Robotic cutting: mechanics and knife control

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Abstract

Skills of cutting natural foods are important for robots looking to play a bigger role in kitchen assistance. The basic objective of cutting is to achieve material fracture via smooth movements of a kitchen knife, which in the process performs work to overcome material toughness, acts against blade-material friction, and generates shape deformation. This paper investigates how a robotic arm drives the knife to cut through an object in a sequence of three moves: pressing, touching, and slicing. To cope with evolving contacts with the material and cutting board, position, force, and impedance controls are applied, either separately or jointly, assisted by force sensing, and/or based on fracture mechanics, so the knife follows a prescribed trajectory to split the object. Force data acquired during the phase of pressing are used for estimating the material's fracture toughness and the coefficient of blade-material friction (together with pressure distribution), all specific to the food item being cut and varying with its freshness. These estimated values are promptly used for control purpose to execute the phase of slicing. Experiments over several types of fruits and vegetables have exhibited natural cutting movements like those performed by a human hand.

Keywords

Dexterous cutting, fracture mechanics, knife pressing, slicing, robot control

1 Introduction

Automation of kitchen skills is an important step towards the advent of multipurpose home robots, which have long been a public fascination. Until today, robotic kitchen assistance has been limited to peripheral tasks such as washing and sorting dishes (Srinivasa et al., 2012), carrying food trays (Iwata and Sugeno, 2009), and making pancakes and noodles (Bleetz et al., 2011), burgers, etc., from prepared raw materials in very structured settings. Typically in food industry, robots are each capable of only one task, whether cutting meat, deboning, or butchering chicken. Industrial food cutting enjoys high efficiency that benefits from specially designed tools for slicing or holding foods. In our life, not so coincidentally, specialized kitchen tools sold at stores or online are also for single operations such as slicing lettuce, peeling potatoes, chopping fruits and vegetables, and so on. To prepare a meal with vegetables, meats, and fruits with no hand involvement in cutting would require a large collection of such specialized machines. This would be unrealistic given the high cost and limited kitchen space. It would also be clumsy and inefficient to constantly switch between foods and corresponding cutting machines. Robots can play a bigger role in the kitchen by becoming more versatile — even within the domain of one kitchen skill.

Food cutting, as an integral part of automatic meal preparation, stands out as one of the ultimate tests on human-level dexterity for robots, which today still lack basic skills such as chop, slice, and dice. There are multiple reasons for the slow technical advance in robotic cutting. Grasping and stabilization of soft and irregularly-shaped food items aside, one main technical challenge is how to plan and control a knife’s movement through a material while reacting to forces of different natures (fracture, friction, viscosity, and contact) exerted by the material and cutting board. A smooth knife movement needs to make use of some estimates of these forces as well as shape deformations, much like they are felt by a knife-holding human hand. Elasticity theories (Saada, 1993; Novozhilov, 1999) and fracture mechanics (Anderson, 2005) can be drawn upon for the purpose of force and deformation modeling.

Dexterous robotic cutting needs to resolve a range of issues, from fracture modeling, to knife holding and control, to object stabilization and maneuver, and to coordination of two hands/arms. We intend to take up these challenges one at a time with increasing complexity in this endeavor for years to come. This paper investigates a cutting task in which the knife is rigidly mounted on a robotic arm. To concentrate on knife skill realization, we make the following four assumptions throughout the paper:

(A1) The object being cut deforms negligibly.

(A2) The object remains stable during cutting.

(A3) The knife moves in a vertical plane.

(A4) The knife’s blade has negligible thickness.

The first assumption arises from that some vegetables and fruits such as potatoes, onions, and apples barely deform.
during cutting. The second assumption can be realized by having the object fixed or held by a robotic hand. The third assumption reflects the most common way of cutting in the kitchen by the human. The fourth one is reasonable since a kitchen knife’s blade typically has thickness in the range 0.4–2.0mm from edge to spine.

As in the home kitchen setup, a cutting board is used. One way of cutting used by the human hand is to move the knife downward against the cutting board, and upon contact, pull its edge on the board to slice the item into two pieces. In this paper, we decompose the cutting action into three consecutive phases (illustrated in Fig. 1): pressing, in which the knife moves fast downward along a prescribed trajectory until its edge makes contact with the cutting board; touching, in which the knife softens its impact on the board; and slicing, in which the knife separates the object completely with its edge sliding/rolling on the board across the object’s bottom.

In our setup, the knife, fixed on the robotic arm, is maneuvered by the latter at a fast speed. The manipulated object barely moves under assumption A2 regardless of the force exerted on it by the knife. For these two reasons the cutting task is not treated as a quasi-static manipulation task in which the manipulated object moves at a low speed, the inertial force is negligible, and the dissipative forces are parallel to velocity and of constant magnitude (Peshkin and Sanderson, 1988). Also, in quasi-static manipulation, feedback control is conducted in an ad hoc manner because it is the velocity of the robotic arm, not its exerted force/torque that is used for adjusting the object’s velocity (Trinkle and Zeng, 1995). Meanwhile, cutting is not a dynamic manipulation since the object is motionless and the corresponding inertial force is negligible.

We here treating cutting as a manipulation task with a focus on controlling the robotic arm’s movements to resolve several key issues which include path following, impact handling, and force regulation. The introduced pressing, touching, and slicing phases of cutting, partitioned according to changing contacts and path constraints, are carried out under position control, hybrid position/impedance control, and hybrid position/force control, respectively. These controls are based on the dynamics of the arm, while interactions between the “motionless” object and the knife are considered quasi-static. The goal is steady progress leading up to a complete separation of the object into two pieces.

Skilful cutting is often evidenced by precision and speed. To keep the knife on a downward cutting trajectory, an open-loop control strategy would be inadequate for generating a fast cutting motion when a large force is needed due to high material toughness and friction. Second, such a strategy would require high accuracy in the arm and knife’s combined Jacobian — often impossible to ensure due to their nonrigid connection — in order to relate its joint velocities to the knife’s velocity. Third, neither an open-loop strategy nor a closed-loop one with position control alone would be able to implement slicing, which relies on maintaining adequate contact force between the knife and the cutting board to achieve material separation. Last, a fast downward movement will inevitably result in the knife colliding with the cutting board, generating an impulsive force that must be mitigated for smoothness and swiftness and for prevention of any damage to the arm. The above considerations all suggest force-based feedback control to play a key role in the cutting task.

1.1 Contributions and outline of the paper

Our work on cutting to be presented bears a number of distinct characteristics and technical novelties:

1. It investigates an under-researched form of manipulation which alters the structure of the manipulated object in the process.
2. It studies a complex skill of maneuvering a kitchen knife via decomposing it into three phases: pressing, touching, and slicing, and carrying them out under different control policies.
3. To realize slicing, fracture mechanics are used for modeling the forces of fracture and friction needed by a controller to regulate the contact force on the cutting board.
4. Physical parameter values estimated in the initial phase of cutting are promptly used for modeling to control the knife movement in the ending phase. This robustly deals with variations of these values among natural objects especially foods, even of the same type, and variations with the same object’s degree of freshness.
5. Cutting experiments are performed on fruits and vegetables (rather than on synthetic objects as often studied in fracture mechanics).

The rest of the paper is organized as follows. Section 1.2 reviews works in related areas including fracture mechanics, cutting of soft tissues, and robot control. Section 1.3 presents the notation used in the paper. Section 2 formulates the technical problem of cutting and presents its underlying mechanics. Section 3 focuses on cutting control of a robotic arm, which has n degrees of freedom (DOFs) in the cutting plane, to adjust to changing contact situations. Section 4 describes how the cutting scheme can be tailored to work for the cutting of soft tissues, and robot control.
a 2-DOF arm, which cannot achieve an arbitrary knife pose locally. Section 5 tackles real-time estimation of fracture toughness, pressure distribution, and coefficient of friction for an object as it is being cut open. Section 6 describes experiments conducted on cutting fruits and vegetables with a 4-DOF WAM Arm utilizing its two DOFs in a vertical plane. Summary, discussion, and future work follow in Section 7.

1.2 Related work
Fracture mechanics (Atkins, 2009) builds on a balance between the work done by the knife and the total amount spent on crack propagation, transformed into other energy forms (strain, kinetic, chemical, etc.), and dissipated by friction. Methods for measuring cutting force and fracture toughness were studied for ductile materials (Atkins, 2004) and live tissues (Chanthasophepan et al., 2003; Gokgol et al., 2012). Fracture force and torque could be obtained via integration along the knife’s edge (Atkins and Xu, 2005). In our paper, this approach has been extended to also account for blade-material friction in modeling. A “slice-push ratio” introduced by Atkins et al. (2004) quantified the amounts of work done along two orthogonal directions in an effort to characterize the dramatic decrease in the fracture force when the knife was simultaneously pressing and slicing the material. A different explanation (Reyssat et al., 2012) for such decrease stated that pushing caused global deformation while slicing yielded local deformation (and thus required less effort to create fracture).

Surgical training makes use of realistic haptic display of soft tissue cutting. Stress and fracture force analyses, supported by simulation and experiment, were performed for robotic cutting of biological materials, accounting for factors such as blade sharpness and slicing angle (Zhou et al., 2006a, 2006b; Zhou and McMurray, 2010; Jamdagni and Jia, 2021). Haptic models, mostly empirical, were developed for animal tissue cutting with a scissor (Chial et al., 2002), soft tissue deformation prior to fracture (Chanthasophepan et al., 2007), as well as needle insertion into soft tissues (Khadem et al., 2016). We refer to Takabi and Tai (2017) for a survey on mechanics and modeling of cutting biological materials.

Cutting is carried out with the knife following some trajectory through contact between its blade and the material, and often, also between its edge and the cutting board. Due to the varying nature of contact experienced by the knife during a complete cutting action, it would be natural to employ multiple control policies. Position control (Murray et al., 1994: pp. 190–199) can realize trajectory following during the pressing phase; impedance control (Hogan, 1985), which adjusts the contact force from a motion deviation like some intended mass-spring-damper, is suitable for the touching phase to reduce the impact between the knife and cutting board caused by a fast downward knife movement; and hybrid force/position control (Raibert and Craig, 1981), which simultaneously regulates position and force in orthogonal directions, is a default choice for the slicing phase, during which the knife’s edge moves on the cutting board. In order for the entire action of cutting to look natural, smooth transitions between these policies would be desirable, similar to switching between position and force controls to regulate the contact force during an impact (Tarn et al., 1996).

To deal with contact constraints, controls of force and position are more effectively conducted in the work space (Khatib, 1987) using a reduced set of coordinates (Lewis et al. 2004: pp. 501–510). While we can apply position control to keep the knife’s orientation during a period of cutting, as sometimes desired, an alternative would be control of a constrained manipulator (Murray et al., 1994: pp. 202–203).

Robotic cutting has been investigated in a number of ways: adaptive control based on position and velocity histories to learn the applied force (Zeng and Hemami, 1997), adaptive force tracking via impedance control (Jung and Hsia, 1999), visual servoing coupled with force control (Long et al., 2014), and impedance control for cooperation between a cutting robot and a pulling robot (Long et al., 2013). Adaptive impedance control was carried out to minimize force errors while cutting a nonhomogeneous workpiece (Jung and Hsia, 1999), employing a gain yielded from stability analysis. A 2-DOF robot (Hu et al., 2012) was even able to debone a bird by following some cutting path determined from x-ray imaging based on force feedback, with the help of a passive mechanism for fixation. Yamaguchi and Atkeson (2016) developed an optical multimodal-sensing skin for the robotic gripper to prevent unexpected motions of the knife when cutting vegetables.

Kinematic redundancy occurs when the robotic arm has more degrees of freedom than required by the task. As in other tasks, in cutting this redundancy can be exploited to gain flexibility and advantage in dealing with kinematic singularities, avoiding collisions, etc. To find a good path out of the solution subspace, the Moore-Penrose pseudoinverse (Whitney, 1972) and damped least-square inverse (Nakamura and Hanafusa, 1986), both of the arm’s Jacobian matrix, can be used. Another way of dealing with redundancy is the gradient projection method (Siciliano, 1990), which maps joint velocities obtained through optimization to the null space of the Jacobian matrix. Task space augmentation, a third approach, imposes additional constrained tasks to be executed alongside the original task with lower priorities (Dietrich et al., 2015).

Artificial intelligence and machine learning also have been applied in robotic cutting. Bogoni and Bajcsy (1995) developed a system that could represent a task such as piercing. Data driven robotic cutting has become another active research topic. In Gemici and Saxena (2014), predefined actions were chosen based on a generative model. Learning algorithms were proposed by Lenz et al. (2015) to estimate the optimal input force for control during cutting, and later extended by Mitsioni et al. (2019), who made use of force/torque sensor readings and applied velocity control for the robot.

1.3 Notation
In this paper, a vector is represented by a lowercase letter in bold, e.g., $\mathbf{a} = (a_x, a_y)^T$, with its $x$- and $y$-coordinates denoted by the same (non-bold) letter with subscripts $x$ and $y$, respectively. A unit vector has a hat, e.g., $\hat{a} = a/\|a\|$. The cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is treated as a scalar. The subscripts $d$ and $e$ refer to the desired value and error, respectively. For example, $a_{gd}$ and $a_{ge}$ are respectively...
Differentiation with respect to time.

$\mathcal{W}$ Transpose of matrix.

Arm frame.

Knife frame.

Origin of $\mathcal{W}$.

Arm’s open end (origin of $A$).

Knife’s tip point (origin of $K$).

Rotation of $A$ from $\mathcal{W}$.

Rotation of $K$ from $\mathcal{W}$.

Arm’s $i$th joint angle.

Arm’s joint angle vector.

Knife’s angular velocity.

Knife’s edge curve in $K$.

Knife’s spine curve in $K$.

Knife’s spine curve in $\mathcal{W}$.

Object’s contour curve in the cutting plane.

Object’s fracture toughness.

Coefficient of blade-object friction.

Pressure distribution of blade-object contact.

Region of blade-object contact.

Fracture force.

Table 1. Nomenclature.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$\mathcal{W}$</td>
<td>World frame.</td>
</tr>
<tr>
<td>$A$</td>
<td>Arm frame.</td>
</tr>
<tr>
<td>$K$</td>
<td>Knife frame.</td>
</tr>
<tr>
<td>$o$</td>
<td>Origin of $\mathcal{W}$.</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Arm’s open end (origin of $A$).</td>
</tr>
<tr>
<td>$p$</td>
<td>Knife’s tip point (origin of $K$).</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Rotation of $A$ from $\mathcal{W}$.</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Rotation of $K$ from $\mathcal{W}$.</td>
</tr>
<tr>
<td>$\theta_i$</td>
<td>Arm’s $i$th joint angle.</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Arm’s joint angle vector.</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Knife’s angular velocity.</td>
</tr>
<tr>
<td>$\beta'$</td>
<td>Knife’s edge curve in $K$.</td>
</tr>
<tr>
<td>$\gamma'$</td>
<td>Knife’s spine curve in $K$.</td>
</tr>
<tr>
<td>$\beta_\theta$</td>
<td>Knife’s edge curve in $\mathcal{W}$.</td>
</tr>
<tr>
<td>$\gamma_\theta$</td>
<td>Knife’s spine curve in $\mathcal{W}$.</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Object’s contour curve in the cutting plane.</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Object’s fracture toughness.</td>
</tr>
<tr>
<td>$P$</td>
<td>Pressure distribution of blade-object contact.</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Region of blade-object contact.</td>
</tr>
<tr>
<td>$f_C$</td>
<td>Fracture force.</td>
</tr>
<tr>
<td>$f_F$</td>
<td>Force due to blade-material friction.</td>
</tr>
<tr>
<td>$f_S$</td>
<td>Force reading by an F/T sensor at $a$.</td>
</tr>
<tr>
<td>$\tau_C$</td>
<td>Torque on $a$ due to fracture.</td>
</tr>
<tr>
<td>$\tau_F$</td>
<td>Torque on $a$ due to friction.</td>
</tr>
<tr>
<td>$\tau_n$</td>
<td>Arm’s joint torque vector.</td>
</tr>
<tr>
<td>$\rho_n$</td>
<td>Wrench exerted at $a$.</td>
</tr>
<tr>
<td>$M$</td>
<td>Combined inertia matrix of arm, sensor, and knife.</td>
</tr>
<tr>
<td>$C$</td>
<td>Combined Coriolis matrix.</td>
</tr>
<tr>
<td>$N$</td>
<td>Combined gravity term.</td>
</tr>
</tbody>
</table>

2 Mechanics of cutting

In this section, we present the geometry of the cutting task, and analyze different forces encountered by the knife as it creates fracture.

2.1 Task geometry

As shown in Fig. 2, cutting of an object takes place in the vertical $x$-$y$ plane, referred to as the world frame $\mathcal{W}$ which is located at some point $o$ on the cutting board. The knife is rigidly mounted on the open end $\alpha$ of a robotic arm whose base is located at $b$. Both the knife and arm as modeled lie and move in the $x$-$y$ plane. The arm has $n$ revolute joints, $n \geq 3$, with angles $\theta_1, \theta_2, \ldots, \theta_n$, and $n$ links with lengths $l_1, l_2, \ldots, l_n$. All the joint angles form a vector

$$\theta = (\theta_1, \theta_2, \ldots, \theta_n) ^\top.$$
kitchen knife considered here has both curved edge and spine, in part because knives of this type are quite common, and in part because straight edge and spine can be considered as degenerate cases of curved ones. As depicted in Fig. 3, the knife’s edge and spine are described in the frame $K$ by two curves $\beta'(u) = (\beta'_x, \beta'_y)\top$ and $\gamma'(q) = (\gamma'_x, \gamma'_y)\top$, respectively, such that $\beta'(0)$ and $\gamma'(0)$ coincide with the frame’s origin at the knife point $p$. In the world frame $W$, they are thus described by the following two curves:

$$
\begin{align*}
\beta_\theta(u) &= p + R(\psi)\beta'(u), \\
\gamma_\theta(q) &= p + R(\psi)\gamma'(q).
\end{align*}
$$

(5)

These two curves, each with three degrees of freedom as a part of the knife, depend on $\theta$, which determines $p$ and $\psi$ according to (2)–(4). More specifically, they are from two families of curves parameterized by $\theta$ but generated respectively by $\beta'$ and $\gamma'$. Substituting (3) into (5), we obtain

$$
\begin{align*}
\beta_\theta(u) &= \left(\beta_{x\theta}(u), \beta_{y\theta}(u)\right) = a(\theta) + R(\psi) p'' + R(\psi) \beta'(u).
\end{align*}
$$

(6)

When $\theta$ is allowed to vary, $\beta_\theta$ is a function with $n + 1$ variables $\theta$ and $u$.

As cutting proceeds, the edge intersects the object at a section of $\beta_\theta(u)$ over some interval $[u_1, u_2]$, $u_1 \leq u_2$, as illustrated in Fig. 3. The section is denoted $\beta_\theta[u_1, u_2]$ for convenience. A section $\gamma_{\theta}[q_1, q_2]$ of the spine $\gamma_\theta(q)$ over $[q_1, q_2]$, for some $q_1$ and $q_2$, may also be inside the object.

Since the object does not deform under assumption A1, we let the curve $\sigma(r)$ describe its non-varying contour of the cross section intersected by the $x$-$y$ plane. The knife’s edge intersects the curve at $\sigma(r_1)$ and $\sigma(r_2)$ from left to right. Thus,

$$
\begin{align*}
\beta_\theta(u_1) &= \sigma(r_1), \\
\beta_\theta(u_2) &= \sigma(r_2).
\end{align*}
$$

(7)

The segments $\beta_\theta[u_1, u_2]$ and $\sigma[r_2, r_1]$ enclose the region of fracture. When a section of the spine $\gamma_\theta[q_1, q_2]$ is inside the cross section, it is bounded by $\sigma[r_1]$ and $\sigma(r_3)$ such that

$$
\begin{align*}
\gamma_\theta(q_1) &= \sigma(r_3), \\
\gamma_\theta(q_2) &= \sigma(r_3).
\end{align*}
$$

(8)

The four segments $\beta_\theta[u_1, u_2]$, $\sigma[r_2, r_3]$, $\gamma_\theta[q_1, q_2]$, and $\sigma[r_3, r_1]$ bound the region $\Omega$ of contact between the blade and the object.

### 2.2 Forces during cutting

For the present, we suppose that the knife’s edge is not in contact with the cutting board. New fracture is being created by the edge segment $\beta_\theta[u_1, u_2]$, which is experiencing a force $\mathbf{f}_C$ due to material fracture. In other words, the work done by $-\mathbf{f}_C$ is yielding fracture. Meanwhile, the blade is experiencing a force $\mathbf{f}_B$ due to its friction with the object’s material inside the contact region $\Omega$. If the object were deformable, the force exerted by the knife would have a third component $-\mathbf{f}_U$ that does work to cause an increase (or decrease) of the object’s strain energy. Under assumption A1, however, we have $\mathbf{f}_U = 0$.

Consider an infinitesimal element of length $ds$ on the knife’s edge starting at $u \in [u_1, u_2]$. See Fig. 4. The element may or may not be generating fracture due to the knife’s rotation. We need only consider the former case here. Let $\omega = \theta_1 + \theta_2 + \cdots + \theta_n$ be the knife’s angular velocity. The element exerts the force $-df_C$ in the direction of its velocity $\mathbf{v} = d\beta_\theta/dt$, obtained from differentiating (6) as

$$
\mathbf{v} = \mathbf{a} + \frac{d}{dt}\left(R(\psi)\right)p'' + \frac{d}{dt}R(\psi)\beta'.
$$

(9)

A movement by the element over distance $d\eta$ generates an area of fracture that is a parallelogram (shown in Fig. 4). Its four sides are parallel to either the edge tangent

$$
\mathbf{t} = \left(\frac{\partial \theta_{x\theta}}{\partial u}, \frac{\partial \theta_{y\theta}}{\partial u}\right)\top / \left\|\left(\frac{\partial \theta_{x\theta}}{\partial u}, \frac{\partial \theta_{y\theta}}{\partial u}\right)\top\right\|
$$

or the velocity $\mathbf{v}$. Now we make use of the material’s fracture toughness $\kappa$, which is defined to be the energy required to propagate a crack by unit area (Atkins, 2009: p. 16). The work done by $-df_C$ over the movement is equal to the energy needed for creating the parallelogram of fracture:

$$
(-df_C \cdot \mathbf{v}) d\eta = -\kappa(\mathbf{v} \cdot \hat{n}) d\eta ds,
$$

where $\mathbf{v} = \mathbf{v}/\|\mathbf{v}\|$ and $\hat{n}$ is the unit inward normal at $\beta_\theta(u)$. From the above and that $df_C$ and $\mathbf{v}$ are collinear, we obtain

$$
df_C = \kappa(\mathbf{v} \cdot \hat{n}) \mathbf{v} ds
$$

(10)

Integration over the segment $S = \beta_\theta[u_1, u_2]$ yields the total fracture force:

$$
f_C = \int_S df_C.
$$

(11)

Since the knife is rigidly attached to the robotic arm’s open end $a$, the fracture force yields a torque at the point:

$$
\tau_C = \int_S (\beta_\theta(u) - a) \times df_C.
$$
Coulomb friction is assumed in the contact region \( \Omega \) on both sides of the blade. Let the unit vector \( \hat{v}(x, y) \) denote the direction of the velocity of an area element at \((x, y)^T \in \Omega \). The force received at the open end \( \alpha \) due to friction is

\[
f_F = -2\mu \int_{\Omega} P(x, y) \hat{v} \, dx dy,
\]

where \( \mu \) is the coefficient of friction and \( P(x, y) \) is the pressure distribution of the blade at \((x, y)^T \). Similarly, the torque at \( \alpha \) due to friction is

\[
\tau_F = -2\mu \int_{\Omega} P(x, y) \left( \frac{x}{y} - \alpha \right) \times \hat{v} \, dx dy.
\]

The wrenches \((f_C, \tau_C)^T\) and \((f_F, \tau_F)^T\) can be evaluated given the knife’s pose \((p_c^T, \phi)^T\) and velocities \((v_c^T, \omega)^T\). If the knife is translating, they have simple forms that are derived in the Appendix. In the general case, the points on the blade within the region \( \Omega \) do not have the same velocity, which implies that the friction and frictional forces and torques can only be calculated numerically.

Subtracting these two wrenches \((f_C^T, \tau_C)^T\) and \((f_F^T, \tau_F)^T\) from the reading by a force/torque sensor at \( \alpha \), we will be able to determine forces of other sources, in particular, the contact force between the knife and the cutting board. This information will be used later for knife control during the last phase of cutting in Section 3.3.

### 3 Dynamics and control of cutting for planar robotic arm

Cutting of an object proceeds in three phases that were previously illustrated in Fig. 1. The first phase is pressing, during which the arm translates the knife downward until its edge contacts the cutting board. The second phase, translational, is touching, during which the arm quickly slows down the knife’s vertical motion to soften its contact with the board. The third phase is slicing, during which the arm translates and rotates the knife to move its contact point with the cutting board across the object’s bottom segment \(p_r, p_c\) (see Fig. 5) in the cutting plane. By now the object has been split into two parts. In this section, we will describe control strategies for carrying out the above three phases of cutting.

Denote by \( \rho_a \) the wrench (force and torque) exerted at the arm’s open end \( \alpha \) due to the knife’s interactions with the object and the cutting board. This wrench can be estimated from the reading of a force/torque (F/T) sensor mounted at \( \alpha \) after compensating for the gravitational effects of the sensor and knife. The open end has generalized coordinates \( \mathbf{x} = (x, y, \phi)^T \), which has the derivative

\[
\dot{x} = J_a \dot{\theta},
\]

where

\[
J_a = \frac{\partial \mathbf{x}}{\partial \theta}
\]

is a \( 3 \times n \) Jacobian matrix derived from (1) and (2).

The kitchen knife, driven by the \( n \)-DOF arm \((n \geq 3)\), can follow any given trajectory in the cutting plane (as long as no joint limit is exceeded). Since both the F/T sensor and the knife are rigidly mounted on the arm, they are treated as parts of the arm’s distal link. The arm dynamics can be represented in the joint space as follows:

\[
\tau = M(\theta) \dot{\theta} + C(\theta, \dot{\theta}) \dot{\theta} + N(\theta) - J_a^T \rho_a,
\]

where \( \tau \) is the arm’s joint torque vector, \( M(\theta) \) is an \( n \times n \) mass matrix accounting for the sensor and knife as well, \( C(\theta, \dot{\theta}) \dot{\theta} \) includes the Coriolis and centrifugal terms, and \( N(\theta) \) is the gravity term.

Assuming that \( J_a \) has full row rank, we obtain the joint acceleration from differentiating equation (14):

\[
\ddot{\theta} = J_a^T (\ddot{x} - \dot{J}_a \dot{\theta}),
\]

where

\[
J_a^T = J_a^T (J_a J_a^T)^{-1}
\]

is the Moore-Penrose inverse of \( J_a^T \). The arm dynamics (15) can be rewritten in the task space:

\[
\tau = MJ_a^T \ddot{x} + \tau_a,
\]

where

\[
\tau_a = (C - MJ_a^T J_a) \dot{\theta} + N - J_a^T \rho_a
\]

contains the Coriolis, centrifugal, gravitational, and external forces.

#### 3.1 Pressing

During pressing, the wrench \( \rho_a \) exerted at the arm’s open end \( \alpha \) is due to the knife-material interaction only. For a desired cutting path \( \mathbf{x}_d = (a, d, a, d, \phi_d)^T \), we propose the following position controller:

\[
\tau = MJ_a^T \lambda + \tau_a,
\]

where, letting \( \mathbf{x}_c = \mathbf{x}_d - \mathbf{x} \),

\[
\lambda = \begin{pmatrix}
\lambda_x \\
\lambda_y \\
\lambda_\phi
\end{pmatrix}
= \dot{\mathbf{x}}_d + K_{v|a} \dot{\mathbf{x}}_c + K_{p|a} \mathbf{x}_c + K_{i|a} \int \mathbf{x}_c \, dt.
\]

(21)

Here, \( K_{v|a} = k_{p|a} I_3 \), \( K_{i|a} = k_{i|a} I_3 \), and \( K_{v|a} = k_{v|a} I_3 \), where \( k_{p|a}, k_{i|a}, k_{v|a} > 0 \) and \( I_3 \) is the \( 3 \times 3 \) identity matrix, respectively represent proportional, integral, and derivative (PID) gains.

Subtracting the dynamics (18) from the controller (20) and left multiplying the resulting equation by \( J_a M^{-1} \), we obtain the following error dynamics:

\[
\ddot{\mathbf{x}}_e + K_{v|a} \dot{\mathbf{x}}_e + K_{p|a} \mathbf{x}_e + K_{i|a} \int \mathbf{x}_e \, dt = 0.
\]

(22)

This is a third order linear time invariant (LTI) system, whose stability can be guaranteed by choosing positive values of \( k_{v|a}, k_{p|a}, \) and \( k_{i|a} \) such that \( k_{v|a} k_{p|a} > k_{i|a} \) (Dorf and Bishop, 2010: pp. 394).

Since a fast knife movement better demonstrates the cutting skill, \( \dot{x}_d \) especially its component \( \dot{a}_d \) should not be set with a small magnitude. The pressing phase ends when contact between the knife and the cutting board is detected from a sudden increase in the force reading of the F/T sensor. Such increase is due to an impact between the knife and the cutting board.

---

**Figure 5.** Knife in contact with the cutting board.
3.2 Touching

Upon establishing their contact at a point \( c \), the cutting board exerts an impulsive force \( f = (f_x, f_y)^T \) on the knife, which is transmitted to the robotic arm. Since an excessive \( f \) value could cause some damage to the arm, there is a need to mitigate it as soon as possible. Meanwhile, the desired \( x \)-directional position \( x_{\text{des}} \) and orientation \( \phi_{\text{des}} \) of the arm frame \( A \) should not be changed. We thus apply impedance control in the \( y \)-direction over \( y_\alpha \) (equivalently on the contact point \( c \)'s \( y \)-coordinate due to the rigid attachment of the knife at \( a \)), while position control over \( x_\alpha \) and \( \phi \). The state is the same \( x \) in the pressing phase, and therefore, adopts its task space dynamics (18). Since the desired velocity is zero, all forces and torques due to material fracture and friction can be ignored. The wrench exerted at \( a \) is entirely due to \( f \):

\[
\rho_a = \begin{pmatrix} f \\ (c - a) \times f \end{pmatrix}.
\]

From the sensor reading, we can extract \( f \), in particular, its \( y \)-component \( f_y \). Then apply hybrid impedance/position control as follows:

\[
\tau = MJ^a \begin{pmatrix} \lambda_x \\ \lambda_y \\ \lambda_\phi \end{pmatrix} + \tau_a,
\]

where the servos \( \lambda_x \) and \( \lambda_\phi \) are as in (21), now with zero desired velocity and acceleration, while the servo in the \( y \)-direction is

\[
\tilde{\lambda}_y = \ddot{a}_{yd} + \frac{k_y a_{yc} + d_y a_{ye} + f_y}{b_y},
\]

where stiffness \( k_y \), damping \( d_y \), and inertia \( b_y \) are designed to achieve the desired behavior.

Stabilities of \( \dot{a}_x \) and \( \phi \) can be established just like for the pressing phase. The close-loop form for \( a_y \) is

\[
b_y \ddot{a}_{yc} + k_y a_{yc} + d_y \dot{a}_{ye} + f_y = 0.
\]

This second order LTI system’s stability is ensured by positive gains (Dorf and Bishop, 2010: pp. 394) and bounded \( f_y \) (Kailath, 1980: pp. 177).

In this phase, impedance control makes the knife-board interaction behave like a mass-spring-damper, which is what we need to soften the impact resulting from the fast knife movement in the pressing phase. Force control, on the other hand, would be unable to generate a fast response in this situation.

3.3 Slicing

Under impedance control, the contact force will experience a significant decrease until it starts to have small fluctuations. However, impedance control cannot precisely regulate the knife-board contact force to ensure complete separation of the material. In the slicing phase, we would like the \( y \)-component of this contact force to be maintained at a certain level to be sure that the object is being split as the knife moves on the cutting board.

The point \( e \) of knife-board contact, as illustrated in Fig. 6, has coordinates \( (c_x, 0)^T \) if derived from the cutting board and \( \beta_\theta = (\beta_{\theta,x}, \beta_{\theta,y})^T \) if from the knife’s edge as described in the world frame \( \mathcal{W} \). Here, we let \( u \) be the parameter value of the contact point on the curve \( \beta_\theta \). The two equations

\[
\begin{align*}
\beta_{\theta,x}(u) &= c_x, \\
\beta_{\theta,y}(u) &= 0,
\end{align*}
\]

hold as along as the contact is maintained. In addition, the tangent at \( \beta_\theta(u) \) is parallel to the cutting board, i.e., to the \( x \)-axis, yielding

\[
\frac{\partial \beta_{\theta,y}}{\partial u} = 0.
\]

Applying (6) and \( \beta_{\theta,y} = (0, 1) \beta_\theta \), the above equation becomes

\[
(sin \psi, \cos \psi) \frac{d\beta'}{du} = 0.
\]

Given the convexity of the knife’s edge curve \( \beta' \), equation (28) defines the location \( u \) of the contact point on the edge as a function of the knife’s orientation \( \psi \) and thus a function of \( \theta \) according to (4) and (2); that is, \( u = u(\theta) \). Constraints (26) and (27) are now rewritten as

\[
\beta_\theta(u(\theta)) = \begin{pmatrix} c_x \\ 0 \end{pmatrix}.
\]

Since the knife point’s position is completely determined as

\[
p = \begin{pmatrix} c_x \\ 0 \end{pmatrix} - R(\psi) \beta'(u(\theta)),
\]

the knife’s configuration during slicing is thus described by

\[
y = \begin{pmatrix} c_x \\ 0 \\ \psi \end{pmatrix}.
\]

The slicing knife is in contact with both the object and the cutting board. The wrench \( \rho_a \) at the arm’s open end \( a \) accordingly has two components:

\[
\rho_a = \rho_a^c + \left( f \right) \times \left( c - a \right),
\]

where

\[
\rho_a^c = \begin{pmatrix} f_c \\ \tau_c \end{pmatrix}
\]

is the wrench due to fracture and friction with its force and torque components obtainable using (10)–(13). The knife-board contact force \( f \) can be obtained by subtracting the calculated force \( f_c + f_F \) from the sensed force \( f_S \) at \( a \). With an objective to maintain the normal contact force \( f_y \), we rewrite the dynamics in (15) as

\[
\tau = MJ \ddot{\theta} + C \dot{\theta} + N - J_a^T \rho_a - J_c^T \begin{pmatrix} f_c \\ 0 \end{pmatrix},
\]

Figure 6. Slicing along the cutting board.
where
\[ J_c = \frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta} \left( \beta_\theta(u) \right) \]  
(34)
(treating the parameter \( u \) as constant so the contact is viewed as a fixed point on the edge) is the \( 3 \times n \) Jacobian matrix at \( c \).

Our objective is to control \( c_x \) and \( f_y \), to ensure separation of the object into two pieces when the slicing phase ends. For this purpose we need to rewrite the dynamics in (33) in terms of \( y \) in the task space. This requires us to make use of the following coordinate transformation:
\[ \dot{y} = L_c \hat{\theta}, \]  
(35)
where, by (30) and (29),
\[ L_c = \frac{\partial}{\partial \theta} \left( \beta_\theta(u) \right) \]  
= \[ J_c + \left( R(\psi) \frac{\partial}{\partial \theta} \left( \beta'(u) \right) \right). \]  
(36)

The second equation above followed from differentiating of equation (6).

We have
\[ \frac{\partial}{\partial \theta} \left( \beta'(u) \right) = \frac{d\beta'}{du} \cdot \nabla u(\theta). \]

The partial derivative \( \partial u/\partial \theta_i \), for \( 1 \leq i \leq n \), can be obtained from differentiating the constraint (28) with respect to \( \theta_i \). They are all identical, leading to the gradient form:
\[ \nabla u = \left( -\cos \psi, \sin \psi \right) \frac{d\beta'}{du} \left( 1, 1, \ldots, 1 \right). \]

The dynamics given in (33), via the substitution \( \ddot{\theta} = L_c^\dagger \left( \ddot{y} - \dot{L_c} \ddot{\theta} \right) \) from differentiating (35), become
\[ \tau = ML_c^\dagger \ddot{y} + \tau_c - J_c^\dagger \left( F - \dot{p}_n \right), \]  
(37)
where
\[ \tau_c = \left( C - ML_c^\dagger \dot{L}_c \right) \ddot{\theta} + N - J_c^\dagger \dot{p}_n. \]  
(38)

Motion planning for the slicing phase is to set a desired time trajectory of \( c_x \) and \( \psi \). If the object is easy to cut, a pure translation of the knife (see Fig. 7(a)) can be used. In this motion, \( \dot{\psi}_d \) is constant and the contact point does not move on the knife’s edge. If the object has a high fracture toughness, a pure rolling motion (see Fig. 7(b)) is used so the action becomes “chopping”. More commonly, the knife translates and rotates simultaneously, as shown in Fig. 7(c).

Different from the touching phase, the cutting board is now viewed as rigid enough to prevent penetration by the knife. By choosing a large enough desired normal contact force \( f_{ct} \), we can ensure the actual contact force \( f_y \) to be positive despite control errors, namely, the knife to be always in contact with the cutting board. Meanwhile, the \( x \)-direction cutting velocity \( \dot{c}_x \) and the orientation \( \psi \) are under position control. Let \( c_{xd}(t) \) be some desired time trajectory of \( c_x \) and \( c_{yd}(t) \) be its error. Let \( \psi_d \) be the desired orientation of the knife so \( \psi_e = \psi_d - \psi \) is the orientation error. The contact force error is represented by \( f_c = f_{ct} - f_y \). We apply the following hybrid control law:

\[ \tau = ML_c^\dagger \begin{pmatrix} \dot{c}_{xd} + k_{v_1} \dot{c}_{xe} + k_{p|e} c_{xe} + k_{i|e} \int c_{xe} \, dt \\ \dot{v}_e + k_{v_1} \dot{v}_e + k_{p|e} \psi_e + k_{i|e} \int \psi_e \, dt \\ f_{dx} + k_{f_1} \int f_{dx} \, dt \end{pmatrix} \]  
+ \tau_c - J_c^\dagger \begin{pmatrix} f_{dx} + k_{f_1} \int f_{dx} \, dt \\ 0 \end{pmatrix}, \]  
(39)
where the parameters \( k_{v_1}, k_{p|e}, k_{i|e} \) are the PID gains, respectively, and \( k_{f_1} \) is the integral force gain.

A close-loop system is obtained by subtracting (37) from (39):
\[ ML_c^\dagger (\alpha_1, 0, \alpha_2)^T - J_c^\dagger (0, \alpha_3, 0)^T = 0, \]  
(40)
where the error items are
\[ \alpha_1 = \dot{c}_{xe} + k_{v_1} \dot{c}_{xe} + k_{p|e} c_{xe} + k_{i|e} \int c_{xe} \, dt, \]
\[ \alpha_2 = \dot{\psi}_e + k_{v_1} \dot{\psi}_e + k_{p|e} \psi_e + k_{i|e} \int \psi_e \, dt, \]
\[ \alpha_3 = f_{dx} + k_{f_1} \int f_{dx} \, dt. \]

Multiplication of \( (\alpha_1, 0, \alpha_2)(L_c^T)^\dagger \) with both sides of (40) to eliminate \( J_c^\dagger (0, \alpha_3, 0)^T \) after applying (36) and (28). This yields
\[ (\alpha_1, 0, \alpha_2)(L_c^T)^\dagger ML_c^\dagger (\alpha_1, 0, \alpha_2)^T = 0. \]
Since the inertia matrix $M$ is positive definite, the above equation holds if and only if $L_{1}^T(\alpha_{1}, 0, \alpha_{2}) = 0$, which implies $\alpha_{1} = 0$ and $\alpha_{2} = 0$ because $L_{1}^T$ has independent columns as a result of $L_{c}$’s full row rank. Convergences of the errors $c_{xx}$ and $\psi_{c}$ to zero are guaranteed by choosing the controller gains based on the third order LTI system stability. Substituting $\alpha_{1} = 0$ and $\alpha_{2} = 0$ into (40), we have $\alpha_{3} = 0$. Diminishing of the error $f_{c}$ is guaranteed with the gain $k_{fi} > 0$.

4 Cutting with a 2-DOF planar robotic arm

In this section, we look at how to adapt the three-phase cutting strategy from Section 3 to a 2-DOF robotic arm that is underactuated for the task. An example is the 4-DOF WAM Arm, which has only two DOFs in a vertical plane.

In the pressing phase, the position of the arm’s open end is simplified from (1) given $n = 2$:

$$a = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = l_{1} \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} + l_{2} \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix},$$

(41)

where $\phi = \theta_1 + \theta_2$. To keep a constant orientation $\psi$ defined in (4), the knife, the trajectory of $a$ must be a circle centered at $(l_{2} \cos \phi, l_{2} \sin \phi)^{\top}$ and with radius $l_{1}$. The knife’s initial orientation uniquely determines the desired trajectory $a = (a_{xd}, a_{yd})^{\top}$.

The arm’s open end has the velocity

$$\dot{a} = J_{a} \dot{\theta},$$

(42)

where the $2 \times 2$ Jacobian matrix is given as $J_{a} = \partial a / \partial \theta$ to include only the first two rows of $J_{a}$. We differentiate (42) and substitute the resulting equation for $\dot{\theta}$ into (15). This results, after some cleanup, in the following dynamic equation which is similar to (18):

$$\tau = M \ddot{J}_{a}^{\top} \dot{a} + (C - M \ddot{J}_{a}^{\top} J_{a}) \ddot{\theta} + N - J_{a}^{\top} \rho_{a}.$$  

(43)

The controller to be applied has the form (20), with the state $\mathbf{x}$, the Jacobian matrix $J_{a}$, and the vector $\mathbf{\lambda}$ replaced by $a$, $J_{a}$, and $\mathbf{\lambda} = (\lambda_{x}, \lambda_{y})^{\top}$, respectively.

The touching phase, described in Section 3.2, aims to soften the contact between the knife and cutting board. With the same state $a$, the task space dynamics equation (43) still holds. The hybrid position and impedance controller (24) is simply adapted for the 2-DOF arm by replacing $a$ with $a$ and $J_{a}$ with $J_{c}$, and by removing $\lambda_{\phi}$.

In the slicing phase, the contact constraints (26)–(28) uniquely determine $\theta_{1}$, $\theta_{2}$, and $u$ as functions of $c_{xx}$, hence the path of the knife. We write $\theta = \theta(c_{xx})$ and $u = u(c_{xx})$. In general, the knife’s motion on the cutting board is not rolling (which is needed for chopping). This is reasoned below. Rolling would require the $x$-velocity of the instantaneous contact point on the knife’s edge to be zero, equivalently, the following condition to hold:

$$(1, 0) J_{c}(\theta) \dot{\theta} = 0,$$

where $J_{c} = \partial \mathbf{\theta}_{c} / \partial \mathbf{\theta}$ includes only the first two rows of $J_{c}$ given in (34). With $\dot{\theta} = (\partial \dot{\theta} / \partial c_{xx}) \dot{c}_{xx}$ substituted in, the above condition becomes

$$(1, 0) J_{c}(\theta) \frac{d \dot{\theta}}{dc_{xx}} \dot{c}_{xx} = 0.$$  

To make steady progress toward cutting open the object, either $\dot{c}_{xx} > 0$ throughout cutting or $\dot{c}_{xx} < 0$ throughout cutting must hold. Hence, rolling would require

$$(1, 0) J_{c}(\theta) \frac{d \dot{\theta}}{dc_{xx}} = 0.$$  

(44)

Equation (44) is independent from equations (26)–(28) derived from the contact geometry. Together the four equations cannot be simultaneously satisfied by the three variables $\theta_{1}$, $\theta_{2}$, and $u$.

The knife’s motion during slicing, driven by the 2-DOF arm, is neither pure translation nor pure rolling, with the contact point moving less on the knife’s edge than on the cutting board.

The state is now $\hat{y} = (c_{xx}, 0)^{\top}$. Denote by $\bar{L}_{c}$ the $2 \times 2$ matrix which consists of the first two rows of $L_{c}$ given in (36). The task space dynamics equation (37) now takes the form

$$\tau = M \bar{L}_{c}^{\top} \ddot{\hat{y}} + \bar{\tau}_{c} - J_{c}^{\top} (f_{c})$$

where

$$\bar{\tau}_{c} = (C - M \bar{L}_{c}^{\top} \bar{L}_{c}) \ddot{\theta} + N - J_{c}^{\top} \rho_{a}.$$  

(45)

A hybrid position/force controller is then adapted from (39):

$$\tau = M \bar{L}_{c}^{\top} \begin{pmatrix} \ddot{c}_{xx} + k_{i1} \dot{c}_{xx} + k_{p1} c_{xx} + k_{i2} \int c_{xx} \, dt \\ 0 \end{pmatrix}$$

$$+ \bar{\tau}_{c} - J_{c}^{\top} \begin{pmatrix} f_{c} \\ 0 \end{pmatrix}.$$  

(46)

5 Dynamic estimation of physical parameters

The object’s fracture toughness $\kappa$ varies with the knife’s sharpness. Its coefficient $\mu$ of friction with the blade depends on the latter’s material. The values of both parameters also vary among natural foods, whether or not of the same type, and for the same food, with its degree of freshness. Pressure on the object being cut depends on the thickness of the blade.

It may vary from point to point inside the region $\Omega$ of the blade-material contact, and at the same point, from one time instant to another during the cutting action.

On the other hand, the parameters $\kappa$, $\mu$, and the pressure distribution function $P(x, y)$ are only used in the last phase (slicing) for modeling the frictional force and frictional forces in order to estimate the knife-board contact force $f$ for control purpose. We can conveniently make use of the force data accumulated during the first phase (pressing) to estimate $\kappa$, $\mu$, and $P(x, y)$ for the object, and apply these values in modeling during the last phase (slicing).

Some simplifications are necessary for such real-time estimation. First, it is reasonable for us to treat $\kappa$ and $\mu$ as constants throughout cutting. Next, $P(x, y)$ is approximated as a uniform (but time varying) pressure distribution over the contact region $\Omega$. The frictional force (12) is now simplified to

$$f_{F} \approx -2 \mu P \int_{\Omega} \hat{\nu} \, dx \, dy.$$  

(47)
From experimental data, we have found that $P$ decreases during the pressing phase. The decrease may be attributed to release of some stress due to fracture. Let $P_{\text{max}}$ be its maximum value. Let $\Phi$ be the area of fracture so far. (Its maximum value $\Phi_{\text{max}}$ corresponds to the area of intersection between the object and the cutting plane.) The pressure distribution at the time instant is modeled linearly as

$$P = r P_{\text{max}},$$

where

$$r = \frac{\Phi_{\text{max}} - \Phi}{\Phi_{\text{max}}}$$

is between 0 and 1. Observing in equation (47) that $\mu$ and $P$ are multiplied together, we can treat $\mu$ and $P_{\text{max}}$ as a single term $\mu P_{\text{max}}$ to be determined.

The F/T sensor mounted at the robotic arm’s open end acquires its readings at equally spaced time instants. Let $f_S$ and $\tau_S$ be the force and torque estimates extracted from the sensor reading after compensation for the gravitational effects of the knife and sensor. During the pressing phase, the sensor reading after compensation for the gravitational effect of the squared difference between the sensed force $f_S$ and its modeled value $\kappa f_C + f_f$ is

$$f_S = f_C + f_f.$$ The first reading used for estimation is obtained after the knife has cut into the object for some distance and the $y$-directional velocity has reached some desired constant value. The last reading is obtained when the lowest point on the knife-edge is several millimeters above the cutting board. The time instant of the first reading is indexed 1 and that of the last reading used for estimation is indexed $n_S$.

Then we estimate $\kappa$ and $\mu P_{\text{max}}$ via minimizing the sum of the squared difference between the sensed force $f_{S,j}$ and its modeled value $f_{C,j} + f_{F,j}$ over all the time instants $j$, $1 \leq j \leq n_S$. It follows from (9) and (10) that $f_{C,j} = \kappa X_{C,j}$, and from (47) and (48) that $f_{F,j} = r_j \mu P_{\text{max}} X_{F,j}$, where $X_{C,j}$ and $X_{F,j}$ are two vectors that can be evaluated through integration, and $r_j$ is evaluated according to (49) at the $j$-th time instant. In summary, $\kappa$ and $\mu P_{\text{max}}$ are estimated via least-square fitting:

$$\min_{\kappa, \mu P_{\text{max}}} \sum_{j=1}^{n_S} (f_{S,j} - \kappa X_{C,j} - r_j \mu P_{\text{max}} X_{F,j})^2.$$ (50)

Closed forms for the optimal values of $\kappa$ and $\mu P_{\text{max}}$ can be easily obtained from vanishing of the gradient of the cost function in (50). Occasionally, $\mu P_{\text{max}}$ may end up with a negative optimal value. When this happens, we apply Newton’s method (Boyd and Vandenberghe, 2004: pp. 484–487) with initial values such as $\kappa = 300$ N/m and $\mu P_{\text{max}} = 2000$ N/m$^2$. The method will terminate its iterations when $\mu P_{\text{max}}$ gets close to 0.

The knife does not change its orientation throughout pressing, which makes it possible to calculate $X_{C,j}$ and $X_{F,j}$, $1 \leq j \leq n_S$, using closed forms as shown in the Appendix.

6 Experiments and simulation

Cutting experiments were carried out with a 4-DOF WAM Arm shown in Fig. 8(a). Its joints 1 and 3 were fixed so the robot effectively had two DOFs. Mounted on the arm’s end effector was a 6-axis Delta IP65 F/T sensor from ATI Industrial Automation. Rigidly attached to the F/T sensor through a metal adapter was a kitchen knife, whose kinematics were in terms of the robot’s joint angles. To apply the dynamics equation (15), we calculated the combined mass, Coriolis, and gravity terms ($M$, $C$, and $N$, respectively) of the arm, sensor, and knife based on the arm’s specification and available sensor information.

To model the kitchen knife, we placed it on a sheet of paper, and drew its contour. After choosing the $x'$- and $y'$-axes of the knife frame $K$ at the knife point, we reconstructed the knife’s edge $\beta(u)$ by setting $u$ to be the $x'$-coordinate and fitting a quartic curve $\beta_u$ to the $y'$-coordinates of the measured points on the edge. Similarly, the knife’s spine was reconstructed through fitting as a quartic curve $\gamma(q) = (q, \gamma_u(q))^T$ with $q$ identified with $x'$.

A Microsoft Kinect sensor supported by a tripod was used in acquiring some densely distributed points on the object’s surface. Those points close enough to the cutting plane were fit over to reconstruct the contour $\sigma(r)$ of the object’s cross section in the plane. A 6-DOF Servo Motor Arm held the object to stabilize it during cutting.

Onions, potatoes, apples, and cucumbers were used in the experiments. In a cutting trial, half of an object (precut by
the human hand) was placed on the cutting board with its flat face down (see the onion and potato in Fig. 8(a) and (b), respectively).

Control gains used in all the three phases of cutting are listed in Table 2. Applied to all four food types, these gains were chosen to keep the robotic arm from generating excessive torque responses at its joints. Such an undesired situation could occur, for instance, with impedance control during touching if a very large value for the gain $d_d$ in (25) was to be chosen, commanding torques to exceed the limits for the arm.

The two terms $f_S$ and $\tau_S$ extracted from the sensor data estimate all the interaction forces and torques of the knife with the object and cutting board, as received at the arm’s open end $a$. During pressing and touching, in which modeling is not needed, the wrench estimate $(f_S, \tau_S)^T$ is substituted for $p_a$ in the expression (19) of $\tau_a$ used by the controllers (20) and (24). During slicing, the knife-board contact force, whose normal component $f_y$ is regulated, is estimated by subtracting from $f_S$ the modeled fracture force $f_C$ and frictional force $f_F$.

### 6.1 Control validation for pressing and touching

Fig. 9(a)–(d) shows the experimental results from cutting an onion. Pressing, touching, and slicing lasted over the time periods $[0, t_1], [t_1, t_2]$, and $[t_2, t_3]$, respectively. The pressing phase, during which the knife was translating, started with the knife slightly above the object and ended when the knife-board impact was detected from sensor readings. The touching phase softened this impact within a short time period (around 0.07 s) before it smoothly transitioned into the slicing phase.

Included in Fig. 9(a) are four snapshots respectively at the start and during the three phases. With a slight abuse of notation we let $c = (c_x, c_y)^T$ also denote the lowest point on the knife’s edge during the pressing phase (when the knife is not in contact with the board). In (b), the ordinate $c_y$ follows the desired trajectory $c_{yd}$ over $[0, t_1]$ and the abscissa $c_x$ follows the desired trajectory $c_{xd}$ over $[0, t_2]$. Since the knife was rigidly connected to the robot and its orientation was kept constant during the pressing period $[0, t_1)$, the actual and desired trajectories of $c$ were simply translated from those of $a$. Fig. 9(c), plotted over the period $[t_1, t_3]$, shows that the contact force decreased from 75 N to 11 N within 0.068 s under impedance control during touching, and then approached the desired value with small variations under force control. As seen in (d), in the pressing phase the modeled frictional force $f_F$ (between the blade and the material) and fracture force $f_C$, even though not used, adds up close to the sensed force $f_S$.

As observed from Fig. 9(d), during slicing the sum $f_{Cy} + f_{Fy}$ of the $y$-directional fracture and frictional forces was almost negligible. This was in fact caused by the WAM Arm’s lack of one DOF to achieve an arbitrary pose of the knife in the cutting plane. An explanation is given below. To maintain the knife-board contact (i.e., $c_y = 0$) equivalently took away one of the two DOFs of the arm in the cutting plane. During slicing, the remaining DOF carried out a path of knife poses which was completely determined. (How fast the knife moved along the path could still be realized via controlling the change rate of this second DOF.) Meanwhile, to cut through the object, this path was also subjected to the constraint that the knife-board contact point $c$ had to move across the bottom segment of the object’s cross section in the cutting plane, say, from left to right. As $c$ was moving in this direction, the knife was rotating counterclockwise and the contact point was moving on its edge towards the tip. This required $c$ to be located several centimeters away from the tip at the start of slicing, resulting in the knife just slightly tilted.

### Table 2. Control gains for individual phases of cutting in the experiments.

<table>
<thead>
<tr>
<th>Pressing &amp; Touching</th>
<th>Slicing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{p_{x,a}}$</td>
<td>$k_{i_{x,a}}$</td>
</tr>
<tr>
<td>500</td>
<td>800</td>
</tr>
</tbody>
</table>

### Figure 9. Cutting of an onion with the setup in Fig. 8(a). (a) Snapshots at the start and during the three phases: pressing, touching, and slicing. The three phases started at $0 s$, $t_1 = 0.326 s$, and $t_2 = 0.394 s$, respectively. The cutting action ended at $t_3 = 0.048 s$. The dashed lines draw out the occluded portions of the knife’s edge, $c$ marks the lowest point on the knife’s edge, and the white arrow in the last snapshot shows the knife’s counterclockwise rotation during the slicing phase. We used $a_{yd} = -0.2 m/s$ and $c_{yd} = 0.17 m/s$ as the desired values in pressing and slicing, respectively. Trajectories of the (b) lowest point $c$ on the knife’s edge, (c) the $y$-component $f_y$ of the contact force between the knife and cutting board, and (d) sensor reading $f_S = (f_{Sx}, f_{Sy})^T$ (drawn in solid lines) of the force exerted on the knife and the sum $f_C + f_F = (f_{Cy} + f_{Fy}, f_{Cx} + f_{Fx})^T$ (drawn in dashed lines) of modeled fracture and frictional forces. The parameter values estimated at the end of pressing and used in slicing were $\kappa = 302.13 N/m$ and $\mu_{P_{max}} = 15961.8 N/m^2$. 

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The knife consequently was having a very small vertical velocity component relative to its horizontal one. Under the knife’s counterclockwise rotation, only those points on its edge to the right of the contact point were generating fracture. The rotation also resulted in partial canceling of the y-components of the frictional forces exerted on points within the knife-blade contact region \( \Omega \). This was why \( f_{Cy} + f_{Fy} \) turned out to be negligible in Fig. 9(d).7

### 6.2 Control validation for slicing

To generate a large downward cutting force \( f_{Cy} \), the knife needed to perform an action close to “chopping”, by starting the third phase in a more tilted pose and rotating clockwise so its edge would be ideally rolling on the cutting board. This was impossible with the arm’s two DOFs in the cutting plane. So we switched to the second setup in Fig. 8(b), where a linear guide was used to translate the board leftward at some speed \( v(t) \).

The world frame \( W \) was fixed and not moving with the board. The contact motion \( c_z(t) \) was relative to the still \( x \)-axis not the translating board. As discussed in Section 4, \( \theta = \theta(c_x) \) for a 2-DOF arm. Rolling of the knife on the board would be achieved by setting the board’s \( x \)-velocity \( v \) to be equal to that of the instantaneous contact point on the knife’s edge:

\[
v = v_c = (1, 0) \frac{d\theta}{dc_x}(c_x) \hat{c}_x,
\]

where \( \dot{\theta} = \partial_{\theta} / \partial \theta \) was introduced in Section 4. However, the obtained \( v \), constantly changing with time, could not be generated accurately enough by the linear guide. Instead, we kept \( v \) at some constant value (−0.04 m/s) and constructed a desired contact motion \( c_{zd}(t) \) to satisfy equation (51) approximately. More specifically, we evaluated the derivative:

\[
c_{zd} = \frac{v}{(1, 0) J_c(\theta(c_x)) \frac{d\theta}{dc_x}(c_x)}
\]

at the time instant, integrated over a time step to update \( c_{zd} \) at the next time instant, and so on.

Fig. 10 focuses on the slicing phase of cutting a potato in the setup of Fig. 8(b) using the \( c_{zd}(t) \) constructed from (52), with three snapshots in (a)–(c). The near-rolling motion of the knife is evidenced in (d) from a small range of difference between the time trajectories of the two opposite sides of (51). The fluctuations of \( v_c \) were in a degree caused by the uneven board which was made of bamboo. As shown in Fig. 10(e), the modeled \( y \)-directional fracture force \( f_{Cy} \) was almost 30 N at the beginning of slicing. It first increased slightly as \( c \) was moving into the object and then gradually decreased to 0 when the object was cut open. The contact force component \( f_{Fy} \) was maintained close to the desired value of 5 N. The modeled frictional force, with its \( y \)-component \( f_{Fy} \) at 2.0 N when the slicing phase started, kept decreasing as the pressure distribution decreased.

### 6.3 Comparison with cutting under position commands

We also tried cutting with the built-in joint space position controller of the WAM arm. More specifically, a trajectory for the arm was planned through inverse kinematics in advance and served as an input for the robot to follow. In the pressing phase, cutting was carried out smoothly. In the touching phase, as shown for the period \([t_1, t_2]\) in Fig. 11(a), the knife-board contact force increased again after a deep drop. Its trajectory is not monotone in contrast with one under impedance control (cf. Fig. 9(c)).

In the slicing phase, the knife-board contact force is better maintained at a certain level to ensure material separation. With the built-in position controller under both setups (see Fig. 8), the desired vertical position of the knife-board contact point was set below the board surface at a constant value. As in Fig. 9(a), the knife was slicing the onion backward. Under the built-in position control, the \( y \)-directional contact force kept increasing from time \( t_2 \) to the
end of cutting (see Fig. 11(a)). Making use of the setup in Fig. 8(b), the slicing knife simulated pure rolling on the cutting board. One snapshot from cutting a potato (see Fig. 11(b)) shows that the knife had broken its contact with the cutting board in the slicing phase. The potato was not fully cut open as a result.

To summarize, position control cannot handle the knife-board impact in the touching phase or maintain the knife board contact force in the slicing phase of the cutting action. There are several reasons. First, high accuracy of the knife’s position is not easy to achieve, considering both its not so rigid connection to the moving robotic arm and its estimated edge curve for calculation of the contact point. Second, in order to avoid a large impact with the cutting board, the velocity of cutting would have to be slow under position control, which is not desired for swiftness of cutting. Third, breaking of the knife-board contact can happen to result in the object not being cut through. Finally, the contact force between the knife and the cutting board can change dramatically with a tiny position error (especially when the board has a hard surface), which would be unavoidable and potentially harmful to the robotic arm.

### 6.4 Repeated cutting and physical parameters

We conducted more experiments with the three-phase cutting strategy. After each cutting action, the small 6-DOF Servo Motor Arm (see Fig. 8(a)) could push the uncut portion of the object forward for a small distance on the cutting board. This made it possible to cut the object into pieces by repeating the same three-phase knife movement. Fig. 12 includes some snapshots from cutting an apple and a cucumber into pieces.

### 6.5 Simulation with a 3-DOF planar arm

To be freed from the WAM Arm’s DOF limitation, we have simulated the general cutting scheme described in Section 3. Our choice is a 3-DOF planar robotic arm, which is the simplest model that can position and orient the knife arbitrarily to realize this cutting scheme. Widely used robotic arms such as the UR10, 7-DOF WAM, and KUKA LBR IIWA 7 all have at least three revolute joints with parallel axes, and thus can operate like a 3-DOF planar arm.

The arm used in the simulation (refering to Fig. 2 for an n-DOF arm) has three links that assume uniform mass distributions. The object is an onion whose physical parameters are set as \( \kappa = 500 \text{ N/m} \) and \( \mu P_{\text{max}} = 7000 \text{ N/m}^2 \). The knife has exactly the same shape of the object contour in the world frame and the knife’s edge and spine curves are obtained in the knife frame.

#### Table 3. Physical parameters in simulation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base of the arm (m)</td>
<td>( b = (0.6, 0.3)^T )</td>
</tr>
<tr>
<td>Link length (m)</td>
<td>( l_1 = 0.6, l_2 = 0.3, l_3 = 0.3 )</td>
</tr>
<tr>
<td>Link mass (kg)</td>
<td>( m_1 = 3.5, m_2 = 1.5, m_3 = 1.0 )</td>
</tr>
<tr>
<td>Link inertia (kg · m²)</td>
<td>( I_{x1} = 0.0817, I_{x2} = 0.0122, I_{x3} = 0.0081 )</td>
</tr>
<tr>
<td>Object contour (m)</td>
<td>( y = -8392x^4 + 1606x^3 - 135x^2 + 5x )</td>
</tr>
<tr>
<td>Knife tip (m)</td>
<td>( p'' = (0, -0.2)^T )</td>
</tr>
<tr>
<td>Knife edge (m)</td>
<td>( y' = 74.0x^4 - 36.15x^3 + 7.09x^2 - 1.78x )</td>
</tr>
<tr>
<td>Knife spine (m)</td>
<td>( y'' = -37.23x^4 + 19.7x^3 - 3.82x^2 + 0.33x )</td>
</tr>
</tbody>
</table>

Notes: Base of the arm is in the world frame, while the knife tip position is in the arm frame. Link inertia is taken at the center of mass. The object contour is taken in the world frame and the knife’s edge and spine curves are obtained in the knife frame.

#### Figure 13. Complete cutting action (simulated, \( t_1 = 0.564 \text{ s}, t_2 = 0.636 \text{ s}, \) and \( t_3 = 1.30 \text{ s} \)) with a 3-DOF planar arm: (a)–(c) desired and actual position and orientation of the arm frame during the pressing phase, (d) knife-board contact force in the touching phase, and (e)–(f) desired and actual knife-board contact position and force in the slicing phase. Same as the experiments shown in Fig. 9, we use \( \theta_{\text{act}} = -0.15 \text{ m/s} \) during pressing and \( \dot{c}_{\text{act}} = 0.15 \text{ m/s} \) in slicing as desired values.
Table 2 which were used in the experiments, except for the three gains $k_{\tau_a} = k_{\tau_c} = 60$ and $b_r = 5$. A spring model is created for the cutting board with stiffness 5000 N/m to calculate $f_{\tau}$ while $f_x$ is set to be 0. Since the cutting board cannot be treated as a rigid environment, a derivative term with gain 0.04 is added in the force controller (39) for the slicing phase. All the three phases of cutting are subjected to force and torque disturbances that are uniformly distributed in the range $[-2.5 \text{ N}, 2.5 \text{ N}]$ and $[-0.25 \text{ N \cdot m}, 0.25 \text{ N \cdot m}]$, respectively.

Shown in Fig. 13(a)–(c) are the actual and desired positions and orientations of the arm frame $A$ during the pressing phase, in which the maximum gaps are 0.002 m, 0.001 m, and 0.005 rad, respectively. The plot in (d) shows that the touching phase lasts for 0.072 s over which the knife-board contact force has dropped from 28 N to 8 N. In (e), the gap between the actual and desired $x$-directional positions of $c$ is almost negligible. In (f), the $y$-component $f_y$ of the knife-board contact force is maintained around 5 N.

7 Discussion and future work

This paper is about how to enable a robotic arm to perform a natural and smooth cutting action. We have sequenced the entire action into three phases (pressing, touching, and slicing), drawing an inspiration from kitchen knife maneuvers by the human hand. Given the action’s complexity and the heavy presence of contacts encountered by the knife (with the object and cutting board), a single policy is clearly inadequate for carrying out the task of cutting robustly. Instead, combinations of control policies are employed sequentially to accommodate specific subgoals and contact constraints for different phases of the action. As demonstrated in our experiments (with the setup in Fig. 8(a)), smooth transitions between these controls ensure a fruit/vegetable to be cut open within 1.0 s (see the submitted video). All the controllers are presented in the task space rather than the joint space, so the knife-board contact constraint is conveniently expressed and inverse kinematics are not needed.

The three-phase cutting strategy can be easily applied to a different robotic arm based on its own model of dynamics. We noticed that, since cutting was carried out in a small operational space, the WAM Arm’s inertial matrix $M$ did not vary much during the action. In fact, additional experiments, conducted with constant $M$ values chosen at the middle of the three phases and the $\theta$ terms removed from (19) and (38) for $\tau_a$ and $\tau_c$, respectively, claimed similar outcomes as those from using the arm’s accurate dynamic model. The situation, however, is expected to be different for a cutting motion with a faster speed in a larger range.

Modeling based on fracture mechanics allows us to separate, from the F/T sensor reading, forces of different sources such as fracture, knife-material contact, and knife-board contact. In particular, we are able to estimate the knife-board contact force and carry out hybrid control during the slicing phase to ensure complete separation of the object.

An immediate extension of this work will be to an object undergoing small deformations from cutting. In addition to fracture and frictional forces, the force that creates deformation of the object is also needed for cutting control. These forces, along with the areas of fracture and contact and the object’s shape and strain energy, can be modeled using the finite element method (FEM). This will be done through “solving” an equation that describes a balance between the work conducted by the knife and the total amount spent on creation of fracture, dissipated through friction, and converted into or from strain energy. A big obstacle will be the high computational cost with an additional level of discretization (of the knife’s motion) needed to model the continuous phenomenon.

For cutting trajectory planning and real-time plan adjustment, it is quite important to efficiently generate reliable force and shape predictions along a hypothesized trajectory. We hope to compensate modeling inaccuracies with force sensing, vision, and improved knife control. In the longer term, we would like to investigate cutting of objects with large deformations and viscosities, where modeling of strain energy and viscous force is essential.

Smoothness of the human hand’s cutting move also comes from its energy efficiency. In our presented robotic cutting strategy, the total amount of energy expended on material fracture and dissipated through friction does not vary too much. Energy expense is dominated by the consumed kinetic energy of the robotic arm. Minimization of this energy could make use of null space control, in case the arm has more than three DOFs in the cutting plane, by using a weighted pseudoinverse in terms of acceleration (Hollerbach and Suh, 1987; Khatib, 1983). Investigation on optimal cutting in the energy sense will inevitably deal with planning of cutting paths, which may very likely lead to improvement or reshaping of the current three-phase strategy to be more humanlike.

A third direction of extension is to have the kitchen knife, instead of rigidly attached to the arm, held by a robotic hand driven by the arm. The system will become more autonomous since the hand can pick up and regrasp the knife. There will be more dexterity because control of the knife is directly done by the hand. More challenges will be faced in the same time. Cutting control will have to take into account higher degrees of freedom, compliance of contact between the knife’s handle and the hand, and (even possibly) finger gaits for adjusting a grasp on the knife. Involvement of a second robotic arm (with a mounted hand) to stabilize and maneuver the object being cut will bring up the challenging issue of arm-arm and arm-hand coordinations, which requires the development of control strategies to allow force interactions between the arms and hand.

At a higher level, we will look at how to plan trajectories to implement different knife skills including chop, slice, dice, roll cut, etc. Realization of a composite skill such as dice needs to tackle the subproblem of cutting in a non-vertical plane (which can be a direct extension to the current work).

8 Acknowledgment

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Notes
1. https://www.youtube.com/watch?v=5TNwh7U1AU
3. https://www.youtube.com/watch?v=lmANsZIZuKQ
4. If a real robotic arm has DOFs out of the cutting plane, the dynamics equation should be recalculated by fixing the corresponding fixed joints.
5. Note that when $n > 3$, on the right hand side of (16) we could add a term $(I - JT^TJ)^{-1}J$ in the null space of $J_T$, where $\zeta$ is set to carry out some lower-level objective such as not exceeding joint angle or torque limits, singularity avoidance, or energy minimization (Hollerbach and Suh, 1987; Siciliano, 1990). The issues of limits and singularity are not of serious concern as arm movements during a cutting action are typically within a small range.
7. There would have been no change to the situation had the knife been moving along the path in the opposite direction.
8. Some preliminary work (Jamdagni and Jia, 2019) was carried out in the authors’ lab.

References
A Computation of fracture and frictional forces for a translating knife

When the knife keeps a constant orientation, all the points on its blade are moving at the same velocity $\mathbf{v}$. Denote $\mathbf{\hat{v}} = \mathbf{v}/\|\mathbf{v}\| = (\hat{v}_x, \hat{v}_y)^T$. Calculation of fracture and frictional forces and torques can be simplified. The integral (10) for the fracture force now has a closed form:

$$f_C = \int_{u_1}^{u_2} \kappa \hat{v} \cdot \left( -\frac{d\beta_y}{du} \right) \mathbf{v} \, du$$

$$= \kappa \left( \hat{v} \cdot -\hat{\beta}_{y,x} \right) \mathbf{v} \bigg|_{u_1}^{u_2}$$

(53)

The integral (11) for the torque due to fracture can be simplified:

$$\tau_C = \int_{u_1}^{u_2} \left( \beta \mathbf{v}(u) - \alpha \right) \times \left( \kappa \hat{v} \cdot \left( -\frac{d\beta_y}{du} \right) \mathbf{v} \right) \, du$$

$$= \kappa \hat{v} \cdot \int_{u_1}^{u_2} \left( \beta \mathbf{v}(u) - \alpha \right) \times \left( -\frac{d\beta_y}{du} \right) \mathbf{v} \, du$$

$$= \kappa \hat{v} \cdot \left( -\hat{\beta}_y \right) \left( \mathbf{a} \times \mathbf{\hat{v}} \right) + \frac{1}{2} \left\{ \int_{u_1}^{u_2} \left( \beta y_{y,y} \mathbf{v}_v^T \right) \, du \right\}$$

$$- \int_{u_1}^{u_2} \left[ \mathbf{v} \cdot \left( \beta_y \right) \right] \, du$$

(54)

---


Under Green’s theorem, the area $A$ of the knife-material contact can be calculated along its boundary $\partial \Omega$ as follows:

$$A = \iint_{\Omega} dx \, dy = \oint_{\partial \Omega} x \, dy$$

$$= \int_{u_1}^{u_2} \beta_{\theta, x} \, d\beta_{\theta, y} + \int_{r_2}^{r_3} \sigma_x \, d\sigma_y + \int_{q_3}^{q_4} \gamma_{\theta, x} \, d\gamma_{\theta, y}$$

$$+ \int_{r_4}^{r_2} \sigma_x \, d\sigma_y.$$  \hspace{1cm} (55)

The frictional force (12) and its generated torque (13) are

$$\mathbf{f}_F = -2\mu PA \hat{v},$$  \hspace{1cm} (56)

$$\tau_F = -2\mu P \iint_{\Omega} \left( \begin{array}{c} x \\ y \end{array} \right) - \mathbf{a} \times \hat{v} \, dxdy$$

$$= 2\mu P \mathbf{a} \times \hat{v} - 2\mu P \iint_{\Omega} \mathbf{a} \times \hat{v} \, dxdy$$

$$= 2\mu P \mathbf{a} \times \hat{v} - 2\mu P \iint_{\Omega} (xv_y - yv_x) \, dxdy$$

$$= 2\mu P \mathbf{a} \times \hat{v} - 2\mu P \iint_{\partial \Omega} \left( \frac{x^2}{2} - yv_y - xvy_x \right) \, dy.$$  \hspace{1cm} (57)

The integrals (53)–(57) have closed forms when the curves $\beta_\theta$ and $\gamma_\theta$ of the knife’s edge and spine and $\sigma$ bounding the object’s cross section are parameterized using polynomials. Such parameterizations are easy to generate, as already done in our experiments in Section 6.