Trajectory of a Billiard Ball and Recovery of Its Initial Velocities

Yan-Bin Jia
Department of Computer Science
Iowa State University
Ames, IA 50010, USA
jia@iastate.edu

Matthew T. Mason and Michael A. Erdmann
School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213, USA
{mason+,me}@cs.cmu.edu

1 Billiard Trajectory

What trajectory will a billiard ball follow on a pool table? Suppose it has initial velocity $\boldsymbol{v}_0$ and angular velocity $\boldsymbol{\omega}_0$. Assume no bouncing of the ball. It will first slide from its initial location $\boldsymbol{x}_0$ until the contact velocity reduces to zero at some location $\boldsymbol{x}_1$. Then it will roll without slip to a stop at another location $\boldsymbol{x}_2$. In this appendix, we will show that the billiard trajectory consists of a parabolic segment from sliding and a line segment from rolling without slip. The trajectory is illustrated in Figure 1.

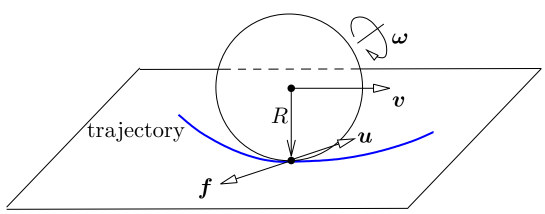

Figure 2: Contact velocity $\boldsymbol{u}$ of a billiard and contact frictional force $\boldsymbol{f}$.

Let m and R be the mass and radius of the billiard ball. The ball has the moment of inertia $\frac{2}{5}mR^2$. Its angular inertia Q is $\frac{2}{5}mR^2$ times the identity matrix. Denote by $g > 0$ the magnitude of the gravitational acceleration.

Let $\boldsymbol{v}$ and $\boldsymbol{\omega}$ be the velocity and angular velocity of the billiard, respectively, during the motion. The velocity at its contact point with the table, also referred to as the *sliding velocity*, is

$$\boldsymbol{u} = \boldsymbol{v} + R\hat{\boldsymbol{z}} \times \boldsymbol{\omega}, \tag{1}$$

where $\hat{\boldsymbol{z}} = (0, 0, 1)$ is the upward normal of the table. Due to its cross product with $\hat{\boldsymbol{z}}$, the z-component of $\boldsymbol{\omega}$ will have no effect over the ball trajectory. So it is ignored. As mentioned earlier, we also ignore any bouncing effect of the ball. So our analysis will make the following assumption:

$\boldsymbol{v}$ and $\boldsymbol{\omega}$ have no components in the z-direction.

1.1 Sliding

The sliding velocity $\boldsymbol{u}$ lies completely in the table plane. Sliding starts when

$$\boldsymbol{u}_0 = \boldsymbol{v}_0 + R\hat{\boldsymbol{z}} \times \boldsymbol{\omega}_0 \tag{2}$$

is not zero. Denote the direction of $\boldsymbol{u}$ by the unit vector $\hat{\boldsymbol{u}}$. The ball is subject to a frictional force $\boldsymbol{f} = -\mu_s mg\hat{\boldsymbol{u}}$, where μ_s is the coefficient of sliding friction. The ball's dynamics are governed by

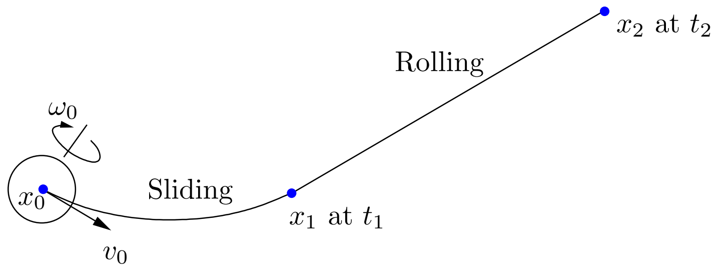

Figure 1: Trajectory of a billiard with initial velocity $\boldsymbol{v}_0$ and angular velocity $\boldsymbol{\omega}_0$.

two equations:

$$\begin{aligned}
m\dot{\boldsymbol{v}} &= -\mu_s mg\hat{\boldsymbol{u}}, \\
Q\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (Q\boldsymbol{\omega}) &= Q\dot{\boldsymbol{\omega}} \\
&= \frac{2}{5}mR^2\dot{\boldsymbol{\omega}} \\
&= (-R\hat{\boldsymbol{z}}) \times (-\mu_s mg\hat{\boldsymbol{u}}) \\
&= \mu_s mgR\hat{\boldsymbol{z}} \times \hat{\boldsymbol{u}}.
\end{aligned}$$

Immediately, we obtain the ball's acceleration and angular acceleration:

$$\dot{\boldsymbol{v}} = -\mu_s g\hat{\boldsymbol{u}}, \tag{3}$$

$$\dot{\boldsymbol{\omega}} = \frac{5}{2R}\mu_s g\hat{\boldsymbol{z}} \times \hat{\boldsymbol{u}}. \tag{4}$$

We derive the contact acceleration by differentiating (1) and substituting (3) and (4) in:

$$\begin{aligned}
\dot{\boldsymbol{u}} &= -\mu_s g\hat{\boldsymbol{u}} + \frac{5}{2}\mu_s g\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{u}}) \\
&= -\mu_s g\hat{\boldsymbol{u}} + \frac{5}{2}\mu_s g((\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{z}} - \hat{\boldsymbol{u}}) \\
&= -\frac{7}{2}\mu_s g\hat{\boldsymbol{u}}, \qquad (5)
\end{aligned}$$

since $\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{u}} = 0$. So we infer that $\dot{\boldsymbol{u}}$ is opposite to the initial contact velocity $\boldsymbol{u}_0$. Consequently, $\boldsymbol{u}$ will never change its direction until it becomes zero. From now on, $\hat{\boldsymbol{u}} = \boldsymbol{u}_0/\|\boldsymbol{u}_0\|$ represents the constant sliding direction. Integration of (5) yields the sliding velocity:

$$\boldsymbol{u} = \boldsymbol{u}_0 - \frac{7}{2}\mu_s gt\hat{\boldsymbol{u}}. \tag{6}$$

Since $\hat{\boldsymbol{u}}$ is constant, we easily integrate (3) and (4):

$$\boldsymbol{v} = \boldsymbol{v}_0 - \mu_s gt\hat{\boldsymbol{u}}, \tag{7}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \frac{5}{2R}\mu_s gt\hat{\boldsymbol{z}} \times \hat{\boldsymbol{u}}. \tag{8}$$

Equations (7) and (8) were also derived in [1, pp. 10–11].

The sliding trajectory is obtained from one more round of integration — of (7):

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad = \quad \boldsymbol{x}_0 + \boldsymbol{v}_0 t - \frac{1}{2}\mu_s g t^2 \hat{\boldsymbol{u}}. \tag{9}$$

Sliding ends when the contact velocity $\boldsymbol{u}$ reduces to zero. From (6) this takes place at time

$$t_1 = \frac{2}{7} \cdot \frac{\|\boldsymbol{u}_0\|}{\mu_s g}. \tag{10}$$

When it happens, the ball has the following velocity from (7):

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{v}_0 - \mu_s g \cdot \frac{2}{7}\frac{\|\boldsymbol{u}_0\|}{\mu_s g}\hat{\boldsymbol{u}} \\ &= \boldsymbol{v}_0 - \frac{2}{7}\boldsymbol{u}_0 \\ &= \frac{5}{7}\boldsymbol{v}_0 - \frac{2}{7}R\hat{\boldsymbol{z}} \times \boldsymbol{\omega}_0, \end{aligned} \tag{11}$$

by (2). Let $\boldsymbol{\omega}_1$ be the angular velocity at t_1. We take the cross product of $\hat{\boldsymbol{z}}$ with both sides of $\boldsymbol{v}_1 + R\hat{\boldsymbol{z}} \times \boldsymbol{\omega}_1 = 0$, obtaining

$$\boldsymbol{\omega}_1 = \frac{1}{R}\boldsymbol{z} \times \boldsymbol{v}_1, \tag{12}$$

Substitution of (10) into (9) indicates that the ball will end sliding at the position

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \frac{2}{49} \cdot \frac{\|\boldsymbol{v}_0 + R\hat{\boldsymbol{z}} \times \boldsymbol{\omega}_0\|}{\mu_s g}(6\boldsymbol{v}_0 - R\hat{\boldsymbol{z}} \times \boldsymbol{\omega}_0). \tag{13}$$

1.2 Pure Rolling

That $\boldsymbol{u} = 0$ at time t_1 implies $\boldsymbol{v}_1 \cdot \boldsymbol{\omega}_1 = 0$ via cross product of $\boldsymbol{v}$ with (1). It is possible that $t_1 = 0$. At t_1, since $\boldsymbol{\omega}_1$ is orthogonal to $\boldsymbol{v}_1$, the frictional force opposes the direction of $\boldsymbol{v}_1$. Hence neither $\boldsymbol{\omega}$ nor $\boldsymbol{v}$ will change its direction from t_1 and on. The ball begins pure rolling without slip until it comes to rest; and $\boldsymbol{u} = 0$ holds as it rolls.

The rolling trajectory is a straight line segment. To derive the deceleration of the ball, we adopt Marlow's treatment [1, p. 12] based on the principle of energy dissipation. Because the ball rolls on a line, we can now denote its velocity and angular velocity by their magnitudes v and ω with no ambiguity. Since $v = R\omega$ under pure rolling, the energy of the ball is

$$\begin{aligned} E &= \frac{1}{2}mv^2 + \frac{1}{2} \cdot \left(\frac{2}{5}mR^2\right)\omega^2 \\ &= \frac{7}{10}mv^2. \end{aligned}$$

Differentiate the above equation:

$$\dot{E} = \frac{7}{5}mv\dot{v}. \tag{14}$$

Let μ_r be the coefficient of rolling friction. The energy dissipation due to rolling friction is

$$\dot{E} = -\mu_r m g v. \tag{15}$$

Comparing (14) and (15), we obtain the deceleration of the rolling ball:

$$\dot{v} = -\frac{5}{7}\mu_r g. \tag{16}$$

Thus the velocity magnitude varies as the ball rolls according to

$$v = \|\boldsymbol{v}_1\| - \frac{5}{7}\mu_r g(t - t_1).$$

The ball will come to stop at time

$$t_2 = t_1 + \frac{7}{5}\frac{\|\boldsymbol{v}_1\|}{\mu_r g}. \tag{17}$$

In the vector forms, the ball velocities during the rolling over time $[t_1, t_2]$ are given as

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{v}_1 - \frac{5}{7}\mu_r g(t - t_1)\hat{\boldsymbol{v}}_1, \\ \boldsymbol{\omega} &= \frac{1}{R}\hat{\boldsymbol{z}} \times \boldsymbol{v}, \end{aligned}$$

where $\hat{\boldsymbol{v}}_1$ is the unit vector in the direction of $\boldsymbol{v}_1$. Integrating the velocity equation above yields the line trajectory of the rolling ball:

$$\boldsymbol{x} = \boldsymbol{x}_1 + (t - t_1)\boldsymbol{v}_1 - \frac{5}{14}\mu_r g(t - t_1)^2\hat{\boldsymbol{v}}_1. \tag{18}$$

Plugging (17) into the above equation, the ball will come to rest at the location

$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \frac{7}{10} \cdot \frac{\|\boldsymbol{v}_1\|}{\mu_r g}\boldsymbol{v}_1. \tag{19}$$

1.3 Parabolic Sliding Trajectory

Under sliding, $\boldsymbol{v}_0 \cdot \boldsymbol{\omega}_0 \neq 0$; hence $\boldsymbol{v}_0 \times \boldsymbol{u}_0 \neq 0$ by (2). We take the cross products of $\hat{\boldsymbol{u}}$ with both sides of the sliding trajectory equation (9), obtaining the time when the ball reaches the location $\boldsymbol{x}$:

$$t = \frac{\Delta\boldsymbol{x} \times \hat{\boldsymbol{u}}}{\boldsymbol{v}_0 \times \hat{\boldsymbol{u}}},$$

where $\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_0$. Here, cross products are treated as scalars along the z-axis. Substitution of t back into (9) gives rise to a pair of equations:

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{v}_0 \cdot \frac{\Delta\boldsymbol{x} \times \hat{\boldsymbol{u}}}{\boldsymbol{v}_0 \times \hat{\boldsymbol{u}}} - \frac{1}{2}\mu_s g\hat{\boldsymbol{u}}\left(\frac{\Delta\boldsymbol{x} \times \hat{\boldsymbol{u}}}{\boldsymbol{v}_0 \times \hat{\boldsymbol{u}}}\right)^2. \tag{20}$$

The first equation of the pair is an identity over the x-coordinate. The second equation of the pair, $y = \boldsymbol{x} \cdot (0, 1)^T$ relates the y-coordinate to the x-coordinate of every point on the trajectory. Hence it is the implicit equation of the sliding trajectory, which is a parabola as stated in the following claim.

Theorem 1 *The sliding trajectory of a billiard ball that starts at the location* $\boldsymbol{x}_0$ *with initial velocity* $\boldsymbol{v}_0$ *and angular velocity* $\boldsymbol{\omega}_0$*, all planar and* $\boldsymbol{v}_0 \cdot \boldsymbol{\omega}_0 \neq 0$*, is part of a parabola that results from the parabola*

$$y = \frac{\mu_s g}{2\|\boldsymbol{v}_0 \times \hat{\boldsymbol{u}}\|^2} x^2$$

via a rotation about the origin through $\phi + \pi/2$*, where* $\hat{\boldsymbol{u}} = (\cos\phi, \sin\phi)^T$*, followed by a translation of*

$$\boldsymbol{x}_0 + \frac{\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}}}{\mu_s g}\left(\boldsymbol{v}_0 - \frac{1}{2}(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}\right).$$

Proof We first determine the point on the trajectory with the maximum curvature. This would be the vertex if the curve turns out to be a parabola. The curvature function computed from the parametric form (9) is

$$\kappa = \frac{\dot{\boldsymbol{x}} \times \ddot{\boldsymbol{x}}}{\|\dot{\boldsymbol{x}}\|^3} = -\frac{\mu_s g \boldsymbol{v}_0 \times \hat{\boldsymbol{u}}}{\|\boldsymbol{v}_0 - \mu_s g t \hat{\boldsymbol{u}}\|^3}. \tag{21}$$

We need only find the minimum of the denominator, or equivalently, the minimum speed of the billiard since $\boldsymbol{v} = \boldsymbol{v}_0 - \mu_s g t \hat{\boldsymbol{u}}$ according to (7). Minimization of

$$\boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{v}_0 \cdot \boldsymbol{v}_0 - 2\mu_s g(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})t + \mu_s^2 g^2 t^2$$

yields the parameter value

$$t^* = \frac{\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}}}{\mu_s g}. \tag{22}$$

Substitution of t^* into (9) then yields

$$\boldsymbol{x}^* = \boldsymbol{x}_0 + \frac{\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}}}{\mu_s g}\left(\boldsymbol{v}_0 - \frac{1}{2}(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}\right).$$

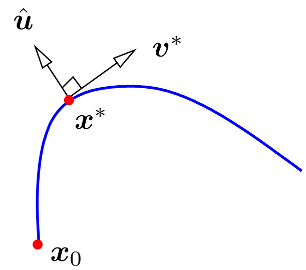

Figure 3: Parabolic sliding trajectory with vertex $\boldsymbol{x}^*$.

Here, $\boldsymbol{x}^*$ is the translation we need.

The orientation of the curve is indicated by the velocity vector (7) at t^* determined as $\boldsymbol{v}^* = \boldsymbol{v}_0 - (\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}$, which is tangent to the curve. See Figure 3. In other words, the tangent is orthogonal to $\hat{\boldsymbol{u}} = (\cos\phi, \sin\phi)$. Furthermore, since

$$(\boldsymbol{x}^* - \boldsymbol{x}_0) \cdot \hat{\boldsymbol{u}} = \frac{1}{2\mu_s g}(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})^2 > 0,$$

the curve bends away from $\boldsymbol{u}$. This implies that $\phi + \pi/2$ is the desired rotation. Applying the rotation to the implicit equation (i.e., second in (20)) yields the ordinate as a quadratic function of the abscissa. Thus the curve is a parabola with vertex at $\boldsymbol{x}^*$ and axis of symmetry in the direction $-\hat{\boldsymbol{u}}$.

Finally, we decide the canonical form $y = ax^2$ of the parabola. Its curvature $2a$ at the vertex must be equal to the curvature value (21) at t^*. So we obtain

$$\begin{aligned} a &= \frac{\mu_s g \|\boldsymbol{v}_0 \times \hat{\boldsymbol{u}}\|}{2\|\boldsymbol{v}_0 - (\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}\|^3} \\ &= \frac{\mu_s g}{2} \cdot \frac{\|\boldsymbol{v}_0 \times \hat{\boldsymbol{u}}\|}{\|\boldsymbol{v}_0 \times \hat{\boldsymbol{u}}\|^3} \\ &= \frac{\mu_s g}{2\|\boldsymbol{v}_0 \times \hat{\boldsymbol{u}}\|^2}. \end{aligned}$$

□

1.4 Straight Sliding Trajectory

When $\boldsymbol{v}_0 \perp \boldsymbol{\omega}_0$, the ball slides along a straight trajectory. It is more convenient to treat the case in terms of two orthogonal directions $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ such that $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{z}}$. The direction $\hat{\boldsymbol{i}}$ is of $\boldsymbol{v}_0$ if $\boldsymbol{v}_0 \neq 0$ and otherwise it is of $\boldsymbol{\omega}_0 \times \hat{\boldsymbol{z}}$. For convenience, we write $\boldsymbol{\omega}_0 = \omega_0 \hat{\boldsymbol{j}}$, $\boldsymbol{v}_k = v_k \hat{\boldsymbol{i}}$, for $k = 0, 1$, and $\boldsymbol{x}_l = x_l \hat{\boldsymbol{i}}$ for $l = 0, 1, 2$. Note that $v_0 \geq 0$. Without loss of generality, we let $\hat{\boldsymbol{i}}$ point to the right. Then $\hat{\boldsymbol{j}}$ points inward to make $\hat{\boldsymbol{z}}$ upward. Hence $\omega < 0$ for a counterclockwise rotation; and $\omega > 0$ for a clockwise rotation. The contact velocity and the path of the ball are also collinear with $\boldsymbol{i}$. The initial sliding velocity by (2) becomes

$$\boldsymbol{u}_0 = u_0 \hat{\boldsymbol{i}} = (v_0 - R\omega_0)\hat{\boldsymbol{i}}. \tag{23}$$

If $v_0 < R\omega_0$, $\hat{\boldsymbol{u}}_0$ opposes the initial velocity which consequently increases under sliding friction.

Recall that t_1 and t_2 are the times at which sliding and rolling ends, respectively. Equations (13), (11), (10), (19), and (17) now respectively become:

$$x_1 - x_0 = \frac{2}{49\mu_s g}|v_0 - R\omega_0|(6v_0 + R\omega_0), \tag{24}$$

$$v_1 = \frac{1}{7}(5v_0 + 2R\omega_0), \tag{25}$$

$$t_1 = \frac{2}{7} \cdot \frac{|v_0 - R\omega_0|}{\mu_s g}, \tag{26}$$

$$x_2 - x_1 = \frac{1}{70\mu_r g}|5v_0 + 2R\omega_0|(5v_0 + 2R\omega_0), \tag{27}$$

$$t_2 = t_1 + \frac{1}{5\mu_r g}|5v_0 + 2R\omega_0|. \tag{28}$$

Suppose the ball's velocity $\boldsymbol{v}$ becomes zero at time t_3 and location $\boldsymbol{x}_3 = x_3 \hat{\boldsymbol{i}}$. If this happens during sliding, then we infer from (7) that $\boldsymbol{v}_0$ and $\hat{\boldsymbol{u}}$ are in the same direction, namely, $v_0 \geq R\omega_0$. In this case, from (7) and (9) we have

$$t_3 = \frac{v_0}{\mu_s g}, \tag{29}$$

$$x_3 - x_0 = \frac{1}{2}\frac{v_0^2}{\mu_s g}. \tag{30}$$

The rest follows from a case-based analysis that measures $R\omega_0$ in terms of v_0. The five cases are illustrated in Figure 4.

(1) $R\omega_0 < -\frac{5}{2}v_0$. Hence $R\omega_0 > v_0$. This implies $0 < t_3 < t_1$ by (26) and (29), and therefore $x_1 < x_3$. From (23) $\boldsymbol{v}_0$ and $\boldsymbol{u}_0$ are in the same direction. The ball will reverse its motion at $x_3 = v_0^2/2\mu_s g$ before sliding ends. We also have $x_2 < x_1$ by (27).

(2) $R\omega_0 = -\frac{5}{2}v_0$. The ball has zero velocity and thus zero angular velocity when sliding ends. There is no pure rolling phase with $x_1 = x_2 = x_3$ and $t_1 = t_2 = t_3 = v_0/\mu_s g$.

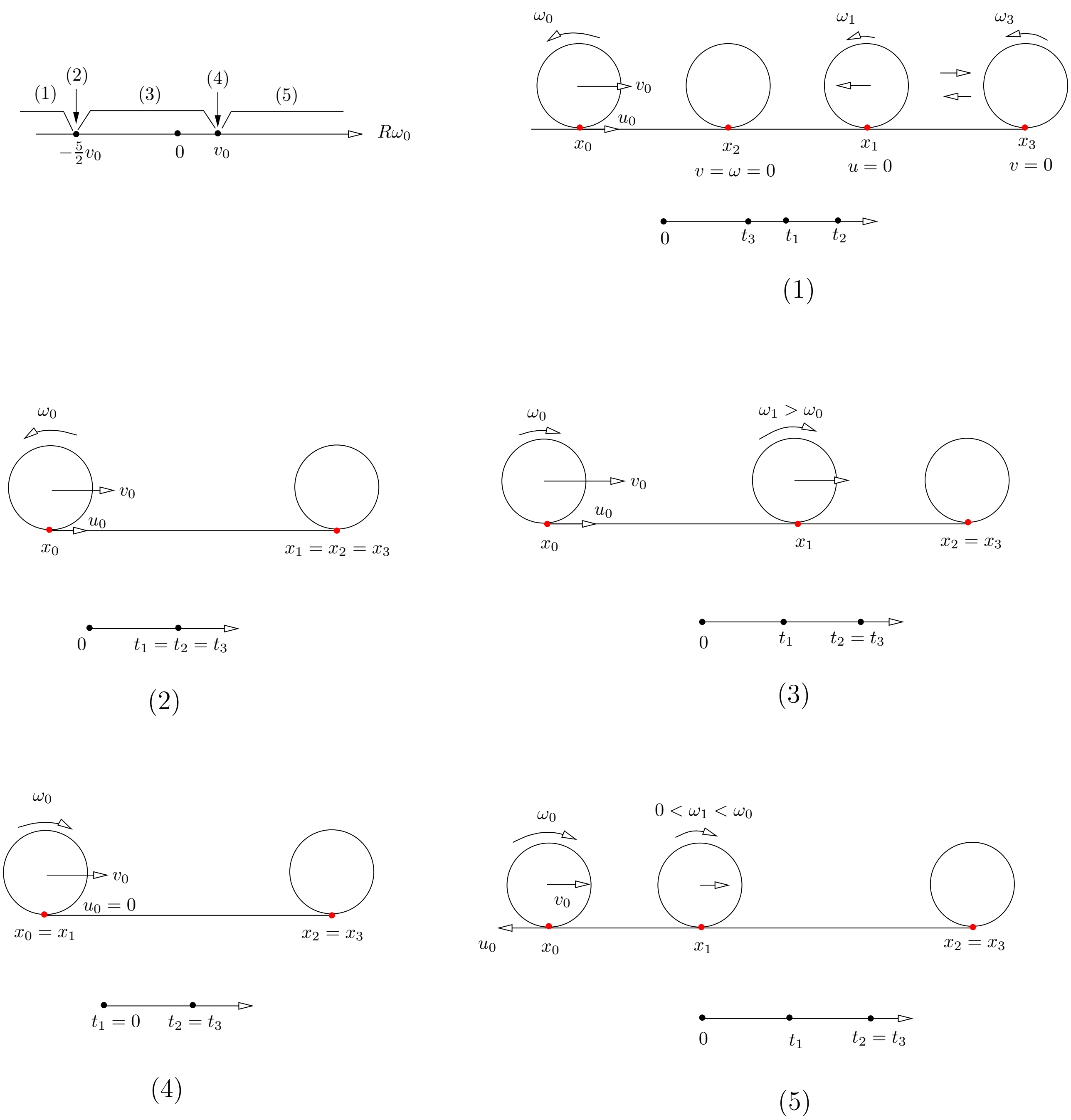

Figure 4: Five scenarios (1)–(5) of straight billiard motion based on $R\omega_0$ relative to v_0 shown in the upper left corner.

(3) $-\frac{5}{2}v_0 < R\omega_0 < v_0$. So the initial sliding velocity $u_0 > 0$. In this case, $t_3 > t_1$; that is, the ball's velocity reduces to zero after sliding ends. This implies that $t_2 = t_3$. The frictional force increases ω during sliding.

(4) $R\omega_0 = v_0$. In this case, $t_1 = 0$. The ball starts pure rolling right away.

(5) $R\omega_0 > v_0$. Here $\boldsymbol{v}_0$ and $\boldsymbol{u}_0$ are in opposite directions. The ball's velocity increases while angular velocity decreases until $v = R\omega$ when pure rolling starts.

2 Recovering Initial Velocities from a Billiard Trajectory

From the video of a billiard motion we can reconstruct its trajectory, and estimate the locations $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ at which the ball begins the motion and ends sliding. Based on the information, we would like to solve for the ball's initial velocity $\boldsymbol{v}_0$ and angular velocity $\boldsymbol{\omega}_0$ assumed to have only x- and y-components.[1] Both velocities are in the xy-plane. The implicit equation for the trajectory, whether a parabolic segment or a line segment, is known, from fitting, for example. When the trajectory is a line segment, the final resting location $\boldsymbol{x}_2$ of the ball is also needed to determine the initial velocities.

2.1 Parabolic Segment

We infer that $\boldsymbol{v}_0 \cdot \boldsymbol{\omega}_0 \neq 0$. Applying standard coordinate transformation to its equation, we can determine that the parabola is generated from a rotation of $y = ax^2$, $a > 0$, through θ followed by a translation of $\boldsymbol{x}^* = (x^*, y^*)$. Here, a, θ, and $\boldsymbol{x}^*$ are known. The sliding trajectory has the parametric form (9) dependent on $\boldsymbol{v}_0$, $\boldsymbol{\omega}_0$ and $\boldsymbol{x}_0$. Take the cross product of $\hat{\boldsymbol{z}}$ with the initial contact velocity $\boldsymbol{u}_0 = \boldsymbol{v}_0 + R\hat{\boldsymbol{z}} \times \boldsymbol{\omega}_0$:

$$\begin{aligned} \hat{\boldsymbol{z}} \times \boldsymbol{u}_0 &= \hat{\boldsymbol{z}} \times \boldsymbol{v}_0 + R((\hat{\boldsymbol{z}} \cdot \boldsymbol{\omega}_0)\hat{\boldsymbol{z}} - \boldsymbol{\omega}_0) \\ &= \hat{\boldsymbol{z}} \times \boldsymbol{v}_0 - R\boldsymbol{\omega}_0. \end{aligned}$$

This leads to

$$\boldsymbol{\omega}_0 = \frac{1}{R}\hat{\boldsymbol{z}} \times (\boldsymbol{v}_0 - \boldsymbol{u}_0). \tag{31}$$

We will first determine the direction $\hat{\boldsymbol{u}}$ of $\boldsymbol{u}_0$, and then $\boldsymbol{v}_0$ and $\boldsymbol{u}_0$. Afterward, $\boldsymbol{\omega}_0$ will be known from (31). Let $\hat{\boldsymbol{u}} = (\cos\phi, \sin\phi)$. Under Theorem 1, $\phi + \pi/2 = \theta$. We immediately have

$$\hat{\boldsymbol{u}} = \left(\cos\left(\theta - \frac{\pi}{2}\right), \sin\left(\theta - \frac{\pi}{2}\right)\right) = (\sin\theta, -\cos\theta). \tag{32}$$

Next, we recover the initial velocity $\boldsymbol{v}_0$. Denote $\boldsymbol{p} = \boldsymbol{x}^* - \boldsymbol{x}_0$. Theorem 1 states that

$$\boldsymbol{p} = \frac{\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}}}{\mu_s g}\left(\boldsymbol{v}_0 - \frac{1}{2}(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}\right).$$

Take the dot products of $\hat{\boldsymbol{u}}$ with both sides of the above equation, and rearrange the terms:

$$(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})^2 = 2\mu_s g \boldsymbol{p} \cdot \hat{\boldsymbol{u}}. \tag{33}$$

[1] We cannot recover the z-components of $\boldsymbol{v}_0$ and $\boldsymbol{\omega}_0$ since they do not affect the trajectory in the horizontal plane.

Meanwhile, Theorem 1 also states that

$$a = \frac{\mu_s g}{2\|\boldsymbol{v}_0 \times \boldsymbol{u}\|^2}.$$

Hence

$$\|\boldsymbol{v}_0 \times \boldsymbol{u}\|^2 = \frac{\mu_s g}{2a}. \tag{34}$$

Add up equations (33) and (34):

$$\begin{aligned} \|\boldsymbol{v}_0\|^2 &= (\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})^2 + \|\boldsymbol{v}_0 \times \hat{\boldsymbol{u}}\|^2 \\ &= \mu_s g \left(2\boldsymbol{p} \cdot \hat{\boldsymbol{u}} + \frac{1}{2a}\right) \end{aligned}$$

This gives us the magnitude of the initial ball velocity:

$$\|\boldsymbol{v}_0\| = \sqrt{\mu_s g \left(2\boldsymbol{p} \cdot \hat{\boldsymbol{u}} + \frac{1}{2a}\right)}. \tag{35}$$

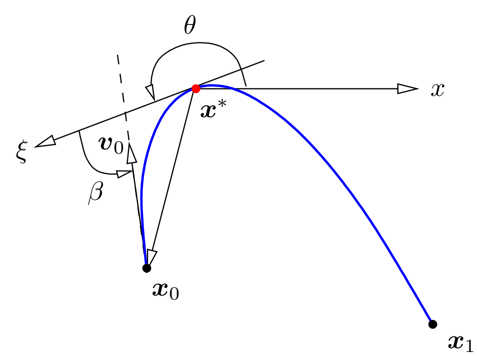

Figure 5: Determining the orientation of initial ball velocity.

The direction $\hat{\boldsymbol{v}}_0$ of $\boldsymbol{v}_0$ must be tangent to the curve at $\boldsymbol{x}_0$, as shown in Figure 5. We project the vector $-\boldsymbol{p} = \boldsymbol{x}_0 - \boldsymbol{x}^*$ onto the direction $\hat{\boldsymbol{\xi}} = (\cos\theta, \sin\theta)$ of the "x-axis" of the local frame of the parabola, obtaining the abscissa

$$\xi = -\boldsymbol{p} \cdot \hat{\boldsymbol{\xi}}.$$

In the local frame, the tangent vector at a point $(\xi, a\xi^2)$ on the parametrized parabola has the direction $(1, 2a\xi)$. Let $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be the tangential angle at $\boldsymbol{x}_0$ with respect to the ξ-axis such that

$$(\cos\beta, \sin\beta) = \frac{(1, 2a\xi)}{\sqrt{1 + 4a^2\xi^2}}.$$

The initial ball velocity is in the direction

$$\hat{\boldsymbol{v}}_0 = \begin{cases} (\cos(\theta+\beta), \sin(\theta+\beta)), & \text{if } (\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \hat{\boldsymbol{\xi}} > 0, \\ -(\cos(\theta+\beta), \sin(\theta+\beta)), & \text{otherwise.} \end{cases}$$

Combining this with the magnitude (35), we have determined the initial ball velocity:

$$\boldsymbol{v}_0 = \sqrt{\mu_s g \left(2\boldsymbol{p} \cdot \hat{\boldsymbol{u}} + \frac{1}{2a}\right)}\hat{\boldsymbol{v}}_0. \tag{36}$$

Now we use $\hat{\boldsymbol{u}}$ and $\boldsymbol{v}_0$ to derive the magnitude u_0 of the initial contact velocity $\boldsymbol{u}_0$. This will give us $\boldsymbol{u}_0 = u_0\hat{\boldsymbol{u}}$, and subsequently $\boldsymbol{\omega}_0$ from (31). Since $\boldsymbol{u}_0 = \boldsymbol{v}_0 + R\hat{\boldsymbol{z}} \times \boldsymbol{\omega}_0$, equation (13) can be rewritten as

$$\boldsymbol{x}_1 - \boldsymbol{x}_0 = \frac{2}{49}\frac{u_0}{\mu_s g}(7\boldsymbol{v}_0 - \boldsymbol{u}_0). \tag{37}$$

Take the cross products of $\hat{\boldsymbol{u}}$ with both sides of the above equation:

$$(\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \hat{\boldsymbol{u}} = \frac{2}{49} \frac{u_0}{\mu_s g} \left(7(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}}) - u_0\right).$$

Rewrite the above as a quadratic equation in u_0:

$$u_0^2 - 7(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})u_0 + \frac{49}{2}\mu_s g(\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \hat{\boldsymbol{u}} = 0.$$

The solution to the equation is

$$u_0 = \frac{7}{2}\left(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}} \pm \sqrt{(\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}})^2 - 2\mu_s g(\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \hat{\boldsymbol{u}}}\right). \tag{38}$$

In (38), the sign '+' is chosen if $\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}} \leq 0$. If $\boldsymbol{v}_0 \cdot \hat{\boldsymbol{u}} > 0$, both signs '+' and '−' are possible. In this case, we need to verify the solution using (37). It can be shown by contradiction that a unique solution exists as long as $\boldsymbol{v}_0$ and $\boldsymbol{\omega}_0$ are not orthogonal (which is true given the parabolic trajectory).

2.2 Line Segment

When the sliding trajectory is a line, we infer that $\boldsymbol{v}_0$ and $\boldsymbol{\omega}_0$ must be orthogonal to each other. Let the unit vector $\hat{\boldsymbol{i}}$ be in the direction of $\boldsymbol{x}_2 - \boldsymbol{x}_0$, and $\hat{\boldsymbol{j}}$ such that $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{z}}$. Write $\boldsymbol{v}_0 = v_0\hat{\boldsymbol{i}}$, $\boldsymbol{\omega}_0 = \omega_0\hat{\boldsymbol{j}}$, and $\boldsymbol{x}_k = x_k\hat{\boldsymbol{i}}$ for $k = 0, 1, 2$. Equations (24) and (27) hold for $x_1 - x_0$ and $x_2 - x_1$, respectively.

To solve for v_0 and ω_0, first from (27) we obtain

$$5v_0 + 2R\omega_0 = C, \tag{39}$$

where

$$C = \begin{cases} \sqrt{70\mu_r g(x_2 - x_1)}, & \text{if } x_2 - x_1 \geq 0; \\ -\sqrt{70\mu_r g(x_1 - x_2)}, & \text{if } x_2 - x_1 < 0. \end{cases} \tag{40}$$

From (24) a constant $D \equiv \frac{49}{2}\mu_s g(x_1 - x_0)$ is rewritten as follows:

$$\begin{aligned} D &= |v_0 - R\omega_0|(6v_0 + R\omega_0) \\ &= \left|v_0 - \frac{C}{2} + \frac{5}{2}v_0\right|\left(6v_0 + \frac{C}{2} - \frac{5}{2}v_0\right) \qquad \text{by (39)} \\ &= \left|\frac{7}{2}v_0 - \frac{C}{2}\right|\left(\frac{7}{2}v_0 + \frac{C}{2}\right) \qquad (41) \\ &= \begin{cases} \frac{49}{4}v_0^2 - \frac{1}{4}C^2, & \text{if } R\omega_0 \leq v_0; \\ \frac{1}{4}C^2 - \frac{49}{4}v_0^2, & \text{if } R\omega_0 > v_0. \end{cases} \qquad (42) \end{aligned}$$

Hence we solve the above for v_0:

$$v_0 = \begin{cases} \frac{1}{7}\sqrt{C^2 + 4D}, & \text{if } R\omega_0 \leq v_0; \\ \frac{1}{7}\sqrt{C^2 - 4D}, & \text{if } R\omega_0 > v_0. \end{cases} \tag{43}$$

The initial angular velocity follows from (39):

$$\omega_0 = \frac{1}{2R}(C - 5v_0). \tag{44}$$

We need to hypothesize $R\omega_0 \leq v_0$ and $R\omega_0 > v_0$ respectively. Then check if the values of v_0 and ω_0 according to (43) and (44) satisfy the original hypothesis. Multiple solutions may exist.

References

[1] Wayland C. Marlow. *The Physics of Pocket Billiards*. Marlow Advanced Systems Technologies, 1994.