Straight Sliding Trajectory of a Ball

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This manuscript analyzes the special case where the ball's initial velocity $\boldsymbol{v}_0$ and angular velocity $\boldsymbol{\omega}_0$, all in the supporting plane under Assumptions (A1) and (A2), are orthogonal to each other. Under Proposition 1, the ball will move along a straight trajectory. It is more convenient to use two orthogonal directions $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ such that $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{z}}$. If $\boldsymbol{v}_0 \neq 0$, chose $\hat{\boldsymbol{i}}$ to be in the direction of $\boldsymbol{v}_0$; otherwise, chose $\hat{\boldsymbol{j}}$ to be in the direction of $\boldsymbol{\omega}_0$. For convenience, we write ω_0 for the signed magnitude of $\boldsymbol{\omega}_0$ along $\hat{\boldsymbol{j}}$, and v_0, v_r, x_r, x_e respectively as the signed magnitudes of $\boldsymbol{v}_0, \boldsymbol{v}_r, \boldsymbol{x}_r, \boldsymbol{x}_e$ along $\hat{\boldsymbol{i}}$. Note that $v_0 \geq 0$. The initial sliding velocity by (1) becomes

$$\boldsymbol{s}_0 = u_0\hat{\boldsymbol{i}} = (v_0 - \rho\omega_0)\hat{\boldsymbol{i}}. \tag{53}$$

If $v_0 < \rho\omega_0$, $\hat{\boldsymbol{s}}_0$ opposes the initial velocity which consequently increase under sliding friction.

Recall that t_r and t_e are the times at which sliding and rolling ends, respectively. Equations (13), (14), (16), (21), and (23) have become:

$$t_r = \frac{2|v_0 - \rho\omega_0|}{7\mu_s g}, \tag{54}$$

$$v_r = \frac{1}{7}(5v_0 + 2\rho\omega_0), \tag{55}$$

$$x_r = \frac{2}{49\mu_s g}|v_0 - \rho\omega_0|(6v_0 + \rho\omega_0), \tag{56}$$

$$t_e = t_r + \frac{1}{5\mu_r g}|5v_0 + 2\rho\omega_0|, \tag{57}$$

$$x_e - x_r = \frac{1}{70\mu_r g}|5v_0 + 2\rho\omega_0|(5v_0 + 2\rho\omega_0). \tag{58}$$

Suppose that the ball's velocity $\boldsymbol{v}$ becomes zero at time t_m and location $\boldsymbol{x}_m = x_m\hat{\boldsymbol{i}}$. If this happens during sliding, then we infer from (9) that $\boldsymbol{v}_0$ and $\hat{\boldsymbol{s}}$ are in the same direction, namely, $v_0 \geq \rho\omega_0$ by (53). In this case, from (9) and (12) we obtain

$$t_m = \frac{v_0}{\mu_s g}, \tag{59}$$

$$x_m = \frac{v_0^2}{2\mu_s g}. \tag{60}$$

The rest of the analysis follows from a case-based reasoning that measures $\rho\omega_0$ in terms of v_0. The five scenarios are illustrated in Figure 19.

(1) $\rho\omega_0 < -\frac{5}{2}v_0$. Hence $\rho\omega_0 < v_0$. This implies $0 < t_m < t_r$ by (54) and (59). From (53), $\boldsymbol{v}_0$ and $\boldsymbol{s}_0$ are in the same direction. The ball will reverse its motion at $x_m = v_0^2/2\mu_s g$ before sliding ends. Therefore, $x_r < x_m$. We also have $x_e < x_r$ by (58).

(2) $\rho\omega_0 = -\frac{5}{2}v_0$. The ball has zero velocity when sliding ends, and thus zero angular velocity by (17). There is no pure rolling phase. We have $x_r = x_e = x_m$ and $t_r = t_e = t_m = v_0/\mu_s g$.

(3) $-\frac{5}{2}v_0 < \rho\omega_0 < v_0$. So the initial sliding velocity $u_0 > 0$. In this case, $t_m > t_r$; that is, the ball's velocity reduces to zero after sliding ends. This implies $t_e = t_m$. The frictional force increases ω during sliding.

(4) $\rho\omega_0 = v_0$. In this case, $t_r = 0$. The ball starts pure rolling right away.

(5) $\rho\omega_0 > v_0$. Here $\boldsymbol{v}_0$ and $\boldsymbol{s}_0$ are in opposite directions. The ball's velocity increases while angular velocity decreases until $v = \rho\omega$ when pure rolling starts.

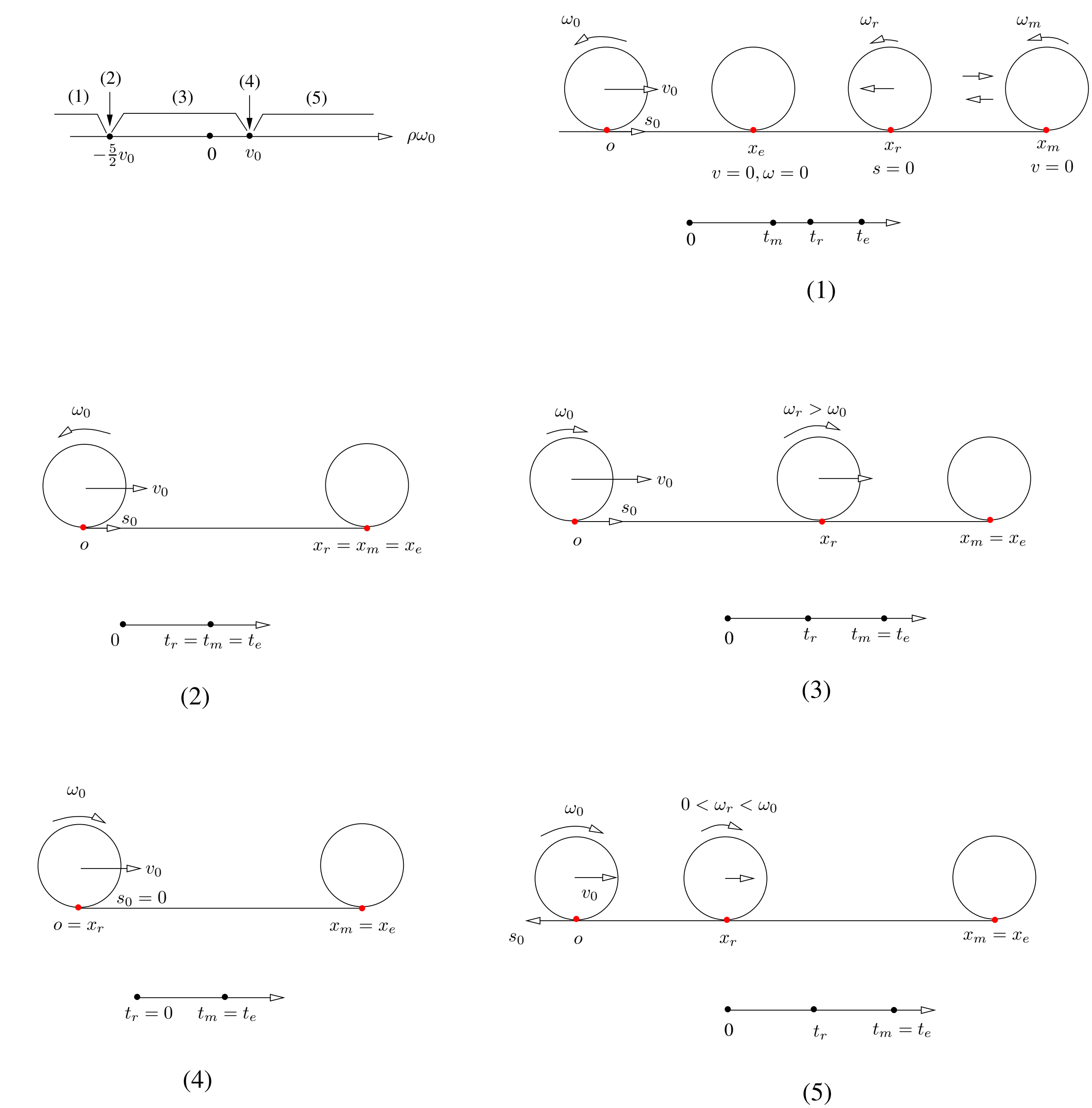

Fig. 19. Five scenarios (1)–(5) of a straight ball motion based on $\rho\omega_0$ relative to v_0 shown in the upper left corner.