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On Computing Optimal Planar Grasps*

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Abstract

The quality of a grasp can often be measured as the magnitude within which any external wrench is resistible by "unit grasp force". In this paper, we present a numerical algorithm to compute the optimal grasp on a simple polygon, given contact forces of unit total magnitude. Forces are compared with torques over the radius of gyration of the polygon. We also address a grasp optimality criterion for resisting an adversary finger located possibly anywhere on the polygon boundary. The disparity between these two grasp optimality criteria are demonstrated by simulation with results advocating that grasps should be measured task-dependently. The paper assumes non-frictional contacts.

1 Introduction

In this paper, we are concerned with finding the optimal grasps on planar shapes, particularly polygons, under two different grasp metrics. The first metric, called $\mathcal{M}_w$, measures the quality of a grasp by the magnitude of the minimum, over all directions, of the maximum wrench resistible by the fingers exerting forces of unit total magnitude. This metric involves comparing a force with a torque, which is feasible by dividing the latter by the radius of gyration of the shape. The second metric, called $\mathcal{M}_f$, measures, under the same finger force constraint, the maximum external force applicable at the worst location on the shape boundary without breaking the grasp.

It is known that a grasp optimal under one metric is usually not very good under another ([8], [5]). In a real task, the space of possible external wrenches is often *reduced* due to the task specification or the working environmental constraints. For instance, consider the task of hammering a nail into wood. The reaction force to the hammer acts only on the head of the hammer; so the external wrenches constitute a ray if point contact is assumed. In another instance, a basketball player keeps the ball in the best possession of his hands to prevent his defender from knocking (and stealing) it away; here the external wrench would mostly result from a quick hit at some part of the ball by the defender's hand. *The full external wrench space is often too strong to assume for many real tasks, and it is thus more adequate to seek grasps optimal for the (reduced) wrench spaces specified by individual tasks.* The metric $\mathcal{M}_f$ reflects such a philosophy.

Throughout the paper we assume non-frictional contacts. We will focus on how to compute the optimal grasps under metrics $\mathcal{M}_w$ and $\mathcal{M}_f$. We will see that the optimization turns out to be difficult under both metrics. Section 2 reduces the optimal grasp problem under metric $\mathcal{M}_w$ to constrained non-linear programming and then solves it numerically; Section 3 analyzes the structure of grasp optimization under metric $\mathcal{M}_f$, unveiling the difficulty in the search for an efficient algorithm; Section 4 discusses simulation results on both metrics; and Section 5 concludes the paper by outlining the future work.

1.1 Previous Work

A grasp on an object is *force (form) closure* if and only if arbitrary force and torque can be exerted on the object through the set of contacts. Salisbury and Roth [19] identified acceptable hand designs as those which could immobilize a grasped object with the finger joints locked while also having the ability to impart arbitrary grasping forces and displacements to the object.

Mishra *et al.* [13] gave upper bounds on the numbers of frictionless fingers that are sufficient for equilibrium and force-closure grasps respectively on objects with piecewise smooth boundaries. Tighter bounds were later obtained by Markenscoff et al. [10] for force-closure grasps on any 2-D or 3-D object, for frictional as well as frictionless contacts.

Based on the work of [17], Nguyen [16] viewed a force-closure grasp as the vector-closure of its contact wrenches. He offered simple algorithms for synthesizing the independent grasp regions for polygons (with/without friction) and for polyhedra (without friction). This work was later extended in [18] to a numerical cell-decomposition algorithm

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assuming frictional contacts for 2-D objects with piecewise polynomial parametric boundaries.

One early optimality measure was introduced in [7] which considers the optimal selection of internal grasp forces to be the one furthest from violating any of the force-closure, friction and joint torque limit constraints. Trinkle [20] formulated the test of force closure as a linear program whose optimal objective value measures how far a grasp is from losing the closure.

Li and Sastry [8] also argued that the choice of a grasp should be based on its capacity to generate body wrenches that are relevant to the task. They were among the first to formulate this idea by introducing the notion of task ellipsoid based on which a task-oriented quality measure was then defined. The optimal grasp problem was also addressed but no algorithm was described.

[11] presents an $O(n^3)$ algorithm to compute the optimal three-finger equilibrium grasp on an n-gon to balance through friction its weight along the third dimension, as well as an $O(n^4)$ algorithm to compute the optimal grasp against any worst-case unit force through the center of gravity of the polygon. Both algorithms have assumed zero external torque while the second one can be viewed as a simplification of the optimization under $\mathcal{M}_f$. Assuming non-frictional contacts, [14] offers an $O(n^2 \log n)$ algorithm to find a three-finger grasp on an n-gon to resist the maximum external force acting through the center of gravity in any direction.

Note the definition of grasp metric $\mathcal{M}_w$ also appear in [5] and [12] where the normalization of finger forces under L_∞ and other metrics are also addressed. Optimal grasp algorithms are given in the first paper for two-jaw and three-jaw grippers to grasp polygons, in which case only finite number of good grasps need to be considered. By decoupling force and torque, the second paper is able to develop an easily computable optimality measure.

The paper [15] summarizes various existing grasp metrics with extensive discussion on the trade-offs among the goodness of a grasp, the geometry of the object, the number of fingers, and the computational complexity of the grasp synthesis algorithms.

Grasp metrics also apply to the design of modular fixtures where round locators and clamps act as fingers to constrain parts. [3] describes an algorithm that, given an arbitrary polygonal part, enumerates all force-closure modular fixtures and then ranks them according to some user-specified quality measure.

The grasp optimization algorithms to be presented in this paper, however, do not make any assumption on forces and torques, nor on the set of grasps to be considered. So the optimizations, inherently harder, are performed over a 4-D configuration space of force-closure grasps (assuming four fingers) with respect to the 3-D wrench space.

2 The Metric $\mathcal{M}_w$

Let D be a non-circular 2-D object with smooth boundary ∂D. Let $\boldsymbol{w}(\boldsymbol{p}) = (\hat{\boldsymbol{n}}(\boldsymbol{p}), \boldsymbol{p} \times \hat{\boldsymbol{n}}(\boldsymbol{p}))$ be the wrench generated by unit force at some point $\boldsymbol{p} \in \partial D$, where $\hat{\boldsymbol{n}}(\boldsymbol{p})$ is the inward normal at $\boldsymbol{p}$, and let $W(D)$ be the set of such wrenches:

$$W(D) = \{\, \boldsymbol{w}(\boldsymbol{p}) \mid \boldsymbol{p} \in \partial D \,\}. \tag{1}$$

It is shown in [10] that, without friction, four fingers are sufficient and necessary to achieve force-closure on D. Let $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4 \in \partial D$ denote the finger positions of a force-closure grasp $\mathcal{G}$ on D. Then the wrenches $\boldsymbol{w}(\boldsymbol{p}_1), \ldots, \boldsymbol{w}(\boldsymbol{p}_4)$ must *positively span* the wrench space $\Re^3$ [16], that is,

$$\sum_{i=1}^{4} \lambda_i \boldsymbol{w}(\boldsymbol{p}_i) = 0, \qquad \text{for some } \lambda_i > 0. \tag{2}$$

Geometrically, the origin O is in the interior of the convex hull of $\boldsymbol{w}(\boldsymbol{p}_1), \ldots, \boldsymbol{w}(\boldsymbol{p}_4)$, that is, $0 \in \operatorname{Int}\operatorname{conv}(\boldsymbol{w}(\boldsymbol{p}_1), \ldots, \boldsymbol{w}(\boldsymbol{p}_4))$. Mechanically, for any external wrench $\boldsymbol{w}$, grasp $\mathcal{G}$ is able to generate its negative wrench $-\boldsymbol{w}$ by exerting adequate forces at $\boldsymbol{p}_i$s.

For $1 \le i \le 4$ let f_i be the magnitude of force by finger i. The *quality* $s(\mathcal{G})$ of $\mathcal{G}$ under metric $\mathcal{M}_w$ is defined to be the minimum magnitude of any external wrench that breaks the grasp, given that the fingers apply unit magnitude of force, that is, $\sum_{i=1}^{4} f_i = 1$.[1] More precisely, we define

$$s\Big(\mathcal{G}(\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4\})\Big) = \max_{\kappa\mathcal{B} \subseteq \operatorname{conv}(\mathrm{w}(\mathrm{p}_1), \ldots, \mathrm{w}(\mathrm{p}_4))} \kappa,$$

where $\mathcal{B}$ is the unit ball centered at the origin. The problem of finding the optimal grasp under metric $\mathcal{M}_w$ can thus be formulated as

$$\max_{p_1, \ldots, p_4 \in \partial D} s\Big(\mathcal{G}(\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4\})\Big).$$

Note a subtlety occurs at defining the norm of a wrench since forces and torques have different units. To cope with this issue, we borrow an idea from [4] which compares a force with a torque over the *radius of gyration* ρ of D. Hence we redefine $\boldsymbol{w}(\boldsymbol{p}) = (\hat{\boldsymbol{n}}(\boldsymbol{p}), \boldsymbol{p} \times \hat{\boldsymbol{n}}(\boldsymbol{p})/\rho)$. If ∂D is not smooth, another subtlety arises for the boundary points where the normals are undefined. If D is a polygon, we regard a finger at some vertex as that finger at a point infinitesimally close to the vertex on one of its adjacent edges. [2]

[1] Note this condition subsumes the condition $\sum_{i=1}^{4} f_i < 1$, following (the force-closure) condition (2). Namely, a force-closure grasp applying less than unit force can easily be shown to be equivalent to the same grasp applying unit force.

[2] Some other papers such as [12] have assumed a rounded fingertip model to handle this issue, in which case the grasp wrench varies continuously at a vertex.

In the remainder of this section we focus on the case that D is a polygon P. Figure 1 shows a force-closure grasp on a 5-gon. The set $W(e)$ of possible wrenches generated by a finger with unit force on an edge e is a non-degenerate line segment, called the *wrench segment*, in the f_x-f_y-τ wrench space (see Figure 2(a)); this wrench segment projects to the inward normal of e in the f_x-f_y force plane (see Figure 2(b)).

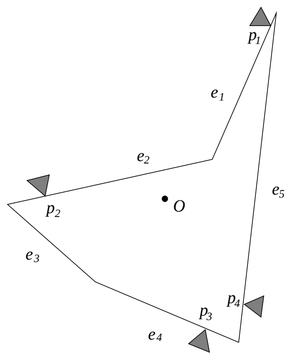
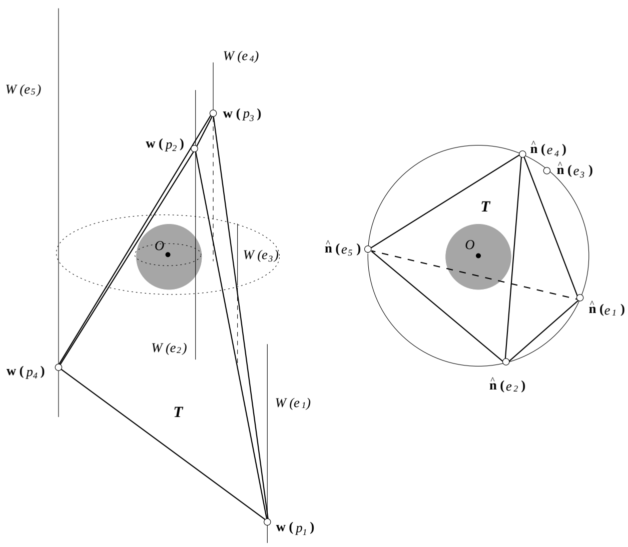


Figure 1: A force-closure grasp on a 5-gon.



(a) (b)

Figure 2: The force-closure grasp in Figure 1 as illustrated in the wrench space. The tetrahedron $\mathcal{T}$ with vertices $\mathbf{w}(\mathbf{p}_1), \ldots, \mathbf{w}(\mathbf{p}_4)$ consists of all wrenches that can be generated by the grasp exerting unit force. The radius of the largest sphere centered at the origin and contained in $\mathcal{T}$ measures the minimum wrench to break the grasp.

Now we can rephrase computing the optimal grasp on a polygon P as selecting four points (wrenches)

$$\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4 \in \bigcup_{e \text{ an edge of } P} W(e)$$

such that

- the origin O lies in the interior of the *grasp tetrahedron* $\mathcal{T}$ defined by $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_4$;
- the minimum distance from O to the four facets of $\mathcal{T}$ is maximized.

To solve this optimization problem, it suffices to look at the subproblem in which every finger i can only move on one edge, say e_i, with unit normal $\hat{\boldsymbol{n}}_i$. For $1 \leq i \leq 4$ let $\boldsymbol{w}_i = \hat{\boldsymbol{n}}_i + t_i \hat{\boldsymbol{\tau}}$ denote the wrench by finger i exerting unit force. Thus an one-to-one correspondence exists between (t_1, t_2, t_3, t_4) and a grasp so from now on we identify them with each other. Note that at least three of these $\hat{\boldsymbol{n}}_i$s must differ from each other, otherwise the grasp cannot be a closure on pure forces.

In the below we only look at the case that $\hat{\boldsymbol{n}}_1, \ldots, \hat{\boldsymbol{n}}_4$ are all different, as the other case is relatively simple. Without loss of generality, let us assume throughout this section that $\hat{\boldsymbol{n}}_1, \ldots, \hat{\boldsymbol{n}}_4$ are in clockwise order, and viewed from $+\infty$ on the $\hat{\boldsymbol{\tau}}$ axis, $\mathcal{T}$ has the topology that *edge* $\boldsymbol{w}_1\boldsymbol{w}_3$ *is above edge* $\boldsymbol{w}_2\boldsymbol{w}_4$.

The equation of the plane determined by three non-collinear points $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ is given as

$$\boldsymbol{n} \cdot \boldsymbol{x} = d,$$

where

$$\begin{aligned} \boldsymbol{n} &= \boldsymbol{q}_1 \times \boldsymbol{q}_2 + \boldsymbol{q}_2 \times \boldsymbol{q}_3 + \boldsymbol{q}_3 \times \boldsymbol{q}_1, \\ d &= \boldsymbol{q}_1 \times \boldsymbol{q}_2 \cdot \boldsymbol{q}_3. \end{aligned}$$

Here $\boldsymbol{n}$ is the plane normal and $\frac{|d|}{\|\boldsymbol{n}\|}$ is the distance from O to the plane; so $\boldsymbol{n}$ points to the plane if $d > 0$ and away from the plane otherwise. The force-closure condition $O \in \operatorname{Int} \mathcal{T}$ requires that O lie at the interior sides of all facets of $\mathcal{T}$. Therefore the following conditions hold:

$$\begin{aligned} \boldsymbol{w}_1 \times \boldsymbol{w}_2 \cdot \boldsymbol{w}_3 &> 0; \\ \boldsymbol{w}_2 \times \boldsymbol{w}_3 \cdot \boldsymbol{w}_4 &< 0; \\ \boldsymbol{w}_3 \times \boldsymbol{w}_4 \cdot \boldsymbol{w}_1 &> 0; \\ \boldsymbol{w}_4 \times \boldsymbol{w}_1 \cdot \boldsymbol{w}_2 &< 0. \end{aligned} \tag{3}$$

The above linear inequalities (in $\boldsymbol{t}$), along with those defining the wrench segments $W(e_i)$:

$$l_i < t_i < u_i, \qquad \text{for} \quad i = 1, 2, 3, 4, \tag{4}$$

define a open convex 4-polytope $\mathcal{P}$ that consists of all force-closure grasps.

Denote by F_α the facet of $\mathcal{T}$ with vertices $\boldsymbol{w}_1, \boldsymbol{w}_2$, and $\boldsymbol{w}_3$, by F_β the facet with vertices $\boldsymbol{w}_2, \boldsymbol{w}_3$, and $\boldsymbol{w}_4$, by F_γ the facet with vertices $\boldsymbol{w}_3, \boldsymbol{w}_4$, and $\boldsymbol{w}_1$, and by F_δ the facet with vertices $\boldsymbol{w}_4, \boldsymbol{w}_1$, and $\boldsymbol{w}_2$. The quality of grasp $\boldsymbol{t}$ is defined by

$$s(\boldsymbol{t}) = \min(s_\alpha(\boldsymbol{t}), s_\beta(\boldsymbol{t}), s_\gamma(\boldsymbol{t}), s_\delta(\boldsymbol{t})). \qquad (5)$$

Here $s_\alpha(\boldsymbol{t})$ is the distance from O to facet F_α:

$$s_\alpha(\boldsymbol{t}) = \frac{d_\alpha}{\|\boldsymbol{n}_\alpha\|},$$

where

$$\begin{aligned} d_\alpha &= \boldsymbol{w}_1 \times \boldsymbol{w}_2 \cdot \boldsymbol{w}_3 \\ &= (t_1 \hat{\boldsymbol{n}}_2 \times \hat{\boldsymbol{n}}_3 + t_2 \hat{\boldsymbol{n}}_3 \times \hat{\boldsymbol{n}}_1 + t_3 \hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{n}}_2) \cdot \hat{\boldsymbol{\tau}}, \qquad (6) \\ \boldsymbol{n}_\alpha &= \boldsymbol{w}_1 \times \boldsymbol{w}_2 + \boldsymbol{w}_2 \times \boldsymbol{w}_3 + \boldsymbol{w}_3 \times \boldsymbol{w}_1 \\ &= (\hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{n}}_2 + \hat{\boldsymbol{n}}_2 \times \hat{\boldsymbol{n}}_3 + \hat{\boldsymbol{n}}_3 \times \hat{\boldsymbol{n}}_1) + \\ &\quad \Big(t_1(\hat{\boldsymbol{n}}_3 - \hat{\boldsymbol{n}}_2) + t_2(\hat{\boldsymbol{n}}_1 - \hat{\boldsymbol{n}}_3) + t_3(\hat{\boldsymbol{n}}_2 - \hat{\boldsymbol{n}}_1)\Big) \times \hat{\boldsymbol{\tau}}; \qquad (7) \end{aligned}$$

and $s_\beta(\boldsymbol{t}), s_\gamma(\boldsymbol{t})$, and $s_\delta(\boldsymbol{t})$ are defined analogously.

Before presenting a numerical algorithm to maximize $s(\boldsymbol{t})$, let us look at how $s_\alpha(\boldsymbol{t})$, $s_\beta(\boldsymbol{t})$, $s_\gamma(\boldsymbol{t})$, and $s_\delta(\boldsymbol{t})$ vary with $\boldsymbol{t}$. This would suggest writing out the gradients of these functions, which seems to be too cumbersome. However, there is a much simpler way of viewing these gradients geometrically.

Lemma 1 *Let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ be a force-closure grasp, and $\mathcal{T}$ the grasp tetrahedron thus defined. If at ξ the line through the origin O and perpendicular to facet F_α of $\mathcal{T}$ intersects F_α in its interior, then*

$$\frac{\partial s_\alpha}{\partial t_1}, \frac{\partial s_\alpha}{\partial t_2}, \frac{\partial s_\alpha}{\partial t_3}\Big|_\xi \quad > \quad 0.$$

Accordingly, we have

$$\begin{aligned} \frac{\partial s_\beta}{\partial t_2}, \frac{\partial s_\beta}{\partial t_3}, \frac{\partial s_\beta}{\partial t_4}\Big|_\xi &\quad < \quad 0, \\ \frac{\partial s_\gamma}{\partial t_3}, \frac{\partial s_\gamma}{\partial t_4}, \frac{\partial s_\gamma}{\partial t_1}\Big|_\xi &\quad > \quad 0, \\ \frac{\partial s_\delta}{\partial t_4}, \frac{\partial s_\delta}{\partial t_1}, \frac{\partial s_\delta}{\partial t_2}\Big|_\xi &\quad < \quad 0 \end{aligned}$$

if the perpendicular lines through O to facets F_β, F_γ and F_δ pass through their interior respectively.

Proof. Let q be an interior point of F_α such that $Oq \perp F_\alpha$, as shown in Figure 3. Now look at the facet F'_α determined by $\xi' = (\xi'_1, \xi_2, \xi_3, \xi_4)$, where $\xi'_1 = \xi_1 + \Delta\xi_1 > \xi_1$. For

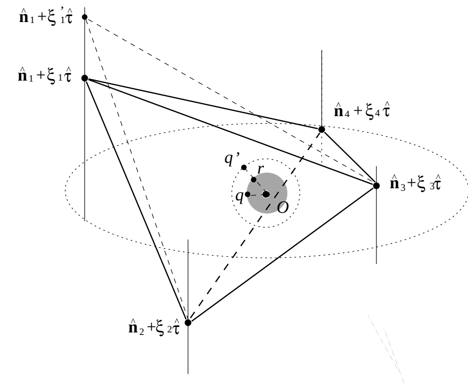


Figure 3: The proof of Lemma 1.

small enough $\Delta\xi_1$, the perpendicular line from O to F'_α intersects F_α and F'_α in their interior at r and q' respectively. We have

$$\begin{aligned} s_\alpha(\xi) &= |Oq| < |Or| \\ &< |Oq'| = s_\alpha(\xi'), \qquad \text{for some } \Delta\xi_1 > 0, \end{aligned}$$

which proves that $\frac{\partial s_\alpha}{\partial t_1}|_\xi > 0$. The rest inequalities in the lemma follow similarly. □

Let $\Omega = \{\, \boldsymbol{t} \mid \boldsymbol{t} \in \mathcal{P} \text{ and } s(\boldsymbol{t}) = s^* \,\}$ be the set of grasps on polygon P that maximizes function (5). We classify polygon P into one of the following four types based on the structure of Ω:

Type 1 $s_i(\boldsymbol{t}) = s^* < s_j(\boldsymbol{t}), s_k(\boldsymbol{t}), s_l(\boldsymbol{t})$, where $ijkl$ denotes some permutation of α, β, γ, and δ, for all $\boldsymbol{t} \in \Omega$;

Type 2 $s_i(\boldsymbol{t}) = s_j(\boldsymbol{t}) = s^*$ for some $\boldsymbol{t} \in \Omega$, but $s_k(\boldsymbol{t})$, $s_l(\boldsymbol{t}) > s^*$ for all such $\boldsymbol{t}$;

Type 3 $s_i(\boldsymbol{t}) = s_j(\boldsymbol{t}) = s_k(\boldsymbol{t}) = s^*$ for some $\boldsymbol{t} \in \Omega$, but $s_l(\boldsymbol{t}) > s^*$ for all such $\boldsymbol{t}$;

Type 4 $s_i(\boldsymbol{t}) = s_j(\boldsymbol{t}) = s_k(\boldsymbol{t}) = s_l(\boldsymbol{t}) = s^*$ for some $\boldsymbol{t} \in \Omega$.

The numerical algorithm hypothesizes every type above, finding its optimum whenever it exits. Finally the optimal grasp is selected as the maximum of the optima under all hypotheses.

2.1 Type 1 Polygon

The optimization on a type 1 polygon turns out to be fairly easy, as stated in the following theorem.

Theorem 2 *Every optimal grasp on a type 1 polygon positions three fingers at some vertices; and one optimal grasp positions all four fingers at some vertices.*

Proof. Without loss of generality, let $\boldsymbol{t} = (t_1, t_2, t_3, t_4)$ with $s_\alpha(\boldsymbol{t}) < s_\beta(\boldsymbol{t}), s_\gamma(\boldsymbol{t}), s_\delta(\boldsymbol{t})$ be an optimal grasp. It follows that F_α must intersect its perpendicular line from the origin O in the interior; hence $t_i = u_i$, for $i = 1, 2, 3$, as shown in Figure 4. For otherwise, by Lemma 1, we

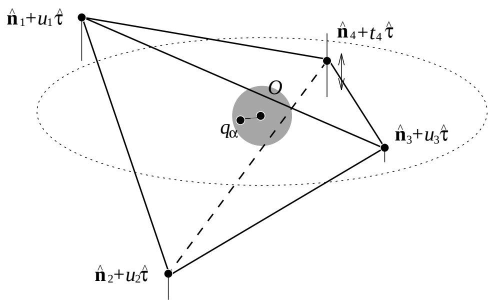


Figure 4: An optimal grasp on a type 1 polygon with $s^* = s_\alpha(\mathbf{t}) < s_\beta(\mathbf{t}),\ s_\gamma(\mathbf{t}), s_\delta(\mathbf{t})$: Fingers 1, 2 and 3 must be at the vertices of e_1, e_2, and e_3 to generate the maximum torques $u_1\hat{\tau}, u_2\hat{\tau}$ and $u_3\hat{\tau}$ respectively, while finger 4 is free to move along e_4 without affecting s^*.

could increase $s_\alpha(\boldsymbol{t})$, hence s^*, by increasing t_1, t_2, or t_3 infinitesimally.

On the other hand, t_4 may increase (decrease) monotonically to u_4 (l_4) without decreasing $s(\boldsymbol{t})$. Suppose this is not true. Then at some $\boldsymbol{t}' = (t_1, t_2, t_3, \xi)$, one of $s_\beta(\boldsymbol{t}), s_\gamma(\boldsymbol{t})$, and $s_\delta(\boldsymbol{t})$ must decrease to s^*. But this implies that the problem is not of type 1, a contradiction. □

The above proof also implies that a type 1 polygon has an optimal grasp that positions all fingers to generate either all maximum torques or all minimum torques. Thus to find the optimum, it suffices to evaluate two grasps: $t_i = u_i$, for all i, and $t_i = l_i$, for all i.

2.2 Type 2 Polygon

When a polygon is of type 2, an optimal grasp can position two fingers at vertices. The optimization reduces to constrained non-linear programming solvable by the Newton-Raphson method for root finding.

Theorem 3 *Every optimal grasp on a type 2 polygon positions at least two fingers at some vertices.*

Proof. Let $\boldsymbol{t}$ be an optimal grasp on some type 2 polygon, and $\mathcal{T}$ its grasp tetrahedron. There are 6 cases according as which two facets determine the optimal grasp quality s^*. The four cases $s_\alpha = s_\beta = s^*, s_\beta = s_\gamma = s^*, s_\gamma = s_\delta = s^*$, and $s_\delta = s_\alpha = s^*$ are similar; so we only need to look at one.

Suppose $s_\alpha = s_\beta = s^*$, the perpendicular lines from the origin to facets F_α and F_β of $\mathcal{T}$ must pass through their interior. So we have $\frac{\partial s_\alpha}{\partial t_1}, \frac{\partial s_\alpha}{\partial t_2}, \frac{\partial s_\alpha}{\partial t_3} > 0$ and $\frac{\partial s_\beta}{\partial t_2}, \frac{\partial s_\beta}{\partial t_3}, \frac{\partial s_\beta}{\partial t_4} < 0$ by Lemma 1. Since $\boldsymbol{t}$ is optimal, no $\Delta\boldsymbol{t}$ with $\boldsymbol{t} + \Delta\boldsymbol{t} \in \mathcal{P}$ exists such that

$$\begin{pmatrix} \nabla s_\alpha \\ \nabla s_\beta \end{pmatrix} \Delta\boldsymbol{t} > 0,$$

where $\nabla s_\alpha = \left(\frac{\partial s_\alpha}{\partial t_1}, \frac{\partial s_\alpha}{\partial t_2}, \frac{\partial s_\alpha}{\partial t_3}, 0\right)$ and $\nabla s_\beta = \left(0, \frac{\partial s_\beta}{\partial t_2}, \frac{\partial s_\beta}{\partial t_3}, \frac{\partial s_\beta}{\partial t_4}\right)$ are the gradients of s_α and s_β respectively. Now suppose $t_1 \neq u_1$ and $t_4 \neq l_4$. Letting $\ell = (1, 0, 0, -1)^T$, we can easily verify that the directional derivatives $\frac{\partial s_\alpha}{\partial \ell} = \nabla s_\alpha \cdot \ell > 0$ and $\frac{\partial s_\beta}{\partial \ell} = \nabla s_\beta \cdot \ell > 0$. Namely, both s_α and s_β can be increased at $\boldsymbol{t}$ along ℓ, a contradiction with the optimality of $\boldsymbol{t}$. Hence either $t_1 = u_1$ or $t_4 = l_4$. (See Figure 5.)

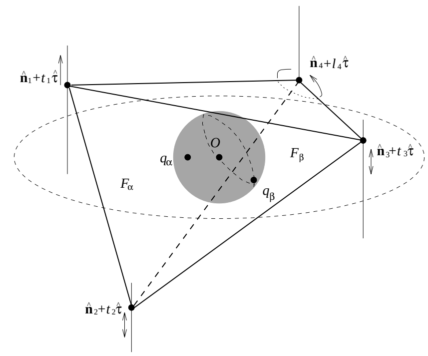


Figure 5: An optimal grasp on a type 2 polygon with $s^* = s_\alpha(\mathbf{t}) = s_\beta(\mathbf{t}) < s_\gamma(\mathbf{t}), s_\delta(\mathbf{t})$: Either $t_1 = u_1$ or $t_4 = l_4$ (as shown) holds. If $t_1 \neq u_1$, then $t_i = l_i$ for $i = 2, 3, 4$; if $t_4 \neq l_4$, then $t_i = u_i$ for $i = 1, 2, 3$.

If $t_4 = l_4$, then $\nabla s_\alpha = (\frac{\partial s_\alpha}{\partial t_1}, \frac{\partial s_\alpha}{\partial t_2}, \frac{\partial s_\alpha}{\partial t_3})$ and $\nabla s_\beta = (0, \frac{\partial s_\beta}{\partial t_2}, \frac{\partial s_\beta}{\partial t_3})$. Let $\ell_1 = (a_1, 0, -1)^T$ for some $a_1 > \frac{\partial s_\alpha}{\partial t_3} / \frac{\partial s_\alpha}{\partial t_1}$. Hence $\frac{\partial s_\alpha}{\partial \ell_1}$, $\frac{\partial s_\beta}{\partial \ell_1} > 0$, which implies that $t_1 = u_1 \vee t_3 = l_3$, by the optimality of $\boldsymbol{t}$. Now let $\ell_2 = (a_2, -1, 0)^T$ for some $a_2 > \frac{\partial s_\alpha}{\partial t_2} / \frac{\partial s_\alpha}{\partial t_1}$. Hence $\frac{\partial s_\alpha}{\partial \ell_2}$, $\frac{\partial s_\beta}{\partial \ell_2} > 0$, which implies that $t_1 = u_1 \vee t_2 = l_2$. So we can infer the following from $t_4 = l_4$:

$$\begin{aligned} &(t_1 = u_1 \vee t_2 = l_2) \wedge (t_1 = u_1 \vee t_3 = l_3) \\ \equiv \quad & t_1 = u_1 \vee (t_2 = l_2 \wedge t_3 = l_3). \end{aligned}$$

Similarly, $t_4 = l_4 \vee (t_2 = u_2 \wedge t_3 = u_3)$ holds if $t_1 = u_1$. Combining the above two conditions, we have shown

$$\begin{aligned} (t_1 = u_1 \wedge t_4 = l_4) \quad & \bigvee \quad (t_1 = u_1 \wedge t_2 = u_2 \wedge t_3 = u_3) \\ & \bigvee \quad (t_2 = l_2 \wedge t_3 = l_3 \wedge t_4 = l4), \end{aligned}$$

under $s_\alpha = s_\beta = s^*$.

Finally, it is easy to show that

$$\begin{aligned} t_1 = u_1 \wedge t_3 = u_3 \wedge (t_2 = u_2 \vee t_4 = u_4); \\ t_2 = l_2 \wedge t_4 = l_4 \wedge (t_1 = l_1 \vee t_3 = l_3) \end{aligned}$$

hold under $s_\alpha = s_\gamma = s^*$ and $s_\beta = s_\delta = s^*$ respectively. □

With Theorem 3 we are able to reduce the optimal grasp problem to a nonlinear programming problem. For the case $s_\alpha = s_\beta = s^*$ with $t_1 = u_1$ and $t_4 = l_4$, the problem takes the form

$$\max_{t_2,t_3} s_\alpha(u_1, t_2, t_3)$$

subject to

$$s_\alpha(u_1, t_2, t_3) \quad = \quad s_\beta(t_2, t_3, l_4), \tag{8}$$

in addition to the force closure constraints (3) and (4) which now define a polygon $\mathcal{P}$. This problem can be numerically solved by introducing a Lagrange multiplier [9], but we offer a simpler method here. Note that equation (8) defines t_3 as an implicit function of t_2 so that $s_\alpha(t_2, t_3) = s_\alpha(t_2, t_3(t_2))$ attains its maximum only if $\frac{ds_\alpha}{dt_2} = 0$. The Newton-Raphson method can be applied to find the zeros of $\frac{ds_\alpha}{dt_1}$.

The directives needed in the iteration, $\frac{ds_\alpha}{dt_2}$ and $\frac{d^2 s_\alpha}{dt_2^2}$, can be solved from differentiating equation (8) and expressed in terms of the first and second order partial derivatives of s_α and s_β. The iteration starts at an interior point on the monotonic curve $s_\alpha = s_\beta$ bounded by $\mathcal{P}$, and ends whenever it converges or reaches the boundary of $\mathcal{P}$. The solution $(t_2^{(k)}, t_3^{(k)})$ is invalid if $s_\alpha < s_\gamma$ or $s_\alpha < s_\delta$ at $(t_2^{(k)}, t_3^{(k)}, u_3, l_4)$.

The other case with three fingers placed at vertices is easy: The location of the fourth finger can be directly solved from the constraint equation.

2.3 Types 3 and 4 Polygons

A type 3 polygon has an optimal grasp $\boldsymbol{t} \in \mathcal{P}$ with, say $s_\alpha(\boldsymbol{t}) = s_\beta(\boldsymbol{t}) = s_\gamma(\boldsymbol{t}) < s_\delta(\boldsymbol{t})$, from which two variables can be eliminated. The following theorem enables the elimination of a third variable so that the optimization eventually reduces to non-linear programming in one variable.

Theorem 4 *Every optimal grasp on a type 3 polygon positions at least one finger at some vertex.*

Proof. Let $\boldsymbol{t}$ be an optimal grasp with $s_\alpha(\boldsymbol{t}) = s_\beta(\boldsymbol{t}) = s_\gamma(\boldsymbol{t}) < s_\delta(\boldsymbol{t})$, without loss of generality. Suppose that none of the fingers is at any vertex, that is, $\boldsymbol{t} \in \operatorname{Int} \mathcal{P}$. The optimality of $\boldsymbol{t}$ implies that no $\Delta\boldsymbol{t}$ exists such that

$$\begin{pmatrix} \nabla s_\alpha \\ \nabla s_\beta \\ \nabla s_\gamma \end{pmatrix} \Delta\boldsymbol{t} \quad = \quad \begin{pmatrix} \frac{\partial s_\alpha}{\partial t_1} & \frac{\partial s_\alpha}{\partial t_2} & \frac{\partial s_\alpha}{\partial t_3} & 0 \\ 0 & \frac{\partial s_\beta}{\partial t_2} & \frac{\partial s_\beta}{\partial t_3} & \frac{\partial s_\beta}{\partial t_4} \\ \frac{\partial s_\gamma}{\partial t_1} & 0 & \frac{\partial s_\gamma}{\partial t_3} & \frac{\partial s_\gamma}{\partial t_4} \end{pmatrix} \Delta\boldsymbol{t} \quad > \quad 0, \tag{9}$$

where $\nabla s_\alpha, \nabla s_\beta$, and ∇s_γ are the gradients of s_α, s_β, and s_γ. Such $\Delta\boldsymbol{t}$ does not exist if and only if there exist some non-negative λ_1, λ_2, and λ_3 with $\lambda_1 + \lambda_2 + \lambda_3 > 0$ such that

$$\lambda_1 \nabla s_\alpha(\boldsymbol{t}) + \lambda_2 \nabla s_\beta(\boldsymbol{t}) + \lambda_3 \nabla s_\gamma(\boldsymbol{t}) = 0. \tag{10}$$

(See [2, p. 27].)

Since $s_\alpha(\boldsymbol{t}) = s_\beta(\boldsymbol{t}) = s_\gamma(\boldsymbol{t}) = s^*$, we can easily show that $\frac{\partial s_\alpha}{\partial t_1}, \frac{\partial s_\alpha}{\partial t_2}, \frac{\partial s_\alpha}{\partial t_3}, \frac{\partial s_\gamma}{\partial t_1}, \frac{\partial s_\gamma}{\partial t_3}, \frac{\partial s_\gamma}{\partial t_4} > 0$ and $\frac{\partial s_\beta}{\partial t_2}, \frac{\partial s_\beta}{\partial t_3}, \frac{\partial s_\beta}{\partial t_4} < 0$, by Lemma 1. Then it is not hard to verify that equation (10) cannot hold for any non-negative $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 + \lambda_2 + \lambda_3 > 0$. So there exists $\Delta\boldsymbol{t}$ satisfying inequality (9). A contradiction. Hence $\boldsymbol{t} \in \partial\mathcal{P}$. □

A type 4 polygon provides three constraints $s_\alpha = s_\beta = s_\gamma = s_\delta$ that eliminate three variables, reducing the optimization again to non-linear optimization in one variable.

3 The Metric $\mathcal{M}_f$

We now move on to the problem of finding the optimal grasp to resist an adversary finger positioned somewhere on the boundary of a 2-D object D. It follows that such a grasp $\mathcal{G}$ at $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_4 \in \partial D$ must be able to generate wrench $-\boldsymbol{w}$ for all $\boldsymbol{w} \in W(D)$. Therefore the set $-W(D) = \{-\boldsymbol{w} \mid \boldsymbol{w} \in W(D)\}$ must be contained in the convex of wrenches $\boldsymbol{w}_1(\boldsymbol{p}_1), \ldots, \boldsymbol{w}_4(\boldsymbol{p}_4)$; so $\mathcal{G}$ is also force-closure.

The metric $\mathcal{M}_f$ on a grasp $\mathcal{G}$ at points $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_4 \in \partial D$ measures how much force by the adversary finger is always resistible by $\mathcal{G}$ with unit force. More precisely, the quality of $\mathcal{G}$ is defined under this metric as

$$s(\mathcal{G}(\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_4\})) = \max_{-\kappa W(D) \subseteq \operatorname{conv}(w(p_1), \ldots, w(p_4))} \kappa.$$

In the below we address the grasp optimization on an n-gon P. Figure 6 illustrates the grasp in Figure 1 under metric $\mathcal{M}_f$. As seen from the figure, only the $2n$ wrenches in $-W(D)$ related to the vertices of P may affect $s(\mathcal{G})$; let the set of these wrenches be denoted by $W_v \subseteq -W(D)$. Again let $\boldsymbol{w}_i = \hat{\boldsymbol{n}}_i + t_i \hat{\boldsymbol{\tau}}$ be the wrench generated by finger i, for $1 \le i \le 4$, and $\mathcal{T}$ the grasp tetrahedron thus defined. As before, we only consider the case that the fingers are constrained on four edges e_1, e_2, e_3, and e_4 with distinct inward normals respectively, because the other case is relatively simple. Furthermore, assume the topology of $\mathcal{T}$ to be that edge $\boldsymbol{w}_1\boldsymbol{w}_3$ is above edge $\boldsymbol{w}_2\boldsymbol{w}_4$, and let $\mathcal{P}$ be the convex 4-polytope defined by inequalities (3) and (4) which consists of all force-closure grasps on P.

Denote by $F_\alpha, F_\beta, F_\gamma$, and F_δ the four facets of the tetrahedron $\mathcal{T}$, $1 \le i \le 4$, and by $s_\alpha, s_\beta, s_\gamma$ and s_δ the maximum scales on W_v for it to be at the interior sides of these

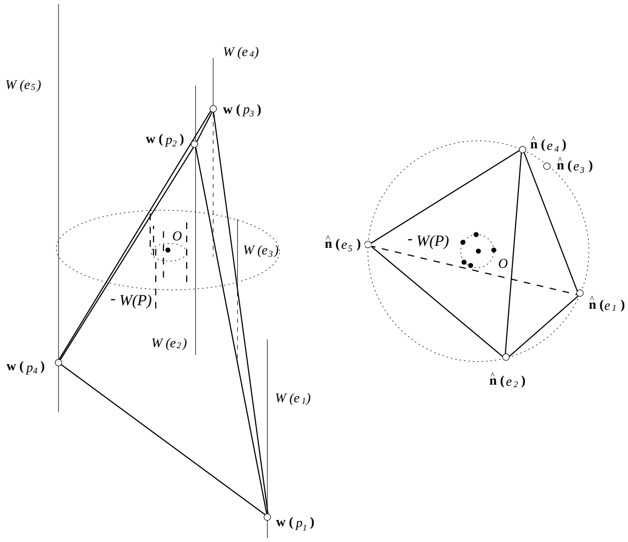


Figure 6: The grasp on the 5-gon in Figure 1 under metric $\mathcal{M}_f$. The quality of the grasp is the largest scale to shrink $-W(P)$ (shown as dashed lines) into the convex hull of $\mathbf{w}(\mathbf{p}_i)$, $1 \le i \le 4$.

facets respectively. Thus we have

$$s_\alpha(\boldsymbol{t}) = \frac{d_\alpha}{\max_{q \in W_v} \boldsymbol{n}_\alpha \cdot \boldsymbol{q}},$$

where d_α and $\boldsymbol{n}_\alpha$ are defined by equations (6) and (7) respectively; and $s_\beta(\boldsymbol{t}), s_\gamma(\boldsymbol{t}), s_\delta(\boldsymbol{t})$ are defined similarly. Now partition $\mathcal{P}$ into cells $C(\boldsymbol{q})$, for all $\boldsymbol{q} \in W_v$, in which $\boldsymbol{q}$ maximizes the dot product with $\boldsymbol{n}_\alpha$. Note the bisector of two adjacent cells $C(\boldsymbol{q}_1)$ and $C(\boldsymbol{q}_2)$ is a hyperplane given by equation

$$\boldsymbol{n}_\alpha \cdot \boldsymbol{q}_1 = \boldsymbol{n}_\alpha \cdot \boldsymbol{q}_2.$$

Since s_α is only related to t_1, t_2 and t_3, the partition forms a power diagram bounded by the projection of $\mathcal{P}$ onto the t_1-t_2-t_3 space and dual to some convex hull in $\Re^3$, all of which can be constructed in $O(n^2)$ time [1]. The power diagrams for other three facets F_β, F_γ and F_δ can be similarly constructed.

Intersecting these four power diagrams yields a cell complex $\mathcal{C}$ bounded by $\mathcal{P}$, in which each cell is associated with some $\boldsymbol{q}_\alpha, \boldsymbol{q}_\beta, \boldsymbol{q}_\gamma$, and $\boldsymbol{q}_\delta \in W_v$ such that

$$s(\mathcal{G}) = \min\Big(\frac{d_\alpha}{\boldsymbol{n}_\alpha \cdot \boldsymbol{q}_\alpha}, \frac{d_\beta}{\boldsymbol{n}_\beta \cdot \boldsymbol{q}_\beta}, \frac{d_\gamma}{\boldsymbol{n}_\gamma \cdot \boldsymbol{q}_\gamma}, \frac{d_\delta}{\boldsymbol{n}_\delta \cdot \boldsymbol{q}_\delta}\Big).$$

We have obtained the following results on the structure of $\mathcal{C}$:

Theorem 5 *If the optimal grasp is attained at* $\boldsymbol{t}^* \in \operatorname{Int} \mathcal{P}$, *then* $s_\alpha(\boldsymbol{t}^*) = s_\beta(\boldsymbol{t}^*) = s_\gamma(\boldsymbol{t}^*) = s_\delta(\boldsymbol{t}^*)$.

Theorem 6 *The optimal grasp is attained on* $\partial\mathcal{P}$ *(i.e., one finger at a vertex) or on the skeleton of the cell complex* $\mathcal{C}$ *defined above.*

The proofs of the above theorems can be found in [6].

So far we have not been able to devise an algorithm to compute $\boldsymbol{t}^*$ due to the structural complexity of $\mathcal{C}$. For the case $\boldsymbol{t}^* \in \partial\mathcal{P}$, the dimension of the optimization is lowered by at least one so that the computation can proceed in a hypothesis-and-verification manner similar to that of the last section. For the case $t^* \in \operatorname{Int}\mathcal{P}$, we suspect that the set

$$\{\, \boldsymbol{t} \mid s_\alpha(\boldsymbol{t}) = s_\beta(\boldsymbol{t}) = s_\gamma(\boldsymbol{t}) = s_\delta(\boldsymbol{t}) \,\}$$

is an one-dimensional piecewise smooth curve so that $\boldsymbol{t}^*$ could be found by traversing along this curve and checking its every intersection with the skeleton of $\mathcal{C}$.

4 Simulations

Simulations on computing the optimal grasps on random polygons under both metrics were conducted. The data were generated as closed random walks on an arrangement of 100 random lines. The radii of gyration of the polygons were computed by a common plane sweeping algorithm. The center of geometry of each polygon was selected as its torque origin. Optimal grasps are computed under both metrics $\mathcal{M}_w$ and $\mathcal{M}_f$ using polygon boundary discretization and under $\mathcal{M}_w$ by a numerical algorithm.

The discretization method proceeded as follows. The boundary of a polygon was first discretized into 25–50 equally spaced points. These points, together with the vertices, were considered as possible finger locations. Next, all edge triples and quadruples were enumerated as possible contact edges for the grasp. Given the contact edges, a quick check was done on whether the edge normals positively expand the force plane. Then all possible finger placements on these edges were evaluated under metrics $\mathcal{M}_w$ and $\mathcal{M}_f$ respectively. The best two grasps, each under one metric, were retained after the iteration over all contact edge choices as the optimal grasps.

Among the 30 polygons tested, only two had the same optimal grasps under both metrics. The optimal grasps under metric $\mathcal{M}_w$ were measured to be 61%–100% of the optima under metric $\mathcal{M}_f$, while the optimal grasps under metric $\mathcal{M}_f$ were measured to be 71%–100% of the optima under metric $\mathcal{M}_w$. These results have sufficiently demonstrated the disparity between the two metrics. An example is show in Figure 7.

The numerical algorithm was implemented to solve for optimal grasps on Types 1 and 2 polygons, and was tested on the same set of data. None of the tested polygons was of type 1, as we had expected. Since our algorithm does not handle types 3 and 4 polygons, we cannot say that all tested polygons were thus of type 2. However, all except one "optimal grasps" found by the numerical algorithm were better than the "optimal grasps" found by the dicretization

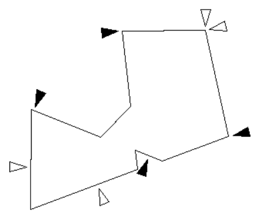


Figure 7: The optimal grasps $\mathcal{G}_w^*$ (white fingers) and $\mathcal{G}_f^*$ (black fingers) on a 10-gon under metrics $\mathcal{M}_w$ and $\mathcal{M}_f$ respectively, computed by discretizing the polygon boundary into 50 points. The grasp $\mathcal{G}_w$ can resist any wrench within magnitude 0.376 but only any force within magnitude 0.261; the grasp $\mathcal{G}_f$ can resist any force within magnitude 0.283 but only any wrench within magnitude 0.35.

method using a resolution of 50. These results suggest that polygons are more likely of type 2. (See Figure 8.)

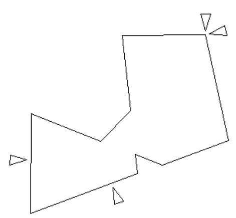


Figure 8: The optimal grasp on the same 10-gon found by the numerical algorithm. It has a quality of 0.388 under metric $\mathcal{M}_w$.

5 Conclusion and Future Work

In this paper we argue that the goodness of a grasp should be measured with respect to the external wrench space specified by each individual task. We have introduced two grasp metrics, $\mathcal{M}_w$ and $\mathcal{M}_f$, as examples for full and reduced wrench spaces respectively. The first metric measures how much external wrench is necessary to break the grasp. The second metric measures how much force by one nasty finger can always be resisted.

We present a numerical algorithm to compute the optimal grasp on a simple polygon under metric $\mathcal{M}_w$. The algorithm essentially reduces the optimization to constrained nonlinear programming. The grasp optimization under metric $\mathcal{M}_f$ has also been addressed. Simulations were conducted on both grasp metrics using polygon boundary discretization, and on $\mathcal{M}_w$ with an implementation of the numerical algorithm. The results have sufficiently demonstrated the disparity between the two metrics.

More work is needed to devise an efficient optimal grasp algorithm under metric $\mathcal{M}_f$. Further along this line of work would be to develop more practical and easily computable grasp metrics as well as to study the associated optimization techniques.

The current work on optimal grasps can be extended to frictional contacts for which the optimization will become conceivably harder despite that fewer fingers are required. Extension of the work to 2-D objects with curved boundaries and 3-D objects may also be of our interest.

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