Randomized Linear Programming

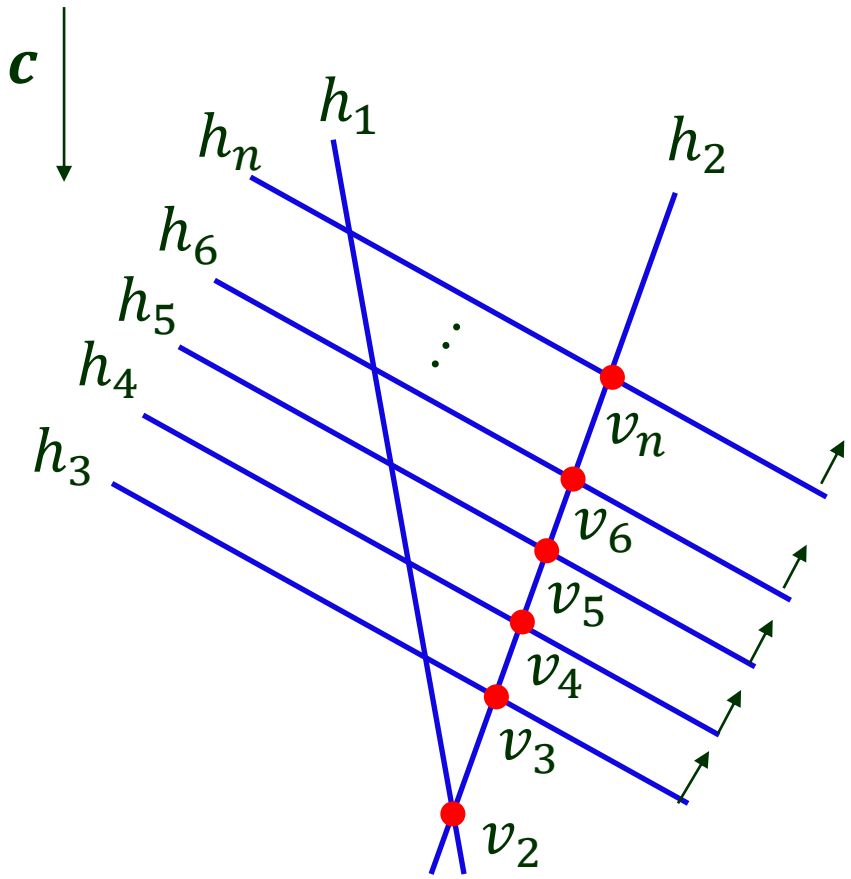
Outline:

I. Generation of a random permutation

II. *Backward analysis* of running time

III. Unbounded linear program

How to Avoid Worst Case?



Pick a *random* ordering of the set H of half-planes.

Bad order still leads to $O(n^2)$.

Most orders lead to a faster algorithm.

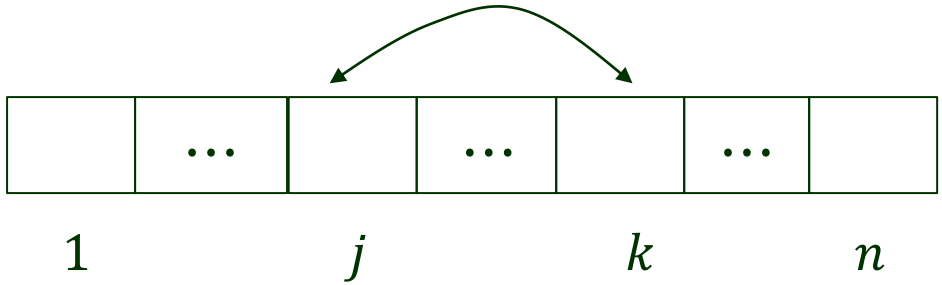
Algorithm Modification

⋮

$h_1, h_2 \leftarrow$ certificate half-planes
$v_2 \leftarrow l_1 \cap l_2$
generate a random permutation of $h_3, h_4, \ldots, h_n$
for $i \leftarrow 3$ to n
    do if $v_{i-1} \in h_i$
        then $v_i \leftarrow v_{i-1}$
        else $v_i \leftarrow$ point p on l_i that maximizes f
                subject to $h_1, h_2, \ldots, h_{i-1}$
            if p does not exist
                then LP is infeasible
return v_n

I. Random Permutation

Array $A[1..n]$



for $k \leftarrow n$ down to 2

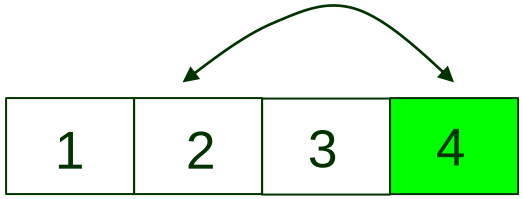
do $j \leftarrow$RANDOM(k)

$A[k] \leftrightarrow A[j]$

// returns a random integer between
//1 and k under uniform distribution.

$O(n)$

How Does It Work?



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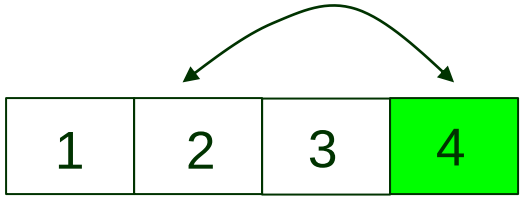

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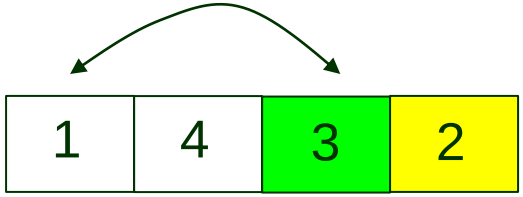

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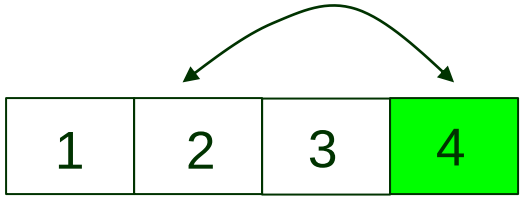

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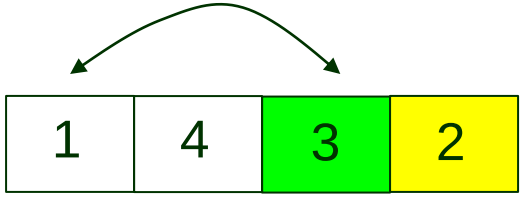

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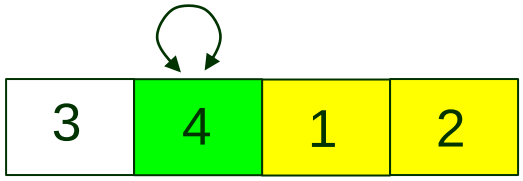

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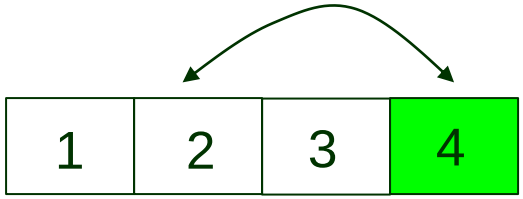

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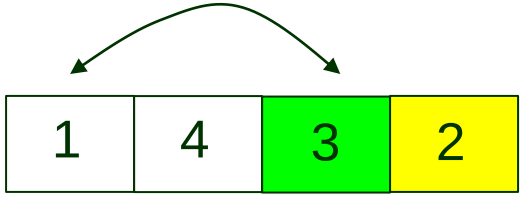

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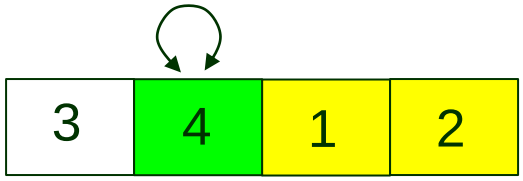

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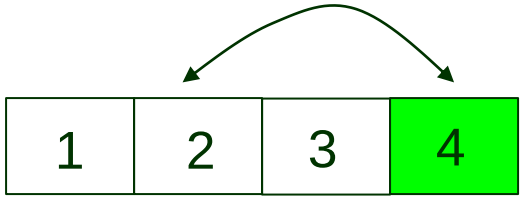

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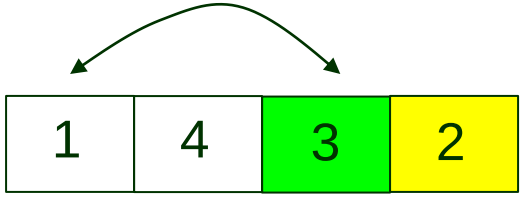

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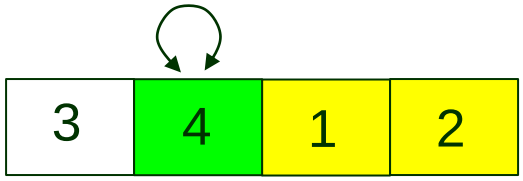

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Every permutation is equally likely to be generated.

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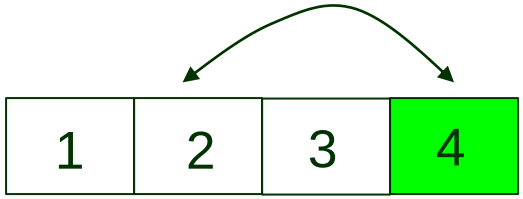

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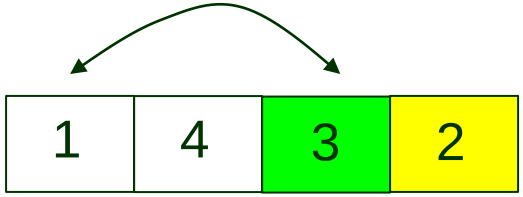

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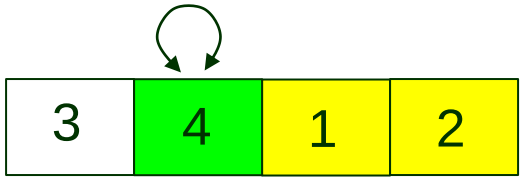

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- ♣ $(n-2)!$ permutations of $h_3, h_4, \ldots, h_n$
- ♣ Average over all of them (expected time).

Start of Analysis

- UNBOUNDEDLP takes time $O(n)$ – to be shown.
- Random permutation takes time $O(n)$.

```
if UNBOUNDEDLP reports that LP is infeasible
  then return // does not detect all infeasible scenarios
  else if LP is unbounded  // to be discussed
      then return a ray along which it is unbounded
      else
        h_1, h_2 ← certificate half-planes
           v_2 ← l_1 ∩ l_2
        generate a random permutation of h_3, h_4, …, h_n
        for i ← 3 to n
            do if v_{i−1} ∈ h_i
                  then v_i ← v_{i−1}
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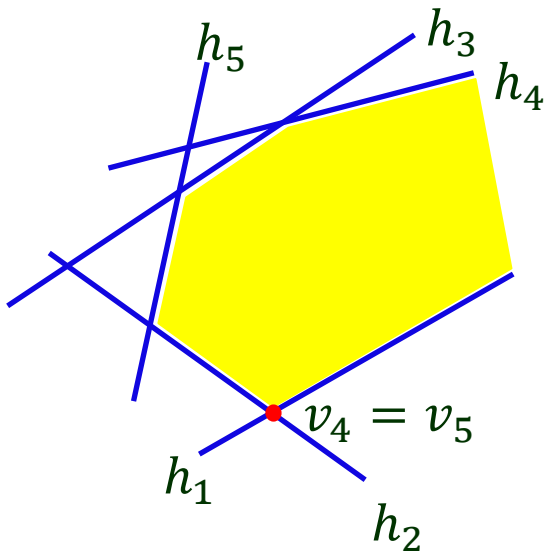
Bounding the average time for solving all 1D LPs.

Random Variable

For stage i

$$X_i = \begin{cases} 0 \text{ if } v_{i-1} \in h_i \\ \\ \\ \\ \end{cases}$$



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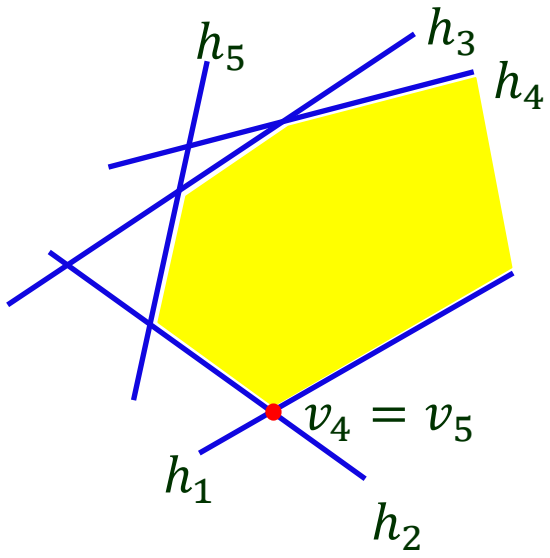

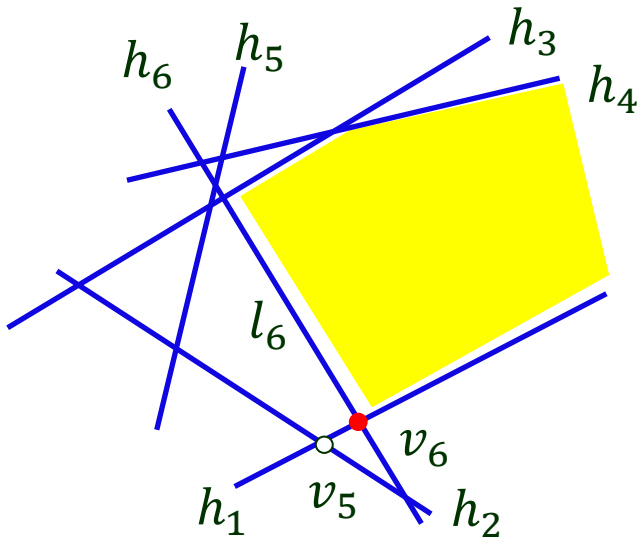

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Total time over all half-planes $h_3, h_4, \ldots, h_n$.

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Too complicated even if possible!

Backward Analysis

Look at the algorithm *backwards*.

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Assume it has finished with the optimal vertex v_n found.

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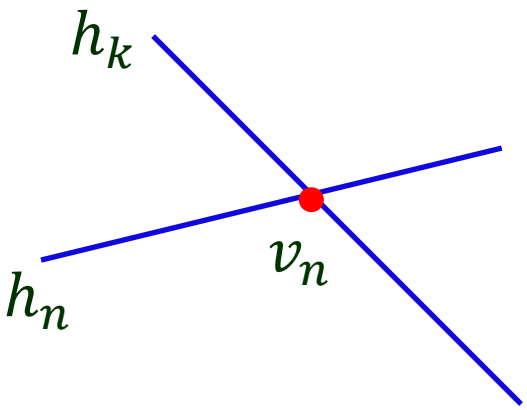

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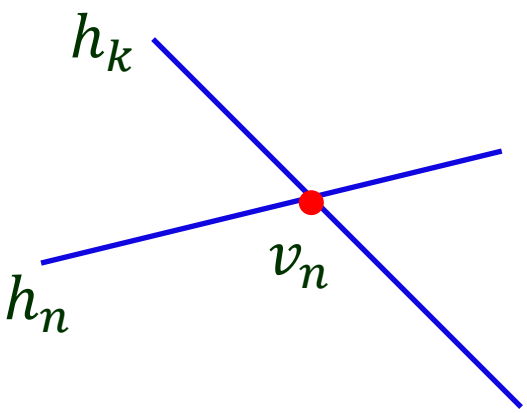

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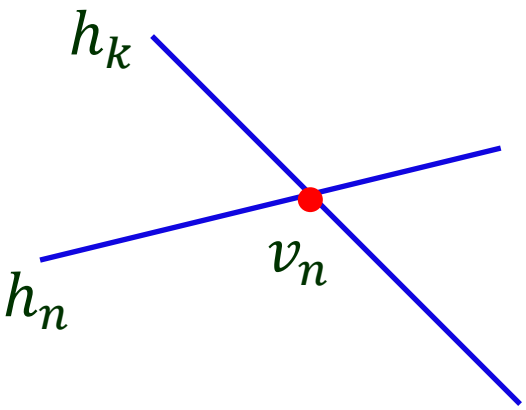


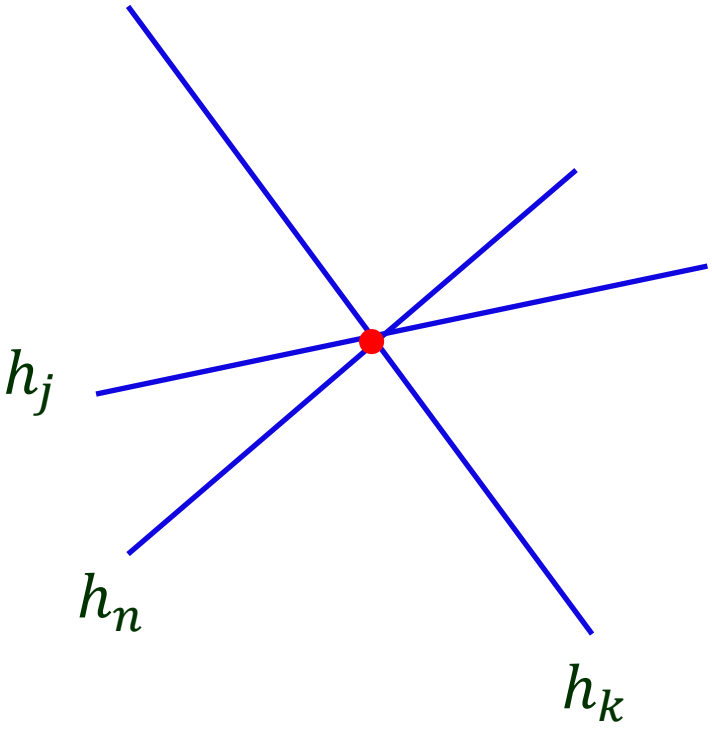
$\Longrightarrow$ h_n defines v_n

h_n a random element of $\{h_1, h_2, \ldots, h_n\}$

$\Longrightarrow$ Pr{h_n is one of v_n's only two defining half-planes}

$$\leq \frac{2}{n}$$

Why $\leq \frac{2}{n}$?



> 2 half-planes may have boundary lines through v_n.

- v_n already existed before addition of h_n.
  ($v_{n-1} \neq v_n$ and $v_{n-1} \notin h_n$)

- removal of h_n does not affect the vertex.

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To bound $E(X_i)$, we fix the set R_i of first i half-planes.

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Thus,

$$E(X_i) \leq \frac{2}{i}$$

Expected Running Time

for solving an LP is

$$\sum_{i=3}^{n} O(i) \cdot \frac{2}{i} = O\left(\sum_{i=3}^{n} \frac{2i}{i}\right)$$

$$= O(n)$$

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Theorem 2D linear programming with n constraints is solvable in $O(n)$ expected time and $O(n)$ storage.

III. Unbounded Linear Program

UNBOUNDEDLP$(H, \boldsymbol{c})$ yields

- a ray $\in \bigcap_{h \in H} h = R$ if LP is unbounded
- $h_1, h_2 \in H$ such that $(\{h_1, h_2\}, \boldsymbol{c})$ is bounded, otherwise.

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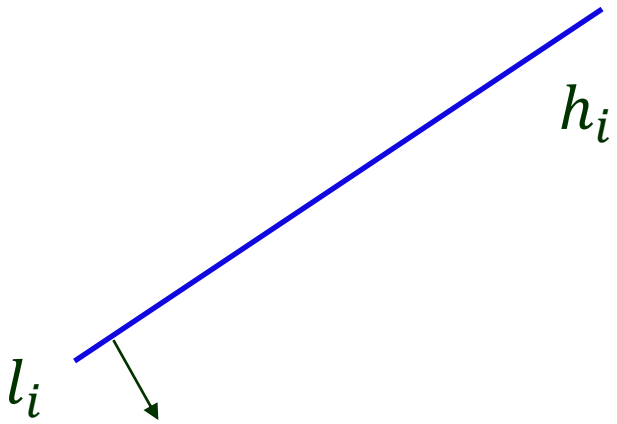

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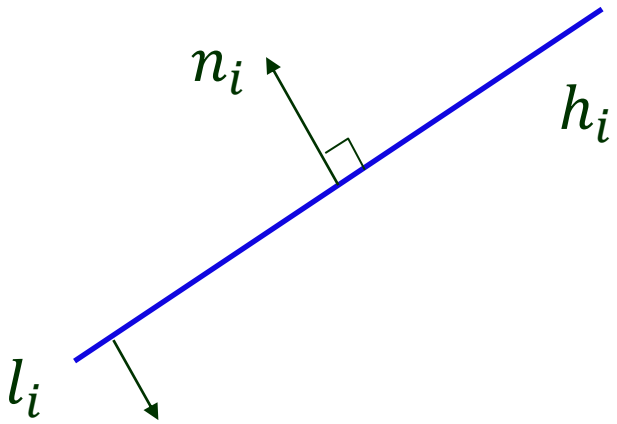

n_i: outward normal

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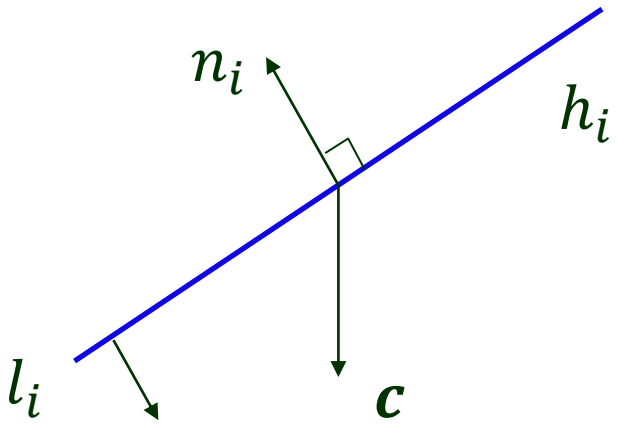

n_i: outward normal

III. Unbounded Linear Program

UNBOUNDEDLP$(H, \boldsymbol{c})$ yields

- a ray $\in \bigcap_{h \in H} h = R$ if LP is unbounded
- $h_1, h_2 \in H$ such that $(\{h_1, h_2\}, \boldsymbol{c})$ is bounded, otherwise.

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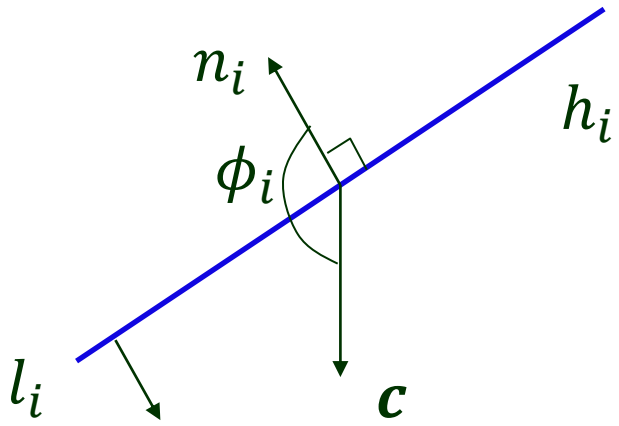

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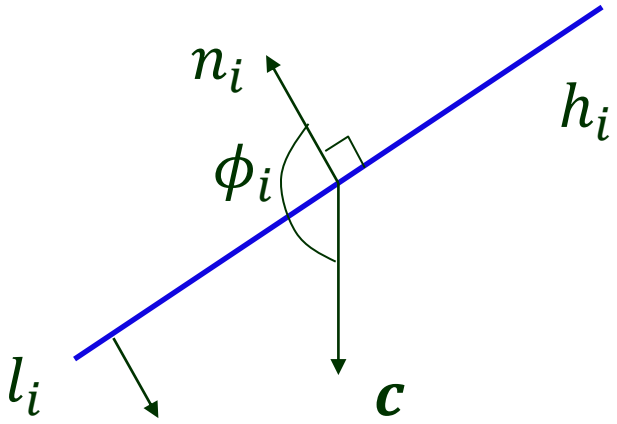
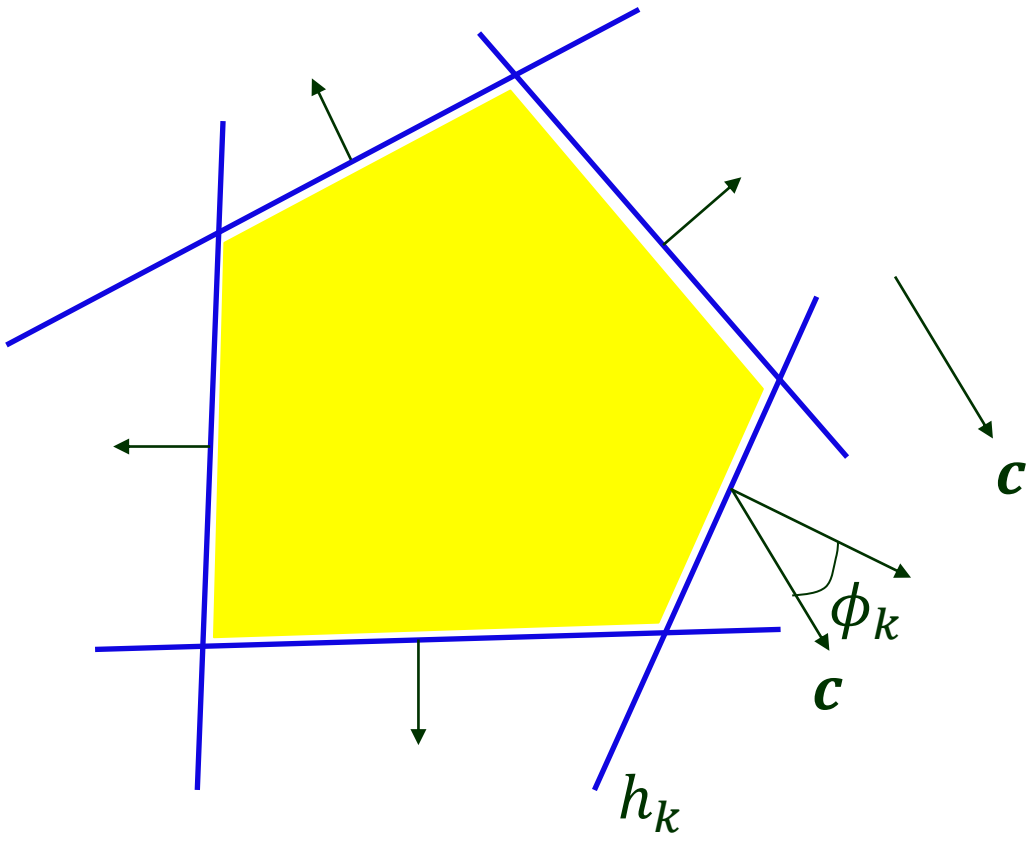


n_i: outward normal

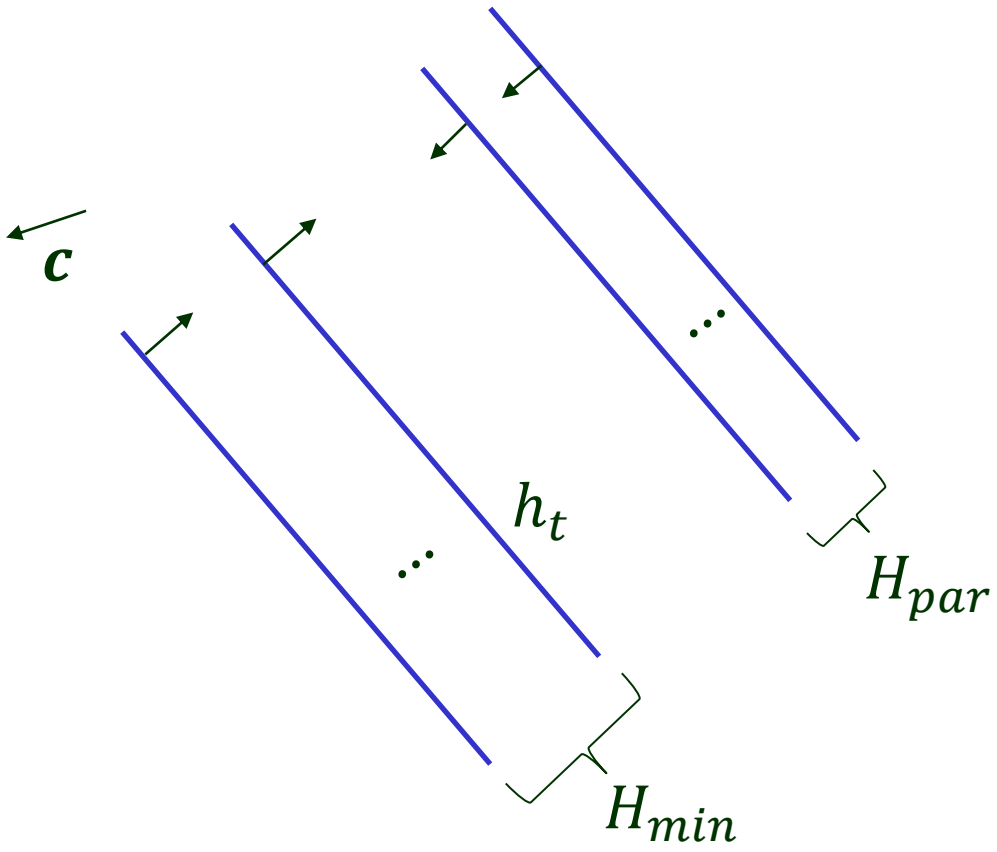
$\phi_i \in [0, \pi]$

Minimum Angle

$$\phi_k = \min_{1 \le i \le n} \phi_i$$



Initial Slab



$$H_{min} = \{h_j \in H \mid n_j = n_k\}$$

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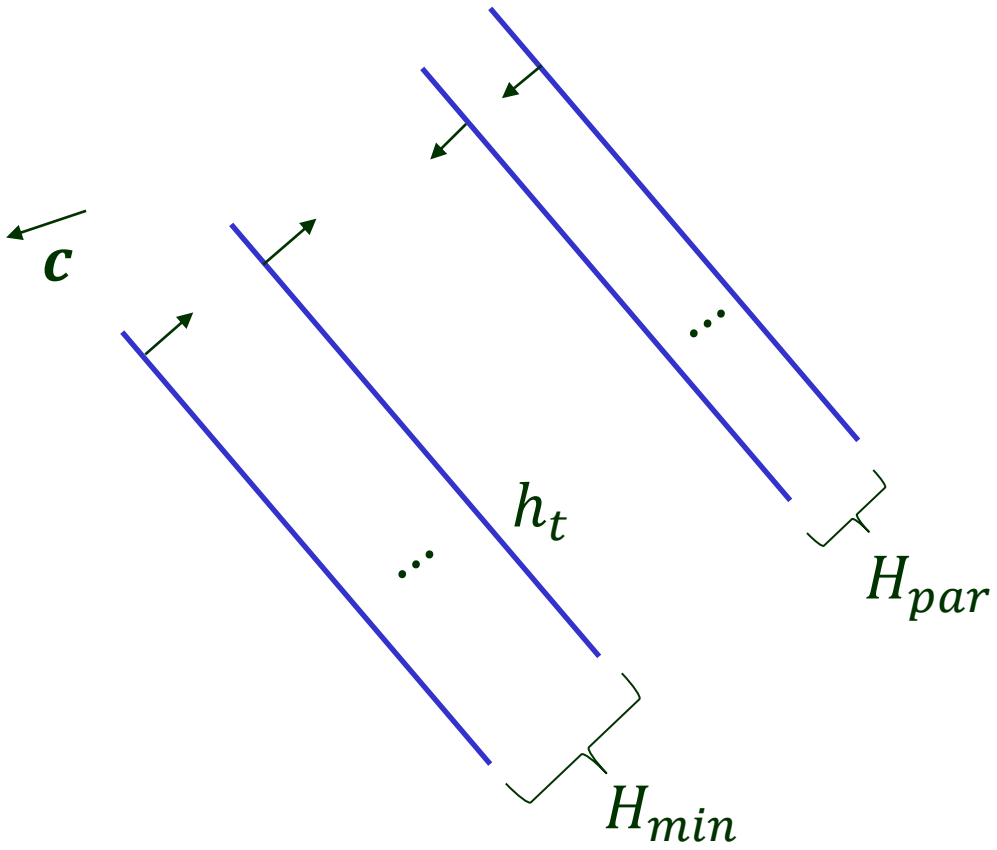
Parallel boundary lines.

Initial Slab



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$$S = \bigcap (H_{min} \cup H_{par})$$

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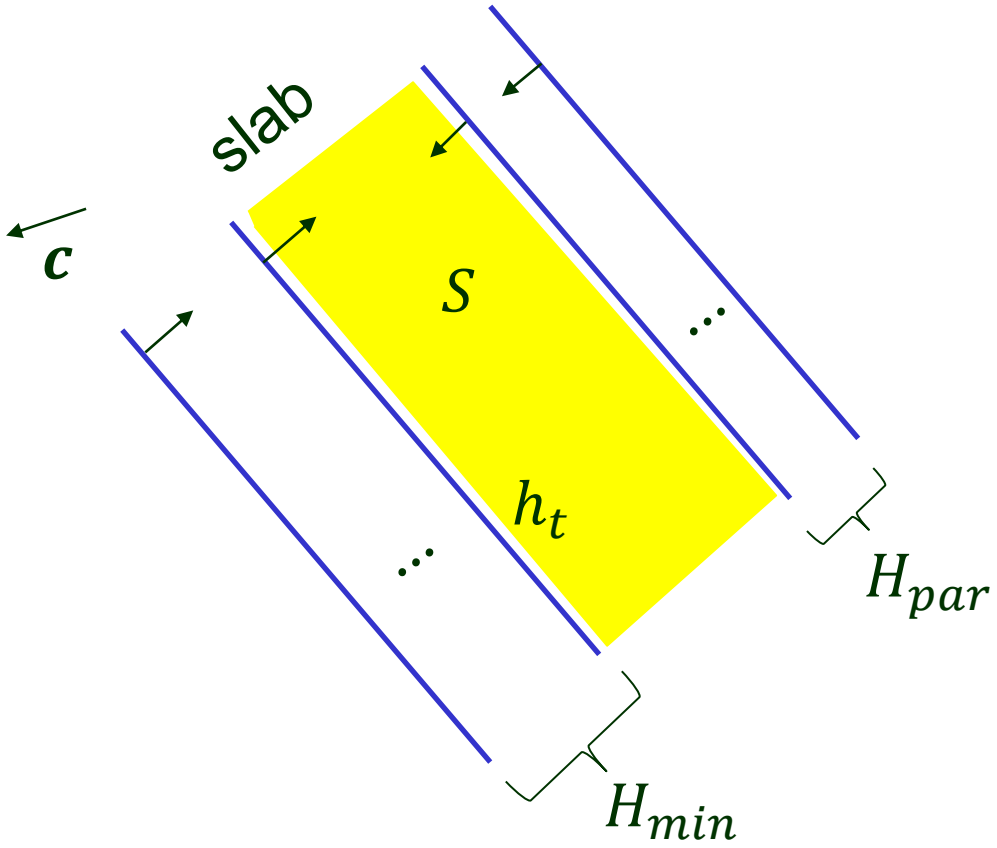

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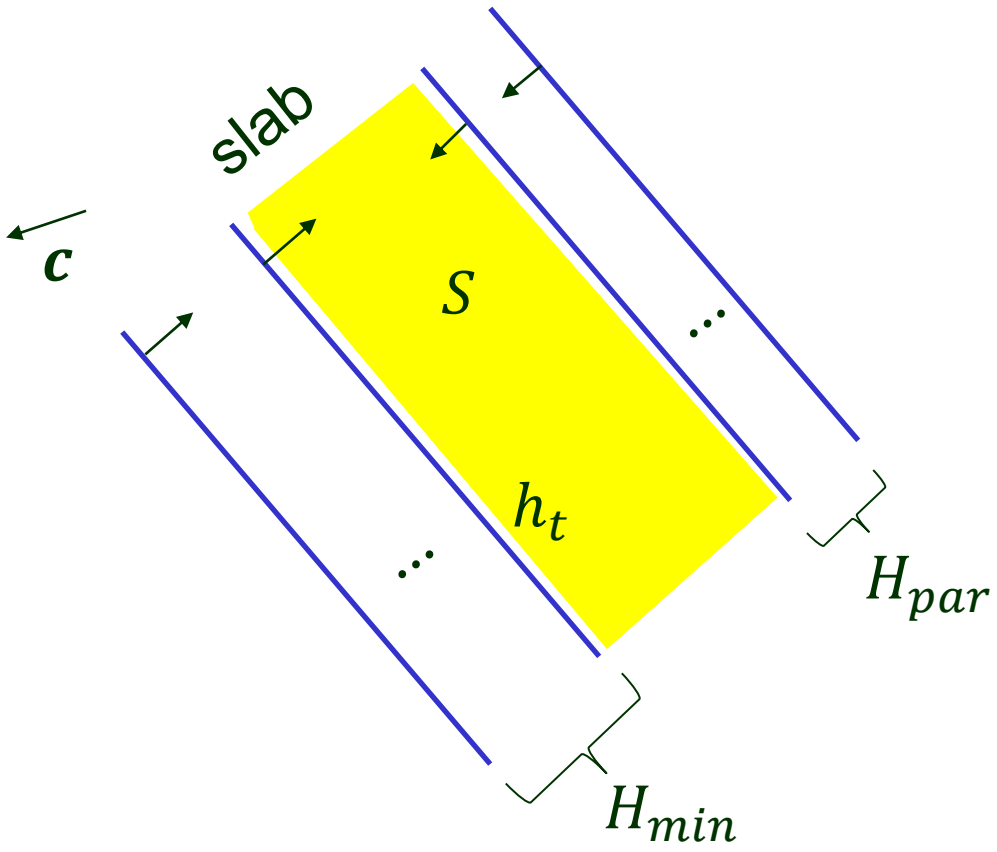

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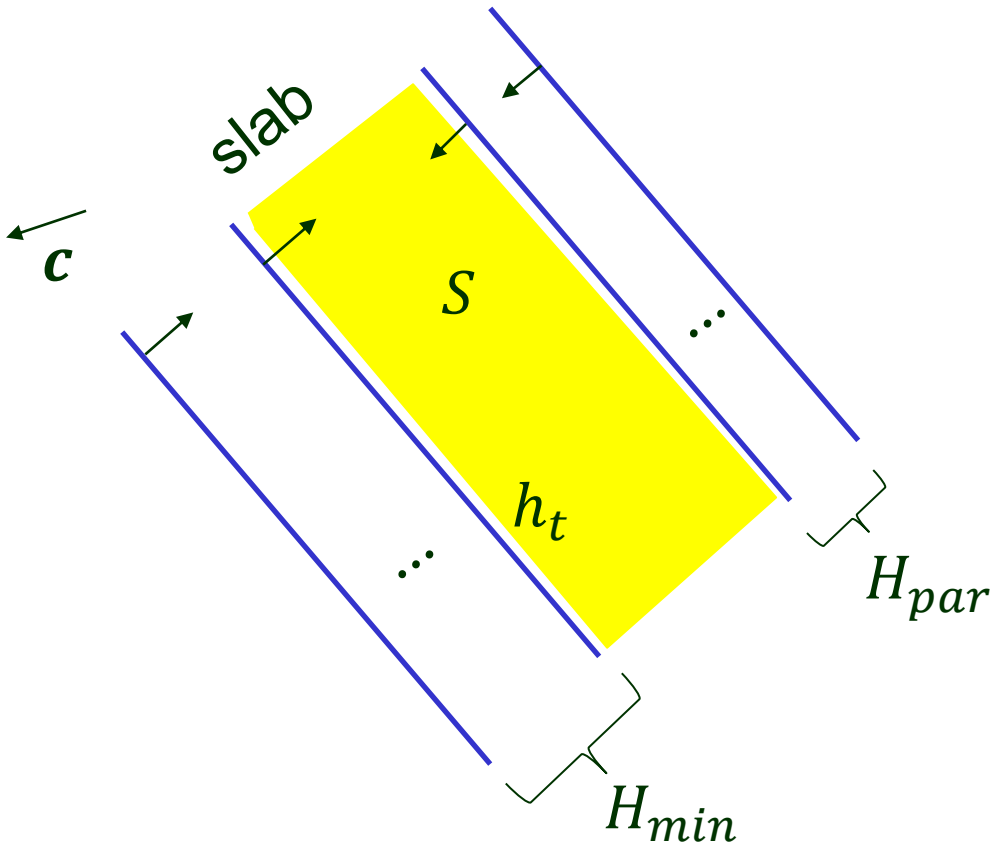

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Parallel boundary lines.

Compute

$$S = \bigcap (H_{min} \cup H_{par})$$

$O(n)$ time

$S = \emptyset \implies$ LP infeasible!

Non-Empty Slab ($S \neq \emptyset$)

S bounded by some line l_t with $h_t \in H_{min}$.

Assumption At least one half-plane not in H_{min} or H_{par}.

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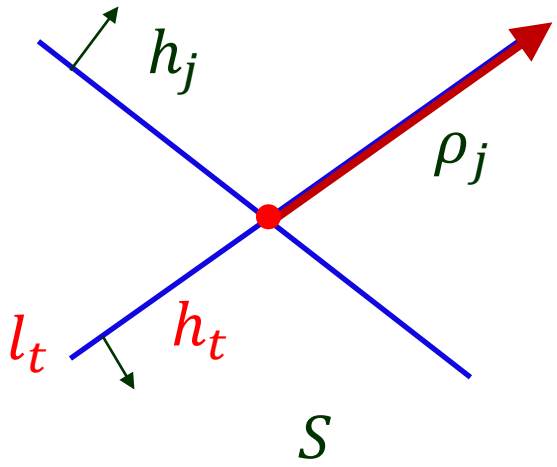
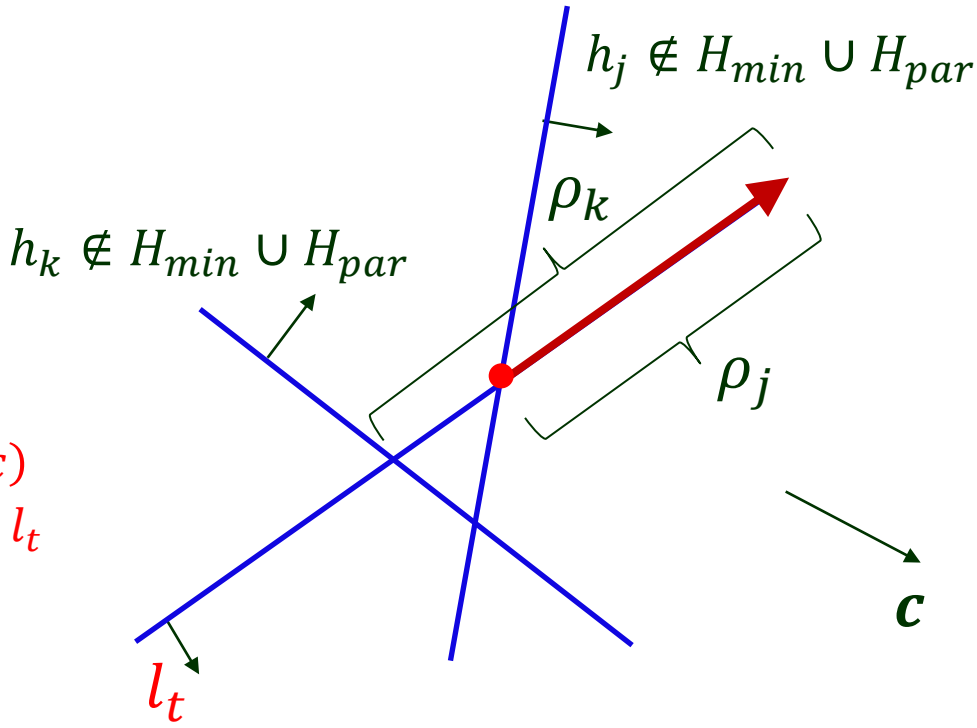


Case 1: Unbounded LP

$\rho_j = l_t \cap h_j$ unbounded in $\boldsymbol{c}$ for all $1 \leq j \leq n, h_j \notin H_{min} \cup H_{par}$.



Lemma 1 If $\rho_j = l_t \cap h_j$ is unbounded in $\boldsymbol{c}$ for every $h_j \notin H_{min} \cup H_{par}$, then (H, c) is unbounded along a ray contained in l_t which can be computed in $O(n)$ time.

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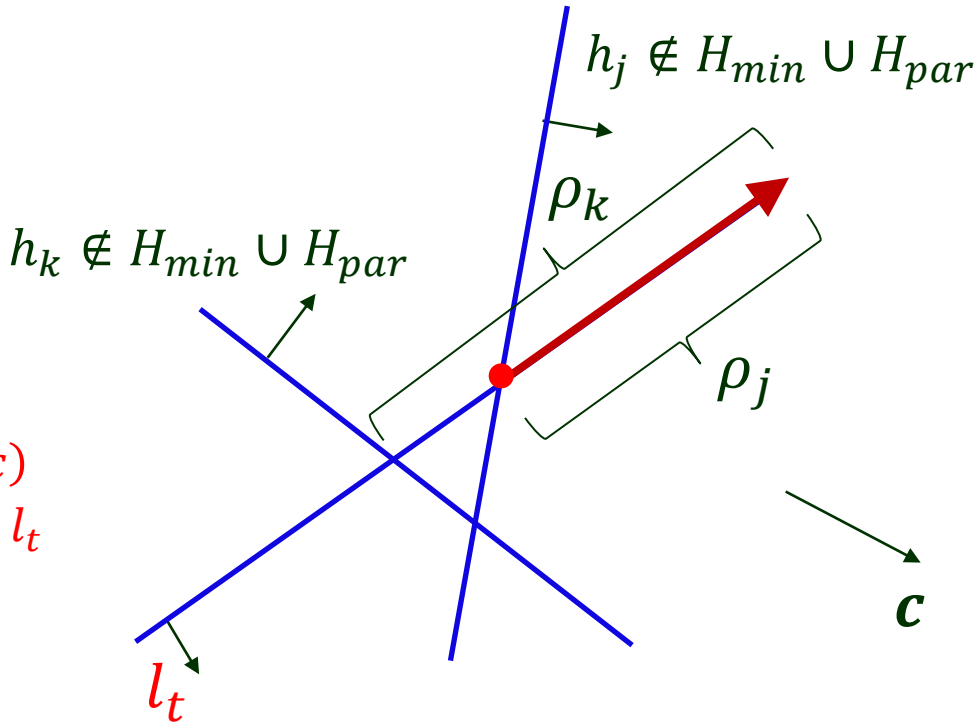
All ρ_j lie on l_t.



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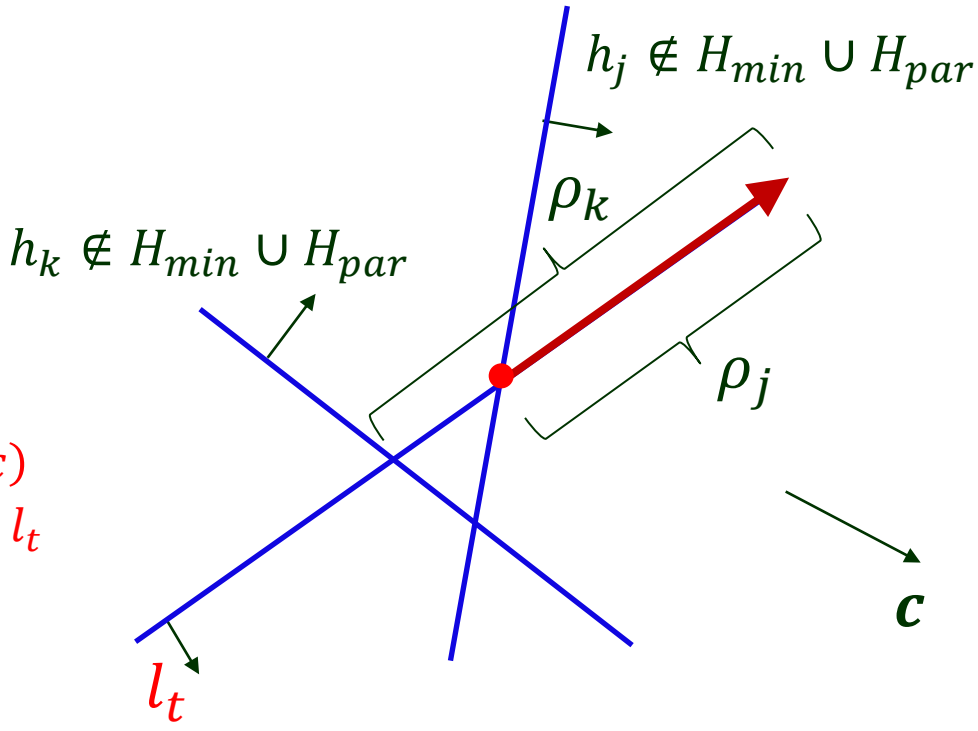
$$\bigcap_{h_j \notin H_{min} \cup H_{par}} \rho_j$$



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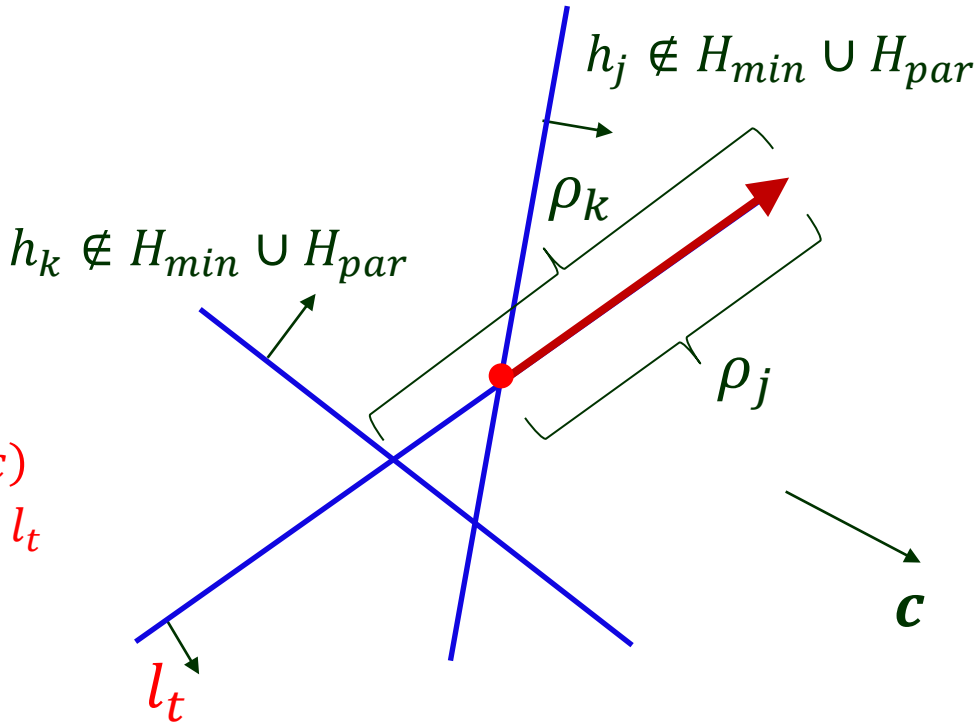

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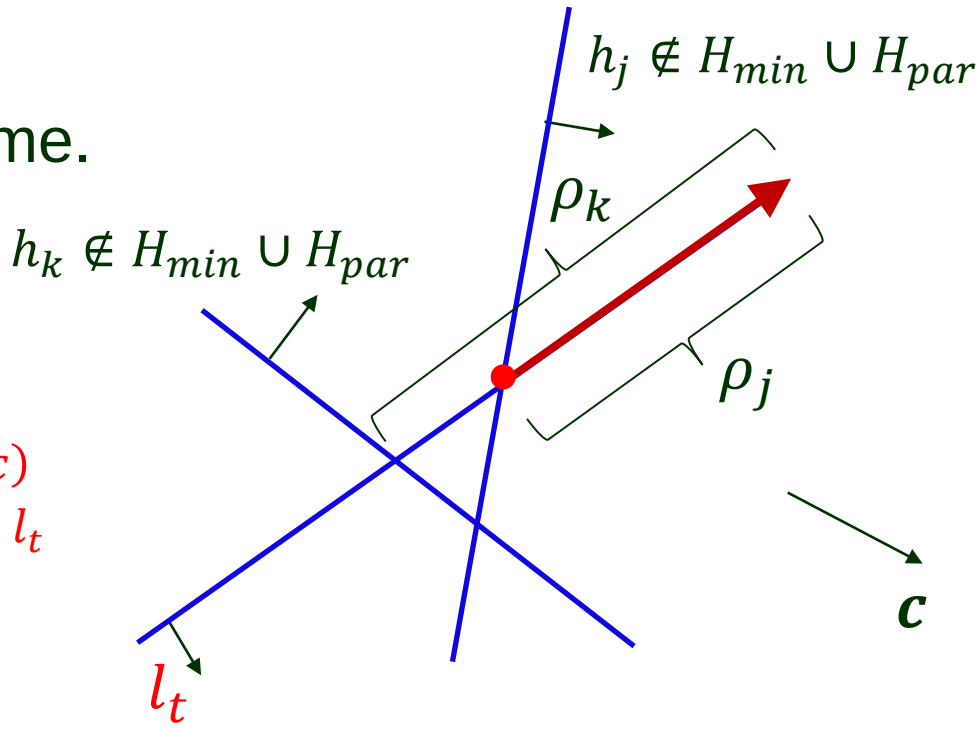
Ray intersection in $O(n)$ time.



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Case 2: Bounded LP

Boundary line l_t with $h_t \in H_{min}$ bounds S.

Lemma 2 If $\rho_j = l_t \cap h_j$ is bounded in $\boldsymbol{c}$ for some $h_j \notin H_{min} \cup H_{par}$, $1 \leq j \leq n$, then LP$\left(\{h_t, h_j\}, c\right)$ is bounded.

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Proof By contradiction. Assume that LP$\left(\{h_t, h_j\}, c\right)$ is unbounded.

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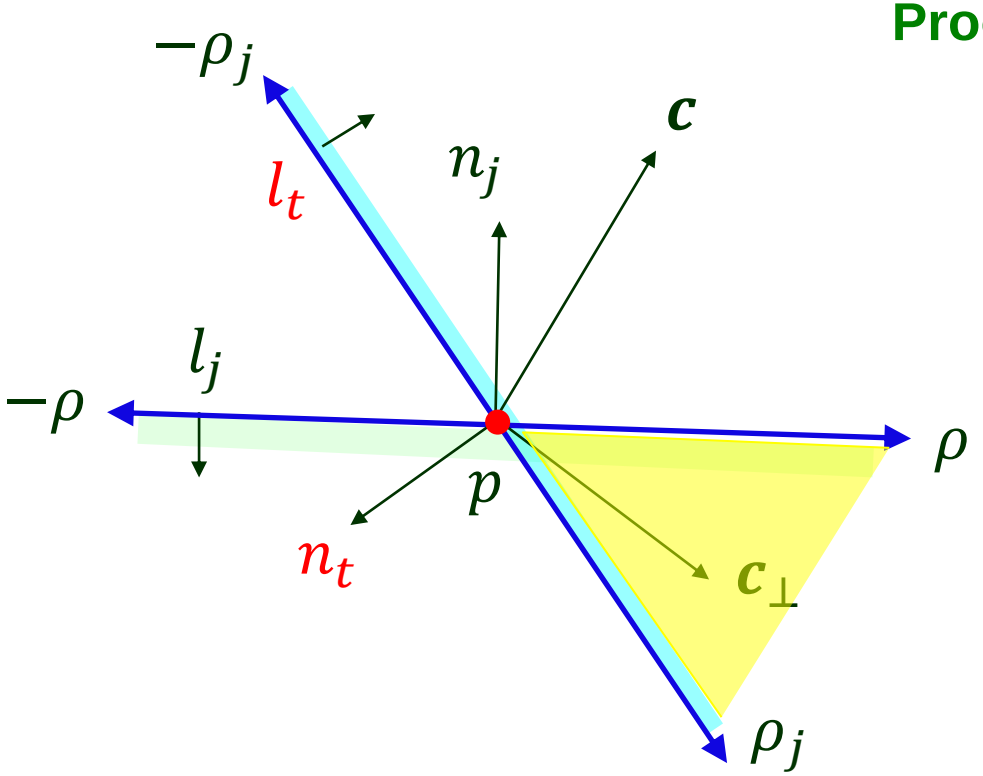

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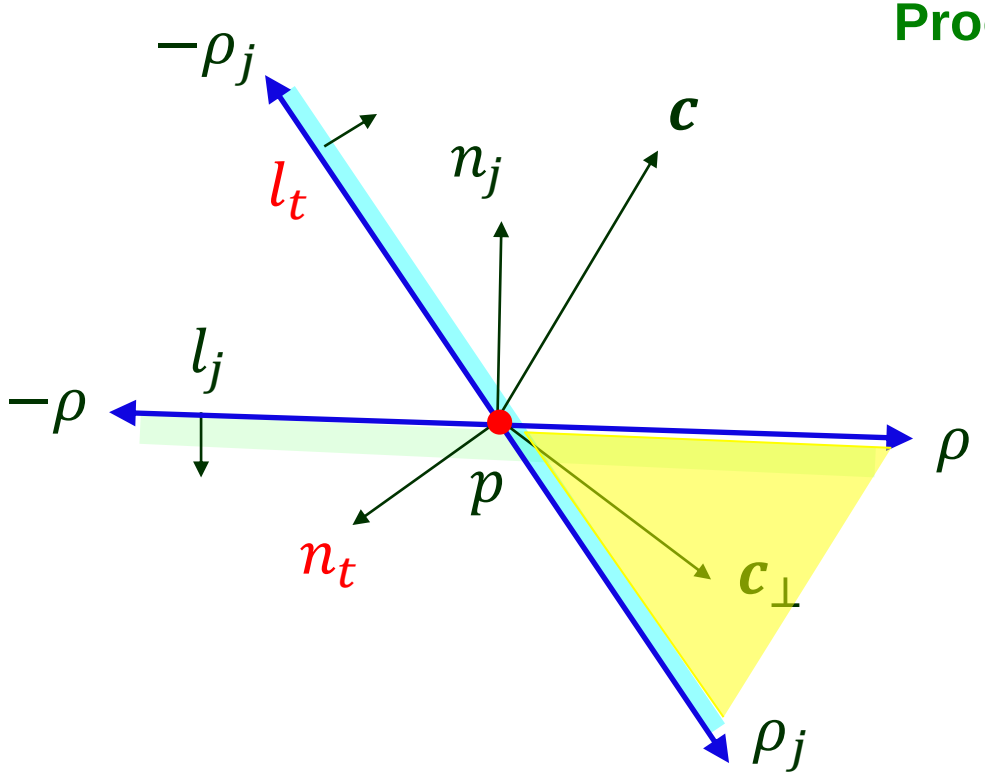

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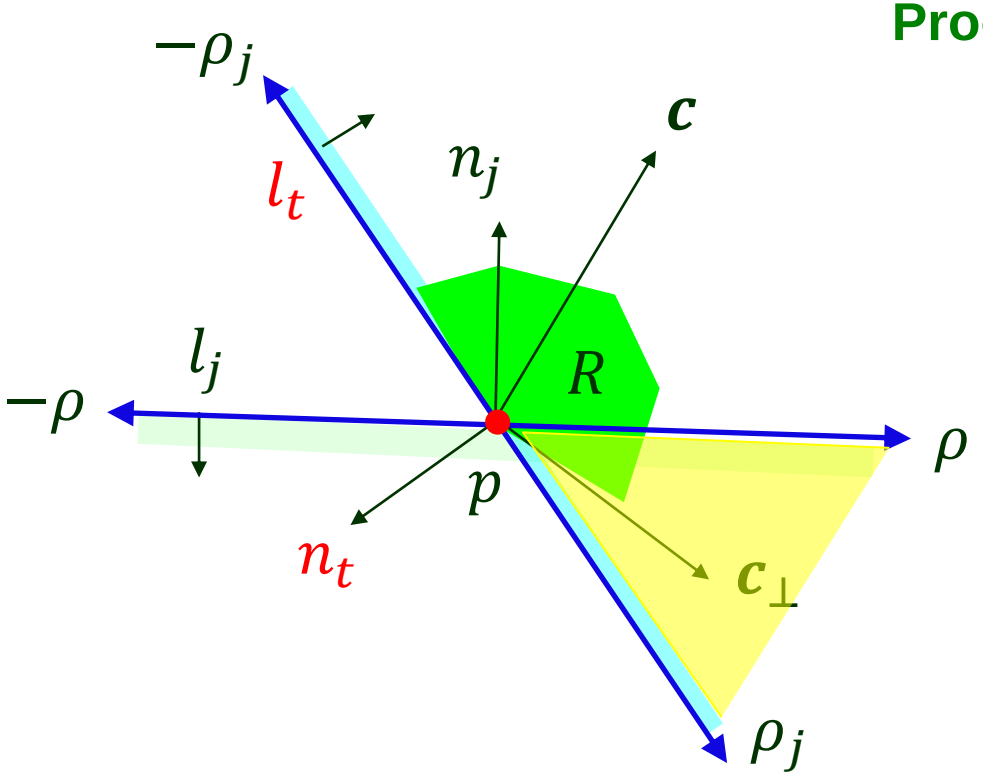

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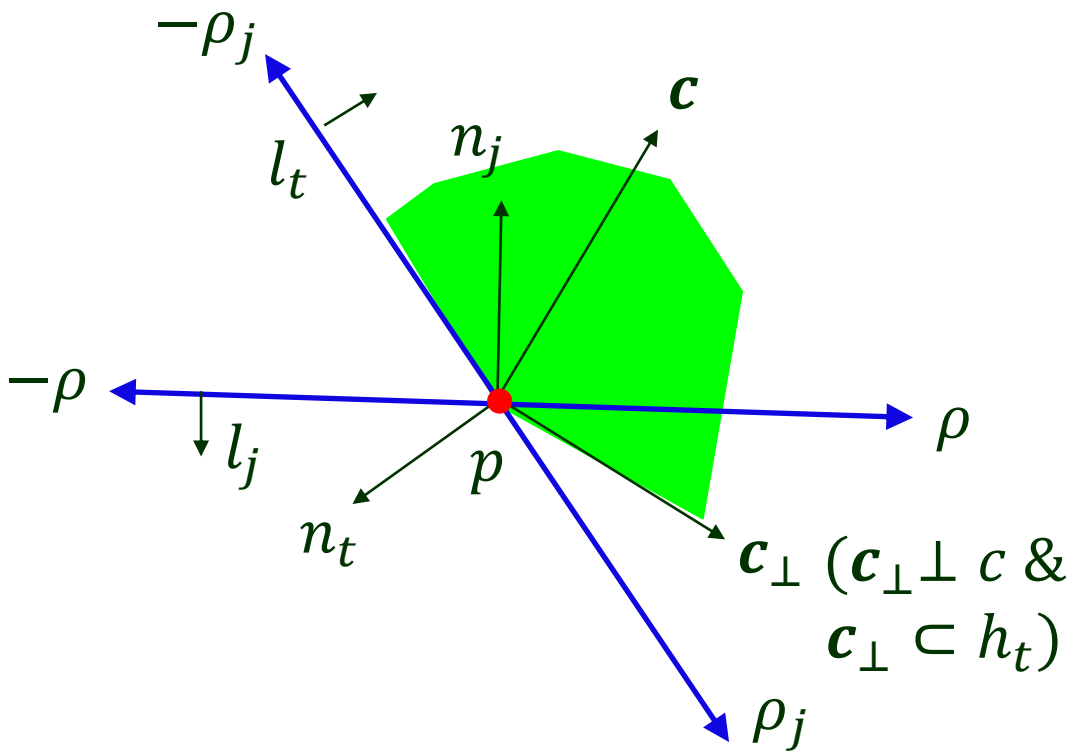
⇩

ρ must be in the region R between $-\rho_j$ and $\boldsymbol{c}_\perp$ ($\boldsymbol{c}_\perp \perp \boldsymbol{c}$ & $\boldsymbol{c}_\perp \subset h_t$) that is contained in h_t.

Contradiction to Choice of h_t

ρ in the region between $\boldsymbol{c}_\perp$ and $-\rho_j$.



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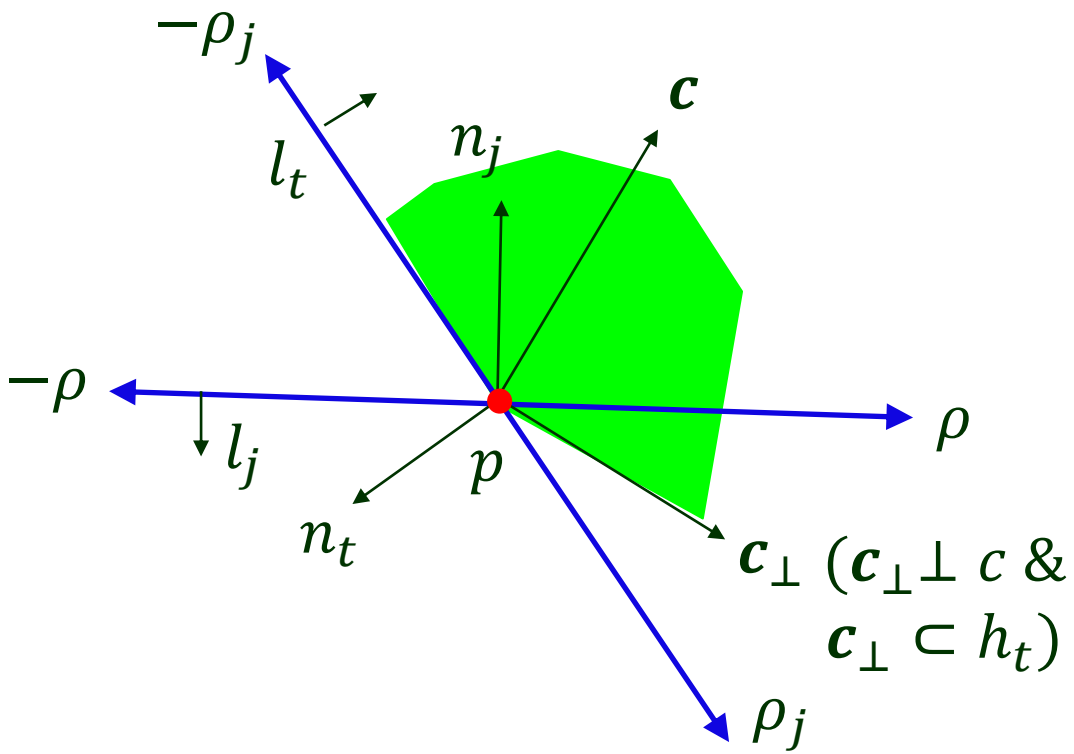
ρ, $\boldsymbol{c}_\perp$, $-\rho_j$ becomes n_j, $\boldsymbol{c}$, n_t, respectively.



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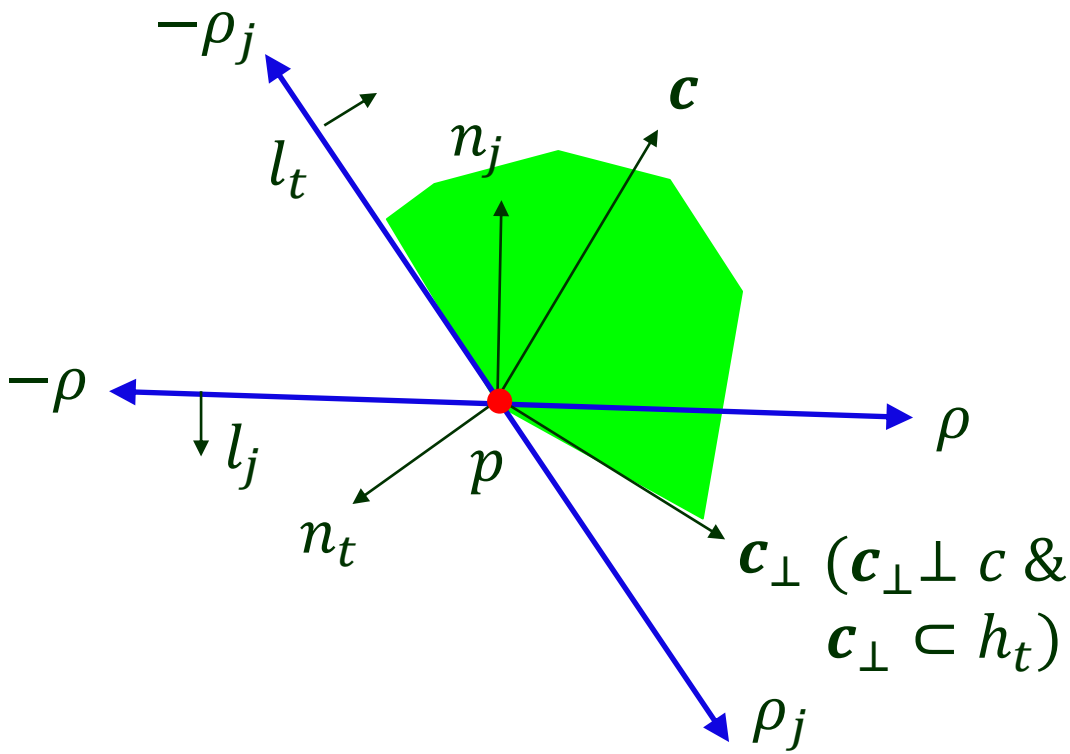
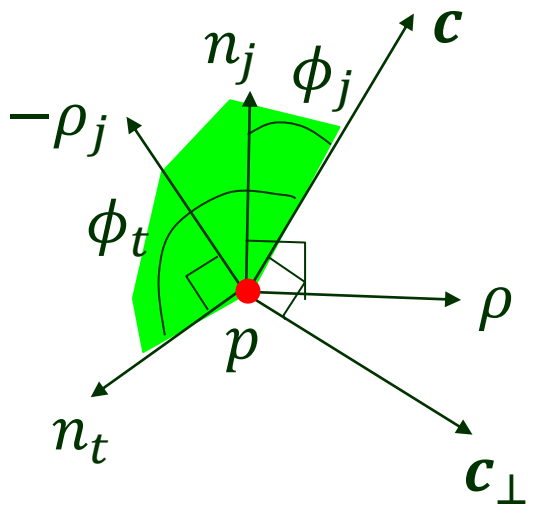
n_j must be in the region between $\boldsymbol{c}$ and n_t (note $n_j \neq n_t$).





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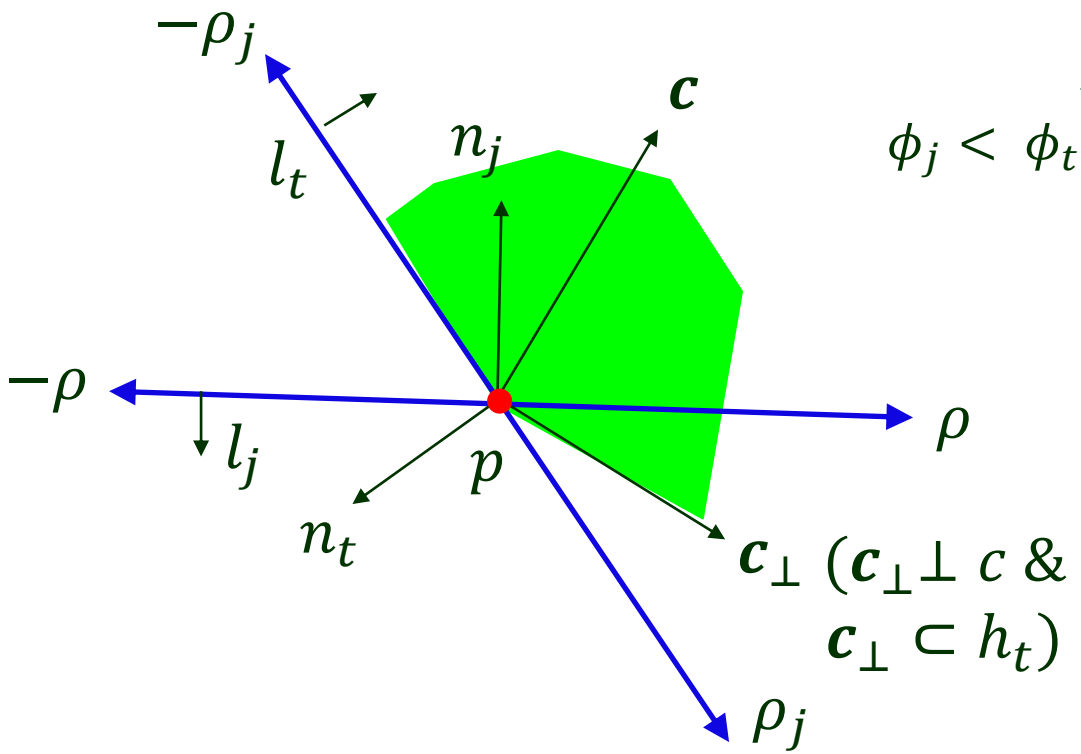

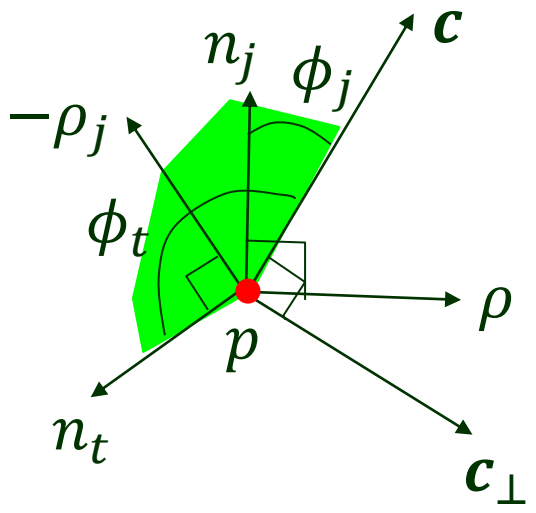

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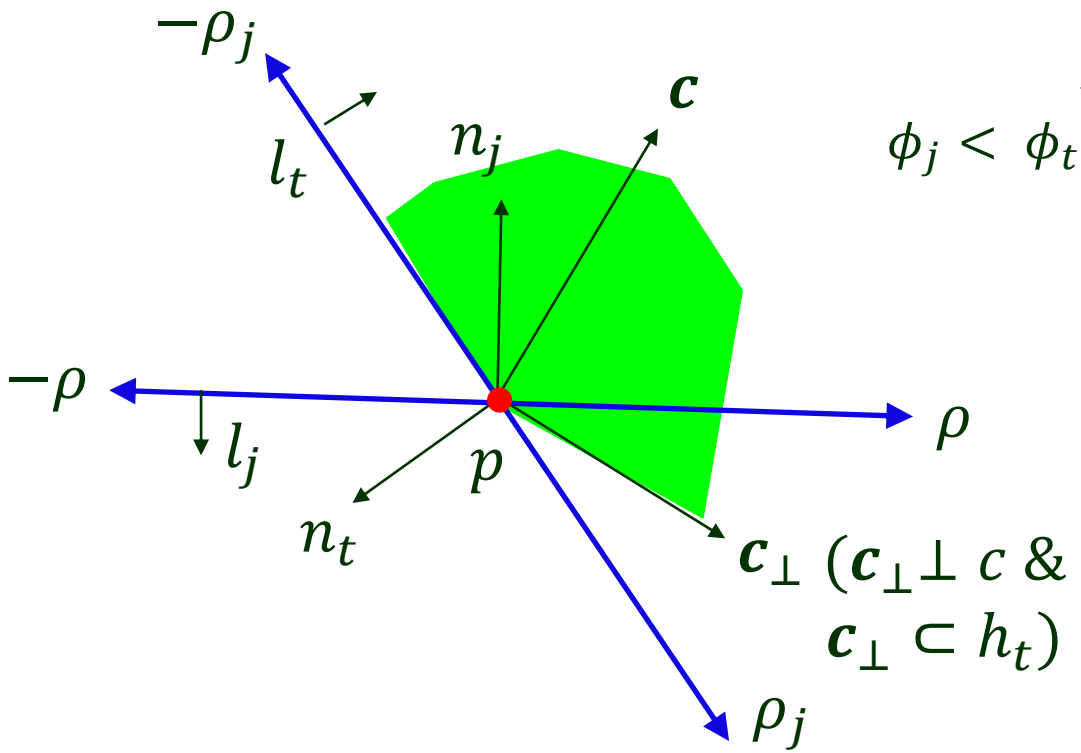
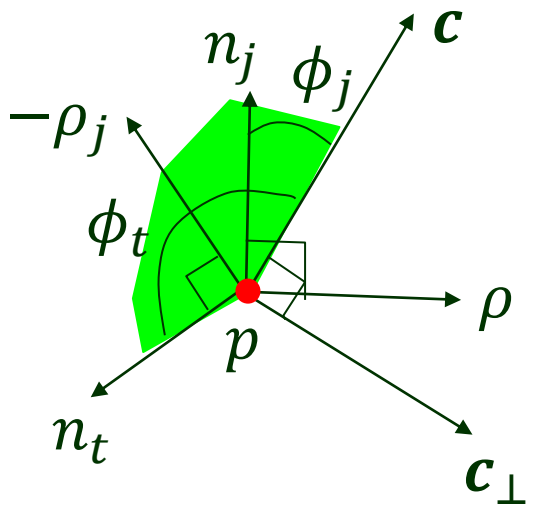
$\phi_j < \phi_t = \min_{1 \le i \le n} \phi_i$ Contradiction!





UNBOUNDEDLP(H, $\boldsymbol{c}$)

Compute ϕ_i for all $1 \le i \le n$

Choose k such that $\phi_k = \min_{1 \le i \le n} \phi_i$

$H_{min} \leftarrow \{h_j \in H \mid n_j = n_k\}$

$H_{par} \leftarrow \{h_j \in H \mid n_j = -n_k\}$

$\widehat{H} \leftarrow H \setminus (H_{min} \cup H_{par})$

$S \leftarrow \cap\, (H_{min} \cup H_{par})$

```
if S = ∅
    then LP(H,c) is infeasible
    else h_t ∈ H_min that bounds S
        if ρ_j = l_t ∩ h_j bounded for some h_j ∈ Ĥ
            then LP(H,c) is bounded // it may still be infeasible
            else LP(H,c) is unbounded along the ray l_t ∩ (∩ Ĥ)
```

Correctness and Running Time

Lemma 3 UNBOUNDEDLP(H, $\boldsymbol{c}$) decides whether a 2D LP with n constrains is unbounded in $O(n)$ time, and if so, returns a ray in the feasible region unbounded in $\boldsymbol{c}$.