Curvature

(Com S 477/577 Notes)

Yan-Bin Jia

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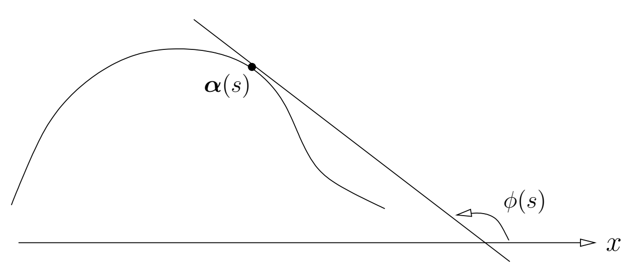
We want to find a measure of how 'curved' a curve is. Since this 'curvature' should depend only on the 'shape' of the curve, it should not be changed when the curve is reparametrized. Further, the measure of curvature should agree with our intuition in simple special cases. Straight lines themselves have zero curvature. Large circles should have smaller curvature than small circles which bend more sharply.

The (signed) curvature of a curve parametrized by its arc length is the rate of change of direction of the tangent vector. The absolute value of the curvature is a measure of how sharply the curve bends. Curves which bend slowly, which are almost straight lines, will have small absolute curvature. Curves which swing to the left have positive curvature and curves which swing to the right have negative curvature. The curvature of the direction of a road will affect the maximum speed at which vehicles can travel without skidding, and the curvature in the trajectory of an airplane will affect whether the pilot will suffer "blackout" as a result of the g-forces involved.

In this lecture we will primarily look at the curvature of *plane curves*. The results will be extended to space curves in the next lecture.

1 Curvature

To introduce the definition of curvature, we consider a unit-speed curve $\boldsymbol{\alpha}(s)$, where s is the arc length. The *tangential angle* ϕ is measured counterclockwise from the x-axis to the unit tangent $T = \dot{\boldsymbol{\alpha}}(s)$, as shown below.



The *curvature* κ of $\boldsymbol{\alpha}$ is the rate of change in the direction of the tangent line at that point with respect to arc length, that is,

$$\kappa = \frac{d\phi}{ds}. \tag{1}$$

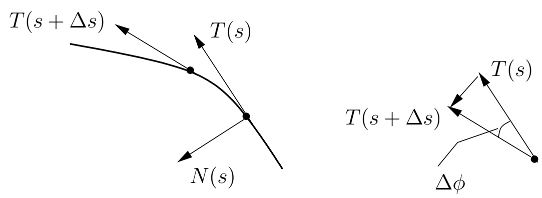
The *absolute curvature* of the curve at the point is the absolute value $|\kappa|$.

Since $\boldsymbol{\alpha}$ has unit speed, $T \cdot T = 1$. Differentiating this equation yields

$$\dot{T} \cdot T = 0.$$

The change of $T(s)$ is orthogonal to the tangential direction, so it must be along the normal direction. The curvature is also defined to measure the turning of $T(s)$ along the direction of the unit normal $N(s)$ where $T(s) \times N(s) = 1$. That is,

$$\dot{T} = \frac{dT}{ds} = \kappa N. \tag{2}$$



We can easily derive one of the curvature definitions (1) and (2) from the other. For instance, if we start with (2), then

$$\begin{aligned}
\kappa &= \dot{T} \cdot N \\
&= \frac{dT}{ds} \cdot N \\
&= \lim_{\Delta s \to 0} \frac{T(s+\Delta s) - T(s)}{\Delta s} \cdot N \\
&= \lim_{\Delta s \to 0} \frac{\Delta\phi \cdot \|T\|}{\Delta s} \\
&= \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} \\
&= \frac{d\phi}{ds}.
\end{aligned}$$

Example 1. Let us compute the curvature of the unit-speed circle

$$\boldsymbol{\alpha}(s) = r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right).$$

We obtain that

$$\begin{aligned}
T &= \dot{\boldsymbol{\alpha}}(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right), \\
N &= \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right), \\
\dot{T} &= \ddot{\boldsymbol{\alpha}}(s) = -\frac{1}{r}\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right) = \frac{1}{r}N.
\end{aligned}$$

Thus

$$\kappa(s) = \frac{1}{r}. \qquad \text{cf. (2)}$$

The curvature of a circle equals the inverse of its radius everywhere.

The next result shows that a unit-speed plane curve is essentially determined once we know its curvature at every point on the curve. The meaning of 'essentially' here is 'up to a rigid motion[1] of $\mathbb{R}^2$'.

[1] A rigid motion consists of a rotation and a translation.

Theorem 1 *Let $\kappa : (a,b) \to \mathbb{R}$ be an integrable function. Then there exists a unit-speed curve $\boldsymbol{\alpha} : (a,b) \to \mathbb{R}^2$ whose curvature is κ. Furthermore, if $\tilde{\boldsymbol{\alpha}} : (a,b) \to \mathbb{R}^2$ is any other unit-speed curve with the same curvature function κ, there exists a rigid body motion that transforms $\tilde{\boldsymbol{\alpha}}$ into $\boldsymbol{\alpha}$.*

Proof Fix $s_0 \in (a,b)$ and define, for any $s \in (a,b)$,

$$\begin{aligned} \phi(s) &= \int_{s_0}^{s} \kappa(u)\, du, \qquad \text{cf. (1)}, \\ \boldsymbol{\alpha}(s) &= \left(\int_{s_0}^{s} \cos\phi(t)\, dt, \int_{s_0}^{s} \sin\phi(t)\, dt \right). \end{aligned}$$

Then, the tangent vector of $\boldsymbol{\alpha}$ is

$$\dot{\boldsymbol{\alpha}}(s) = \Big(\cos\phi(s), \sin\phi(s) \Big),$$

which is a unit vector making an angle $\phi(s)$ with the x-axis. Thus $\boldsymbol{\alpha}$ is unit speed, and has curvature

$$\frac{d\phi}{ds} = \frac{d}{ds} \int_{s_0}^{s} \kappa(u)\, du = \kappa(s).$$

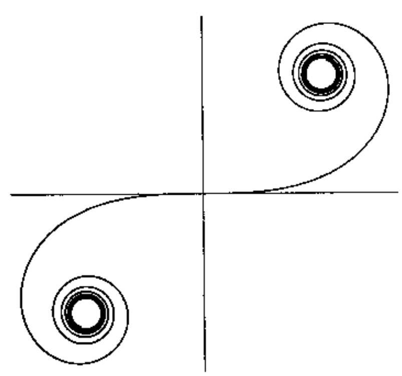
For a proof of the second part, we refer to [3, p. 31]. □

The above theorem shows that we can find a plane curve with any given smooth function as its signed curvature. But simple curvature can lead to complicated curves, as shown in the next example.

Example 2. Let the signed curvature be $\kappa(s) = s$. Following the proof of Theorem 1, and taking $s_0 = 0$, we get

$$\begin{aligned} \phi(s) &= \int_{0}^{s} u\, du = \frac{s^2}{2}, \\ \boldsymbol{\alpha}(s) &= \left(\int_{0}^{s} \cos\frac{s^2}{2}\, ds, \int_{0}^{s} \sin\frac{s^2}{2}\, ds \right). \end{aligned}$$

These integrals can only be evaluated numerically.[2] The curve is drawn in the figure below.[3]



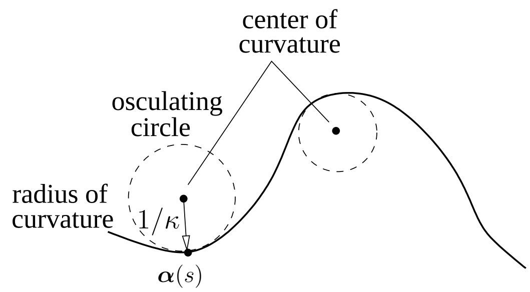
[2] They arise in the theory of diffraction of light, where they are called *Fresnel's integrals*, and the curve is called *Cornu's Spiral*, although it was first considered by Euler.

[3] Taken from [3, p. 33].

2 Radius of Curvature and Total Curvature

When the curvature $\kappa(s) > 0$, the *center of curvature* lies along the direction of $N(s)$ at distance $1/\kappa$ from the point $\boldsymbol{\alpha}(s)$. When $\kappa(s) < 0$, the center of curvature lies along the direction of $-N(s)$ at distance $-1/\kappa$ from $\boldsymbol{\alpha}(s)$. In either case, the center of curvature is located at

$$\boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)} N(s).$$



Here, $1/|\kappa|$ is called the *radius of curvature*. The *osculating circle*, when $\kappa \neq 0$, is the circle at the center of curvature with radius $1/|\kappa|$. It approximates the curve locally up to the second order.

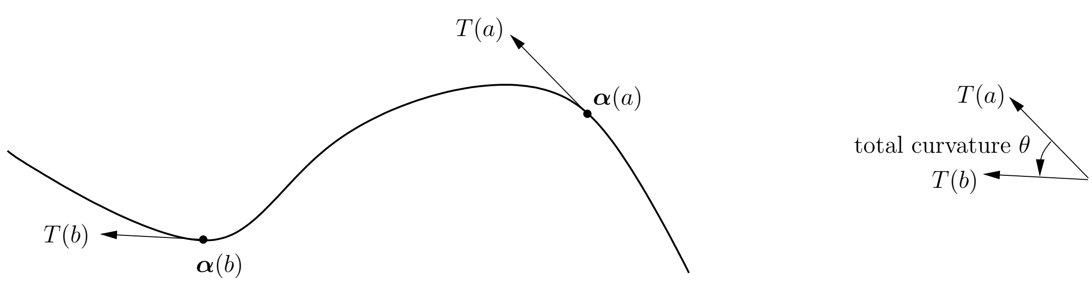
The *total curvature* over a closed interval $[a, b]$ measures the rotation of the unit tangent $T(s)$ as s changes from a to b:

$$\begin{aligned} \Phi(a,b) &= \int_a^b \kappa \, ds \\ &= \int_a^b \frac{d\phi}{ds} \, ds \\ &= \int_a^b d\phi \\ &= \phi(b) - \phi(a). \end{aligned}$$

If the total curvature over $[a, b]$ is within $[0, 2\pi]$, it has a closed form:

$$\Phi(a,b) = \begin{cases} \arccos\Big(T(a) \cdot T(b)\Big), & \text{if } T(a) \times T(b) \geq 0; \\ 2\pi - \arccos\Big(T(a) \cdot T(b)\Big), & \text{otherwise.} \end{cases}$$

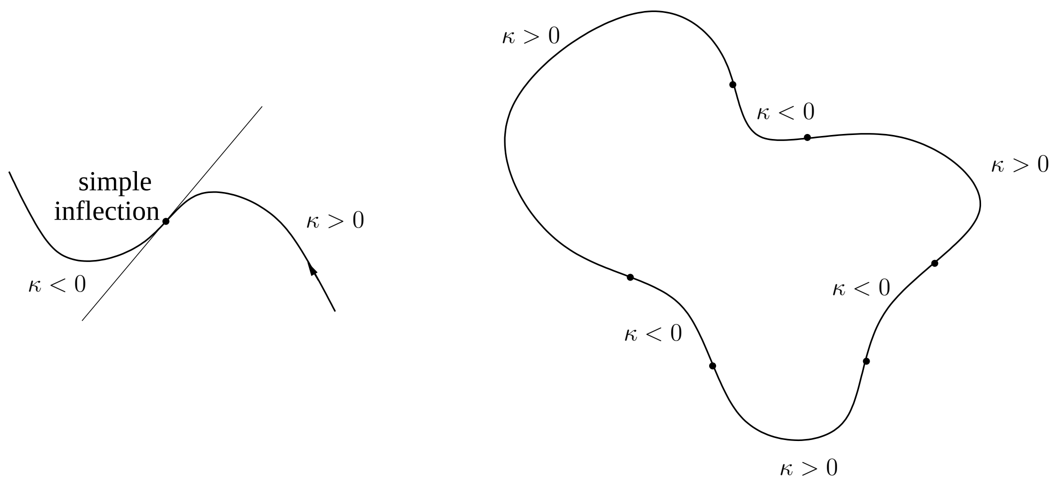
When the tangent makes several full revolutions[4] as s increases from a to b, the total curvature cannot be determined just from $T(a)$ and $T(b)$.



[4]For example, the curve is the Cornu's Spiral.

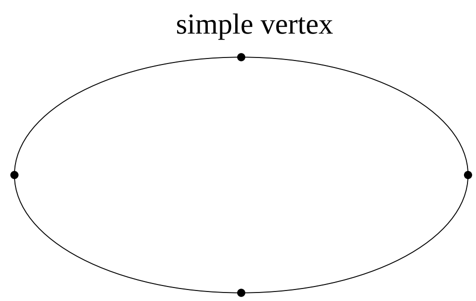
3 Inflection and Vertex

A point s on the curve $\boldsymbol{\alpha}$ is a *simple inflection*, or an *inflection*, if the curvature $\kappa(s) = 0$ but $\dot{\kappa}(s) \neq 0$. Intuitively, a simple inflection is where the curve swing from the left of the tangent at the point to its right or from the right of the tangent to its left; or in the case of simple closed curve, it is where the closed curve $\boldsymbol{\alpha}$ changes from convex to concave or from concave to convex. In the figure below, the curve on the left has one simple inflection while the curve on the right has six simple inflections.



In general, a point s with $\kappa(s) = \dot{\kappa}(s) = \cdots = \kappa^{(j-1)}(s) = 0$ and $\kappa^{(j)}(s) \neq 0$ is an *inflection point of order* j. A second order inflection point, also referred to as a point of *simple undulation*, will not alter the convexity or concavity of its neighborhood on a simple closed curve.

A *simple vertex*, or a *vertex*, of a curve satisfies $\dot{\kappa} = 0$ but $\ddot{\kappa} \neq 0$. Intuitively, a simplex vertex is where the curvature attains a local minimum or maximum. For example, an ellipse has four vertices, on its major and minor axes.



4 Curvature of an Arbitrary-Speed Curve

Let $\boldsymbol{\alpha}(t)$ be a regular but not necessarily unit-speed curve. We obtain the unit tangent as $T = \dot{\boldsymbol{\alpha}}/\|\dot{\boldsymbol{\alpha}}\|$ and the unit normal N as the counterclockwise rotation of T by $\pi/2$. Still denote by $\kappa(t)$

the curvature function. Let $\tilde{\boldsymbol{\alpha}}(s)$ be the unit-speed reparametrization of $\boldsymbol{\alpha}$, where s is an arc-length function for $\boldsymbol{\alpha}$. Let $\tilde{T} = d\tilde{\boldsymbol{\alpha}}/ds$ be the unit tangent and $\tilde{\kappa}(s)$ the curvature function under this unit-speed parametrization. The curvature at a point is *independent* of any parametrization so $\kappa(t) = \tilde{\kappa}(s(t))$. Also by definition $T(t) = \tilde{T}(s)$. Differentiate this equation and apply the chain rule:

$$\dot{T}(t) = \dot{\tilde{T}}(s) \cdot \frac{ds}{dt}. \tag{3}$$

Since $\tilde{\boldsymbol{\alpha}}(s)$ is unit-speed, we know that

$$\dot{\tilde{T}}(s) = \tilde{\kappa}(s)\tilde{N}(s).$$

Substitution of the function s in this equation yields

$$\dot{\tilde{T}}(s) = \tilde{\kappa}\Big(s(t)\Big)\tilde{N}\Big(s(t)\Big) = \kappa(t)N(t) \tag{4}$$

by the definition of κ and N in the arbitrary-speed case. We know that $ds/dt = \|\dot{\boldsymbol{\alpha}}(t)\|$ from the definition of arc length

$$s = \int_{t_0}^{t} \|\dot{\boldsymbol{\alpha}}(u)\|\, du.$$

Denote by $v = \|\dot{\boldsymbol{\alpha}}(t)\|$ the speed function of $\boldsymbol{\alpha}$. Equations (3) and (4) combine to yield

$$\dot{T} = \kappa v N. \tag{5}$$

Now let $\boldsymbol{\alpha}(t) = (x(t), y(t))$. Then

$$\begin{aligned} T &= (\dot{x}, \dot{y})/\|\dot{\boldsymbol{\alpha}}(t)\| = (\dot{x}, \dot{y})\Big/\sqrt{\dot{x}^2 + \dot{y}^2}, \\ N &= (-\dot{y}, \dot{x})\Big/\sqrt{\dot{x}^2 + \dot{y}^2}. \end{aligned}$$

Substituting these terms into (5) yields a formula for evaluating the curvature:

$$\begin{aligned} \kappa &= \frac{\dot{T} \cdot N}{v} \\ &= \left(\frac{(\ddot{x}, \ddot{y})}{\sqrt{\dot{x}^2 + \dot{y}^2}} + \frac{d}{dt}\left(\frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right)(\dot{x}, \dot{y}) \right) \cdot \frac{(-\dot{y}, \dot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big/ \sqrt{\dot{x}^2 + \dot{y}^2} \\ &= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}. \end{aligned} \tag{6}$$

We can write the formula simply as

$$\kappa = \frac{\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}}{\|\dot{\boldsymbol{\alpha}}\|^3}, \tag{7}$$

by treating the cross product as a scalar. The denominator $\|\dot{\boldsymbol{\alpha}}\|^3$ can be regarded as a correction to differentiations of $\boldsymbol{\alpha}$ when the curve is not unit-speed: division by $\|\dot{\boldsymbol{\alpha}}\|$ once for the velocity $\dot{\boldsymbol{\alpha}}$, a second time for the normal, and a third time for the acceleration $\ddot{\boldsymbol{\alpha}}$. When the curve is unit speed, $\dot{\boldsymbol{\alpha}} = T$ and $\ddot{\boldsymbol{\alpha}} = \dot{T} = \kappa N$. The formula (7) becomes an identity under $T \times N = 1$.

Example 3. Find the curvature of the curve $\boldsymbol{\alpha}(t) = (t^3 - t, t^2)$. so we have

$$\begin{aligned} \dot{\boldsymbol{\alpha}}(t) &= (3t^2 - 1, 2t), \\ \ddot{\boldsymbol{\alpha}}(t) &= (6t, 2). \end{aligned}$$

Therefore

$$\begin{aligned} \kappa &= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \\ &= \frac{(3t^2 - 1) \cdot 2 - 2t \cdot 6t}{\left((3t^2 - 1)^2 + (2t)^2\right)^{3/2}} \\ &= -\frac{6t^2 + 2}{(9t^4 - 2t^2 + 1)^{3/2}}. \end{aligned}$$

Finally, let us derive the formula for the total curvature over $[a, b]$. Let $\tilde{\boldsymbol{\alpha}}(s)$ be the unit-speed parametrization of $\boldsymbol{\alpha}$, where s is the arc length function. Let $\tilde{a}$ and $\tilde{b}$ be the parameter values such that

$$\tilde{\boldsymbol{\alpha}}(\tilde{a}) = \boldsymbol{\alpha}(a) \qquad \text{and} \qquad \tilde{\boldsymbol{\alpha}}(\tilde{b}) = \boldsymbol{\alpha}(b).$$

Then the total curvature of $\tilde{\boldsymbol{\alpha}}$ over $[\tilde{a}, \tilde{b}]$ is given by

$$\int_{\tilde{a}}^{\tilde{b}} \tilde{\kappa}(s)\, ds.$$

Since $ds/dt = \|\dot{\boldsymbol{\alpha}}(t)\|$, we substitute t for s in the above equation and obtain the total curvature formula:

$$\Phi(a, b) = \int_a^b \kappa(t)\, \|\dot{\boldsymbol{\alpha}}(t)\|\, dt.$$

References

[1] B. O'Neill. *Elementary Differential Geometry.* Academic Press, Inc., 1966.

[2] J. W. Rutter. *Geometry of Curves.* Chapman & Hall/CRC, 2000.

[3] A. Pressley. *Elementary Differential Geometry.* Springer-Verlag London, 2001.