Parametric Curves

(Com S 477/577 Notes)

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1 Introduction

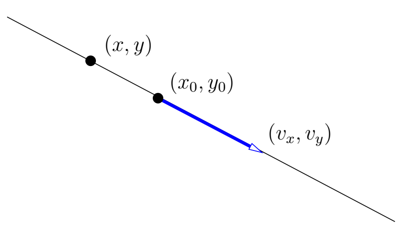
Curves and surfaces are abundant with man-made objects, tools, and machines which are ubiquitous in our daily life. Engineering curves and surfaces have many applications in industry. For example, hyperbolic shapes are used on cooling towers while spiral shapes are used on turbines. Cycloids are the choices for designing roller coasters and in violin plate arching. Involutes are applied to ensure that two intermeshed gears have constant relative rates of rotation. In computer vision, quadrics are a good choice for reconstructing closed shapes. Mobile robots follow Bezier curves as their paths to stay within their acceleration limits, and so on.

A *curve* in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is a differentiable function $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$). The initial point is $\boldsymbol{\alpha}[a]$ and the final point is $\boldsymbol{\alpha}[b]$. The *domain* of the curve is the interval $[a, b]$. A portion of $\boldsymbol{\alpha}$ defined on an interval $[c, d] \subseteq [a, b]$ is called a *curve segment*.

EXAMPLE 1. **Straight Line** The line is the simplest curve in the plane as its coordinate functions are linear. Explicitly, the curve

$$\boldsymbol{\alpha}(t) = \boldsymbol{p} + t\boldsymbol{v}, \qquad \boldsymbol{v} \neq \boldsymbol{0}, \tag{1}$$

is a straight line through the reference point $\boldsymbol{p} = \boldsymbol{\alpha}(0) = (x_0, y_0)$ in the direction $\boldsymbol{v} = (v_x, v_y)$. Here, t is the signed distance from a point $\boldsymbol{\alpha}(t)$ on the line to $\boldsymbol{p}$ as scaled by $\|\boldsymbol{v}\|$.



As shown on the left, the vector from $\boldsymbol{p}$ to a point (x, y) on the line must be either in the direction of (v_x, v_y) or in its opposite direction. Hence, the cross product of the two vectors must be zero, that is,

$$(x - x_0, y - y_0) \times (v_x, v_y) = 0.$$

Expansion of the above cross product yields an *implicit equation* of the line that relates the x and y coordinates of every incident point:

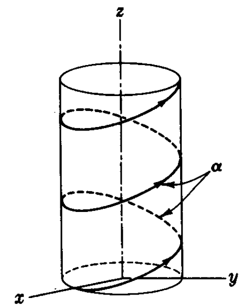
$$v_y x - v_x y - v_y x_0 + v_x y_0 = 0. \tag{2}$$

EXAMPLE 2. **Helix**[1]

[1]The figure is from [1, p. 16].

The curve $t \to (a\cos t, a\sin t, 0)$ travels around a circle of radius $a > 0$ in the x-y plane. If we allow this curve to rise (or fall) at a constant rate, we obtain a helix

$$\boldsymbol{\alpha} = (a\cos t, a\sin t, bt), \qquad \text{where } a > 0 \text{ and } b \neq 0.$$



EXAMPLE 3. The curve $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^3$ such that

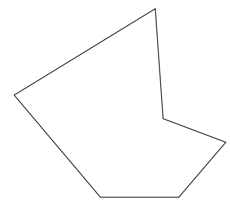
$$\boldsymbol{\alpha}(t) = (e^t, e^{-t}, \sqrt{2}t)$$

shares with the helix in Example 2 the property of rising constantly. However, it lies over the hyperbola $xy = 1$ in the x-y plane instead of a circle.

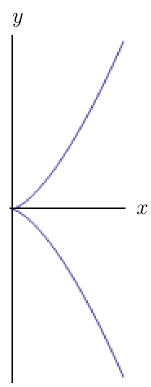
A curve $\boldsymbol{\alpha}(t) = (x(t), y(t))$ is said to be *smooth* at $t = t_0$ if its kth derivative

$$\boldsymbol{\alpha}^{(k)}(t) = \left(x^{(k)}(t), y^{(k)}(t)\right)$$

exists for any integer $k > 0$. A *piecewise smooth curve* $\boldsymbol{\alpha}$ has a domain which is the union of a finite number of subintervals over each of which $\boldsymbol{\alpha}$ is smooth.



EXAMPLE 4. A line $\boldsymbol{\alpha}(t) = \boldsymbol{p} + t\boldsymbol{q}$ is a smooth curve. Here $\dot{\boldsymbol{\alpha}}(t) = \boldsymbol{q}$ and $\boldsymbol{\alpha}^{(k)} = \boldsymbol{0}$ for $k > 1$. A polygon, on the other hand, is a piecewise smooth curve, where each edge determines a subdomain.



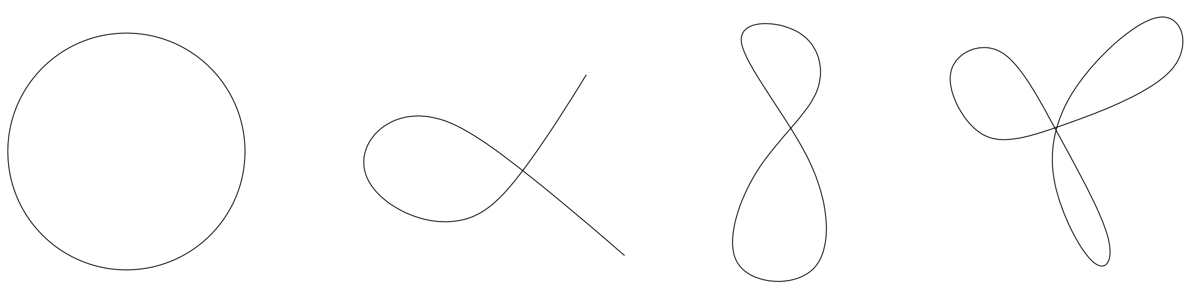
EXAMPLE 5. **Cuspidal cubic** The curve $\boldsymbol{\alpha}(t) = (t^2, t^3)$ is smooth.

We have

$$\begin{aligned}
\dot{\boldsymbol{\alpha}}(t) &= (2t, 3t^2), \\
\ddot{\boldsymbol{\alpha}}(t) &= (2, 6t), \\
\dddot{\boldsymbol{\alpha}}(t) &= (0, 6), \\
\boldsymbol{\alpha}^{(k)}(t) &= \boldsymbol{0}, \qquad k \geq 4.
\end{aligned}$$

Consider a plane curve $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^2$. It is called a *closed parametric curve* if $\boldsymbol{\alpha}(a) = \boldsymbol{\alpha}(b)$. A point of *self-crossing* is a point $\boldsymbol{\alpha}(t_1)$ for which there exist finitely many distinct values $t_1, \ldots, t_n \in [a, b]$, $n \geq 2$, which satisfy $\boldsymbol{\alpha}(t_1) = \boldsymbol{\alpha}(t_2) = \cdots = \boldsymbol{\alpha}(t_n)$, and in the case $n = 2$, $[t_1, t_2] \neq [a, b]$.

EXAMPLE 6. A circle is closed. The other three curves all have self-crossings.



2 Velocity, Speed, and Arc Length

Let $\boldsymbol{\alpha}(t)$ be a curve. The *velocity vector* of $\boldsymbol{\alpha}$ at t is $\dot{\boldsymbol{\alpha}}(t)$. The *speed* at t is the length $\|\dot{\boldsymbol{\alpha}}(t)\|$. The meaning is clear if we see $\boldsymbol{\alpha}(t)$ as the location of a moving point at time t. The parametrization $\boldsymbol{\alpha}(t)$ is *unit-speed* if $\|\dot{\boldsymbol{\alpha}}(t)\| = 1$ for all values of t. A point where $\dot{\boldsymbol{\alpha}}(t) = \mathbf{0}$ is called a *cusp* on the curve.

Example 7. The origin on the cuspidal cubic in Example 5 is a cusp.

The curve $\boldsymbol{\alpha}(t)$ is *regular* if all velocity vectors are different from zero, that is, $\dot{\boldsymbol{\alpha}}(t) \neq \mathbf{0}$ for all t. Intuitively, a point moving on the curve with velocity $\dot{\boldsymbol{\alpha}}(t)$ will never come to a stop or reverse its direction.

Example 8. Consider the curve $\boldsymbol{\alpha}(\theta) = (a\theta\cos\theta, a\theta\sin\theta)$. It has velocity

$$\dot{\boldsymbol{\alpha}}(\theta) = a(\cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta),$$

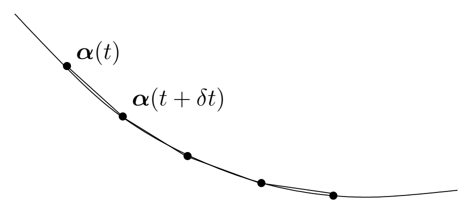
and speed

$$\|\dot{\boldsymbol{\alpha}}(\theta)\| = |a|\sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2} = |a|\sqrt{1+\theta^2} \neq 0.$$

Therefore the parametrization is regular.

The velocity and speed depend on its parametrization. Non-regularity at a point may be just a property of the parametrization, and need not correspond to any special feature of the curve geometry. For a different parametrization the curve may have a non-zero velocity at the same point.

To formulate the length of $\boldsymbol{\alpha}$, we note that the portion over $[t, t+\delta t]$ is nearly a straight line when δt is very small. So the length over $[t, t+\delta t]$ can be approximated by



$$\|\boldsymbol{\alpha}(t+\delta t) - \boldsymbol{\alpha}(t)\|,$$

which again is approximated by

$$\|\dot{\boldsymbol{\alpha}}(t)\|\delta t.$$

We divide $\boldsymbol{\alpha}$ up into segments, each of which corresponds to a small increment δt. As δt tends to zero, we will obtain the exact length. The *arc length* of $\boldsymbol{\alpha}$ from $t = a$ to $t = b$ is thus defined as

$$\int_a^b \|\dot{\boldsymbol{\alpha}}(t)\|\, dt.$$

Example 9. **Logarithmic spiral** The curve

$$\boldsymbol{\alpha}(t) = (e^t\cos t, e^t\sin t),$$

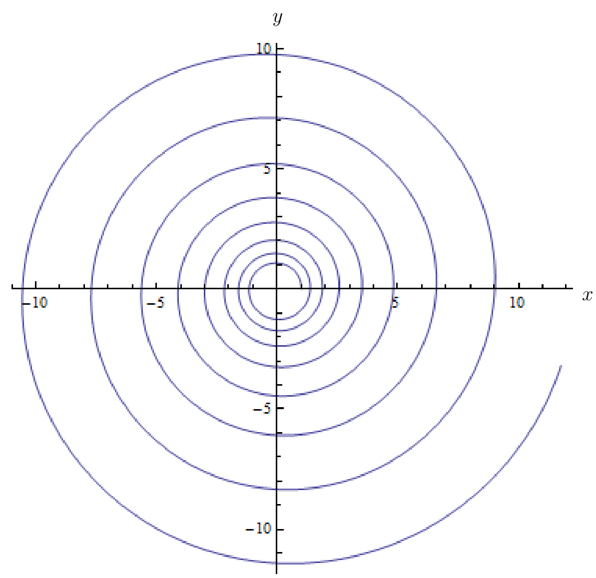
has a spiral motion. We obtain that

$$\begin{aligned} \dot{\boldsymbol{\alpha}}(t) &= \left(e^t(\cos t - \sin t), e^t(\sin t + \cos t)\right), \\ \|\dot{\boldsymbol{\alpha}}(t)\| &= \sqrt{2}e^t. \end{aligned}$$

Hence the arc length of $\boldsymbol{\alpha}$ starting at $\boldsymbol{\alpha}(0) = (1, 0)$, for instance, is

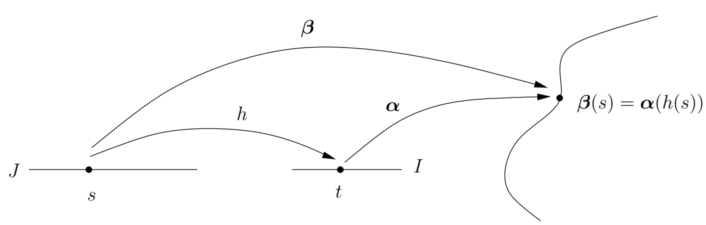
$$s = \int_0^t \sqrt{2} e^u \, du = \sqrt{2}(e^t - 1).$$

The plot below shows a logarithmic spiral $(e^{t/20} \cos t, e^{t/20} \sin t)$ over $[0, 50]$.



3 Reparametrization

Let I and J be intervals. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a curve and h a differentiable function. Then the composite function $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$ is a curve called the *reparametrization* of $\boldsymbol{\alpha}$ by h.



Example 10. Suppose $\boldsymbol{\alpha}(t) = (\sqrt{t}, t\sqrt{t}, 1 - t)$ on $(0, 4)$. If $h(s) = s^2$ on $(0, 2)$, then

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(h(s)) = \boldsymbol{\alpha}(s^2) = (s, s^3, 1 - s^2).$$

The curve $\boldsymbol{\alpha}$ has been reparametrized by h to yield the curve $\boldsymbol{\beta}$.

At each time s in the interval J, the curve $\boldsymbol{\beta}$ is at the point $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(h(s))$ reached by the curve $\boldsymbol{\alpha}$ at time $h(s)$ in the interval. Thus β does follow the route of $\boldsymbol{\alpha}$, but it reaches a given point on the route at a different time than $\boldsymbol{\alpha}$ does.

Sometimes one is interested only in the route followed by a curve and not in the particular speed at which it traverses its route. One way to ignore the speed of a curve $\boldsymbol{\alpha}$ is to reparametrize to a curve $\tilde{\boldsymbol{\alpha}}$ which has unit speed $\|\dot{\tilde{\boldsymbol{\alpha}}}\| = 1$.

Theorem 1 *If $\boldsymbol{\alpha}$ is a regular curve, then there exists a reparametrization $\tilde{\boldsymbol{\alpha}}$ that has unit speed.*

Proof Consider the arc length function

$$s(t) = \int_c^t \|\dot{\boldsymbol{\alpha}}(u)\| \, du,$$

where c is a number in the domain of $\boldsymbol{\alpha}$. It then follows that

$$\dot{s}(t) = \|\dot{\boldsymbol{\alpha}}(t)\|;$$

namely, the derivative of s is the speed function $\|\dot{\boldsymbol{\alpha}}(t)\|$. Since $\boldsymbol{\alpha}$ is regular, $\dot{\boldsymbol{\alpha}} \neq \mathbf{0}$ everywhere; hence $ds/dt > 0$ always holds. By a standard theorem of calculus, the function s has an inverse function $t(s)$, and

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{\|\dot{\boldsymbol{\alpha}}(t)\|}.$$

Now we let $\tilde{\boldsymbol{\alpha}}(s) = \boldsymbol{\alpha}(t(s))$ be the reparametrization of $\boldsymbol{\alpha}$. Then

$$\dot{\tilde{\alpha}}(s) = \dot{\boldsymbol{\alpha}}(t(s)) \, \frac{dt}{ds}.$$

Hence, the speed of $\tilde{\boldsymbol{\alpha}}$ is

$$\|\dot{\tilde{\alpha}}(s)\| = \|\dot{\boldsymbol{\alpha}}(t(s))\| \frac{1}{\|\dot{\boldsymbol{\alpha}}(t(s))\|} = 1.$$

□

The unit-speed curve $\tilde{\boldsymbol{\alpha}}$ is said to have *arc-length parameterization*, since the arc length of $\tilde{\boldsymbol{\alpha}}$ from $s = a$ to $s = b$, $a < b$, is just $b - a$.

Example 11. Let us consider the helix $\boldsymbol{\alpha} = (a\cos t, a\sin t, bt)$ in Example 2 again. It has velocity

$$\dot{\boldsymbol{\alpha}}(t) = (-a\sin t, a\cos t, b).$$

Hence

$$\|\dot{\boldsymbol{\alpha}}(t)\|^2 = \dot{\boldsymbol{\alpha}}(t) \cdot \dot{\boldsymbol{\alpha}}(t) = a^2 \sin^2 t + a^2 \cos^2 t + b^2 = a^2 + b^2.$$

Thus $\boldsymbol{\alpha}$ has *constant* speed:

$$c = \|\dot{\boldsymbol{\alpha}}\| = \sqrt{a^2 + b^2}.$$

The arc length from $t = 0$ is then

$$s(t) = \int_0^t c \, du = ct.$$

Hence, $t(s) = \frac{s}{c}$. Substituting this into the formula for $\boldsymbol{\alpha}$, we get the unit-speed reparametrization

$$\tilde{\boldsymbol{\alpha}}(s) = \boldsymbol{\alpha}\left(\frac{s}{c}\right) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}\right).$$

Although every regular curve has a unit-speed reparametrization, this may be very complicated, or even impossible to write down explicitly, as the examples show.

Example 12. The logarithmic spiral

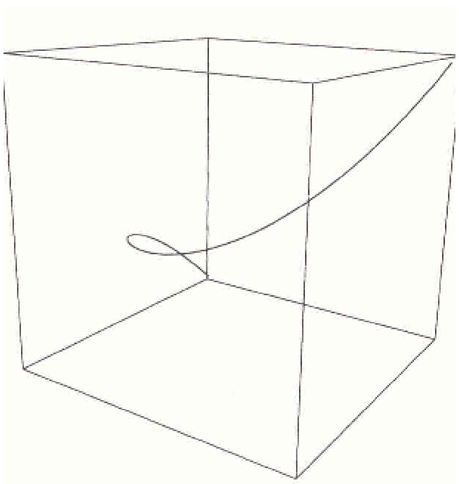
$$\boldsymbol{\alpha}(t) = (e^t \cos t, e^t \sin t),$$

has speed

$$\sqrt{2}e^t > 0.$$

So it is regular. The arc length starting at $(1, 0)$ was found in Example 9 to be $s = \sqrt{2}(e^t - 1)$. Hence, $t = \ln(\frac{s}{\sqrt{2}} + 1)$, so a unit-speed reparametrization of $\boldsymbol{\alpha}$ is given by the rather unwieldy formula

$$\tilde{\boldsymbol{\alpha}}(s) = \left(\left(\frac{s}{\sqrt{2}} + 1 \right) \cos \left(\ln \left(\frac{s}{\sqrt{2}} + 1 \right) \right), \left(\frac{s}{\sqrt{2}} + 1 \right) \sin \left(\ln \left(\frac{s}{\sqrt{2}} + 1 \right) \right) \right).$$



Example 13. **Twisted cubic**[2] This is the space curve given by

$$\boldsymbol{\alpha}(t) = (t, t^2, t^3), \qquad -\infty < t < \infty.$$

We have

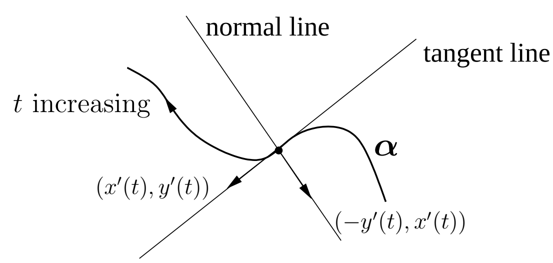
$$\begin{aligned} \dot{\boldsymbol{\alpha}}(t) &= (1, 2t, 3t^2), \\ \|\dot{\boldsymbol{\alpha}}(t)\| &= \sqrt{1 + 4t^2 + 9t^4}. \end{aligned}$$

Since the speed $\|\dot{\boldsymbol{\alpha}}(t)\|$ is not zero everywhere, $\boldsymbol{\alpha}$ is regular. And the arc-length starting at $\boldsymbol{\alpha}(0) = \mathbf{0}$ is

$$s = \int_0^t \sqrt{1 + 4u^2 + 9u^4}\, du.$$

The above integral has a horrendous closed form not in terms of familiar functions.

4 Tangent and Normal



The standard method of studying the geometry of a curve at a point is to attach orthonormal vectors to the point and see how the directions of these vectors change as the point moves on the curve for an infinitesimal distance. We choose tangent and normal vectors at a regular point. Let $\boldsymbol{\alpha}(t) = (x(t), y(t))$ be a curve. At a regular point $\boldsymbol{\alpha}(t)$ there exists a (nonzero) *tangent vector* $\dot{\boldsymbol{\alpha}}(t) = (\dot{x}(t), \dot{y}(t))$. It represents the velocity of the curve at the point. The normal vector $(-\dot{y}(t), \dot{x}(t))$ at $\boldsymbol{\alpha}(t)$ is given by rotating the tangent vector counterclockwise through an angle $\frac{\pi}{2}$. Note that $(\dot{x}(t), \dot{y}(t)) \times (-\dot{y}(t), \dot{x}(t)) = (\dot{x}(t))^2 + (\dot{y}(t))^2 > 0$.

If $\boldsymbol{\alpha}(t)$ is a unit-speed curve, then both the tangent vector and the normal vector are unit vectors. By convention they are denoted as T and N, respectively, with the cross product $T \times N = 1$.

[2] The figure originally appears in [3, p. 14].

For a parametric curve we have a tangent line and a normal line at each regular point $\boldsymbol{\alpha}(t)$. The tangent line to the curve at $\boldsymbol{\alpha}(t)$ passes through $\boldsymbol{\alpha}(t)$ and is parallel to $\dot{\boldsymbol{\alpha}}(t) \neq 0$. So it has the parametric equation

$$\Big(x(s), y(s)\Big) = \boldsymbol{\alpha}(t) + s\dot{\boldsymbol{\alpha}}(t), \qquad s \in (-\infty, \infty),$$

or equivalently, the algebraic equation

$$\Big((x, y) - \boldsymbol{\alpha}(t)\Big) \cdot \Big(-\dot{y}(t), \dot{x}(t)\Big) = 0,$$

given that $(-\dot{y}, \dot{x})$ is orthogonal to $\dot{\boldsymbol{\alpha}}$. The normal line at $\boldsymbol{\alpha}(t)$ passes through the point and is parallel to $(-\dot{y}(t), \dot{x}(t))$. So its equations are of the form

$$\Big(x(s), y(s)\Big) = \boldsymbol{\alpha}(t) + s\Big(-\dot{y}(t), \dot{x}(t)\Big), \qquad s \in (-\infty, \infty),$$

or equivalently,

$$\bigg(\Big(x(s), y(s)\Big) - \boldsymbol{\alpha}(t)\bigg) \cdot \dot{\boldsymbol{\alpha}}(t) = 0.$$

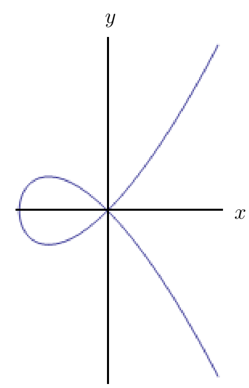
Example 14. **Crunodal cubic** is described as

$$\boldsymbol{\alpha}(t) = \Big(t^2 - 1, t(t^2 - 1)\Big).$$

Find its tangent and normal lines of the curve at the points $t = \pm 1, 0$.

We obtain

$$\begin{aligned} \dot{\boldsymbol{\alpha}}(t) &= (2t, 3t^2 - 1), \\ \dot{\boldsymbol{\alpha}}(1) &= (2, 2), \\ \dot{\boldsymbol{\alpha}}(-1) &= (-2, 2), \\ \dot{\boldsymbol{\alpha}}(0) &= (0, -1), \\ \boldsymbol{\alpha}(\pm 1) &= (0, 0). \end{aligned}$$



Here $\boldsymbol{\alpha} = (0, 0)$ is referred to as a *double point* since it is attained at both $t = 1$ and $t = -1$. The tangent lines at this double point are respectively

$$(x, y) = s(1, 1), \qquad \text{or equivalently,} \qquad y = x,$$

and

$$(x, y) = s(-1, 1), \qquad \text{or equivalently,} \qquad y = -x.$$

The normal lines at the double point are respectively

$$(x, y) = s(-1, 1), \qquad \text{or equivalently,} \qquad y = -x,$$

and

$$(x, y) = s(-1, -1), \qquad \text{or equivalently,} \qquad y = x.$$

At $t = 0$, we have $\dot{\boldsymbol{\alpha}}(0) = (0, -1)$, and the tangent line at $\boldsymbol{\alpha}(0)$ is

$$(x, y) = (-1, 0) + s(0, -1), \qquad \text{or equivalently,} \qquad x = -1.$$

The normal line at $\boldsymbol{\alpha}(0)$ is

$$(x, y) = (-1, 0) + s(1, 0), \qquad \text{or equivalently,} \qquad y = 0.$$

References

[1] B. O'Neill. *Elementary Differential Geometry*. Academic Press, Inc., 1966.

[2] J. W. Rutter. *Geometry of Curves*. Chapman & Hall/CRC, 2000.

[3] A. Pressley. *Elementary Differential Geometry*. Springer-Verlag London, 2001.