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Local Observability of Rolling*

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Abstract

This paper investigates local observability of the pose and motion of a curved three-dimensional object rolling on a rough horizontal plane. The plane models a controllable robotic palm imbued with tactile sensors. The palm can accelerate in arbitrary translational directions and the tactile sensors can determine the contact location between the palm and the rolling object at every instant in time. The object and contact motions are governed by a nonlinear system derived from the kinematics and dynamics of rolling. Through cotangent space decomposition, a sufficient condition on local observability of the system is obtained. This condition depends only on the differential geometry of contact and on the object's angular inertia matrix; it is satisfied by all but some degenerate shapes such as a sphere.

The above result demonstrates that the geometric and dynamic information of a manipulation task is often encoded in a small amount of tactile data.

1 Introduction

Geometry and mechanics are always closely tied to each other in a manipulation task. The states (or configurations) of an object and its manipulator evolve under the laws of mechanics and subject to the geometric constraints of contact. Such interaction often yields simple information into which the geometry and motions of the object and manipulator are encoded by the mechanics of the manipulation. For example, the contact between the object and the manipulator is a form of interaction through which the object's pose and motion may be conveyed to the manipulator in an implicit way. The reaction force is viewed as another form of interaction. From these kinds of encoded information we may be able to recover

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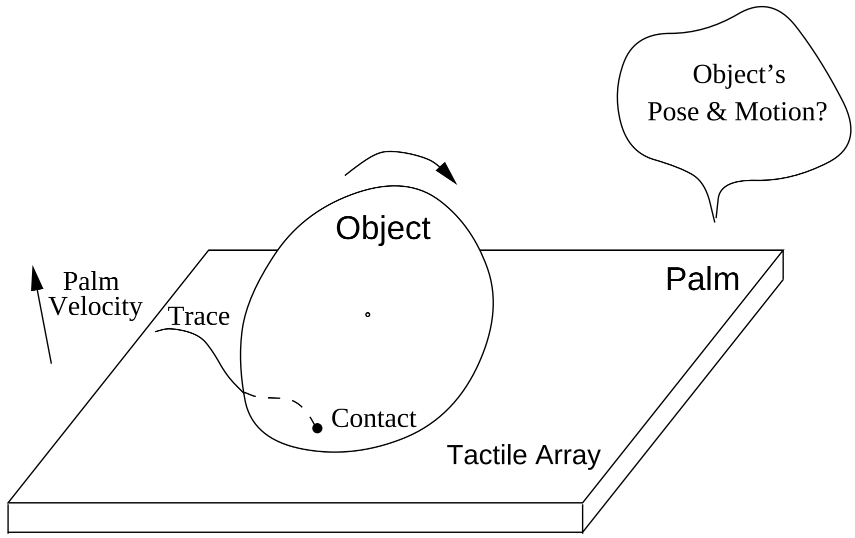

Figure 1: Palm sensing.

some essential unknowns about the task, especially the configuration (pose and motion) of the object.

In this paper we will look into the contact information and see how much about a task can be revealed from it. More specifically, we would like to know whether the contact between an object and a manipulator could be enough for the latter to "observe", at least locally, the pose and motion of the former.

The task chosen for our study involves a horizontal plane, or a palm, which can translate in arbitrary directions, and a smooth object which can only roll on the palm. As shown in Figure 1, the palm is trying to control the object's motion by executing certain motion of its own. The state of the object, comprising its pose and motion, however, is unknown at the moment. So the palm needs to determine the state before purposefully manipulating the object. We refer to this state estimation problem as *palm sensing*.

As the object rolls, its contact traces out a curve as a function of time in the plane (see Figure 1). Suppose the palm is covered by an array of tactile cells which are able to detect the contact location at any time instant. In this paper we would like to know if this curve of contact contains enough information for the palm to know about the configuration of the object. Our approach is to study local observability of rolling from the contact curve. Through this investigation we hope to touch on the more general issue of mechanics-based sensing and information retrieval in robotic tasks.

Section 2 derives a nonlinear system from the kinematics and dynamics of rolling that describes the object's motion as well as the contact motions upon the surfaces of the object and palm. Section 3 establishes a sufficient condition on local observability for the configuration of the rolling object. This condition depends only on the object's angular inertia matrix and its local shape around the contact. Section 4 summarizes the result and addresses two research directions in the future.

1.1 Related Work

We relate our work to previous work in rigid body dynamics, contact kinematics, sensing, and parts orienting. The work also draws upon the part of nonlinear control theory concerned with observability.

The first general discussion on the motion of a rigid body was due to Poisson in 1838, though the special case of a sphere was treated by Coriolis earlier in 1835. A few years later, friction was introduced into rigid body dynamics by Cournot in volumes 5 and 8 of *Crelle's Journal*. In 1861, Slesser gave the equations of a rigid body constrained to roll and pivot without sliding on a horizontal plane. This method was followed by Routh who discussed the rolling of a sphere on any surface.

In his comprehensive introduction to rigid body dynamics [20], MacMillan coped with sliding in the case of linear pressure distributions over the planar base of contact, and rolling in the case of a sphere on a smooth surface.

Observing the duality between a force and point, Brost and Mason [2] described a graphical method for the analysis of rigid body motion with multiple frictional contacts in the plane. The dynamic problem of predicting the accelerations and contact forces of multiple rigid bodies in contact with Coulomb friction have a unique solution if the the system Jacobian matrix has full column rank and the coefficients of friction are small enough. This result was obtained by Pang and Trinkle [24] who introduced complementarity formulations of the problem.

Montana [21] derived a set of differential equations that govern the motion of a contact point in response to a relative motion of the objects in contact, and applied these equations to local curvature estimation and contour following. The kinematics of spatial motion with point contact was also studied by Cai and Roth [3] who assumed a tactile sensor able to measure the relative motion at the contact point. The special kinematics of two rigid bodies rolling on each other was treated by Li and Canny [17] in view of path planning in the contact configuration space.

Assuming rolling contact only, Kerr and Roth [16] combined the forward kinematics of multifingered hands and the contact kinematics into a system of nonlinear, time-varying ordinary differential equations which can be solved for the finger joint velocities that are necessary for achieving a desired object motion.

Orienting mechanical parts was early studied by Grossman and Blasgen [10]. They used a vibrating box to constrain a part to a small finite number of possible stable poses and then determined the particular pose by a sequence of probes using a tactile sensor. Inspired by their result, Erdmann and Mason [6] constructed a planner that employs sensorless tilting operations to orient planar objects randomly dropped into a tray, based on a simple model of the quasi-static mechanics of sliding. Erdmann [5] also developed a working system for orienting parts using two palms, together with frictional contact analysis tools that can predict relative slip between parts. The work demonstrates the feasibility of automatic nonprehensile palm manipulation.

Utilizing the theory of limit surfaces [8], Böhringer *et al.* [1] developed a geometric model for the mechanics of an array of microelectromechanical structures and showed how this structure can be used to uniquely and efficiently align a part up to symmetry. Approximating the limit surface of an object as an ellipsoid, Lynch *et al.* [19] designed a control system to translate the object by pushing with tactile feedback; this control system may be used for active sensing of the object's center of mass.

Salisbury [25] first proposed the concept of fingertip force sensing with an approach for determining contact locations and orientations from force and moment measurements. Howe and Cutkosky [12] introduced dynamic tactile sensing in which sensors capture fine surface features during motion, presenting mechanical analysis and experimental performance measurements for the stress rate sensor.

Grimson and Lozano-Pérez [9] used tactile measurements of positions and surface normals on a 3D object to identify and locate it from a set of known 3D objects, based on the geometric constraints imposed by these tactile data. Gaston and Lozano-Pérez [7] showed how to identify and locate a polyhedron on a known plane using local information from tactile sensors which includes the position of contact points and the ranges of surface normals at these points.

Hermann and Kerner [11] first studied observability using the observation space. A result due to Crouch [4] shows that an analytic system is observable if and only if the observation space distinguishes points in the state space. Luenberger-like asymptotic observers, first constructed by Luenberger [18] for linear systems, remain likely the most commonly used observer forms for nonlinear systems today.

Jia and Erdmann [14, 15] investigated how to "observe" a planar object being pushed by a finger. They established the local observability of the object's pose and motion from the contact motion on the finger; and introduced two nonlinear observers as sensing strategies.

2 The Rolling Motion

The motion of an object rolling on a plane without slipping is subject to two constraints: (1) The point in contact has zero velocity relative to the plane; and (2) the object has no rotation about the contact normal. In this section we shall study the rolling motion of an object and the resulting motions of the contact on the object and in the plane, resulting from the plane translation. We shall derive a system of nonlinear differential equations that describes the object's kinematics and dynamics. We shall see that the kinematics are affected by the local geometry at the contact, while the dynamics are affected by the position of the contact and the object's angular inertia.

Before we begin, the problem needs to be formally defined. As shown in Figure 1, it concerns an object $\mathcal{B}$ (with or without any initial velocity) moving on a horizontal plane $\mathcal{P}$, which is translating at velocity $\boldsymbol{v}_{\mathcal{P}}$. In order to maintain contact, the plane acceleration $\dot{\boldsymbol{v}}_{\mathcal{P}}$ must not exceed the gravitational acceleration g downward. The friction between $\mathcal{P}$ and $\mathcal{B}$ is assumed to be large enough to allow only the pure rolling motion of $\mathcal{B}$ (without slipping).

For the sake of simplicity and clarity, we make a few additional assumptions:

1. Object $\mathcal{B}$ has uniform mass distribution.

2. Object $\mathcal{B}$ is bounded by an orientable surface.

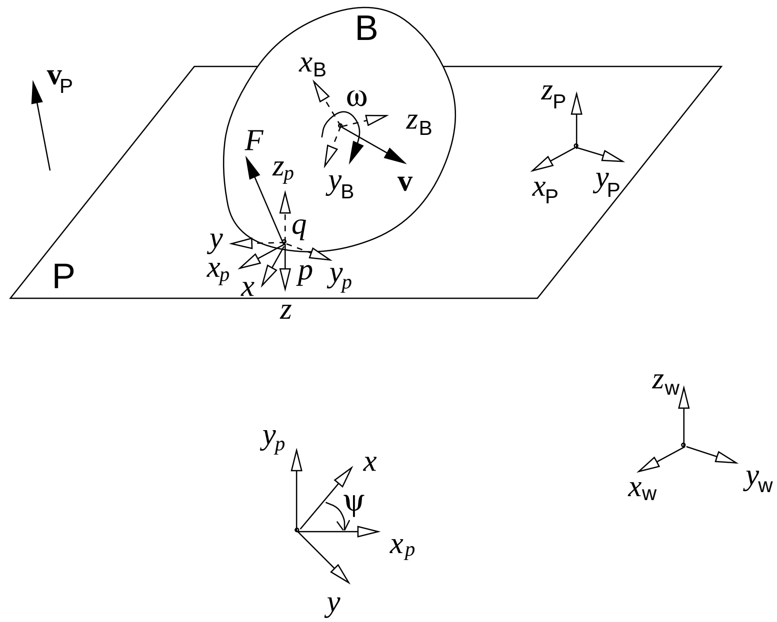

Figure 2: The coordinate frames for rolling. The body frame $\boldsymbol{x}_\mathcal{B}$-$\boldsymbol{y}_\mathcal{B}$-$\boldsymbol{z}_\mathcal{B}$ of object $\mathcal{B}$ defined by its principal axes; the frame $\boldsymbol{x}_\mathcal{P}$-$\boldsymbol{y}_\mathcal{P}$-$\boldsymbol{z}_\mathcal{P}$ of plane $\mathcal{P}$; the frame $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ at the contact $\boldsymbol{q}$ on $\mathcal{B}$; the frame $\boldsymbol{x}_p$-$\boldsymbol{y}_p$-$\boldsymbol{z}_p$ at the contact $\boldsymbol{p}$ in $\mathcal{P}$; and the world frame $\boldsymbol{x}_\mathcal{W}$-$\boldsymbol{y}_\mathcal{W}$-$\boldsymbol{z}_\mathcal{W}$. The frames $\boldsymbol{x}_\mathcal{P}$-$\boldsymbol{y}_\mathcal{P}$-$\boldsymbol{z}_\mathcal{P}$, $\boldsymbol{x}_p$-$\boldsymbol{y}_p$-$\boldsymbol{z}_p$, and $\boldsymbol{x}_\mathcal{W}$-$\boldsymbol{y}_\mathcal{W}$-$\boldsymbol{z}_\mathcal{W}$ have the same orientation.

3. The contact stays in one proper patch $\boldsymbol{\beta}$ in the surface of $\mathcal{B}$ during the period of time when local observability is concerned.

4. The patch $\boldsymbol{\beta}$ is convex and has positive Gaussian curvature everywhere.

5. The patch $\boldsymbol{\beta}$ is principal.

The third assumption makes sense because local observability is concerned with an infinitesimal amount of time. The fourth assumption restricts the contact to a point. The fifth assumption is justified because every point in a surface has a neighborhood that can be reparametrized as a principal patch. This fact is stated in Appendix A and proved in [27, pp. 320-323].

2.1 Kinematics of Rolling

To describe the motion of $\mathcal{B}$ and the motions of contact on both $\mathcal{P}$ and $\mathcal{B}$, we here set up several coordinate frames. Let $\boldsymbol{o}$ be the center of mass of $\mathcal{B}$. Let the coordinate frame $\Pi_\mathcal{B}$ as shown in Figure 2 be centered at $\boldsymbol{o}$ and defined by $\mathcal{B}$'s principal axes $\boldsymbol{x}_\mathcal{B}$, $\boldsymbol{y}_\mathcal{B}$, and $\boldsymbol{z}_\mathcal{B}$. The angular inertia matrix I with respect to $\Pi_\mathcal{B}$ is thus diagonal. Frame $\Pi_\mathcal{B}$ is moving constantly relative to the plane $\mathcal{P}$ due to the rolling of $\mathcal{B}$.

A frame $\Pi_\mathcal{P}$ with axes $\boldsymbol{x}_\mathcal{P}$, $\boldsymbol{y}_\mathcal{P}$, and $\boldsymbol{z}_\mathcal{P}$ is attached to the plane $\mathcal{P}$ such that axis $\boldsymbol{z}_\mathcal{P}$ is an upward normal to $\mathcal{P}$. The world coordinate frame, denoted by $\Pi_\mathcal{W}$, has the same orientation as frame $\Pi_\mathcal{P}$.

It is often convenient to describe the motion of $\mathcal{B}$ in terms of its body frame $\Pi_{\mathcal{B}}$ by velocity $\boldsymbol{v}$ and angular velocity $\boldsymbol{\omega}$. In the meantime, denote by $\boldsymbol{v}_{\mathcal{B}}$ and $\boldsymbol{\omega}_{\mathcal{B}}$ the velocities of $\mathcal{B}$ relative to the fixed frame that currently coincide with $\Pi_{\mathcal{B}}$. Hence $\boldsymbol{v}_{\mathcal{B}}$ and $\boldsymbol{v}$ are identical at the moment, same are $\omega_{\mathcal{B}}$ and $\boldsymbol{\omega}$. But their derivatives have quite different meanings, as will be discussed in Section 2.2.

The contact point on $\mathcal{B}$ is denoted by $\boldsymbol{q} = \boldsymbol{\beta}(\boldsymbol{s}) = \boldsymbol{\beta}(s_1, s_2)$ in $\Pi_{\mathcal{B}}$. Since $\boldsymbol{\beta}$ is a principal patch, the normalized Gauss frame Π at $\boldsymbol{q}$ is well-defined by axes

$$
\begin{aligned}
\boldsymbol{x} &= \frac{\boldsymbol{\beta}_{s_1}}{\|\boldsymbol{\beta}_{s_1}\|}, \\
\boldsymbol{y} &= \frac{\boldsymbol{\beta}_{s_2}}{\|\boldsymbol{\beta}_{s_2}\|}, \\
\boldsymbol{z} &= \frac{\boldsymbol{\beta}_{s_1} \times \boldsymbol{\beta}_{s_2}}{\|\boldsymbol{\beta}_{s_1}\| \cdot \|\boldsymbol{\beta}_{s_2}\|},
\end{aligned}
$$

all with respect to the body frame $\Pi_{\mathcal{B}}$. The parameters $\boldsymbol{s}$ of $\boldsymbol{\beta}$ are chosen such that $\boldsymbol{z}$ is the outward normal. The orientation of the contact frame Π relative to the body frame $\Pi_{\mathcal{B}}$ is thus given by a 3×3 rotation matrix

$$
R = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}). \tag{1}
$$

Meanwhile, the contact point in the plane $\mathcal{P}$ is denoted by $\boldsymbol{p} = (\boldsymbol{u}, 0)^T = (u_1, u_2, 0)^T$ in frame $\Pi_{\mathcal{P}}$. We attach to $\boldsymbol{p}$ a frame Π_p (with axes $\boldsymbol{x}_p, \boldsymbol{y}_p, \boldsymbol{z}_p$) with the same orientation as frames $\Pi_{\mathcal{P}}$ and $\Pi_{\mathcal{W}}$. Let ψ be the angle of rotation needed to align axis $\boldsymbol{x}$ with axis $\boldsymbol{x}_p$ (see Figure 2). The matrix

$$
R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{pmatrix}
$$

therefore relates axes $\boldsymbol{x}, \boldsymbol{y}$ to axes $\boldsymbol{x}_p, \boldsymbol{y}_p$. Accordingly, Π is related to Π_p by the 3×3 rotation matrix

$$
R_{pq} = \begin{pmatrix} R_\phi & 0 \\ 0 & -1 \end{pmatrix}.
$$

Consequently, the orientation of the body frame $\Pi_{\mathcal{B}}$ relative to the contact frame Π_p is given by the matrix

$$
\begin{aligned}
R_{po} &= R_{pq} R^T \\
&= \begin{pmatrix} R_\phi & 0 \\ 0 & -1 \end{pmatrix} (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})^T \\
&= \begin{pmatrix} R_\phi (\boldsymbol{x}, \boldsymbol{y})^T \\ -\boldsymbol{z}^T \end{pmatrix}.
\end{aligned} \tag{2}
$$

Now we see that the orientation of $\mathcal{B}$ relative to $\mathcal{P}$ (and thereby to the world frame $\Pi_{\mathcal{W}}$) is completely determined by $\boldsymbol{s}$ and ψ, which represent the three degrees of freedom of $\mathcal{B}$ in order to maintain contact with $\mathcal{P}$. The velocity and angular velocity of $\mathcal{B}$ in $\Pi_{\mathcal{W}}$ are given by $R_{po}\boldsymbol{v}$ and $R_{po}\boldsymbol{\omega}$, respectively.

The contact kinematics depends on the relative motion between the two contact frames Π and Π_p. Denote by $(\omega_x, \omega_y, \omega_z)^T$ the angular velocity of Π relative to Π_p and in terms of Π. Since Π is fixed relative to $\Pi_{\mathcal{B}}$ and the angular velocity of $\Pi_{\mathcal{B}}$ relative to Π_p is $\boldsymbol{\omega}$, we have

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = R^T\boldsymbol{\omega} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})^T\boldsymbol{\omega}. \tag{3}$$

Rolling without slipping imposes a constraint on the angular velocity $\boldsymbol{\omega}$:

$$\omega_z = \boldsymbol{z} \cdot \boldsymbol{\omega} = 0. \tag{4}$$

Also due to rolling the relative velocity of Π to Π_p is zero:

$$v_x = v_y = v_z = 0.$$

Therefore the absolute velocity of $\boldsymbol{q}$ equals $\boldsymbol{v}_{\mathcal{P}}$; in other words,

$$R_{po}(\boldsymbol{v}_{\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}} \times \boldsymbol{\beta}) = \boldsymbol{v}_{\mathcal{P}}, \tag{5}$$

or equivalently,

$$R_{po}(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{\beta}) = \boldsymbol{v}_{\mathcal{P}}. \tag{6}$$

Thus the velocity $\boldsymbol{v}$ is determined by $\boldsymbol{v} = R_{po}^T\boldsymbol{v}_{\mathcal{P}} - \boldsymbol{\omega} \times \boldsymbol{\beta}$.

The local geometry of $\mathcal{B}$ and $\mathcal{P}$ at the contact plays a major role in the contact kinematics. We will utilize Montana's kinematic equations of contact [21] which relates the contact motions to the relative motion of the two frames (in this case Π and Π_p) at the contact. First of all, we compute the shape operator, geodesic curvatures, and metric of $\mathcal{B}$ as

$$\begin{aligned} S &= (\boldsymbol{x}, \boldsymbol{y})^T(-\nabla_x\boldsymbol{z}, -\nabla_y\boldsymbol{z}) \\ &= \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \\ K_g &= \boldsymbol{y}^T(\nabla_x\boldsymbol{x}, \nabla_y\boldsymbol{x}) \\ &= (\kappa_{g_1}, \kappa_{g_2}), \\ V &= \begin{pmatrix} \|\boldsymbol{\beta}_{s_1}\| & 0 \\ 0 & \|\boldsymbol{\beta}_{s_2}\| \end{pmatrix}, \end{aligned}$$

and those corresponding invariants of $\mathcal{P}$ as

$$\begin{aligned} S_{\mathcal{P}} &= \mathbf{0}, \\ K_{g_{\mathcal{P}}} &= \mathbf{0}, \\ V_{\mathcal{P}} &= I_2, \end{aligned}$$

where κ_1 and κ_2 are the principal curvatures of $\boldsymbol{\beta}$ at $\boldsymbol{q}$, κ_{g_1} and κ_{g_2} the geodesic curvatures of the principal curves at $\boldsymbol{q}$, and I_2 the 2×2 identity matrix. Plugging the above forms, (4), and (6) into Montana's equations yields the kinematics of rolling:

$$\dot{\boldsymbol{u}} = R_\psi S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T \boldsymbol{\omega}; \tag{7}$$

$$\dot{\boldsymbol{s}} = V^{-1} S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T \boldsymbol{\omega}; \tag{8}$$

$$\dot{\psi} = K_g S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T \boldsymbol{\omega}. \tag{9}$$

Since the patch $\boldsymbol{\beta}$ has Gaussian curvature $K = \det S \neq 0$, the shape operator S is invertible.

Equations (7), (8), and (9) shall be better understood in the following geometric way. Let ρ_1 and ρ_2 be the radii of curvature of the normal sections in the principal directions $\boldsymbol{x}$ and $\boldsymbol{y}$ at the contact $\boldsymbol{q}$, respectively. From the convexity of $\boldsymbol{\beta}$ we have $\rho_1 = -\frac{1}{\kappa_1}$ and $\rho_2 = -\frac{1}{\kappa_2}$. For simplicity, let us assume the principal curves to be unit-speed; that is, $\|\boldsymbol{\beta}_{s_1}\| = \|\boldsymbol{\beta}_{s_2}\| = 1$. Hence equation (8) reduces to

$$\begin{aligned} \dot{\boldsymbol{s}} &= I_2 \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \begin{pmatrix} -\omega_y \\ \omega_x \end{pmatrix} \\ &= \begin{pmatrix} -\rho_1 \omega_y \\ \rho_2 \omega_x \end{pmatrix}. \end{aligned}$$

Figure 3 shows the normal sections of $\boldsymbol{\beta}$ in the $\boldsymbol{x}$ and $\boldsymbol{y}$ directions. Since $\boldsymbol{q}$ has zero relative velocity to $\boldsymbol{p}$, the angular velocity component ω_y generates a contact velocity component of $-\omega_y \rho_1$ along the s_1-parameter curve at $\boldsymbol{q}$, while the component ω_x generates a contact velocity component of $\omega_x \rho_2$ along the s_2-parameter curve at $\boldsymbol{q}$. Under rolling, the velocity $\dot{\boldsymbol{u}}$ is related to the velocity $\dot{\boldsymbol{s}}$ by matrix R_ψ. Equation (9), now written as,

$$\dot{\psi} = (\kappa_{gx}, \kappa_{gy}) \dot{\boldsymbol{s}},$$

has a similar explanation from the definition of geodesic curvature.

2.2 Dynamics of Rolling

To apply Newton's second law, we need to use a fixed reference frame with respect to which the velocity and angular velocity can be properly differentiated. Let the reference frame be the fixed frame that instantaneously coincides with frame $\Pi_{\mathcal{B}}$. Therefore we shall use the absolute velocities $\dot{\boldsymbol{v}}_{\mathcal{B}}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}}$ in this fixed frame.

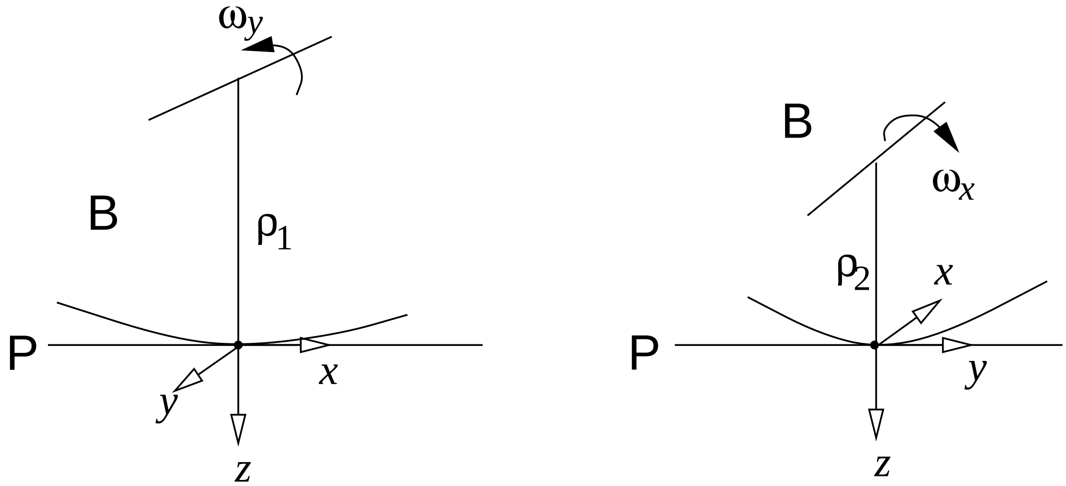

Figure 3: The normal sections of a pure rolling object $\mathcal{B}$'s surface in the principal directions $\boldsymbol{x}$ and $\boldsymbol{y}$ at the contact. The radii of curvature are ρ_1 and ρ_2, respectively. The angular velocities at the centers of curvature about $\boldsymbol{y}$ and $\boldsymbol{x}$ are ω_y and ω_x, respectively. It is clear that the velocities of contact in $\boldsymbol{x}$ and $\boldsymbol{y}$ are $-\rho_1\omega_y$ and $\rho_2\omega_x$, respectively.

Although $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are relative velocities in terms of the moving frame $\Pi_{\mathcal{B}}$, it is not difficult to derive that [1]

$$\dot{\boldsymbol{v}}_{\mathcal{B}} = \boldsymbol{\omega} \times \boldsymbol{v} + \dot{\boldsymbol{v}}; \tag{10}$$

$$\dot{\boldsymbol{\omega}}_{\mathcal{B}} = \boldsymbol{\omega} \times \boldsymbol{\omega} + \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}. \tag{11}$$

Let F be the contact force on object $\mathcal{B}$. The dynamics of the object obey Newton's and Euler's equations:

$$\begin{aligned} F + mg\boldsymbol{z} &= m\dot{\boldsymbol{v}}_{\mathcal{B}} = m(\dot{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{v}); \\ \boldsymbol{\beta}(\boldsymbol{s}) \times F &= I\dot{\boldsymbol{\omega}}_{\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}} \times I\boldsymbol{\omega}_{\mathcal{B}} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times I\boldsymbol{\omega}. \end{aligned}$$

Immediately from the above equations we can eliminate the contact force F which may be anywhere inside the contact friction cone:

$$\boldsymbol{\beta} \times m(\dot{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{v} - g\boldsymbol{z}) = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times I\boldsymbol{\omega}. \tag{12}$$

Next, differentiate the kinematic constraint (5) and plug (10) and (11) in:

$$(R_{po}\boldsymbol{\omega}) \times R_{po}(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{\beta}) + R_{po}\Big(\dot{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{v} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\beta} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\beta} + \dot{\boldsymbol{\beta}})\Big) = \boldsymbol{a}_{\mathcal{P}},$$

where $\boldsymbol{a}_{\mathcal{P}}$ is the acceleration of the plane $\mathcal{P}$. The above equation is simplified to

$$2\boldsymbol{\omega} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{\beta}) + \dot{\boldsymbol{v}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\beta} + \boldsymbol{\omega} \times \dot{\boldsymbol{\beta}} = R_{po}^T \boldsymbol{a}_{\mathcal{P}}, \tag{13}$$

[1] The reader may either try a derivation himself or refer to MacMillan [20, pp. 175-176].

since

$$(R_{po}\boldsymbol{\omega}) \times R_{po}(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{\beta}) \;=\; R_{po}\Big(\boldsymbol{\omega} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{\beta})\Big).$$

From (12) and (13) we obtain

$$\dot{\boldsymbol{\omega}} \;=\; D^{-1}\Big(\boldsymbol{\beta} \times R_{po}^T \boldsymbol{a}_{\mathcal{P}} - \boldsymbol{\beta} \times \Big(g\boldsymbol{z} + \boldsymbol{\omega} \times \boldsymbol{v} + 2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\beta}) + \boldsymbol{\omega} \times \dot{\boldsymbol{\beta}}\Big) - \boldsymbol{\omega} \times \frac{I}{m}\boldsymbol{\omega}\Big), \tag{14}$$

where

$$D \;=\; \frac{I}{m} + \|\boldsymbol{\beta}\|^2 I_3 - \boldsymbol{\beta}\boldsymbol{\beta}^T, \tag{15}$$

$$\boldsymbol{v} \;=\; R_{po}^T \boldsymbol{v}_{\mathcal{P}} - \boldsymbol{\omega} \times \boldsymbol{\beta}, \qquad \text{from (6).}$$

In the above I_3 denotes the 3×3 identity matrix. The matrix D is positive definite, following that the angular inertia matrix I is positive definite. More specifically, for any 3×1 vector $\boldsymbol{e} \neq \mathbf{0}$, we have

$$\begin{aligned} \boldsymbol{e}^T D \boldsymbol{e} &= \boldsymbol{e}^T(\frac{I}{m} + \|\boldsymbol{\beta}\|^2 I_3 - \boldsymbol{\beta}\boldsymbol{\beta}^T)\boldsymbol{e} \\ &= \boldsymbol{e}^T \frac{I}{m}\boldsymbol{e} + \|\boldsymbol{\beta}\|^2 \|\boldsymbol{e}\|^2 - (\boldsymbol{e} \cdot \boldsymbol{\beta})^2 \\ &> 0 \end{aligned}$$

Theorem 1 *The nonlinear system consisting of equations*

$$\begin{aligned} \dot{\boldsymbol{u}} &= R_\psi S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T \boldsymbol{\omega}, \\ \dot{\boldsymbol{s}} &= V^{-1} S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T \boldsymbol{\omega}, \\ \dot{\psi} &= K_g S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T \boldsymbol{\omega}, \\ \dot{\boldsymbol{\omega}} &= D^{-1}\boldsymbol{\beta} \times R_{po}^T \boldsymbol{a}_{\mathcal{P}} - D^{-1}\Big(\boldsymbol{\beta} \times \Big(g\boldsymbol{z} + \boldsymbol{\omega} \times \boldsymbol{v} + 2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\beta}) + \boldsymbol{\omega} \times \dot{\boldsymbol{\beta}}\Big) + \boldsymbol{\omega} \times \frac{I}{m}\boldsymbol{\omega}\Big), \end{aligned} \tag{16}$$

governs the object motion as well as the contact motions on the object and in the plane.

Both rolling constraints (4) and (6) were used in deriving system (16) and are therefore implicitly contained in the system. Nevertheless, (6) indicates that $\boldsymbol{v}$ is a redundant state variable of the system as it depends on $\boldsymbol{\omega}$ and $\boldsymbol{s}$; and (4) induces another redundancy among ω_1, ω_2, and ω_3.

A state of the nonlinear system (16) has eight (scalar) variables including $\boldsymbol{u} = (u_1, u_2)^T$, $\boldsymbol{s} = (s_1, s_2)^T$, ψ, and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$. Together they determine the object's position and orientation relative to frame $\Pi_{\mathcal{P}}$. Thus given the plane motion, they determine the object's motion relative to frame $\Pi_{\mathcal{B}}$. The one redundancy among ω_1, ω_2, ω_3 will be taken care of when we investigate the local observability of system (16) in the next section. Note there is no need to treat $\boldsymbol{v}_{\mathcal{P}}$ as a state variable since it is known.

The output of the system is $\boldsymbol{u} = (u_1, u_2)^T$. The input of the system is $\boldsymbol{a}_{\mathcal{P}} = (a_{\mathcal{P}_1}, a_{\mathcal{P}_2}, a_{\mathcal{P}_3})^T$ whose components control the input fields (see Appendix B)

$$\boldsymbol{h}_1 = \begin{pmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{h}}_1 \end{pmatrix}, \quad \boldsymbol{h}_2 = \begin{pmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{h}}_2 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{h}_3 = \begin{pmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{h}}_3 \end{pmatrix},$$

respectively, where the vector fields $\tilde{hb}_1, \tilde{\boldsymbol{h}}_2, \tilde{\boldsymbol{h}}_3$ together define a 3×3 matrix:

$$\begin{aligned} (\tilde{\boldsymbol{h}}_1, \tilde{\boldsymbol{h}}_2, \tilde{\boldsymbol{h}}_3) &= D^{-1}\boldsymbol{\beta} \times R_{po}^T \\ &= D^{-1}\boldsymbol{\beta} \times \Big((\boldsymbol{x}, \boldsymbol{y})R_{\psi}, -\boldsymbol{z}\Big), \end{aligned} \tag{17}$$

Denoted by $\boldsymbol{f}$ the drift field of the system, which is composed of the right sides of the equations (16) when $\boldsymbol{a}_{\mathcal{P}} = 0$. It measures the rate of state change under no input.

3 Local Observability

In the previous section we studied the kinematics and dynamics of rolling and derived the nonlinear system (16) that describes the object and contact motions. The unknowns include the point of contact $\boldsymbol{u}$ in the plane, the point of contact on the object as determined by $\boldsymbol{s}$, the rotation ψ of the object about the contact normal, and the object's angular velocity $\boldsymbol{\omega}$. These variables constitute a state in system (16)'s state space manifold M. This section will look into whether knowing $\boldsymbol{u}$ is sufficient for locally determining $\boldsymbol{s}, \psi$, and $\boldsymbol{\omega}$.

Our approach is to investigate the local observability of system (16). Denote by ξ the current state of rolling so that $\xi = (\boldsymbol{u}, \boldsymbol{s}, \psi, \boldsymbol{\omega})^T$. Essentially, we need to determine if the observability codistribution $d\mathcal{O}$ at ξ equals the cotangent space $T_\xi^* M$ spanned by the differentials $d\boldsymbol{u}, d\boldsymbol{s}, d\psi, d\boldsymbol{\omega}$ at ξ. The codistribution $d\mathcal{O}$ (as defined in Appendix B) consists of the differentials of u_1 and u_2 and their higher order Lie derivatives with respect to $\boldsymbol{f}$, $\boldsymbol{h}_1$, $\boldsymbol{h}_2$, $\boldsymbol{h}_3$. The linear space $\mathcal{O}$ spanned by these functions is called the observation space of the system.

We decompose the cotangent space $T_\xi^* M$ at state ξ into orthogonal subspaces $T_\xi^* M_u$, $T_\xi^* M_s$, $T_\xi^* M_\psi$, $T_\xi^* M_\omega$, with bases $d\boldsymbol{u}, d\boldsymbol{s}, d\psi, d\boldsymbol{\omega}$, respectively. Obviously, $T_\xi^* M_u$ is spanned since $d\boldsymbol{u} \in d\mathcal{O}$. First, we shall show that the cotangent subspace $T_\xi^* M_\omega$ is spanned. Second, we shall derive a sufficient condition about the contact geometry that guarantees the cotangent subspace $T_\xi^* M_s$ to be spanned. Third, we shall extend this sufficient condition for the spanning of $T_\xi^* M_s \times T_\xi^* M_\psi$. Finally, we shall argue that the differentials chosen from the observability codistribution $d\mathcal{O}$ to span these subspaces will span $T_\xi^* M$ under the same sufficient conditions.

3.1 Angular Velocity

As noted before, one of the angular velocity components $\omega_1, \omega_2, \omega_3$ is a redundant state variable under the rolling constraint (4). Let the coordinates of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in frame $\Pi_{\mathcal{B}}$ be

(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3), respectively. Without loss of generality, we assume $z_3 \neq 0$ in some neighborhood of the state. Immediately from (4)

$$\omega_3 = -\frac{z_1\omega_1 + z_2\omega_2}{z_3}. \tag{18}$$

The new system rewritten from system (16) using (18) has seven state variables $\boldsymbol{u}$, $\boldsymbol{s}$, ψ, ω_1, ω_2. The new observation space equals the old one, except that every appearance of ω_3 is now replaced with (18). For this reason, we will not distinguish the new observation space from $\mathcal{O}$ unless we are taking the partial derivatives of functions in $\mathcal{O}$ with respect to ω_1 and ω_2, on which ω_3 has become dependent.

The cotangent subspace $T^*_\xi M_\omega$ is two-dimensional and spanned by $d\omega_1(\xi)$ and $d\omega_2(\xi)$. To show the spanning of $T^*_\xi M_\omega$ by differentials in the codistribution $d\mathcal{O}$, it suffices to prove that the Jacobian matrix $\partial L_f\boldsymbol{u}/\partial(\omega_1, \omega_2)$ has rank 2, where $L_f\boldsymbol{u} = (L_f u_1, L_f u_2)^T$. This Jacobian is given as

$$\begin{aligned}
\frac{\partial L_f\boldsymbol{u}}{\partial(\omega_1,\omega_2)} &= \frac{\partial L_f\boldsymbol{u}}{\partial\boldsymbol{\omega}}\,\frac{\partial\boldsymbol{\omega}}{\partial(\omega_1,\omega_2)} \\
&= \frac{\partial\dot{\boldsymbol{u}}}{\partial\boldsymbol{\omega}}\,\frac{\partial\boldsymbol{\omega}}{\partial(\omega_1,\omega_2)} \\
&= R_\psi S^{-1}\begin{pmatrix} y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{z_1}{z_3} & -\frac{z_2}{z_3} \end{pmatrix} \\
&= R_\psi S^{-1}\begin{pmatrix} -\frac{x_2}{z_3} & \frac{x_1}{z_3} \\ -\frac{y_2}{z_3} & \frac{y_1}{z_3} \end{pmatrix}.
\end{aligned}$$

Computing the determinant of the Jacobian is straightforward:

$$\begin{aligned}
\det\left(\frac{\partial L_f\boldsymbol{u}}{\partial(\omega_1,\omega_2)}\right) &= \det R_\psi\ \det S^{-1}\,\frac{x_1y_2 - x_2y_1}{z_3^2} \\
&= -\frac{1}{Kz_3},
\end{aligned}$$

where $K \neq 0$ is the Gaussian curvature at the contact point $\boldsymbol{q}$.

Proposition 2 *The cotangent subspace $T^*_\xi M_\omega$ is spanned by the differentials $dL_f\boldsymbol{u}$.*

This lemma states that rolling induces a one-to-one mapping from the space of angular velocities to the space of contact velocities in the plane for constant $\boldsymbol{s}$ and ψ. Accordingly, there exists some control that can distinguishes between any two different (but close enough) angular velocities of the object in the same pose.

3.2 Contact on the Object

We now proceed to determine whether the cotangent subspace $T_\xi^* M_s$ is spanned by the differentials in $d\mathcal{O}$. This time we choose from $\mathcal{O}$ six (scalar) functions $L_{h_1} L_f \boldsymbol{u}$, $L_{h_2} L_f \boldsymbol{u}$, $L_{h_3} L_f \boldsymbol{u}$ comprising the following 2×3 matrix:

$$
\begin{aligned}
Q_\psi &= \left(L_{h_1} L_f \boldsymbol{u},\ L_{h_2} L_f \boldsymbol{u},\ L_{h_3} L_f \boldsymbol{u}\right) \\
&= R_\psi S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times \Big((\boldsymbol{x}, \boldsymbol{y}) R_\psi, -\boldsymbol{z} \Big). \qquad (19)
\end{aligned}
$$

The cotangent subspace $T_\xi^* M_s$ is spanned provided the partial derivatives $\partial Q_\psi / \partial s_1$ and $\partial Q_\psi / \partial s_2$, viewed as two 6-dimensional vectors, are linearly independent. Introducing a 2×3 matrix:

$$
Q = S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times R, \qquad (20)
$$

where $R = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ as given by (1), we have

Lemma 3 *The partial derivatives $\partial Q_\psi / \partial s_1$ and $\partial Q_\psi / \partial s_2$ are linearly dependent if and only if the partial derivatives $\partial Q / \partial s_1$ and $\partial Q / \partial s_2$ are linearly dependent.*

Proof Suppose there exist c_1 and c_2, not both zero, such that

$$
c_1 \frac{\partial Q_\psi}{\partial s_1} + c_2 \frac{\partial Q_\psi}{\partial s_2} = \mathbf{0}.
$$

Expand this equation into two:

$$
\begin{aligned}
& c_1 R_\psi \frac{\partial}{\partial s_1} \Big(S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times (\boldsymbol{x}, \boldsymbol{y}) \Big) R_\psi \\
& \quad + c_2 R_\psi \frac{\partial}{\partial s_2} \Big(S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times (\boldsymbol{x}, \boldsymbol{y}) \Big) R_\psi = \mathbf{0}; \qquad (21) \\
& c_1 R_\psi \frac{\partial}{\partial s_1} \Big(S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times (-\boldsymbol{z}) \Big) \\
& \quad + c_2 R_\psi \frac{\partial}{\partial s_2} \Big(S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times (-\boldsymbol{z}) \Big) = \mathbf{0}. \qquad (22)
\end{aligned}
$$

Multiply (21) by R_ψ on both the left and the right and (22) by $-R_\psi$ on the left; and merge the two resulting equations into one:

$$
c_1 \frac{\partial}{\partial s_1} \Big(S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \Big) + c_2 \frac{\partial}{\partial s_2} \Big(S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \Big) = \mathbf{0}.
$$

Hence $\partial Q / \partial s_1$ and $\partial Q / \partial s_2$ are linearly dependent.

Conversely. □

Thus we will focus our investigation on the linear independence of the partial derivatives $\partial Q/\partial s_1$ and $\partial Q/\partial s_2$. Unfortunately, for certain shape $\boldsymbol{\beta}$, these two partial derivatives are linearly dependent. For example, if $\boldsymbol{\beta}$ is a sphere with radius r, one can verify that[2]

$$Q = \begin{pmatrix} -\frac{5}{7} & 0 & 0 \\ 0 & -\frac{5}{7} & 0 \end{pmatrix}.$$

The partial derivatives of Q are zero and obviously linearly dependent. In this example, the contact point can be anywhere on the sphere whatever path the sphere rolls along in the supporting plane.

But objects of more general shape are to our real interest. So we aim at establishing some condition on the linear independence of $\partial Q/\partial s_1$ and $\partial Q/\partial s_2$ that can be satisfied by most shapes.

To compute these two partial derivatives, we first obtain the partial derivatives of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$:

$$\begin{aligned} \frac{\partial}{\partial s_1}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &= \|\boldsymbol{\beta}_{s_1}\|(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \begin{pmatrix} 0 & -\kappa_{g_1} & -\kappa_1 \\ \kappa_{g_1} & 0 & 0 \\ \kappa_1 & 0 & 0 \end{pmatrix}; \\ \frac{\partial}{\partial s_2}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &= \|\boldsymbol{\beta}_{s_2}\|(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \begin{pmatrix} 0 & -\kappa_{g_2} & 0 \\ \kappa_{g_2} & 0 & -\kappa_2 \\ 0 & \kappa_2 & 0 \end{pmatrix}. \end{aligned}$$

Let $A = D^{-1}\boldsymbol{\beta}\times$, where

$$\boldsymbol{\beta}\times = \begin{pmatrix} 0 & -\boldsymbol{\beta}_3 & \boldsymbol{\beta}_2 \\ \boldsymbol{\beta}_3 & 0 & -\boldsymbol{\beta}_1 \\ -\boldsymbol{\beta}_2 & \boldsymbol{\beta}_1 & 0 \end{pmatrix}$$

with $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)^T$ in frame $\Pi_{\mathcal{B}}$ is the skew-symmetric matrix representing cross product. Then $DA = \boldsymbol{\beta}\times$. Differentiate this equation with respect to s_1 (s_2, respectively) on both sides and solve for the partial derivatives of A:

$$\begin{aligned} \frac{\partial}{\partial s_1}(D^{-1}\boldsymbol{\beta}\times) &= D^{-1}\left(\|\boldsymbol{\beta}_{s_1}\|\boldsymbol{x}\times - \frac{\partial D}{\partial s_1}D^{-1}\boldsymbol{\beta}\times\right); \\ \frac{\partial}{\partial s_2}(D^{-1}\boldsymbol{\beta}\times) &= D^{-1}\left(\|\boldsymbol{\beta}_{s_2}\|\boldsymbol{y}\times - \frac{\partial D}{\partial s_2}D^{-1}\boldsymbol{\beta}\times\right). \end{aligned}$$

With all the above partial derivatives we can write out the partial derivatives of Q:

$$\frac{\partial Q}{\partial s_1} = \left(\frac{\partial S^{-1}}{\partial s_1}(\boldsymbol{y}, -\boldsymbol{x})^T - \|\boldsymbol{\beta}_{s_1}\|S^{-1}(\kappa_{g_1}\boldsymbol{x},\, \kappa_{g_1}\boldsymbol{y} + \kappa_1\boldsymbol{z})^T\right) D^{-1}\boldsymbol{\beta} \times R$$

[2]Use, for instance, the parametrization $\beta = r(\cos s_1 \cos s_2, -\cos s_1 \sin s_2, \sin s_1)^T$.

$$+ S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T D^{-1}\Big(-\frac{\partial D}{\partial s_1} D^{-1}\boldsymbol{\beta} \times R$$

$$+ \|\boldsymbol{\beta}_{s_1}\|\Big((0, \boldsymbol{z}, -\boldsymbol{y}) + \boldsymbol{\beta} \times (\kappa_{g_1}\boldsymbol{y} + \kappa_1\boldsymbol{z},\ -\kappa_{g_1}\boldsymbol{x},\ -\kappa_1\boldsymbol{x})\Big)\Big); \tag{23}$$

$$\frac{\partial Q}{\partial s_2} = \left(\frac{\partial S^{-1}}{\partial s_2}(\boldsymbol{y}, -\boldsymbol{x})^T - \|\boldsymbol{\beta}_{s_2}\| S^{-1}(\kappa_{g_2}\boldsymbol{x} - \kappa_2\boldsymbol{z},\ \kappa_{g_2}\boldsymbol{y})^T\right) D^{-1}\boldsymbol{\beta} \times R$$

$$+ S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T D^{-1}\Big(-\frac{\partial D}{\partial s_2} D^{-1}\boldsymbol{\beta} \times R$$

$$+ \|\boldsymbol{\beta}_{s_2}\|\Big((-\boldsymbol{z}, 0, \boldsymbol{x}) + \boldsymbol{\beta} \times (\kappa_{g_2}\boldsymbol{y},\ -\kappa_{g_2}\boldsymbol{x} + \kappa_2\boldsymbol{z},\ -\kappa_2\boldsymbol{y})\Big)\Big). \tag{24}$$

It seems very difficult to say anything about the linear independence of $\partial Q/\partial s_1$ and $\partial Q/\partial s_2$ directly from their complicated forms above. So we intend to find a matrix A and a sufficient condition under which $(\partial Q/\partial s_1)A$ and $(\partial Q/\partial s_2)A$ will be linearly independent. The same condition will then ensure the linear independence between $\partial Q/\partial s_1$ and $\partial Q/\partial s_2$. The matrix of our choice is $R^T\boldsymbol{\beta}$, which, when multiplied on (23) and (24) to the right, gets rid of those terms in $\partial Q/\partial s_1$ and $\partial Q/\partial s_2$ that have $\boldsymbol{\beta} \times R$ on the right. We can further simplify the remaining terms in the products using the following equations:

$$\begin{aligned}
(0, \boldsymbol{z}, -\boldsymbol{y})R^T &= \boldsymbol{x}\times; \\
(-\boldsymbol{z}, 0, \boldsymbol{x})R^T &= \boldsymbol{y}\times; \\
(\kappa_{g_1}\boldsymbol{y} + \kappa_1\boldsymbol{z}, -\kappa_{g_1}\boldsymbol{x}, -\kappa_1\boldsymbol{x})R^T &= \kappa_{g_1}\boldsymbol{z}\times\ -\ \kappa_1\boldsymbol{y}\times; \\
(\kappa_{g_2}\boldsymbol{y}, -\kappa_{g_2}\boldsymbol{x} + \kappa_2\boldsymbol{z}, -\kappa_2\boldsymbol{y})R^T &= \kappa_{g_2}\boldsymbol{z}\times\ +\ \kappa_2\boldsymbol{x}\times.
\end{aligned}$$

In the end, we obtain

$$\frac{\partial Q}{\partial s_1}R^T\boldsymbol{\beta} = \|\boldsymbol{\beta}_{s_1}\| S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T D^{-1}\boldsymbol{\beta} \times \boldsymbol{b}_1; \tag{25}$$

$$\frac{\partial Q}{\partial s_2}R^T\boldsymbol{\beta} = \|\boldsymbol{\beta}_{s_2}\| S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T D^{-1}\boldsymbol{\beta} \times \boldsymbol{b}_2, \tag{26}$$

where

$$\boldsymbol{b}_1 = -\boldsymbol{x} + (\kappa_{g_1}\boldsymbol{z} - \kappa_1\boldsymbol{y}) \times \boldsymbol{\beta}; \tag{27}$$

$$\boldsymbol{b}_2 = -\boldsymbol{y} + (\kappa_{g_2}\boldsymbol{z} + \kappa_2\boldsymbol{x}) \times \boldsymbol{\beta}. \tag{28}$$

Lemma 4 *The vectors $(\partial Q/\partial s_1)R^T\boldsymbol{\beta}$ and $(\partial Q/\partial s_2)R^T\boldsymbol{\beta}$ are linearly dependent if and only if $\boldsymbol{\beta} \cdot (I\boldsymbol{z}) = 0$ or $\det(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{\beta}) = 0$.*

Proof We need to conduct a few steps of reasoning:

$$\frac{\partial Q}{\partial s_1}R^T\boldsymbol{\beta} \quad \text{and} \quad \frac{\partial Q}{\partial s_2}R^T\boldsymbol{\beta} \quad \text{are linearly dependent}$$

if and only if

$$(\boldsymbol{y}, -\boldsymbol{x})^T D^{-1}\boldsymbol{\beta} \times \boldsymbol{b}_1 \quad \text{and} \quad (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1}\boldsymbol{\beta} \times \boldsymbol{b}_2 \quad \text{are linearly dependent}$$

if and only if

$$D^{-1}\boldsymbol{\beta} \times \boldsymbol{b}_1, \quad D^{-1}\boldsymbol{\beta} \times \boldsymbol{b}_2, \quad \text{and} \quad \boldsymbol{z} \quad \text{are linearly dependent}$$

if and only if

$$\boldsymbol{\beta} \times \boldsymbol{b}_1, \quad \boldsymbol{\beta} \times \boldsymbol{b}_2, \quad \text{and} \quad D\boldsymbol{z} \quad \text{are linearly dependent,}$$

where

$$\begin{aligned} D\boldsymbol{z} &= \left(\frac{I}{m} + \|\boldsymbol{\beta}\|^2 I_3 - \boldsymbol{\beta}\boldsymbol{\beta}^T\right)\boldsymbol{z} \\ &= \frac{I}{m}\boldsymbol{z} + \boldsymbol{\beta} \times (\boldsymbol{z} \times \boldsymbol{\beta}). \end{aligned}$$

There are two cases: (1) $\boldsymbol{\beta} \cdot (I\boldsymbol{z}) = 0$ and (2) $\boldsymbol{\beta} \cdot (I\boldsymbol{z}) \neq 0$. In case (1), $\boldsymbol{\beta} \times \boldsymbol{b}_1, \boldsymbol{\beta} \times \boldsymbol{b}_2, D\boldsymbol{z}$ are linearly dependent. In case (2), they are linearly dependent if and only if $\boldsymbol{\beta} \times \boldsymbol{b}_1$ and $\boldsymbol{\beta} \times \boldsymbol{b}_2$ are linearly dependent, which happens if and only if $\det(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{\beta}) = 0$ □

Combining Lemmas 3 and 4, we arrive at a sufficient condition for the spanning of $T^*_\xi M_s$.

Proposition 5 *The cotangent subspace $T^*_\xi M_s$ is spanned by the differentials of six functions $L_{h_1}L_f\boldsymbol{u}$, $L_{h_2}L_f\boldsymbol{u}$, $L_{h_3}L_f\boldsymbol{u}$, all in the observation space of system (16), if (1) $\boldsymbol{\beta} \cdot (I\boldsymbol{z}) \neq 0$ and (2) $\det(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{\beta}) \neq 0$.*

When $\boldsymbol{\beta}$ is a sphere, $\boldsymbol{b}_1 = \boldsymbol{b}_2 = \boldsymbol{0}$, violating condition (2) in Proposition 5. For general $\boldsymbol{\beta}$, conditions (1) and (2) in Proposition 5 are satisfied at all but at most a one-dimensional set of contact points.

Up until now we have selected from the observation space $\mathcal{O}$ a total of 10 scalar functions: $\boldsymbol{u}$, $L_f\boldsymbol{u}$, $L_{h_1}L_f\boldsymbol{u}$, $L_{h_2}L_f\boldsymbol{u}$, and $L_{h_3}L_f\boldsymbol{u}$. The first two functions depend on only $\boldsymbol{u}$, while the last six functions on only $\boldsymbol{s}$ and ψ. Hence we have shown that the cotangent subspace $T^*_\xi M_u \times T^*_\xi M_s \times T^*_\xi M_\omega$ is spanned unless $\boldsymbol{\beta} \cdot (I\boldsymbol{z}) = 0$ or $\det(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{\beta}) = 0$.

3.3 Rotation about the Contact Normal

The rolling objects has three degrees of freedom: the point of contact on the object $\boldsymbol{s}$ and the rotation ψ of the object about the contact normal. The last proposition states that the contact point alone can be distinguished under its sufficient condition. Such condition essentially ensures the differentials of $L_{h_1}L_f\boldsymbol{u}$, $L_{h_2}L_f\boldsymbol{u}$, and $L_{h_3}L_f\boldsymbol{u}$ to span the cotangent subspace $T^*_\xi M_s$. Now we shall show that these differentials indeed span the cotangent space $T^*_\xi M_s \times T^*_\xi M_\psi$ except for special shapes such as a sphere. That is, we shall show that the 2×3 matrices $\partial Q_\psi/\partial s_1$, $\partial Q_\psi/\partial s_2$, and $\partial Q_\psi/\partial \psi$ are linearly independent for general shape $\boldsymbol{\beta}$.

Let Q_{12} be the 2×2 matrix formed by the first two columns in Q:

$$Q_{12} = S^{-1}(\boldsymbol{y}, -\boldsymbol{x})^T D^{-1}\boldsymbol{\beta} \times (\boldsymbol{x}, \boldsymbol{y}).$$

Construct a 2×3 matrix:

$$\tilde{Q} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Q + Q_{12} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Lemma 6 *The partial derivatives $\partial Q_\psi/\partial s_1$, $\partial Q_\psi/\partial s_2$, $\partial Q_\psi/\partial \psi$ are linearly dependent if and only if $\tilde{Q}$ and the partial derivatives $\partial Q/\partial s_1$, $\partial Q/\partial s_2$ are linearly dependent.*

The proof of Lemma 6 is similar to that of Lemma 3, and makes use of the following equations:

$$R_\psi \frac{\partial R_\psi}{\partial \psi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$
$$\frac{\partial R_\psi}{\partial \psi} R_\psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This lemma, along with Propositions 2 and 5, suggest the weak dependency, if any, of system (16)'s local observability on the object's rotation angle ψ.

In this section, assuming the linear independence of $\partial Q/\partial s_1$ and $\partial Q/\partial s_2$, we want to determine if $\tilde{Q}$ is linearly independent of them. If not always, we would like to seek some sufficient condition on the object shape $\boldsymbol{\beta}$. Let us start by offering a rather restrictive necessary condition for $\tilde{Q}$ to vanish.

Lemma 7 *The matrix $\tilde{Q}$ is equal to $\mathbf{0}$ only if the following three conditions all hold:*

$$\begin{aligned} \boldsymbol{\beta} \times \boldsymbol{z} &= \mathbf{0}; \\ \kappa_1 \boldsymbol{x}^T D^{-1} \boldsymbol{x} &= \kappa_2 \boldsymbol{y}^T D^{-1} \boldsymbol{y}; \\ \boldsymbol{x}^T D^{-1} \boldsymbol{y} &= 0. \end{aligned} \tag{29}$$

Proof There are two cases: (1) $\boldsymbol{\beta} \times \boldsymbol{z} \neq \mathbf{0}$, and (2) $\boldsymbol{\beta} \times \boldsymbol{z} = \mathbf{0}$.

In the first case, we have $D^{-1}\boldsymbol{\beta} \times \boldsymbol{z} \neq \mathbf{0}$ due to that D^{-1} is positive definite. The third column of $\tilde{Q}$ is not zero, by the following steps of reasoning:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times \boldsymbol{z} = \mathbf{0}$$

if and only if

$$(\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times \boldsymbol{z} = \mathbf{0}$$

if and only if

$$D^{-1}\boldsymbol{\beta} \times \boldsymbol{z} = c\boldsymbol{z}, \qquad \text{for some } c \neq 0$$

if and only if

$$\boldsymbol{\beta} \times \boldsymbol{z} = cD\boldsymbol{z}, \qquad \text{for some } c \neq 0,$$

only if

$$0 = \boldsymbol{z} \cdot (\boldsymbol{\beta} \times \boldsymbol{z}) = \boldsymbol{z}^T D \boldsymbol{z}.$$

But $\boldsymbol{z}^T D \boldsymbol{z} > 0$, as D is positive definite.

In the second case, the third column of Q is zero. Since $\boldsymbol{\beta} = \|\boldsymbol{\beta}\| \boldsymbol{z}$, we have

$$\begin{aligned} Q_{12} &= \|\boldsymbol{\beta}\| S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{z} \times (\boldsymbol{x}, \boldsymbol{y}) \\ &= \|\boldsymbol{\beta}\| S^{-1} (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} (\boldsymbol{y}, -\boldsymbol{x}). \end{aligned}$$

The 2×2 matrix $(\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} (\boldsymbol{y}, -\boldsymbol{x})$ is positive definite, so we write

$$Q_{12} = \|\boldsymbol{\beta}\| S^{-1} \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \qquad a > 0 \text{ and } ad - b^2 > 0.$$

Now we are able to simplify the first two columns of $\tilde{Q}$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Q_{12} + Q_{12} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \|\boldsymbol{\beta}\| \begin{pmatrix} -\frac{b}{\kappa_1} - \frac{b}{\kappa_2} & \frac{a}{\kappa_1} - \frac{d}{\kappa_2} \\ \frac{a}{\kappa_1} - \frac{d}{\kappa_2} & \frac{b}{\kappa_1} + \frac{b}{\kappa_2} \end{pmatrix}.$$

The above 2×2 matrix is zero if and only if $b = 0$ and $\frac{a}{\kappa_1} = \frac{d}{\kappa_2}$, which imply conditions (29). □

Suppose the necessary conditions in Lemma 7 does not hold. Then $\tilde{Q} \neq \mathbf{0}$. Suppose again $\partial Q / \partial s_1$ and $\partial Q / \partial s_2$ are linearly independent. The matrices $\partial Q / \partial s_1$, $\partial Q / \partial s_2$, and $\tilde{Q}$ are linearly dependent if and only if there exists c_1, c_2, not all zero, such that

$$c_1 \frac{\partial Q}{\partial s_1} + c_2 \frac{\partial Q}{\partial s_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Q + Q_{12} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{30}$$

Multiply both sides of (30) by $R^T \boldsymbol{\beta}$ on the right, plug (25) and (26) in, and then multiply both sides of the resulting equation by S on the left:

$$c_1 \|\boldsymbol{\beta}_{s_1}\| (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times \boldsymbol{b}_1 + c_2 \|\boldsymbol{\beta}_{s_2}\| (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times \boldsymbol{b}_2 = (\boldsymbol{y}, -\boldsymbol{x})^T D^{-1} \boldsymbol{\beta} \times (\boldsymbol{\beta} \times \boldsymbol{z}).$$

Solving the above equation for c_1 and c_2 and substituting the solutions into (30), we end up with a system of third order partial differential equations of $\boldsymbol{\beta}$, four of which are independent. We postulate that this PDE system has at most one solution.

Proposition 8 *Assume $dL_{h_1} L_f \boldsymbol{u}$, $dL_{h_2} L_f \boldsymbol{u}$, and $dL_{h_3} L_f \boldsymbol{u}$ span the cotangent subspace $T^*_\xi M_s$. Furthermore, assume the necessary condition in Lemma 7 does not hold. The same differentials will also span the cotangent subspace $T^*_\xi M_s \times T^*_\xi M_\psi$ except for at most one (local) shape of $\boldsymbol{\beta}$.*

3.4 Sufficient Conditions for Local Observability

To summarize, we have chosen from the observation space $\mathcal{O}$ a total of ten functions: two scalar functions $\boldsymbol{u}$, which are the system outputs; two first order Lie derivatives $L_f\boldsymbol{u}$, which involve ω_1, ω_2, s_1, s_2, ψ; and six second order Lie derivatives $L_{h_1}L_f\boldsymbol{u}$, $L_{h_2}L_f\boldsymbol{u}$, $L_{h_3}L_f\boldsymbol{u}$, which involve s_1, s_2, ψ only. The differentials of these functions (except $L_f\boldsymbol{u}$) live in orthogonal subspaces of the cotangent space T_ξ^*M. If these subspaces are spanned, given that $T_\xi^*M_\omega$ is spanned by $dL_f\boldsymbol{u}$, T_ξ^*M is also spanned and local observability of system (16) follows.

Combining Proposition 2 through Proposition 8, we obtain a sufficient condition for the local observability of rolling:

Theorem 9 *The rolling system (16) is locally observable except for at most one object shape* $\boldsymbol{\beta}$ *if*

$$\det(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{\beta}) \neq 0 \tag{31}$$

and

$$\begin{aligned} \boldsymbol{\beta}\cdot(I\boldsymbol{z}) \neq 0, & \qquad \textit{when } \boldsymbol{\beta}\times\boldsymbol{z}\neq\boldsymbol{0}; \qquad (32)\\ \kappa_1\boldsymbol{x}^T D^{-1}\boldsymbol{x} \neq \kappa_2\boldsymbol{y}^T D^{-1}\boldsymbol{y} \quad \textit{or} & \\ \boldsymbol{x}^T D^{-1}\boldsymbol{y} \neq 0, & \qquad \textit{when } \boldsymbol{\beta}\times\boldsymbol{z}=\boldsymbol{0}, \qquad (33) \end{aligned}$$

where $\boldsymbol{b}_1$ *and* $\boldsymbol{b}_2$ *are given by (27) and (28), respectively.*[3]

Condition (31) depends on the differential geometry of contact. It is only violated by at most a one-dimensional set of points in a general patch $\boldsymbol{\beta}$. Condition (32) states that the contact normal transformed by the angular inertia matrix should not be perpendicular the contact location vector. It is only violated by at most a one-dimensional set of points in a general patch. Condition (33), only for $\boldsymbol{\beta}\times\boldsymbol{z}=\boldsymbol{0}$, is hardly seen to be violated by any shape other than a sphere.

4 Summary

This paper deals with local observability of a three-dimensional object rolling on a translating plane. The object is bounded by a smooth surface that makes point contact with the plane, which is free to accelerate in any direction. The plane is rough enough to allow only the pure rolling motion of the object. As the object rolls, the contact traces out a path in the plane, which can be detected by a tactile array sensor embedded in the plane.

Utilizing Montana's equations for contact kinematics, we described the kinematics and dynamics of this task by a nonlinear system, of which the output is the contact location in the plane. Then we established a sufficient condition for the pose and motion of the object to be locally observable. This was done by decomposing the cotangent space at the current state into orthogonal subspaces associated with the object's pose and angular

[3]Note that $\beta\times\boldsymbol{z}=\mathbf{o}$ implies $\beta\cdot(I\boldsymbol{z})\neq 0$ given that I is positive definite.

velocity, respectively, and by later combining the sufficient conditions on the spanning of these subspaces. The combined condition depends only on the object's contact geometry and angular inertia matrix. And it is general enough to be satisfied almost everywhere on any object surface that does not assume certain degeneracies as a sphere does.

From a broader perspective, we expect the paper's result on local observability from contact to generalize to many other manipulation tasks. The presented work is part of our larger goal to understand how much information about a manipulation task is encoded in a small amount of tactile data and to develop methods for retrieving such information.

One direction of future work will be to design a nonlinear observer that can asymptotically estimate the pose and motion of a rolling object from the path of contact. Further along this line of research will be to implement (tactile) sensors and mount them on a palm to estimate poses and motions of real objects.

Another direction is to study the controllability of the rolling object on a palm and to combine the result with the present local observability result for developing parts orienting and dextrous manipulation strategies.

Appendices

A Elements of Differential Geometry

This appendix briefly reviews some basics in differential geometry. We assume the reader's knowledge about tangent vectors, tangent spaces, vector fields, curves; and hence do not go over these definitions. For an elementary introduction to differential geometry, we refer the reader to O'Neill [23]; for a comprehensive introduction, we refer to Spivak's five volume series that begins with [26].

A.1 Normal and Gaussian Curvatures

A mapping $\boldsymbol{f}$ from an open set $D \subset \Re^2$ to $\Re^3$ is *regular* provided that at each point $\boldsymbol{p} \in D$ the Jacobian matrix $\partial \boldsymbol{f} / \partial \boldsymbol{p}$ has rank 2. A *coordinate patch* (or a *patch*) $\boldsymbol{g} : D \to \Re^3$ is a one-to-one regular mapping from an open set $D \subset \Re^2$ to $\Re^3$. A patch $\boldsymbol{g} : D \to \Re^3$ is called a *proper* patch provided its inverse function $\boldsymbol{g}^{-1} : \boldsymbol{g}(D) \to D$ is continuous.

A subset M of $\Re^3$ is a *surface* in $\Re^3$ provided for each point $\boldsymbol{p} \in M$ there exists a proper patch $\boldsymbol{g}$ whose image contains a neighborhood of $\boldsymbol{p}$.

Let Z be a vector field on a surface M in $\Re^3$, $\boldsymbol{v}$ a tangent vector to M, and α a curve in M that has initial velocity $\alpha'(0) = \boldsymbol{v}$. Let $Z_\alpha : t \mapsto Z(\alpha(t))$ be the restriction of Z to α. Then the *covariant derivative* of Z with respect to $\boldsymbol{v}$ is defined to be $\nabla_v Z = (Z_\alpha)'(0)$. Let $\boldsymbol{p}$ be a point of M and U a unit normal vector field on a neighborhood of $\boldsymbol{p}$ in M. The *shape operator* of M at $\boldsymbol{p}$ is a function $S_p : \boldsymbol{v} \mapsto -\nabla_v U$ for each tangent vector $\boldsymbol{v}$ to M at $\boldsymbol{p}$. The shape operator S_p is a *symmetric linear* operator that maps the tangent plane $T_p(M)$ to itself.

Let $\boldsymbol{u}$ be a unit tangent vector to M at a point $\boldsymbol{p}$. The *normal curvature* of M in the $\boldsymbol{u}$ direction is given by $\kappa_n(\boldsymbol{u}) = S(\boldsymbol{u}) \cdot \boldsymbol{u}$. The *normal section* of M at $\boldsymbol{p}$ in the $\boldsymbol{u}$ direction is a curve cut from M by a plane containing $\boldsymbol{u}$ and the surface normal $U(\boldsymbol{p})$; hence its curvature is the normal curvature $\kappa_n(\boldsymbol{u})$. The maximum and minimum values of the normal curvature $\kappa_n(\boldsymbol{u})$ are called the *principal curvatures* of M at $\boldsymbol{p}$ and denoted by κ_1 and κ_2, respectively. The directions in which κ_1 and κ_2 occur are called the *principal directions* of M at $\boldsymbol{p}$. Vectors in these directions are called the *principal vectors* of M at $\boldsymbol{p}$.

The *Guassian curvature* K at a point $\boldsymbol{p}$ of M is the determinant of the shape operator S_p and the *mean curvature* is the function $H = \frac{1}{2}\mathrm{trace}\, S_p$. They can be also expressed in terms of the principal curvatures by $K = \kappa_1\kappa_2$ and $H = \frac{1}{2}(\kappa_1 + \kappa_2)$.

A.2 Patch Computations

A patch $\boldsymbol{f} : D \to M$ is *orthogonal* provided its two partial derivatives are orthogonal to each other, that is, $\boldsymbol{f}_u \cdot \boldsymbol{f}_v = 0$ for each $(u, v) \in D$. An orthogonal patch $\boldsymbol{f} : D \to M$ is *principal* provided $S(\boldsymbol{f}_u) \cdot \boldsymbol{f}_v = S(\boldsymbol{f}_v) \cdot \boldsymbol{f}_u = 0$, where S is the shape operator of M.

A regular curve α in M is a *principal curve* (or *line of curvature*) provided that its velocity α' always points in a principal direction. The parameter curves of a principal patch $\boldsymbol{f}$ are lines of curvature.

If a point $\boldsymbol{p}$ on a surface M is not umbilic, then there exists a one-to-one and regular mapping $\boldsymbol{f} : U \to M$ on an open set $U \subset \Re^2$ with $\boldsymbol{p} \in \boldsymbol{f}(U)$, whose parameter curves are lines of curvature.

Let $\boldsymbol{f} : D \to M$ be an principal patch in surface M. The *normalized Gauss frame* at a point $\boldsymbol{f}(u,v)$ is the coordinate frame with origin at $\boldsymbol{f}(u,v)$ and coordinate axes[4]

$$\begin{aligned}
\boldsymbol{x}(u,v) &= \frac{\boldsymbol{f}_u(u,v)}{\|\boldsymbol{f}_u(u,v)\|};\\
\boldsymbol{y}(u,v) &= \frac{\boldsymbol{f}_v(u,v)}{\|\boldsymbol{f}_v(u,v)\|};\\
\boldsymbol{z}(u,v) &= \boldsymbol{x}(u,v) \times \boldsymbol{y}(u,v).
\end{aligned}$$

The shape operator S with respect to $\boldsymbol{x}, \boldsymbol{y}$ is

$$\begin{aligned}
S &= (\boldsymbol{x}, \boldsymbol{y})^T(-\nabla_x \boldsymbol{z}, -\nabla_y \boldsymbol{z})\\
&= (\boldsymbol{x}, \boldsymbol{y})^T \left(-\frac{\boldsymbol{z}_u}{\|\boldsymbol{f}_u\|}, -\frac{\boldsymbol{z}_v}{\|\boldsymbol{f}_v\|}\right)\\
&= \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},
\end{aligned}$$

where κ_1 and κ_2 are the principal curvatures. At a point $\boldsymbol{f}(u,v) \in D$ the geodesic curvatures κ_{gu} and κ_{gv}, of the u-parameter curve and the v-parameter curve, respectively, are given as

$$\begin{aligned}
\kappa_{gu} &= \boldsymbol{y} \cdot \nabla_x \boldsymbol{x}\\
&= \boldsymbol{y} \cdot \frac{\boldsymbol{x}_u}{\|\boldsymbol{f}_u\|};\\
\kappa_{gv} &= -\boldsymbol{x} \cdot \nabla_y \boldsymbol{y}\\
&= \boldsymbol{y} \cdot \nabla_y \boldsymbol{x}\\
&= \boldsymbol{y} \cdot \frac{\boldsymbol{x}_v}{\|\boldsymbol{f}_v\|}.
\end{aligned}$$

A.3 Manifolds, Cotangent Bundles, Codistributions

An *n-dimensional* manifold M is a set furnished with a collection $\mathcal{C}$ of abstract patches (one-to-one functions $\boldsymbol{f} : D \to M$, D an open set in $\Re^n$) such that

1. M is covered by the images of the (abstract) patches in the collection $\mathcal{C}$.
2. For any two patches $\boldsymbol{f}$ and $\boldsymbol{g}$ in $\mathcal{C}$, the composite functions $\boldsymbol{g}^{-1}\boldsymbol{f}$ and $\boldsymbol{f}^{-1}\boldsymbol{g}$ are differentiable and defined on open sets in $\Re^n$.

[4]We assume the coordinate frame is everywhere right-handed.

A surface of $\Re^3$ is just a two-dimensional manifold. Tangent vectors, tangent spaces, vector fields on an n-dimensional manifold are defined in the same way as in the special case $n = 2$. We need only replace $i = 1, 2$ by $i = 1, 2, \ldots, n$.

Let M be a manifold and T_pM its tangent space at a point $\boldsymbol{p}$ of M. The dual space of T_pM, denoted T_p^*M, is called the *cotangent space* of M at $\boldsymbol{p}$. More specifically, T_p^*M consists of all linear functions on T_pM. An element of T_p^*M is called a *cotangent vector*. The *cotangent bundle* of a manifold M is defined as

$$T^*M = \bigcup_{p \in M} T_p^*M.$$

A *codistribution* D on a manifold M assigns to each point $\boldsymbol{p} \in M$ a linear subspace $D(\boldsymbol{p})$ of the cotangent space T_p^*M.

A *one-form* ϕ on M is a map that assigns to each point $\boldsymbol{p} \in M$ a cotangent vector $\phi(\boldsymbol{p}) \in T_p^*M$. The gradient of a real-valued function f on M is a one-form df called the *differential* of f.

A.4 Lie Derivatives

Let M be an n-dimensional manifold. The *Lie derivative* of a function $h : M \to \Re$ along a vector field X on M, denoted by $L_X h$, is the directional derivative $dh(X) = dh \cdot X$, where the one-form dh is the differential of h. We use the notation $L_{X_1} L_{X_2} \cdots L_{X_l} h$ for the repeated Lie derivative $L_{X_1}(L_{X_2}(\ldots(L_{X_l} h)\ldots))$ with respect to vector fields $X_l, \ldots, X_2, X_1$ on M.

The *Lie bracket* of two vector fields X and Y on M at a point $\boldsymbol{p}$ of M is a vector field defined as

$$[X, Y](\boldsymbol{p}) = \frac{\partial Y}{\partial \boldsymbol{p}}(\boldsymbol{p})\, X(\boldsymbol{p}) - \frac{\partial X}{\partial \boldsymbol{p}}(\boldsymbol{p})\, Y(\boldsymbol{p}).$$

The ad-notation is used for repeated Lie brackets:

$$\begin{aligned} \mathrm{ad}_X^0 Y &= Y; \\ \mathrm{ad}_X^j Y &= [X, \mathrm{ad}_X^{j-1} Y], \qquad \text{for } j > 0. \end{aligned}$$

The bracket $[X, Y]$ can be interpreted in some sense as the "derivative" of the vector field Y along the vector field X. It is therefore also denoted as $L_X Y$, the Lie derivative of Y along X.

B Observability of a Nonlinear System

The theoretical foundation of our work comes from the part of control theory concerned with the observability of nonlinear systems. For a general introduction to nonlinear control theory, we refer the reader to Isidori [13] and Nijmeijer and van der Schaft [22].

Consider a smooth affine (or input-linear) control system together with an output map:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{m} u_i\, \boldsymbol{g}_i(\boldsymbol{x}), \qquad (u_1, \ldots, u_m) \in U \subset \Re^m, \qquad (34) \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}), \end{aligned}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ is the *state* in a smooth n-dimensional manifold $M \subseteq \Re^n$ (called the *state space manifold*), $\boldsymbol{f}, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_m$ are smooth vector fields on M, and $\boldsymbol{h} = (h_1, \ldots, h_k)^T : M \to \Re^k$ is the smooth output map of the system. Here $\boldsymbol{f}$ is called the *drift vector field*, and $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_m$ are called the *input vector fields.* In the system, $u_1, \ldots, u_m$ are the inputs, called the *controls*, over time whose Cartesian product range U defines the system's input space. At state $\boldsymbol{x}$, $\boldsymbol{f}(\boldsymbol{x})$ is a tangent vector to M representing the rate of change of $\boldsymbol{x}$ without any input, while $\boldsymbol{g}_j(\boldsymbol{x})$ for $1 \le j \le m$ is a tangent vector showing the rate of such change due to unit input of u_j.

We are only concerned with the class of controls $\mathcal{U}$ that consists of piecewise constant functions that are continuous from the right.[5] We call these controls *admissible.* The system with constant controls, or no input fields, equivalently, is said to be *autonomous.*

Denote by $\boldsymbol{y}(t, \boldsymbol{x}_0, \boldsymbol{u})$, $t \ge 0$, the output function of the system with initial state $\boldsymbol{x}_0$ and under control $\boldsymbol{u}$. Two states $\boldsymbol{x}_1, \boldsymbol{x}_2 \in M$ are said to be *indistinguishable* (denoted by $\boldsymbol{x}_1 I \boldsymbol{x}_2$) if for every admissible control $\boldsymbol{u}$ the output functions $\boldsymbol{y}(t, \boldsymbol{x}_1, \boldsymbol{u})$ and $\boldsymbol{y}(t, \boldsymbol{x}_2, \boldsymbol{u})$, $t \ge 0$ are identical on their common domain of definition. The system is *observable* if $\boldsymbol{x}_1 I \boldsymbol{x}_2$ implies $\boldsymbol{x}_1 = \boldsymbol{x}_2$.

To derive a condition on nonlinear observability, the above definition of "observable" is localized in the following way. Let $V \subset M$ be an open set containing states $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. These two states are said to be V-indistinguishable, denoted by $\boldsymbol{x}_1 I^V \boldsymbol{x}_2$, if for any $T > 0$ and any constant control $\boldsymbol{u} : [0, T] \to U$ such that $\boldsymbol{x}(t, \boldsymbol{x}_1, \boldsymbol{u}), \boldsymbol{x}(t, \boldsymbol{x}_2, \boldsymbol{u}) \in V$ for all $0 \le t \le T$, it follows that $\boldsymbol{y}(t, \boldsymbol{x}_1, \boldsymbol{u}) = \boldsymbol{y}(t, \boldsymbol{x}_2, \boldsymbol{u})$ for all $0 \le t \le T$ on their common domain of definition. The system is *locally observable* at $\boldsymbol{x}_0$ if there exists a neighborhood W of $\boldsymbol{x}_0$ such that for every neighborhood $V \subset W$ of $\boldsymbol{x}_0$ the relation $\boldsymbol{x}_0 I^V \boldsymbol{x}_1$ implies that $\boldsymbol{x}_0 = \boldsymbol{x}_1$. The system is called *locally observable* if it is locally observable at every $\boldsymbol{x}_0 \in M$. Figure 4 illustrates local observability for the case with one output function.

The *observation space* $\mathcal{O}$ of system (34) is the linear space (over $\Re$) of functions on M that includes $h_1, \ldots, h_k$, and all repeated Lie derivatives

$$L_{X_1} L_{X_2} \cdots L_{X_l} h_j, \qquad j = 1, \ldots, k, \;\; l = 1, 2, \ldots$$

where $X_i \in \{\boldsymbol{f}, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_m\}$, $1 \le i \le l$. It is not difficult to show that $\mathcal{O}$ is also the linear space of functions on M that includes $h_1, \ldots, h_k$, and all repeated Lie derivatives

$$L_{Z_1} L_{Z_2} \cdots L_{Z_l} h_j, \qquad j = 1, \ldots, k, \;\; l = 1, 2, \ldots$$

where

$$Z_i(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \sum_{j=1}^{m} u_{ij}\, \boldsymbol{g}_j(\boldsymbol{x}), \qquad (35)$$

[5] So that $\mathcal{U}$ is closed under concatenation.

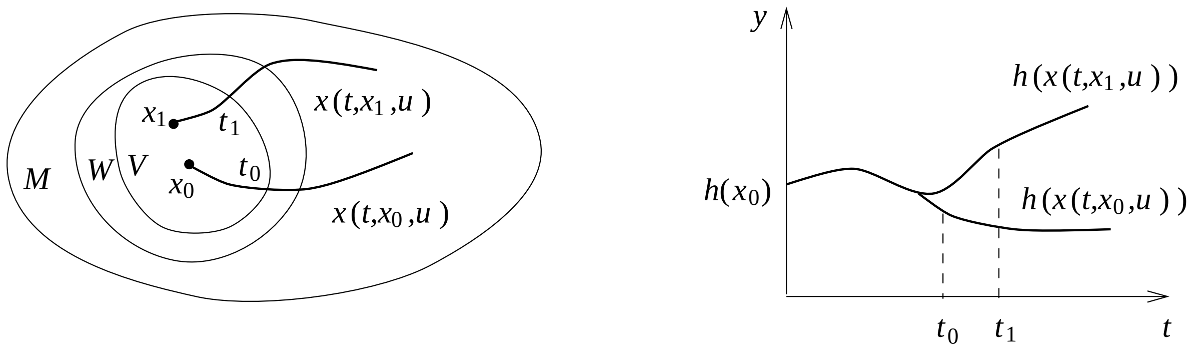

Figure 4: Local observability at state $\boldsymbol{x}_0$. Given the state space M, $W \subset M$ is some neighborhood of $\boldsymbol{x}_0$. For any neighborhood $V \subset W$ of $\boldsymbol{x}_0$, $\boldsymbol{x}_0$ is V-distinguishable from all other states in V. More specifically, for any state $\boldsymbol{x}_1 \neq \boldsymbol{x}_0$ in V, there exists a constant admissible control $\boldsymbol{u}$ such that the two state trajectories $\boldsymbol{x}(t, \boldsymbol{x}_0, \boldsymbol{u})$ and $\boldsymbol{x}(t, \boldsymbol{x}_1, \boldsymbol{u})$ will yield different outputs before one of them exits V (at time t_0).

for some point $\boldsymbol{u}_i = (u_{i1}, \ldots, u_{im}) \in U$.

The observation space shall be better understood with the notion of integral curve. Given a nonlinear system

$$\dot{\boldsymbol{z}} = Z(\boldsymbol{z}),$$

defined by some vector field Z on the state space M, the *integral curve* $\sigma_{z_0}(t)$ is the solution of the system satisfying the initial condition $\sigma_{z_0}(0) = \boldsymbol{z}_0$. For every bounded subset $M_1 \subset M$, there exists an interval $(t_1, t_2) \ni 0$ on which the integral curve $\sigma_{z_0}(t)$ is well-defined for all $t \in (t_1, t_2)$. This allows us to introduce on M_1 a set of maps, called the *flow*,

$$\begin{aligned} Z^t : M_1 &\to M, \qquad t \in (t_1, t_2), \\ \boldsymbol{z}_0 &\mapsto \sigma_{z_0}(t). \end{aligned}$$

Now choose inputs of system (34) such that it is driven by a sequence of vector fields $Z_1, \ldots, Z_l$ of form (35) for small time $t_1, \ldots, t_l$, respectively. The outputs of the system at time $t_1 + \cdots + t_l$ are

$$h_i(Z_l^{t_l} \circ Z_{l-1}^{t_{l-1}} \circ \cdots Z_1^{t_1}(\boldsymbol{x}_0)), \quad \text{for } i = 1, ..., k.$$

Differentiate these outputs sequentially with respect to $t_l, t_{l-1}, \ldots, t_1$ at $t_l = 0, t_{l-1} = 0, \ldots$, $t_1 = 0$ yields $L_{Z_1} L_{Z_2} \ldots L_{Z_l} h_i(\boldsymbol{x}_0)$, for $i = 1, \ldots, k$. Hence we see that the observation space in fact consists of the output functions and their derivatives along all possible system trajectories (in infinitesimal time).

The *observability codistribution* at state $\boldsymbol{x} \in M$, denoted by $d\mathcal{O}(\boldsymbol{x})$, is defined as:

$$d\mathcal{O}(\boldsymbol{x}) = \text{span}\{\, dH(\boldsymbol{x}) \mid H \in \mathcal{O}\,\}.$$

Theorem 10 (Hermann and Krener) *System (34) is locally observable at state $\boldsymbol{x}_0 \in M$ if $\dim d\mathcal{O}(\boldsymbol{x}_0) = n$.*

The equation $\dim d\mathcal{O}(\boldsymbol{x}_0) = n$ is called the *observability rank condition.* Proofs of the above theorem can be found in [11] and [22, pp. 95–96]. Basically, to distinguish between a state and any other state in its neighborhood, it is necessary to consider not only the output functions but also their derivatives along all possible system trajectories. The rank condition ensures the existence of n output functions and/or derivatives that together define a diffeomorphism on some neighborhood of the state, which in turn ensures that the state is locally distinguishable.

References

[1] K.-F. Böhringer, B. R. Donald, R. Mihailovich, and N. C. MacDonald. Sensorless manipulation using massively parallel microfabricated actuator arrays. In *Proceedings of the 1994 IEEE International Conference on Robotics and Automation*, pages 826–833, 1994.

[2] Randy C. Brost and Matthew T. Mason. Graphical analysis of planar rigid-body dynamics with multiple frictional contacts. In *The Fifth International Symposium on Robotics Research*, pages 293–300, 1990.

[3] Chunsheng Cai and Bernard Roth. On the spatial motion of a rigid body with point contact. In *Proceedings of the 1987 IEEE International Conference on Robotics and Automation*, pages 686–695, 1987.

[4] P. E. Crouch. Dynamical realizations of finite Volterra series. *SIAM Journal on Control and Optimization*, 19(1):177–202, 1981.

[5] Michael Erdmann. An exploration of nonprehensile two-palm manipulation: Planning and execution. In *The Seventh International Symposium on Robotics Research*, October 21–25 1995. To appear in *International Journal of Robotics Research.*

[6] Michael Erdmann and Matt Mason. An exploration of sensorless manipulation. *IEEE Journal of Robotics and Automation*, 4(4):369–379, 1988.

[7] Peter C. Gaston and Tomás Lozano-Pérez. Tactile recognition and localization using object models: The case of polyhedra on a plane. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 6(3):257–266, 1984.

[8] Suresh Goyal, Andy Ruina, and Jim Papadopoulos. Planar sliding with dry friction. *Wear*, 143:307–352, 1991.

[9] W. E. L. Grimson and T. Lozano-Pérez. Model-based recognition and localization from sparse range or tactile data. *International Journal of Robotics Research*, 3(3):3–35, 1984.

[10] David D. Grossman and Michael W. Blasgen. Orienting mechanical parts by computer-controlled manipulator. *IEEE Transaction on Systems, Man, and Cybernetics*, 5(5):561–565, 1975.

[11] Robert Hermann and Arthur J. Krener. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, AC-22(5):728–740, 1977.

[12] Robert D. Howe and Mark R. Cutkosky. Dynamic tactile sensing: Perception of fine surface features with stress rate sensing. *IEEE Transactions on Robotics and Automation*, 9(2):140–151, 1993.

[13] A. Isidori. *Nonlinear Control Systems: An Introduction.* Springer-Verlag, 1985.

[14] Yan-Bin Jia and Michael Erdmann. Pose from pushing. In *Proceedings of the 1996 IEEE International Conference on Robotics and Automation*, pages 165–171, 1996.

[15] Yan-Bin Jia and Michael Erdmann. Observing pose and motion through contact. To appear in *Proceedings of the 1998 IEEE International Conference on Robotics and Automation*, 1998.

[16] Jeffrey Kerr and Bernard Roth. Analysis of multifingered hands. *International Journal of Robotics Research*, 4(4):3–17, 1986.

[17] Zexiang Li and John Canny. Motion of two rigid bodies with rolling constraint. *IEEE Transactions on Robotics and Automation*, 6(1):62–72, 1990.

[18] David G. Luenberger. An introduction to observers. *IEEE Transactions on Automatic Control*, AC-16(6), 1971.

[19] Kevin Lynch, Hitoshi Maekawa, and Kazuo Tanie. Manipulation and active sensing by pushing using tactile feedback. In *Proceedings of the 1992 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 416–421, 1992.

[20] William Duncan MacMillan. *Dynamics of Rigid Bodies.* McGraw-Hill Book Company Inc., 1936. Dover reprint, 1960.

[21] David J. Montana. The kinematics of contact and grasp. *International Journal of Robotics Research*, 7(3):17–32, 1988.

[22] Henk Nijmeijer and Arjan van der Schaft. *Nonlinear Dynamical Control Systems.* Springer-Verlag, New York, 1990.

[23] Barrett O'Neill. *Elementary Differtial Geometry.* Academic Press, Inc., 1966.

[24] Jong-Shi Pang and Jeffrey C. Trinkle. Complementary formulations and existence of solutions of dynamic multi-rigid-body contact problems with Coulomb friction. *Mathematical Programming*, 73:199–226, 1996.

[25] Kenneth Salisbury. Interpretation of contact geometries from force measurements. In M. Brady and R. Paul, editors, *Robotics Research*, pages 565–577. The MIT Press, 1984.

[26] Michael Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 1. Publish or Perish, Inc., Boston, 1970.

[27] Michael Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 3. Publish or Perish, Inc., Boston, 1975.