^*) + \boldsymbol{t} \times (r\hat{\boldsymbol{l}}_1 r^*). && (31)
\end{aligned}$$

The first cross product on the right hand side of (31) operates on two rotated vectors. The result is equal to applying the rotation to the their original cross product:

$$\begin{aligned}
(r\boldsymbol{p}r^*) \times (r\hat{\boldsymbol{l}}_1 r^*) &= r(\boldsymbol{p} \times \hat{\boldsymbol{l}}_1) r^* \\
&= r\boldsymbol{m}_1 r^*. && (32)
\end{aligned}$$

To handle the second cross product, we make use of the identity that the cross product of two pure quaternions $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{a} \times \boldsymbol{b} = \frac{1}{2}(\boldsymbol{b}\boldsymbol{a}^* + \boldsymbol{a}\boldsymbol{b})$. Namely, we have

$$\boldsymbol{t} \times (r\hat{\boldsymbol{l}}_1 r^*) = \frac{1}{2}\left(r\hat{\boldsymbol{l}}_1 r^* \boldsymbol{t}^* + \boldsymbol{t} r\hat{\boldsymbol{l}}_1 r^*\right). \quad (33)$$

Substitute (32) and (33) into (31):

$$\boldsymbol{m}_2 = r\boldsymbol{m}_1 r^* + \frac{1}{2}\left(r\hat{\boldsymbol{l}}_1 r^* \boldsymbol{t}^* + \boldsymbol{t} r\hat{\boldsymbol{l}}_1 r^*\right). \quad (34)$$

Next, substitute the above and (30) into the expression of l_2:

$$\begin{aligned}
l_2 &= \hat{\boldsymbol{l}}_2 + \epsilon \boldsymbol{m}_2 \\
&= r\hat{\boldsymbol{l}}_1 r^* + \epsilon\left(r\boldsymbol{m}_1 r^* + \frac{1}{2}\left(r\hat{\boldsymbol{l}}_1 r^* \boldsymbol{t}^* + \boldsymbol{t} r\hat{\boldsymbol{l}}_1 r^*\right)\right). && (35)
\end{aligned}$$

Meanwhile, we have

$$\begin{aligned}
\sigma \otimes l_1 \otimes \sigma^* &= \left(r + \frac{\epsilon}{2}\boldsymbol{t}r\right) \otimes (\hat{\boldsymbol{l}}_1 + \epsilon \boldsymbol{m}_1) \otimes \left(r^* + \frac{\epsilon}{2} r^* \boldsymbol{t}^*\right) && \text{(by (22))}\\
&= \left(r\hat{\boldsymbol{l}}_1 + \epsilon r\boldsymbol{m}_1 + \frac{\epsilon}{2}\boldsymbol{t}r\hat{\boldsymbol{l}}_1\right) \otimes \left(r^* + \frac{\epsilon}{2} r^* \boldsymbol{t}^*\right) \\
&= r\hat{\boldsymbol{l}}_1 r^* + \frac{\epsilon}{2} r\hat{\boldsymbol{l}}_1 r^* \boldsymbol{t}^* + \epsilon r\boldsymbol{m}_1 r^* + \frac{\epsilon}{2}\boldsymbol{t} r\hat{\boldsymbol{l}}_1 r^*. && (36)
\end{aligned}$$

A comparison between (35) and (36) then establishes (29). □

Recall that transformation of a point $\boldsymbol{v}_1$ to another point $\boldsymbol{v}_2$ under the dual quaternion σ is carried out by the product

$$1 + \epsilon \boldsymbol{v}_2 = \sigma \otimes (1 + \epsilon \boldsymbol{v}_1) \otimes \sigma^\diamond,$$

in which the second product involves the third conjugate $\sigma^\diamond$. The transformation applied to the line l_2 has a similar form (29), except the second product involves the second conjugate σ^*. The difference is attributed to the different dual quaternion forms of a point and a line.

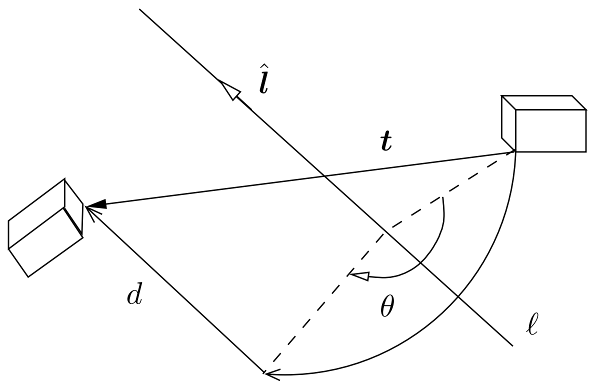

Figure 2: Any rigid displacement is equivalent to a rotation about some screw axis ℓ followed by a translation along the axis.

8.2 Screw Displacement

Chasles' theorem [10], attributed to Michel Chasles in 1830, states that any rigid displacement is equivalent to a rotation about some line, called the *screw axis*, followed by a translation in the direction of the line. The screw axis is represented in Plücker coordinates as $(\hat{\boldsymbol{l}}, \boldsymbol{m})$. See Figure 2. Let θ be the rotation angle and $d = \boldsymbol{t} \cdot \hat{\boldsymbol{l}}$ the distance of the subsequent translation in the axis direction $\hat{\boldsymbol{l}}$. The screw parameters include $\hat{\boldsymbol{l}}, \boldsymbol{m}, \theta, d$ with the ratio d/θ referred to as the *pitch* of the screw.[1]

Given a transformation represented by the dual quaternion $\sigma = p + \epsilon q$, we can recover the screw parameters as follows. First, from the quaternion p we extract the rotation axis $\hat{\boldsymbol{l}}$ and angle θ such that $p = \cos\frac{\theta}{2} + \hat{\boldsymbol{l}}\sin\frac{\theta}{2}$. Next, extract the translation vector $\boldsymbol{t}$ from σ according to (24). The distance of the translation along the screw axis is then found to be

$$\begin{aligned} d &= \boldsymbol{t} \cdot \hat{\boldsymbol{l}} \\ &= (2qp^*) \cdot \hat{\boldsymbol{l}}. \end{aligned}$$

Derivation of the moment vector requires a little effort. It is presented in [2] with the resulting form

$$\begin{aligned} \boldsymbol{m} &= \frac{1}{2}\left(\boldsymbol{t} \times \hat{\boldsymbol{l}} + \hat{\boldsymbol{l}} \times (\boldsymbol{t} \times \hat{\boldsymbol{l}})\cot\frac{\theta}{2}\right). \\ &= \frac{1}{2}\left(\boldsymbol{t} \times \hat{\boldsymbol{l}} + (\boldsymbol{t} - d\hat{\boldsymbol{l}})\cot\frac{\theta}{2}\right). \end{aligned} \tag{37}$$

When $\theta = 0$ or π, $\boldsymbol{m}$ goes to infinity and the screw axis is at infinity.

Conversely, given the screw parameters $\hat{\boldsymbol{l}}, \boldsymbol{m}, \theta, d$, we can construct the corresponding dual quaternion σ as follows. Multiply both sides of equation (37) with $\sin\frac{\theta}{2}$, and rewrite the resulting equation as

$$\sin\frac{\theta}{2}\hat{\boldsymbol{m}} + \frac{d}{2}\cos\frac{\theta}{2}\hat{\boldsymbol{l}} = \frac{1}{2}\left(\boldsymbol{t} \times \hat{\boldsymbol{l}}\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\boldsymbol{t}\right). \tag{38}$$

[1] We refer to [10] for more on screw coordinates.

Meanwhile, we have

$$
\begin{aligned}
\frac{1}{2}\boldsymbol{t}r &= \frac{1}{2}\boldsymbol{t}\left(\cos\frac{\theta}{2} + \hat{\boldsymbol{l}}\sin\frac{\theta}{2}\right) \\
&= -\frac{1}{2}\boldsymbol{t}\cdot\hat{\boldsymbol{l}}\sin\frac{\theta}{2} + \frac{1}{2}\boldsymbol{t}\cos\frac{\theta}{2} + \frac{1}{2}\boldsymbol{t}\times\hat{\boldsymbol{l}}\sin\frac{\theta}{2} \\
&= -\frac{d}{2}\sin\frac{\theta}{2} + \frac{1}{2}\left(\boldsymbol{t}\cos\frac{\theta}{2} + \boldsymbol{t}\times\hat{\boldsymbol{l}}\sin\frac{\theta}{2}\right) \\
&= -\frac{d}{2}\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\boldsymbol{m} + \frac{d}{2}\cos\frac{\theta}{2}\hat{\boldsymbol{l}},
\end{aligned}
$$

after substituting the left side of (38) in. The dual quaternion can now be assembled:

$$
\begin{aligned}
\sigma &= r + \frac{\epsilon}{2}\boldsymbol{t}r \\
&= \left(\cos\frac{\theta}{2} + \hat{\boldsymbol{l}}\sin\frac{\theta}{2}\right) + \epsilon\left(-\frac{d}{2}\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\boldsymbol{m} + \frac{d}{2}\cos\frac{\theta}{2}\hat{\boldsymbol{l}}\right) \\
&= \left(\cos\frac{\theta}{2} - \epsilon\frac{d}{2}\sin\frac{d}{2}\right) + \left(\hat{\boldsymbol{l}}\sin\frac{\theta}{2} + \epsilon\left(\sin\frac{\theta}{2}\boldsymbol{m} + \frac{d}{2}\cos\frac{\theta}{2}\hat{\boldsymbol{l}}\right)\right). \qquad (39)
\end{aligned}
$$

The form (39) is the sum of a dual scalar part and a dual vector part.

Simplification of (39) can be done by making use of the form (10) for a function of dual quaternion. Apply it to the sine and cosine functions at $\theta + \epsilon d$:

$$\cos\frac{\theta+\epsilon d}{2} = \cos\frac{\theta}{2} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}, \qquad (40)$$

$$\sin\frac{\theta+\epsilon d}{2} = \sin\frac{\theta}{2} + \epsilon\frac{d}{2}\cos\frac{\theta}{2}. \qquad (41)$$

Then (39) has a more compact form:

$$\sigma = \cos\frac{\theta+\epsilon d}{2} + \sin\frac{\theta+\epsilon d}{2}(\hat{\boldsymbol{l}} + \epsilon\boldsymbol{m}). \qquad (42)$$

In this new representation, the rotation angle θ about the screw axis $(\hat{\boldsymbol{l}}, \boldsymbol{m})$ and the translation distance d along the axis are separated from the screw axis itself.

Introducing the following dual angle and dual vector:

$$
\begin{aligned}
\bar{\theta} &= \theta + \epsilon d, \\
\bar{\hat{\boldsymbol{l}}} &= \hat{\boldsymbol{l}} + \epsilon\boldsymbol{m},
\end{aligned}
$$

we further rewrite (42) into a very succinct form:

$$\sigma = \cos\frac{\bar{\theta}}{2} + \bar{\hat{\boldsymbol{l}}}\sin\frac{\bar{\theta}}{2}, \qquad (43)$$

which closely resembles the form $\cos\frac{\theta}{2} + \hat{\boldsymbol{l}}\sin\frac{\theta}{2}$ of r that describes the rotation part of the transformation. Just like r is a unit quaternion, σ is a unit dual quaternion.

Another convenience provided by the form (43) is that it allows us to define the power of σ as follows:

$$\sigma^\tau = \cos\frac{\tau\bar{\theta}}{2} + \bar{\hat{\boldsymbol{l}}}\sin\frac{\tau\bar{\theta}}{2}, \qquad \tau > 0. \qquad (44)$$

Intuitively, the power σ^τ keeps the same screw axis $(\hat{\boldsymbol{l}}, \boldsymbol{m})$ but scales both rotation angle and translation distance by the factor τ.

9 Differentiation

Differentiation of a dual quaternion is carried out on its real and dual parts separately. For a unit dual quaternion in the form (22) that describes the composition of a rotation and a translation, its derivative is calculated as follows:

$$\dot{\sigma} = \dot{r} + \frac{\epsilon}{2}(\dot{\boldsymbol{t}}r + \boldsymbol{t}\dot{r}). \tag{45}$$

It is known that $\dot{r} = \frac{1}{2}\boldsymbol{\omega} r$, where $\boldsymbol{\omega}$ is the angular velocity treated as a pure quaternion. Also, $\dot{\boldsymbol{t}} = \boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity also treated as a pure quaternion. Plugging these two expressions into (45), we have

$$\begin{aligned} \dot{\sigma} &= \frac{1}{2}\boldsymbol{\omega} r + \frac{\epsilon}{2}\left(\boldsymbol{v} r + \frac{1}{2}\boldsymbol{t}\boldsymbol{\omega} r\right) \\ &= \frac{1}{2}\boldsymbol{\omega} r + \frac{\epsilon}{2}\boldsymbol{v} r + \frac{\epsilon}{4}\boldsymbol{t}\boldsymbol{\omega} r. \end{aligned}$$

10 Interpolation

Sometimes we need to interpolate a smooth motion between two transformations, represented by unit dual quaternions σ_1 and σ_2, respectively. We can easily convert them into the standard form (43):

$$\begin{aligned} \sigma_1 &= r_1 + \frac{\epsilon}{2}\boldsymbol{t}_1 r_1, \\ \sigma_2 &= r_2 + \frac{\epsilon}{2}\boldsymbol{t}_2 r_2. \end{aligned}$$

We see that the transformation from σ_1 to σ_2 is $\sigma_1^{-1} \otimes \sigma_2$.

The screw linear interpolation (ScLERP) is an extension of the spherical linear interpolation (SLERP) [13] over quaternions. It performs a screw motion with constant rotation and translation speeds from σ_1 to σ_2:

$$\sigma(\tau) = \sigma_1 \otimes (\sigma_1^{-1} \otimes \sigma_2)^\tau, \qquad \tau \in [0, 1]. \tag{46}$$

By (28), $\sigma_1^{-1} = r_1^* - \frac{\epsilon}{2} r_1^* \boldsymbol{t}_1$. Thus, we calculate the transformation as

$$\begin{aligned} \sigma_1^{-1} \otimes \sigma_2 &= \left(r_1^* - \frac{\epsilon}{2} r_1^* \boldsymbol{t}_1\right) \otimes \left(r_2 + \frac{\epsilon}{2}\boldsymbol{t}_2 r_2\right) \\ &= r_1^* r_2 + \frac{\epsilon}{2}(r_1^* \boldsymbol{t}_2 r_2 - r_1^* \boldsymbol{t}_1 r_2) \\ &= r_1^* r_2 + \frac{\epsilon}{2} r_1^* (\boldsymbol{t}_2 - \boldsymbol{t}_1) r_2. \end{aligned}$$

From the above product we apply the method from Section 8.2 to extract the corresponding screw coordinates $\hat{\boldsymbol{l}}, \boldsymbol{m}, \theta, d$. Then we can rewrite the product into the form (43). The interpolation formula (46) is extended below for direct evaluation at τ using (40) and (41) :

$$\begin{aligned} \sigma(\tau) &= \sigma_1 \otimes \left(\cos\frac{\tau\bar{\theta}}{2} + \bar{\hat{\boldsymbol{l}}} \sin\frac{\tau\bar{\theta}}{2}\right) \\ &= \left(r_1 + \frac{\epsilon}{2}\boldsymbol{t}_1 r_1\right) \otimes \left(\cos\frac{\tau\theta}{2} + \hat{\boldsymbol{l}}\sin\frac{\tau\theta}{2} + \epsilon\frac{\tau d}{2}\left(-\sin\frac{\tau\theta}{2} + \hat{\boldsymbol{l}}\cos\frac{\tau\theta}{2}\right)\right). \end{aligned}$$

11 Computational Efficiency and Applications

Consisting of eight scalars, dual quaternions are a compact representation of rigid transformations in the space. The excess of two variables in a dual quaternion than the six degrees of freedom can be regarded as a tradeoff for efficiency of computation and ease of operator concatenation (which are not provided by alternative representations such as orthogonal matrices, Euler angles, homogeneous coordinates, or screw coordinates). Dual quaternions are geometrically meaningful and have no singularities when representing 3D rotations. Extensive analysis [3, 4] comparing dual quaternions with matrix representations and screw coordinates in sequential and parallel executions concluded that dual quaternions were the best representation in terms of compactness and efficiency.

A direct application of dual quaternions is in deriving the kinematics [6] and inverse kinematics [12] of robotic manipulators, where their full potential has not been materialized due to the long tradition of using homogeneous transformations. Neuroscientists are extending this line of application to investigate motor planning of eye or arm movements, or sensorymotor transformations [9].

In computer vision, dual quaternions have been used to solve problems such as registration of multiple views [15] and hand-eye calibration [2]. They are combined with extended Kalman filtering to effectively estimate poses and motions of moving objects [5], and in particular, track aerial vehicles and stars. Applications in graphics and animation [8] have been centered around the ability of dual quaternions to interpolate smoothly between rigid transformations. We refer to [9] for a survey of the applications of dual quaternions.

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