

Algebraic Curves*

(Com S 477/577 Notes)

Yan-Bin Jia

Oct 17, 2017

An *algebraic curve* is a curve which is described by a polynomial equation:

$$f(x, y) = \sum a_{ij}x^i y^j = 0$$

in x and y . The *degree* of the curve is the degree of the polynomial $f(x, y)$. A parametric curve $\alpha(t) = (p(t), q(t))$, where $p(t)$ and $q(t)$ are polynomials in t , is an algebraic curve. This is because we can eliminate t from the two polynomial equations $x = p(t)$ and $y = q(t)$ using the resultant technique¹, and obtain a polynomial equation, called their *resultant*, in x and y . The resultant has a degree no more than $\max(\deg(p), \deg(q))$.

Given a curve $f(x, y) = 0$, we can often derive a simplified equation via rotation and translation. The geometry of the curve will not be affected though. Suppose the curve undergoes a rotation of θ about the origin, followed by a translation (x_0, y_0) . Every point (x, y) on the curve moves to the point

$$(u, v) = (x \cos \theta - y \sin \theta + x_0, x \sin \theta + y \cos \theta + y_0).$$

Conversely, we express (x, y) in terms of (u, v) :

$$(x, y) = \left((u - x_0) \cos \theta + (v - y_0) \sin \theta, -(u - x_0) \sin \theta + (v - y_0) \cos \theta \right).$$

Substitution of the above expressions into the original curve equation $f(x, y) = 0$ yields the equation $g(u, v) = f(x(u, v), y(u, v)) = 0$ for the transformed curve. We rename this new curve as $g(x, y) = 0$.

1 Quadratic Curves — Conic Sections

By properly selecting θ, x_0, y_0 , every quadratic curve in the general form

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y + \zeta = 0,$$

where at least one of α, β , and γ is not zero, can be transformed into one of the following *canonical* forms:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1, & \text{if } \beta^2 - 4\alpha\gamma < 0; \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1, & \text{if } \beta^2 - 4\alpha\gamma > 0; \\ y^2 &= 2ax, & \text{if } \beta^2 - 4\alpha\gamma = 0. \end{aligned}$$

*Most of the materials are taken from [1].

¹Polynomial resultants, not required by the course, are described in the optional notes on September 28.

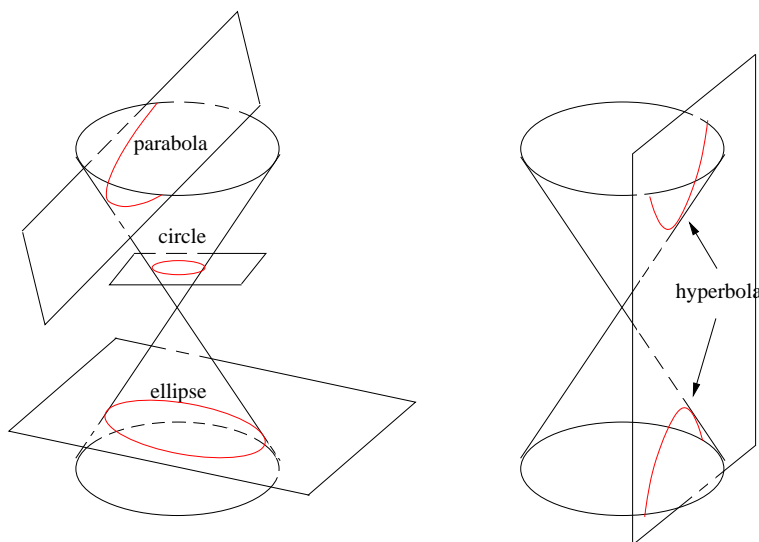


Figure 1: Conic sections.

Thus, all quadratic curves are classified into ellipses, hyperbolas, and parabolas. There are also called *conic sections* because they can be generated by the intersection of a plane with one or two nappes of a double cone (i.e., two cones placed apex to apex). If the plane is perpendicular to the axis of the cone, a circle is generated. Otherwise, an ellipse or a parabola is generated if the plane intersects only one nappe, and a hyperbola is generated if the plane intersects both nappes.

2 Some Cubic Curves

For equations of higher degree there is no such simple classification. In the next section, we look at some examples of curves described by polynomials of degree three. The *semi-cubical parabola* (i.e., a cuspidal cubic) is a cubic curve which has two branches at the origin, but the tangents there to the two branches coincide. See Figure 2. The origin is a cusp of the curve, and the common limiting tangent (i.e., the x -axis) to the two branches at the cusp is called the *cuspidal tangent*.

The algebraic equation of the semi-cubical parabola is

$$y^2 = x^3.$$

The parametric equation is

$$(x, y) = (t^2, t^3),$$

where t takes on any real value. This is a parametrization in that both $x(t)$ and $y(t)$ are polynomials in t . To derive this from the algebraic equation, we consider the intersection of the curve with the line $y = tx$ for each fixed value of t . Substituting $y = tx$ into $y^2 - x^3 = 0$ gives $x^2(t^2 - x) = 0$, which is solved to yield $x = t^2$ and $y = tx = t^3$. Therefore all points satisfying the algebraic equation are given by the parametric equation $(x, y) = (t^2, t^3)$.

The polar equation for the above curve is

$$\rho = \sin^2 \theta \sec^3 \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

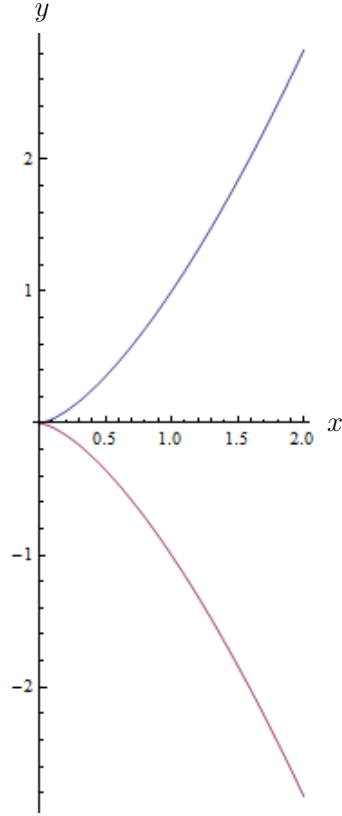


Figure 2: Semi-cubical parabola $y^2 = x^3$.

This equation is obtained by substituting $x = \rho \cos \theta$ and $y = \rho \sin \theta$ into the algebraic equation. We obtain $\rho^2(\sin^2 \theta - \rho \cos^3 \theta) = 0$, which gives $\rho^2 = 0$ and $\sin^2 \theta - \rho \cos^3 \theta = 0$. But the second equation subsumes the first one (when $\theta = 0$). From $\rho = \sin^2 \theta / \cos^3 \theta$ we have

$$\mathbf{r} = (\rho \cos \theta, \rho \sin \theta) = (\tan^2 \theta, \tan^3 \theta),$$

and the substitution $t = \tan \theta$ yields the parametric equation.

The semi-cubical parabola is an isochronous curve; a particle descending the curve under gravity falls equal vertical distances in an equal time (Huygens 1687). It was the first algebraic curve whose arc length was calculated (Neile 1659). Before this the arc length had been found only for certain transcendental curves such as the cycloid and the equi-angular spiral.

Some plane curves appear to have several (disconnected) parts. The simplest example of this is the hyperbola, which has two unbounded parts that do not meet in the plane. We now give an example of a cubic curve with two parts, one of which is bounded and the other of which is unbounded. The curve, shown in Figure 3, has the algebraic equation:

$$y^2 - x^3 + x = 0. \tag{1}$$

It can be parametrized as below:

$$x = t, \quad y = \pm \sqrt{t(t^2 - 1)} \quad -1 \leq t \leq 0 \text{ or } t \geq 1.$$

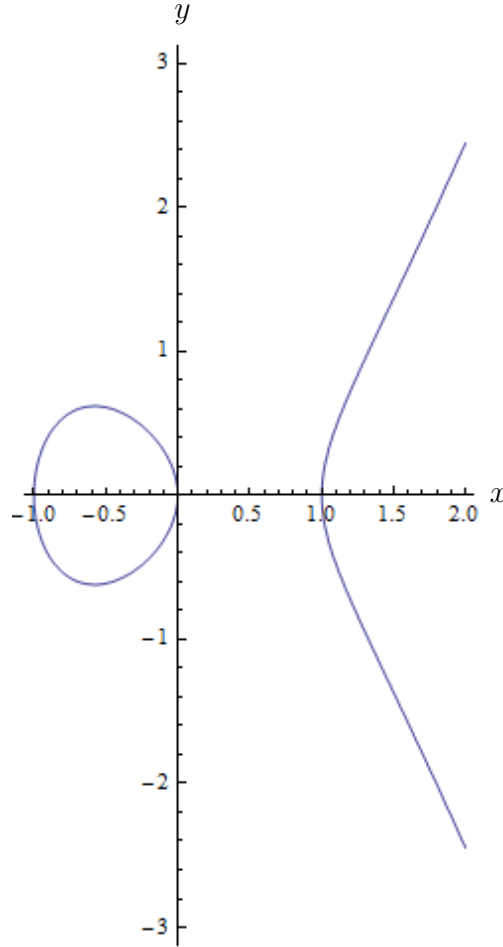


Figure 3: Two-part cubic $y^2 - x^3 + x = 0$.

This parametrization by x is obtained by solving the algebraic equation to give y as a function of x . We determine where the line $x = t$ meets the curve for each fixed value of t . Substituting $x = t$ into the two-part cubic given in (1) gives $y^2 - t^3 + t = 0$. Solving this gives $x = t$ and $y = \pm\sqrt{t(t^2 - 1)}$. This yields the parametrization of four parts of the curves, one in each quadrant. These parametric equations are not differentiable at $t = 0, \pm 1$.

An algebraic curve can be parametrized in many different ways. An alternative parametric equation is given by determining where the line $y = tx$ meets the curve for each fixed value of t . Substituting $y = tx$ into (1) gives $x(t^2x - x^2 + 1) = 0$. Solving this cubic equation in x gives the origin $(x, y) = \mathbf{0}$, the parametrization

$$\mathbf{r}(t) = \frac{1}{2} \left(t^2 - \sqrt{t^4 + 4}, t \left(t^2 - \sqrt{t^4 + 4} \right) \right)$$

of the bounded part of the curve, and the parametrization

$$\mathbf{r}(t) = \frac{1}{2} \left(t^2 + \sqrt{t^4 + 4}, t \left(t^2 + \sqrt{t^4 + 4} \right) \right)$$

of the unbounded part. The origin is not covered by the parametrization of the bounded part, because it corresponds to the value ∞ of the parameter, that is, to the vertical line $x = 0$. These two parametrizations are differentiable for all values of t .

The polar equation of the same curve is

$$-\rho^2 \cos^3 \theta + \rho \sin^2 \theta + \cos \theta = 0.$$

This is obtained by substituting $x = \rho \cos \theta$ and $y = \rho \sin \theta$ into the algebraic equation. There are two values of ρ corresponding to each value of θ .

3 Singular Points

A *singular point* on the algebraic curve $f(x, y) = 0$ is a point (a, b) at which

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0;$$

that is, at which the gradient $\nabla f = \mathbf{0}$. At a non-singular point, $\nabla f \neq \mathbf{0}$. To determine singular points, it is not sufficient to solve $\nabla f = \mathbf{0}$. We must find simultaneous solutions of $\nabla f = \mathbf{0}$ and $f = 0$; that is, we must find those ‘singularities of the polynomial’ which also lie on the curve.

EXAMPLE 1. Find the singular points of the circle $f(x, y) = x^2 + y^2 - 1$. From $\nabla f = (2x, 2y) = \mathbf{0}$ we obtain that $x = y = 0$. But $(0, 0)$ does not lie on the circle, so the circle has no singular points.

EXAMPLE 2. For the cuspidal cubic $g(x, y) = y^2 - x^3 = 0$, we obtain that $\nabla g = (-3x^2, 2y) = \mathbf{0}$ if and only if $x = y = 0$. The only singular point is $(0, 0)$.

4 Parametrization of Algebraic Curves

Consider an algebraic curve given by $f(x, y) = 0$, where f is a polynomial of degree $d \geq 1$. One question is ‘Is it possible to parametrize the curve?’ We now show that algebraic curves can be parametrized locally near non-singular points. In general, algebraic curves, or parts of them, can be parametrized either by x or by y , or by both.

A *local parametrization* of an algebraic curve near a point (a, b) on the curve, is a parametrization $J \rightarrow \mathbb{R}^2$ of a piece of the curve including the point (a, b) . Usually we assume that the point (a, b) corresponds to an interior point of the interval J of parametrization.

Theorem 1 (Local Parametrization Theorem) *Let f be a polynomial of degree $d \geq 1$ satisfying $f(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) \neq 0$. Then there is an interval $J = (a - \epsilon, a + \epsilon)$ where $\epsilon > 0$, and a unique smooth function $\phi : J \rightarrow \mathbb{R}$ such that $\phi(a) = b$ and $f(x, \phi(x)) = 0$ for all $x \in J$. Namely, $(x, \phi(x))$ is a unique regular smooth local parametrization of the curve over $(a - \epsilon, a + \epsilon)$ by the x coordinate. Locally, the curve $f(x, y) = 0$ is the graph of the function $y = \phi(x)$, and*

$$\frac{d\phi}{dx} = -\frac{f_x}{f_y}, \quad x \in J.$$

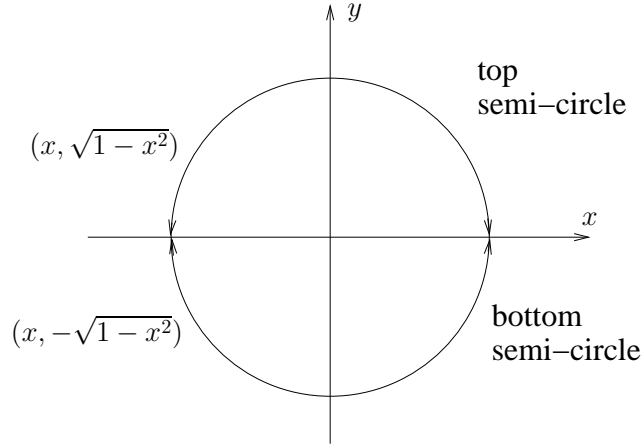


Figure 4: Parametrization of a unit circle.

Similarly, in the case where $\frac{\partial f}{\partial x}(a, b) \neq 0$, there exists a unique regular smooth local parametrization $(\psi(y), y)$ defined near b and satisfying $\psi(b) = a$, $f(\psi(y), y) = 0$, and

$$\frac{d\psi}{dy} = -\frac{f_y}{f_x}$$

for y near b . In the neighborhood of (a, b) the curve $f(x, y) = 0$ is the graph of the function $x = \psi(y)$.

The above theorem is a special case of the implicit function theorem, of which the proof can be found in texts on analysis. The condition $\frac{\partial f}{\partial y}(a, b) \neq 0$ for the first local parametrization implies that the gradient Δf is not parallel to the x -axis, and subsequently the tangent line to the curve, perpendicular to Δf , is not parallel to the y -axis. (Hence, the derivative $\frac{dy}{dx} = \frac{d\phi}{dx}$ is not infinite at the point.) Similarly, the condition $\frac{\partial f}{\partial x}(a, b) \neq 0$ ensures that the tangent at the point is not parallel to the x -axis such that $\frac{dx}{dy} = \frac{d\psi}{dy}$ exists.

The theorem essentially says that the curve $f(x, y) = 0$ can be locally parametrized at any point with non-vanishing gradient $\nabla f(x, y)$. The smooth local parametrization of algebraic curves by x or y as given in Theorem 1 are regular; for example, the derivative of $(x, \phi(x))$ is $(1, \dot{\phi}(x)) \neq 0$.

EXAMPLE 3. Consider the unit circle $f(x, y) = x^2 + y^2 - 1 = 0$. We have $f_y = 2y \neq 0$ when $y \neq 0$. Thus the theorem tells us there is a local smooth parametrization by x near (a, b) where $a^2 + b^2 = 1$ and $b \neq 0$. Indeed the parametrization is given by

$$\left(x, \sqrt{1-x^2}\right) \quad \text{or by} \quad \left(x, -\sqrt{1-x^2}\right), \quad -1 < x < 1$$

depending on whether $b > 0$ or $b < 0$. In this case, for fixed a with $-1 < a < 1$, there are two values for b , and local parametrization are given.

Similarly, for $a \neq 0$, there is a local parametrization by y near (a, b) . Again this is given by

$$\left(\sqrt{1-y^2}, y\right) \quad \text{or by} \quad \left(-\sqrt{1-y^2}, y\right), \quad -1 < y < 1$$

depending on whether $a > 0$ or $a < 0$.

5 Curvature of Algebraic Curves

In general algebraic curves cannot be parametrized in a simple way, that is, by using a simple formula which gives a parametrization of the whole curve. Therefore, the formula for the curvature of parametric curves cannot be used. However, according to Theorem 1 algebraic curves can be parametrized locally near a non-singular point, though not in general using a simple formula. We now use this result to give a parameter-free formula for the curvature of algebraic curves.

We introduce the Hessian matrix of a polynomial $f(x, y)$ as

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

Theorem 2 *The curvature of the algebraic curve $f(x, y) = 0$ at a non-singular point (x, y) on the curve is*

$$\begin{aligned} \kappa &= \pm \frac{(f_y, -f_x)H \begin{pmatrix} f_y \\ -f_x \end{pmatrix}}{\|\nabla f\|^3} \\ &= \pm \frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{(f_x^2 + f_y^2)^{3/2}}, \end{aligned} \quad (2)$$

where the sign ‘ $-$ ’ is chosen in case the motion along the curve is in the direction of the vector $(f_y, -f_x)$ (i.e., the gradient ∇f points to the left during the motion) and where the sign ‘ $+$ ’ is chosen in case the motion along the curve is in the direction of the vector $(-f_y, f_x)$ (i.e., the gradient points to the right).

Proof By Theorem 1, the algebraic curve can be given a regular local parametrization near a non-singular point by $\alpha(t) = (x(t), y(t))$. Differentiating the equation

$$f(x(t), y(t)) = 0,$$

using the chain rule, we obtain

$$\dot{\alpha} \cdot (f_x, f_y) = 0.$$

Therefore $\dot{\alpha}$ is orthogonal to (f_x, f_y) , and we have $\dot{\alpha} = (\dot{x}, \dot{y}) = \lambda(f_y, -f_x)$ for some $\lambda(t) \neq 0$. One more differentiation gives us

$$\ddot{\alpha} \cdot (f_x, f_y) + (\dot{x}, \dot{y})H \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 0.$$

Consequently, we have

$$\begin{aligned} \dot{\alpha} \times \ddot{\alpha} &= \dot{x}\ddot{y} - \dot{y}\ddot{x} \\ &= \ddot{\alpha} \cdot (-\dot{y}, \dot{x}) \\ &= \lambda \ddot{\alpha} \cdot (f_x, f_y) \\ &= -\lambda(\dot{x}, \dot{y})H \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \\ &= -\lambda^3 (f_y, -f_x)H \begin{pmatrix} f_y \\ -f_x \end{pmatrix} \\ &= -\lambda^3 (f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}) \end{aligned}$$

Since $\|\dot{\alpha}\| = |\lambda| \cdot \|(f_y, -f_x)\| = |\lambda|(f_x^2 + f_y^2)^{1/2}$, we have that

$$\begin{aligned}\kappa &= \frac{\dot{\alpha} \times \ddot{\alpha}}{\|\dot{\alpha}\|^3} \\ &= \frac{\lambda^3}{|\lambda|^3} \cdot \frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{(f_x^2 + f_y^2)^{3/2}} \\ &= \pm \frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{\|\nabla f\|^3}.\end{aligned}$$

□

The proof of Theorem 2 does not assume any specific form of f . Hence the theorem and the curvature form (2) hold for any implicit curve $f(x, y) = 0$, as long as f is twice differentiable in the neighborhood of the point of interest.

Corollary 3 *The algebraic curve has a point of inflection at a non-singular point (a, b) if and only if*

$$f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}$$

is zero at (a, b) and changes sign as (x, y) moves through (a, b) along the curve.

Thus, in order to find all points of inflection on an algebraic curve we first determine the points where the curvature is zero by solving the equations

$$\begin{aligned}f(x, y) &= 0, \\ f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy} &= 0,\end{aligned}$$

and then find those solutions that are non-singular points. Next, we determine whether the curvature changes sign as we move along the curve past the point of zero curvature. If it does, the point is an inflection. In practice we can check on the change of sign if, for example, the curve can be parametrized near the point by x , and

$$f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}$$

becomes a function of x after substituting from $f(x, y) = 0$. Such a local parameterization by x exists by Theorem 1 provided that the tangent at the point is not parallel to the y -axis.

EXAMPLE 4. Calculate the curvature of the hyperbola

$$x^2 - 3y^2 = 1$$

at the point $(2, 1)$.

Let $f = x^2 - 3y^2 - 1$. We have

$$\begin{aligned}f_x &= 2x, \\ f_y &= -6y, \\ f_{xx} &= 2, \\ f_{xy} &= 0, \\ f_{yy} &= -6.\end{aligned}$$

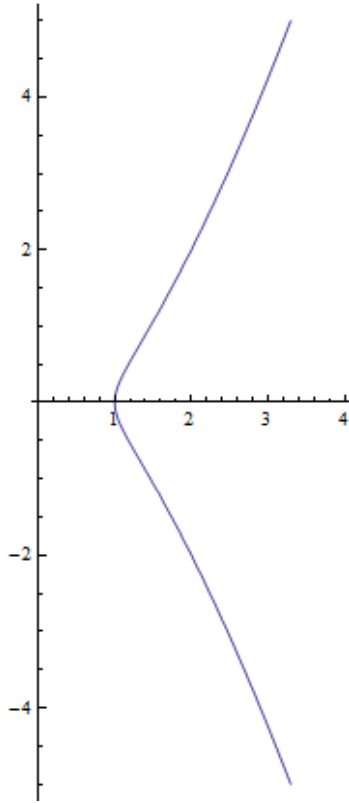


Figure 5: Acnodal cubic.

Therefore

$$\begin{aligned}\kappa &= \pm \frac{(-6y)^2 \cdot 2 + (2x)^2 \cdot (-6)}{(4x^2 + 36y^2)^{\frac{3}{2}}} \\ &= \pm \frac{9y^2 - 3x^2}{(x^2 + 9y^2)^{\frac{3}{2}}}.\end{aligned}$$

In particular,

$$\kappa(2, 1) = \pm \frac{3}{(\sqrt{13})^3}.$$

EXAMPLE 5. Find the points of inflection of the acnodal cubic

$$f(x, y) = y^2 + x^2 - x^3 = 0.$$

We have

$$\begin{aligned}f_x &= 2x - 3x^2, \\ f_y &= 2y, \\ f_{xx} &= 2 - 6x, \\ f_{xy} &= 0, \\ f_{yy} &= 2.\end{aligned}$$

From $f(x, y) = 0$, we have $y^2 = x^3 - x^2$, and

$$\begin{aligned}f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy} &= 4y^2(2 - 6x) + 2(2x - 3x^2)^2 \\ &= 8y^2 - 24xy^2 + 8x^2 - 24x^3 + 18x^4 \\ &= 8(x^3 - x^2) - 24x(x^3 - x^2) + 8x^2 - 24x^3 + 18x^4 \\ &= 8x^3 - 6x^4.\end{aligned}$$

The origin is a singular point. Therefore, there are precisely two points of zero curvature determined from $8x^3 - 6x^4 = 0$. They are $\left(\frac{4}{3}, \pm\frac{4}{3\sqrt{3}}\right)$. At these points, $f_y = 2y \neq 0$. Therefore, the curve can be parametrized locally by x . Also

$$8x^3 - 6x^4 = 6x^3 \left(\frac{4}{3} - x\right)$$

changes sign as we move along the curve past each of the points and therefore the curvature also changes sign. Thus, the two points are points of inflection.

References

- [1] J. W. Rutter. *Geometry of Curves*. Chapman & Hall/CRC, 2000.
- [2] Wolfram MathWorld: <http://mathworld.wolfram.com/topics/AlgebraicCurves.html>.