Arbitrary-Speed Curves*

(Com S 477/577 Notes)

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The Frenet formulas are valid only for unit-speed curves; they tell the rate of change of the orthonormal vectors T, N, B *with respect to arc length*. However, most curves that arise from practice are hardly parameterized with arc length. Also, for numerical computations, reparametrization with arc length is impractical, since it is rarely possible to find an explicit formula.

When a regular curve $\boldsymbol{\alpha}$ is not unit-speed, we can transfer to $\boldsymbol{\alpha}$ the Frenet apparatus of a unit-speed reparametrization $\tilde{\boldsymbol{\alpha}}$ of $\boldsymbol{\alpha}$, with no need of any closed form for $\tilde{\boldsymbol{\alpha}}$. Explicitly, if s is an arc-length function for $\boldsymbol{\alpha}$, then

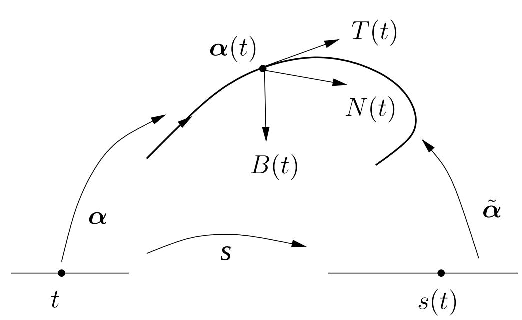
$$\boldsymbol{\alpha}(t) = \tilde{\boldsymbol{\alpha}}\Big(s(t)\Big), \qquad \text{for all } t,$$

or, in function notation, $\boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}}(s)$. Now if $\tilde{\boldsymbol{\alpha}}$ has curvature $\tilde{\kappa} > 0$, $\tilde{\tau}$, $\tilde{T}$, $\tilde{N}$, and $\tilde{B}$ are defined for $\tilde{\boldsymbol{\alpha}}$ as we studied before. We define for $\boldsymbol{\alpha}$ the

$$\begin{array}{ll} \textit{Curvature function:} & \kappa(t) = \tilde{\kappa}(s) \\ \textit{Torsion function:} & \tau(t) = \tilde{\tau}(s) \\ \textit{Unit tangent:} & T(t) = \tilde{T}(s) \\ \textit{Principal normal:} & N(t) = \tilde{N}(s) \\ \textit{Binormal:} & B(t) = \tilde{B}(s) \end{array}$$

1 Frenet Frame Computation

In general κ and $\tilde{\kappa}$ are different functions, defined on different intervals. But they *give exactly the same description of the turning of the common route of* $\boldsymbol{\alpha}$ *and* $\tilde{\boldsymbol{\alpha}}$, since at any point $\boldsymbol{\alpha}(t) = \tilde{\boldsymbol{\alpha}}(s(t))$. The numbers $\kappa(t)$ and $\tilde{\kappa}(s(t))$ are by the definition the same. Since only a change of parameterization is involved, its fundamental geometric meaning is the same as before. Similarly with the rest of the Frenet apparatus. In particular, T, N, B again form an orthonormal basis at every point t on $\boldsymbol{\alpha}$.



The speed v of the curve is the proper correction factor on the rate of change of T, N, B in the general case.

*All the materials are from [1].

Lemma 1 *If $\boldsymbol{\alpha}$ is a regular curve in $\mathbb{R}^3$ with $\kappa > 0$, then*

$$\begin{aligned}\dot{T} &= \quad \kappa v N, \\ \dot{N} &= -\kappa v T \quad + \tau v B, \\ \dot{B} &= \quad -\tau v N\end{aligned}$$

Proof Let $\tilde{\boldsymbol{\alpha}}$ be a unit-speed reparametrization of $\boldsymbol{\alpha}$. Then by definition, $T = \tilde{T}(s)$, where s is an arc-length function for $\boldsymbol{\alpha}$. The chain rule as applied to differentiation gives

$$\dot{T} = \dot{\tilde{T}}(s)\dot{s}.$$

By the usual Frenet equations, $\dot{\tilde{T}} = \tilde{\kappa}\tilde{N}$. Substituting the function s in this equation yields

$$\dot{\tilde{T}}(s) = \tilde{\kappa}(s)\tilde{N}(s) = \kappa N$$

by the definition of κ and N in the arbitrary-speed case. Since $\dot{s}$ is the speed function v of $\boldsymbol{\alpha}$, these two equations combine to yield $\dot{T} = \kappa v N$. The formulas for $\dot{N}$ and $\dot{B}$ are derived in the same way.

□

Let us use the same letter to designate both a curve $\boldsymbol{\alpha}$ and its unit-speed parameterization $\tilde{\boldsymbol{\alpha}}$, and similarly with the Frenet apparatus of these two curves. Differences in derivatives are handled by writing dT/ds for either $\dot{\tilde{T}}$ or its reparametrization $\dot{\tilde{T}}(s)$. With these conventions, the proof above would combine the chain rule $\dot{T} = (dT/ds)\dot{s}$ and Frenet formula $dT/ds = \kappa N$ to give $\dot{T} = \kappa v N$.

Curvature is revealed from the second order derivative, i.e., the acceleration, of the curve. Only for a *constant-speed* curve is acceleration orthogonal to velocity, since $\dot{\boldsymbol{\beta}} \cdot \dot{\boldsymbol{\beta}}$ being constant is equivalent to $\frac{d}{dt}(\dot{\boldsymbol{\beta}} \cdot \dot{\boldsymbol{\beta}}) = 2\dot{\boldsymbol{\beta}} \cdot \ddot{\boldsymbol{\beta}} = 0$. In the general case, we analyze velocity and acceleration by expressing them in terms of the Frenet frame field.

Lemma 2 *If $\boldsymbol{\alpha}$ is a regular curve with speed function v, then the velocity and acceleration of $\boldsymbol{\alpha}$ are given by*

$$\begin{aligned}\dot{\boldsymbol{\alpha}} &= vT, \\ \ddot{\boldsymbol{\alpha}} &= \frac{dv}{dt}T + \kappa v^2 N.\end{aligned}$$

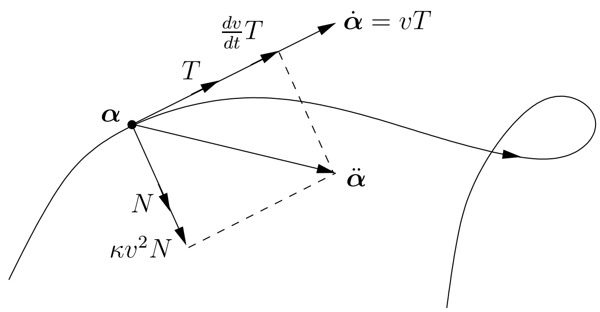
Proof Since $\boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}}(s)$, where s is the arc-length function of $\boldsymbol{\alpha}$, we find that

$$\dot{\boldsymbol{\alpha}} = \dot{\tilde{\alpha}}(s)\frac{ds}{dt} = v\tilde{T}(s) = vT.$$

Then a second differentiation yields

$$\ddot{\boldsymbol{\alpha}} = \frac{dv}{dt}T + v\dot{T} = \frac{dv}{dt}T + \kappa v^2 N$$

according to Lemma 1. □



The formula $\dot{\boldsymbol{\alpha}} = vT$ is to be expected — $\dot{\boldsymbol{\alpha}}$ and T are each tangent to the curve, and T has a unit length while $\|\dot{\boldsymbol{\alpha}}\| = v$. The formula for acceleration is more interesting. By definition, $\ddot{\boldsymbol{\alpha}}$ is the rate of change of the velocity $\dot{\boldsymbol{\alpha}}$, and in general both the length and the direction of $\dot{\boldsymbol{\alpha}}$ are changing. The *tangential component* $\dot{v}T$ of $\ddot{\boldsymbol{\alpha}}$ measures the rate of change of the length of $\dot{\boldsymbol{\alpha}}$ (that is, of the speed of $\boldsymbol{\alpha}$). The *normal component* $\kappa v^2 N$ measures the rate of change of the direction of $\dot{\boldsymbol{\alpha}}$. Newton's laws of motion show that these components may be experienced as forces. For example, in a car that is speeding up or slowing down on a straight road the only force one feels is due to $\dot{v}T$. If one takes an unbanked curve at speed v, the sideways force one feels is due to $\kappa v^2 N$. Here κ measures how sharply the road turns; the effect of speed is given by v^2, so 60 miles per hour is four times as unsettling as 30.

We now find effectively computable expressions for the Frenet apparatus.

Theorem 3 *Let $\boldsymbol{\alpha}$ be a regular curve in $\mathbb{R}^3$ with $\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} \neq 0$. Then*

$$\begin{aligned} T &= \frac{\dot{\boldsymbol{\alpha}}}{\|\dot{\boldsymbol{\alpha}}\|}, \\ B &= \frac{\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|}, && (1) \\ N &= B \times T, \\ \kappa &= \frac{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|}{\|\dot{\boldsymbol{\alpha}}\|^3}, && (2) \\ \tau &= \frac{(\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}) \cdot \dddot{\boldsymbol{\alpha}}}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|^2} = \frac{\det(\dot{\boldsymbol{\alpha}}\, \ddot{\boldsymbol{\alpha}}\, \dddot{\boldsymbol{\alpha}})}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|^2}. && (3) \end{aligned}$$

Proof The equations for T and N follow from their definitions. So here we need only prove (1), (2), and (3). Since $v = \|\dot{\boldsymbol{\alpha}}\| > 0$, the formula $T = \dot{\boldsymbol{\alpha}}/\|\dot{\boldsymbol{\alpha}}\|$ is equivalent to $\dot{\boldsymbol{\alpha}} = vT$. From the preceding lemma we get

$$\begin{aligned} \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} &= (vT) \times \left(\frac{dv}{dt}T + \kappa v^2 N\right) \\ &= v\frac{dv}{dt} T \times T + \kappa v^3 T \times N \\ &= \kappa v^3 B. \end{aligned}$$

Taking norms we find

$$\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\| = \|\kappa v^3 B\| = \kappa v^3$$

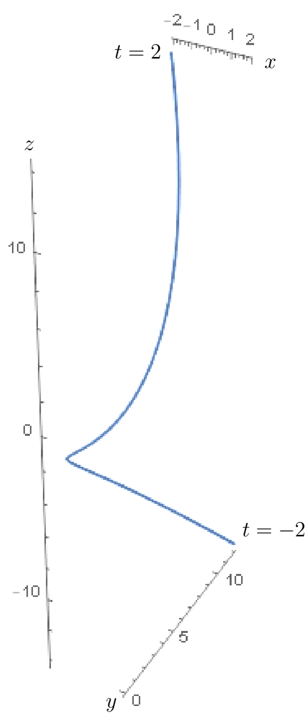
because $\|B\| = 1$, $\kappa \geq 0$, and $v > 0$. This proves (2). Indeed this equation shows that for regular curves, $\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\| > 0$ is equivalent to the usual condition $\kappa > 0$. (Thus for $\kappa > 0$, $\dot{\boldsymbol{\alpha}}$ and $\ddot{\boldsymbol{\alpha}}$ are linearly independent and determine the osculating plane at each point, as do T and N.) Then

$$B = \frac{\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}}{\kappa v^3} = \frac{\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|}.$$

Now only the formula for torsion remains to be proved. To find the dot product $(\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}) \cdot \dddot{\boldsymbol{\alpha}}$, we express everything in terms of T, N, B. We already know that $\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} = \kappa v^3 B$. Since $B \cdot T = B \cdot N = 0$, we need only find the B component of $\dddot{\boldsymbol{\alpha}}$. But by Lemma 2,

$$\begin{aligned} \dddot{\boldsymbol{\alpha}} &= \frac{d}{dt}\left(\frac{dv}{dt}T + \kappa v^2 N\right) \\ &= \frac{d^2 v}{dt^2}T + \frac{dv}{dt}\kappa v N + \frac{d}{dt}\left(\kappa v^2\right)N + \kappa v^2 \dot{N} \\ &= \kappa\tau v^3 B + \left(\frac{d^2 v}{dt^2} - \kappa^2 v^3\right)T + \left(\frac{dv}{dt}\kappa v + \frac{d}{dt}(\kappa v^2)\right)N \end{aligned}$$

following Lemma 2. Consequently $(\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}) \cdot \dddot{\boldsymbol{\alpha}} = \kappa^2 v^6 \tau$. Equation (3) then follows since $\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\| = \kappa v^3$. □



$\boldsymbol{\alpha}(t) = (3t - t^3, 3t^2, 3t + t^3)$

EXAMPLE 1. We compute the Frenet frame of the curve

$$\boldsymbol{\alpha}(t) = (3t - t^3, 3t^2, 3t + t^3).$$

The derivatives are

$$\begin{aligned} \dot{\boldsymbol{\alpha}}(t) &= 3(1 - t^2, 2t, 1 + t^2), \\ \ddot{\boldsymbol{\alpha}}(t) &= 6(-t, 1, t), \\ \dddot{\boldsymbol{\alpha}}(t) &= 6(-1, 0, 1). \end{aligned}$$

And the velocity is

$$v(t) = \|\dot{\boldsymbol{\alpha}}(t)\| = \sqrt{\dot{\boldsymbol{\alpha}}(t) \cdot \dot{\boldsymbol{\alpha}}(t)} = 3\sqrt{2}(1 + t^2).$$

Next, we have

$$\begin{aligned} \dot{\boldsymbol{\alpha}}(t) \times \ddot{\boldsymbol{\alpha}}(t) &= 18(-1 + t^2, -2t, 1 + t^2), \\ \|\dot{\boldsymbol{\alpha}}(t) \times \ddot{\boldsymbol{\alpha}}(t)\| &= 18\sqrt{2}(1 + t^2). \end{aligned}$$

The expressions above for $\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}$ and $\dddot{\boldsymbol{\alpha}}$ yield

$$(\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}) \cdot \dddot{\boldsymbol{\alpha}} = 6 \cdot 18 \cdot 2(1 - t^2 + 1 + t^2) = 216.$$

It remains only to substitute this data into the formulas in Theorem 3, with N being computed by another cross product. The final results are

$$\begin{aligned} T &= \frac{(1-t^2, 2t, 1+t^2)}{\sqrt{2}(1+t^2)}, \\ N &= \frac{(-2t, 1-t^2, 0)}{1+t^2}, \\ B &= \frac{(-1+t^2, -2t, 1+t^2)}{\sqrt{2}(1+t^2)}, \\ \kappa &= \frac{1}{3(1+t^2)^2}, \\ \tau &= \frac{1}{3(1+t^2)^2}. \end{aligned}$$

Example 2. Let us compute the torsion of the helix in its standard parameterization

$$\boldsymbol{\gamma}(\theta) = (a\cos\theta, a\sin\theta, b\theta).$$

First, we have

$$\begin{aligned} \dot{\boldsymbol{\gamma}} &= (-a\sin\theta, a\cos\theta, b), \\ \ddot{\boldsymbol{\gamma}} &= (-a\cos\theta, -a\sin\theta, 0), \\ \dddot{\boldsymbol{\gamma}} &= (a\sin\theta, -a\cos\theta, 0). \end{aligned}$$

Hence,

$$\begin{aligned} \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} &= (ab\sin\theta, -ab\cos\theta, a^2), \\ \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|^2 &= a^2(a^2+b^2), \\ (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}} &= a^2 b, \end{aligned}$$

and so the torsion is

$$\tau = \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}.$$

Let us summarize the situation. We now have the Frenet apparatus for an arbitrary-speed curve $\boldsymbol{\alpha}$. This apparatus satisfies the extended Frenet formulas (with factor v) and may be computed by Theorem 3. If $v = 1$, that is, if $\boldsymbol{\alpha}$ is unit-speed curve, the Frenet formulas in Lemma 1 simplify slightly, but Theorem 3 may be replaced by the much simpler definitions that we learned before.

2 An Application — Spherical Images

Let us consider one application of the results in the previous section. There are a number of interesting ways in which one can assign to a given curve $\boldsymbol{\beta}$ a new curve $\bar{\boldsymbol{\beta}}$ whose geometric properties illuminate some aspects of the behavior of $\boldsymbol{\beta}$. For example, if $\boldsymbol{\beta}$ is a unit-speed curve, the curve $\boldsymbol{\sigma}(s) = T(s) = \dot{\boldsymbol{\beta}}(s)$ is the *spherical image* of $\boldsymbol{\beta}$. Here $\boldsymbol{\sigma}$ is the curve such that each point $\boldsymbol{\sigma}(s)$ has the same Euclidean coordinates as the unit tangent vector $T(s)$. Roughly speaking, $\boldsymbol{\sigma}(s)$ is obtained by translating $T(s)$ to the origin. The spherical image lies entirely on the unit sphere Σ of $\mathbb{R}^3$, since $\|\boldsymbol{\sigma}\| = \|T\| = 1$, and *the motion represents the curving of* $\boldsymbol{\beta}$.

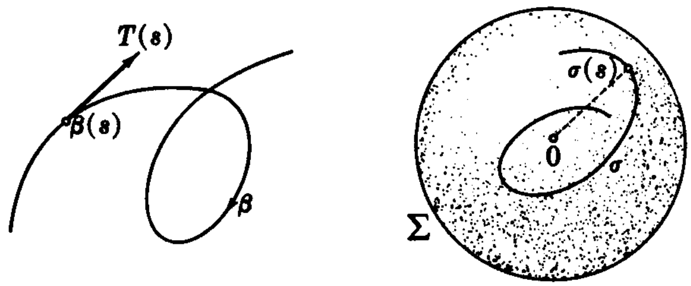


Figure 1: Spherical image, Fig. 2.17 on p. 71 of [1].

Example 3. Suppose $\boldsymbol{\beta}$ is the helix described by

$$\boldsymbol{\beta}(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}\right),$$

where $c = \sqrt{a^2+b^2}$. Thus the spherical image of a helix lies on the circle cut from the unit sphere by the plane $z = \frac{b}{c}$.

There is no loss of generality in assuming that the original curve $\boldsymbol{\beta}$ has unit speed (in the case of the helix given above), but we cannot also expect $\boldsymbol{\sigma}$ to have unit speed. In fact, since $\boldsymbol{\sigma} = T$, we have

$$\dot{\boldsymbol{\sigma}} = \dot{T} = \kappa N.$$

Thus $\boldsymbol{\sigma}$ moves always in the principal normal direction of $\boldsymbol{\beta}$, with speed $\|\dot{\boldsymbol{\sigma}}\|$ equal to the curvature κ of $\boldsymbol{\beta}$.

Next we assume $\kappa > 0$, and use the Frenet formulas for $\boldsymbol{\beta}$ to compute the curvature of $\boldsymbol{\sigma}$. Now

$$\begin{aligned} \ddot{\boldsymbol{\sigma}} &= \frac{d}{dt}(\kappa N) \\ &= \frac{d\kappa}{ds}N + \kappa\dot{N} \\ &= \frac{d\kappa}{ds}N + \kappa(-\kappa T + \tau B) \\ &= -\kappa^2 T + \frac{d\kappa}{ds}N + \kappa\tau B. \end{aligned}$$

Thus

$$\begin{aligned} \dot{\boldsymbol{\sigma}} \times \ddot{\boldsymbol{\sigma}} &= -\kappa^3 N \times T + \kappa^2 \tau N \times B \\ &= \kappa^2(\kappa B + \tau T). \end{aligned}$$

Therefore the curvature of the spherical image $\boldsymbol{\sigma}$ is

$$\begin{aligned} \kappa_\sigma &= \frac{\|\dot{\boldsymbol{\sigma}} \times \ddot{\boldsymbol{\sigma}}\|}{\|\dot{\boldsymbol{\sigma}}\|^3} \\ &= \frac{\sqrt{\kappa^2+\tau^2}}{\kappa} \\ &= \left(1 + \left(\frac{\tau}{\kappa}\right)^2\right)^{1/2} \\ &> 1 \end{aligned}$$

and thus depends only on the ratio of torsion to curvature for the original curve $\boldsymbol{\beta}$.

References

[1] B. O'Neill. *Elementary Differential Geometry.* Academic Press, Inc., 1966.