Linear Programming

Outline:

I. Introduction to linear programming (LP)

II. Incremental solution to LP in 2D

The Diet Problem*

How much money to spend in order to get what Polly needs every day?

- energy (2,000 kcal)
- protein (55 g)
- calcium (800 mg)

* V. Chvatal. *Linear Programming.* W. H. Freeman and Company, 1983.

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Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (cents)
Oat meal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

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Daily Serving Limits

	Servings at most per day
Oatmeal	4
Chicken	3
Eggs	2
Milk	8
Cherry pie	2
Pork with beans	2

Task: Design the *most economical* menu.

Formulating the Problem

x_1: servings of oatmeal

x_2: servings of chicken

x_3: servings of eggs

x_4: servings of whole milk

x_5: servings of cherry pie

x_6: servings of pork with beans

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subject to

and

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Servings-per-day limits

and

$$\begin{aligned} \text{energy} \quad & 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \ge 2000 \\ \text{protein} \quad & 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \ge 55 \\ \text{calcium} \quad & 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \ge 800 \end{aligned}$$

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Linear Program!

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Linear Programming (LP)

Max $c_1 x_1 + c_2 x_2 + \cdots + c_d\ x_d$

subject to

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1d} x_d \leq b_1$$
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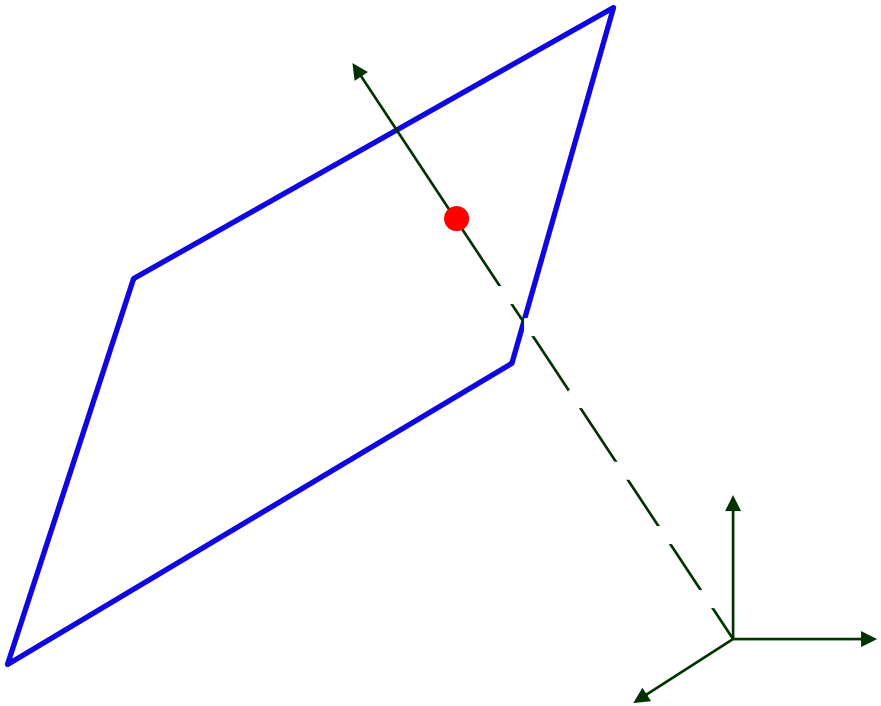

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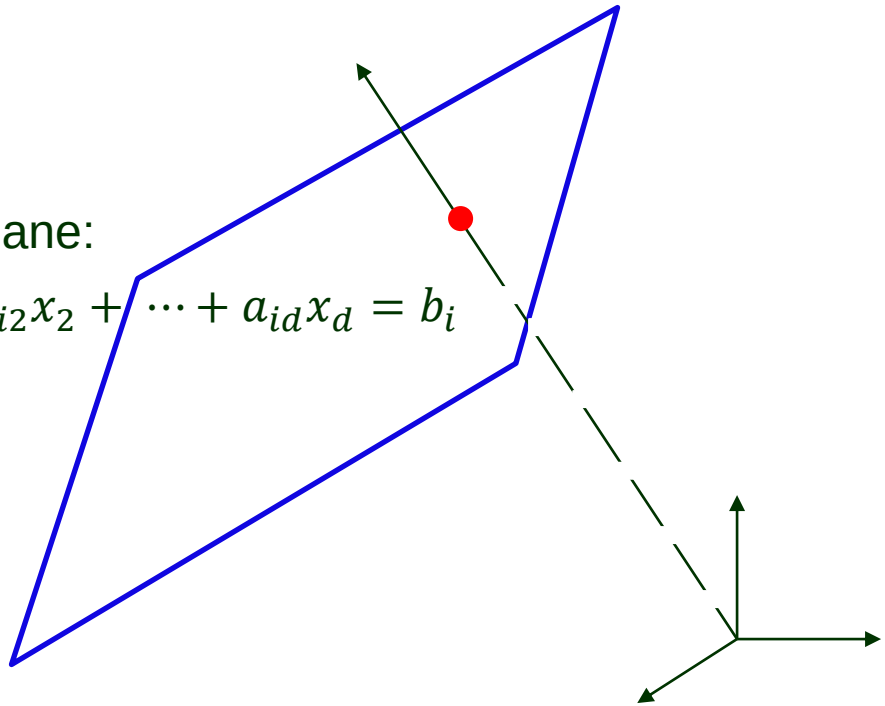

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$$\Downarrow$$

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Unit normal to hyperplane:

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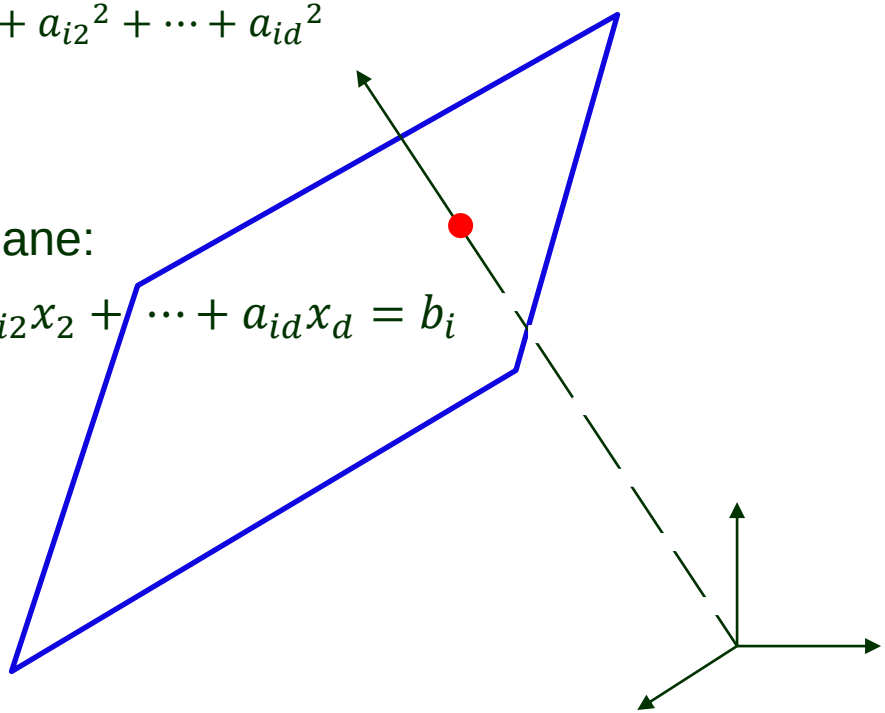

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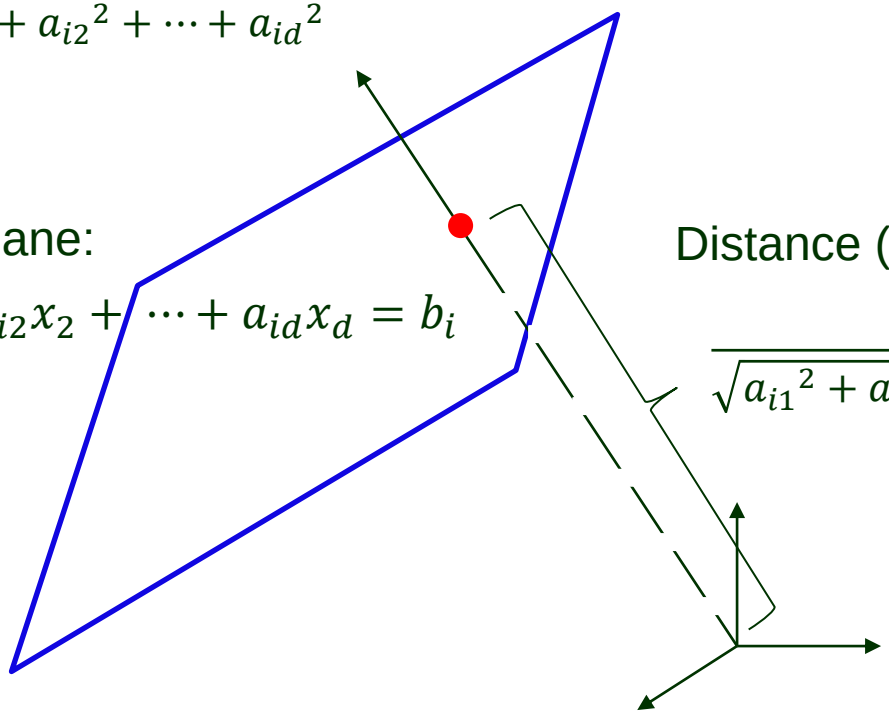

Hyperplane:

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Distance (signed) from the origin:

$$\frac{b_i}{\sqrt{{a_{i1}}^2 + {a_{i2}}^2 + \cdots + {a_{id}}^2}}$$

General LP

High dimensions ($d > 2$): Simplex algorithm (George Dantzig, 1947)

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Polynomial time algorithms:

- Ellipsoid algorithm (Leonid Khachivan, 1979)
- Karmarka's algorithm (Narenda Karmarkar, 1984) $O(d^{3.5}L)$

Running time is a high order polynomial in d variables and #bits L in input.

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In practice, the simplex algorithm is a better choice because

- its average running time is polynomial under various probability distributions, and
- it is numerically more stable.

2-Dimensional (2D) LP

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We will focus on $d = 2$:

- cost $\boldsymbol{c} = (c_x, c_y)$
- cost $f(p) = c_x p_x + c_y p_y$ at a point (p_x, p_y)
- constraint set H of n half-planes
- *feasible region*: $R = \cap_{h \in H} h$

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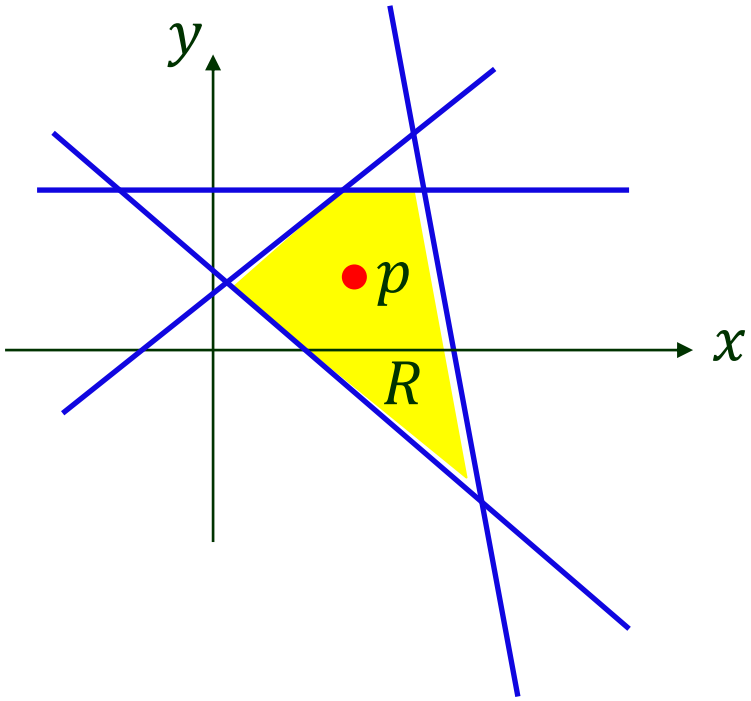
e.g., $H = \{x \geq 0, x + y \leq 1, x - y \geq 2\}$

Feasible Region



Intersection of all half-planes.

feasible solution: a point p in the region

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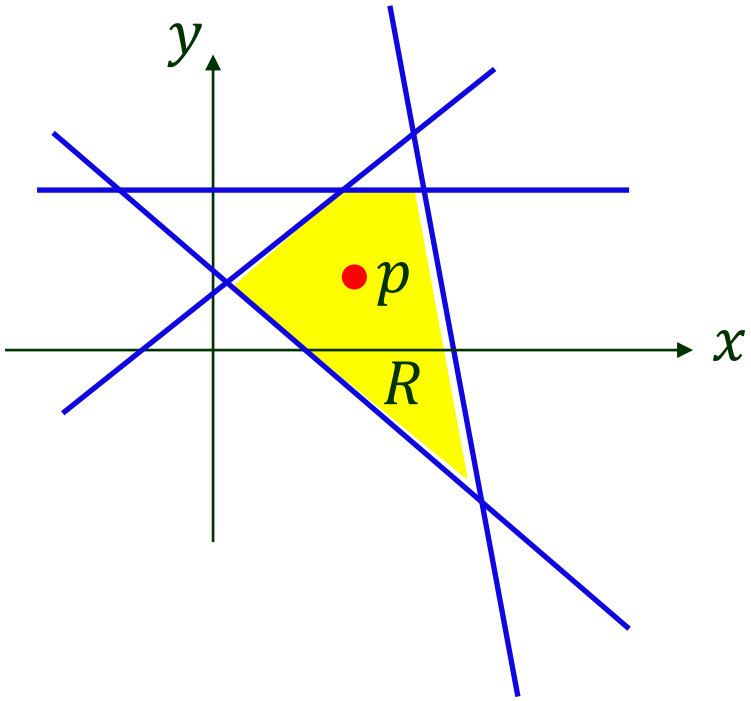

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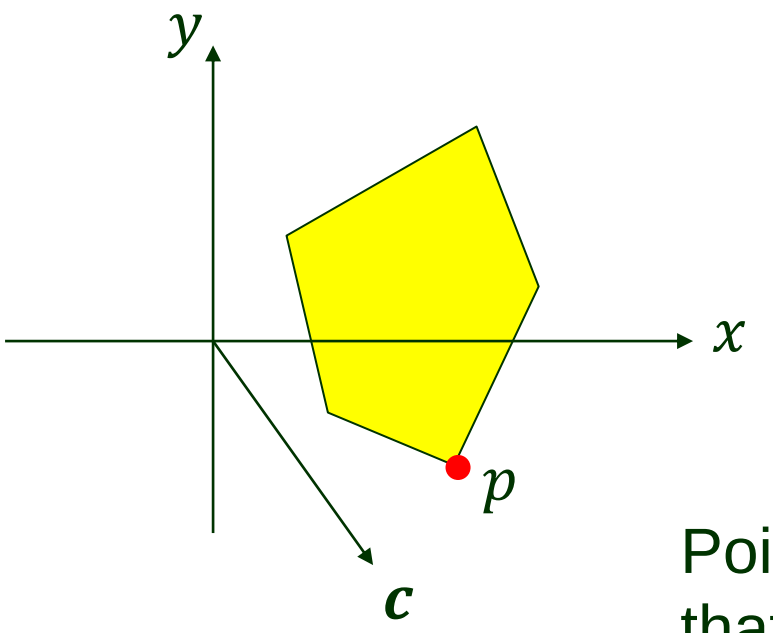
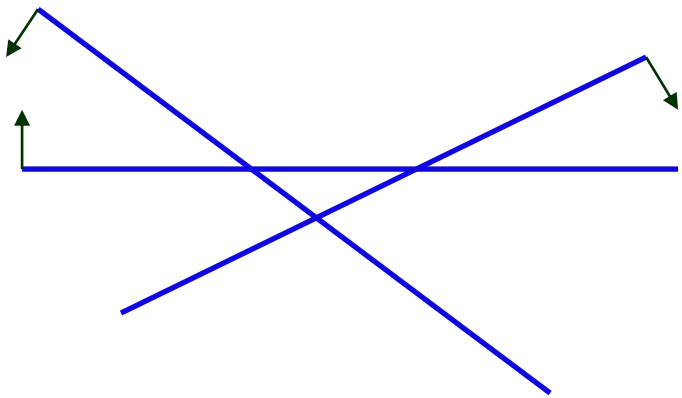
The LP is *infeasible* if $\bigcap_{h \in H} h = \emptyset$.

Optimal Solution for $(H, \boldsymbol{c})$



Point in the feasible region
that is *extreme* in the direction $\boldsymbol{c}$.

Two Possibilities



(a) infeasible LP

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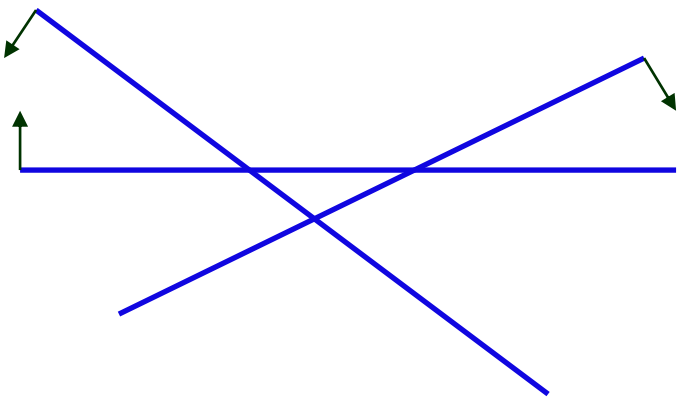

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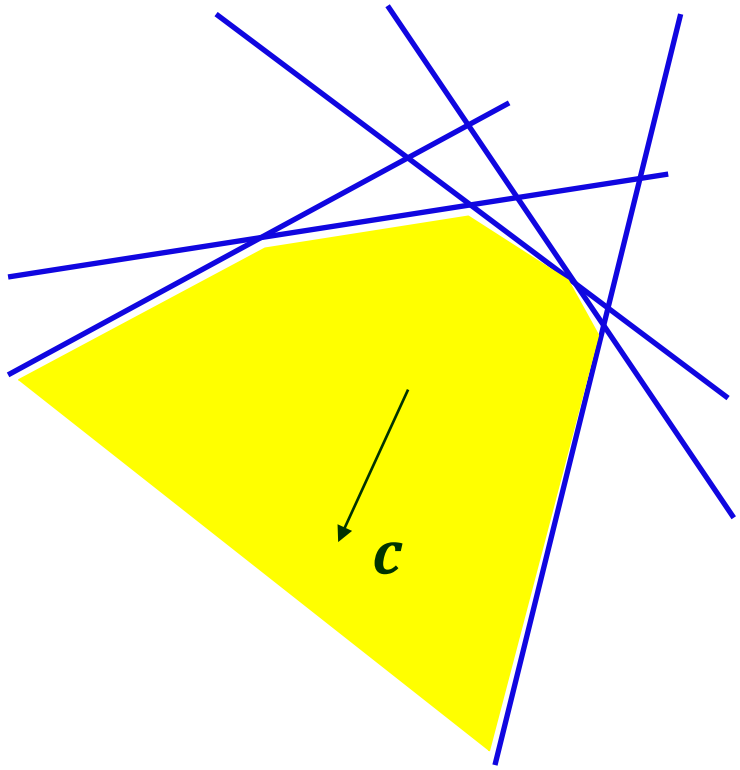
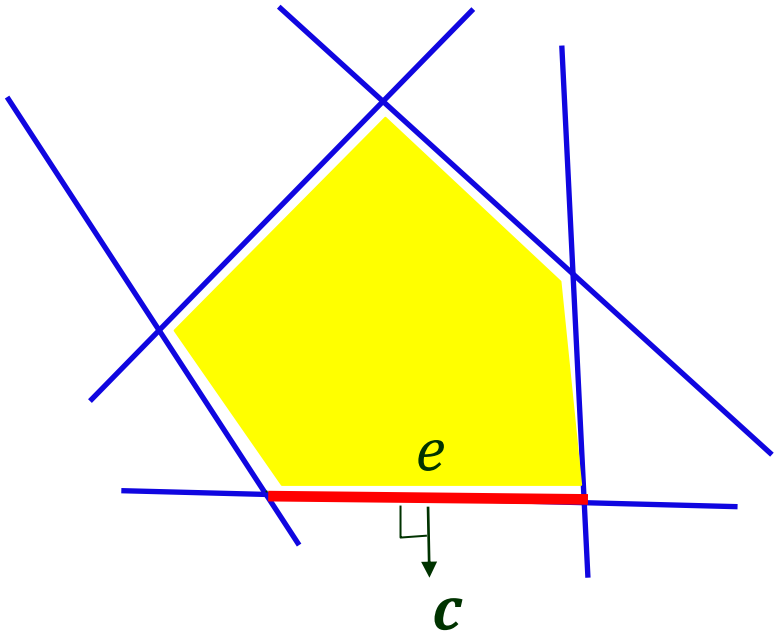


(b) unbounded LP

(f arbitrarily large)

Two More Possibilities

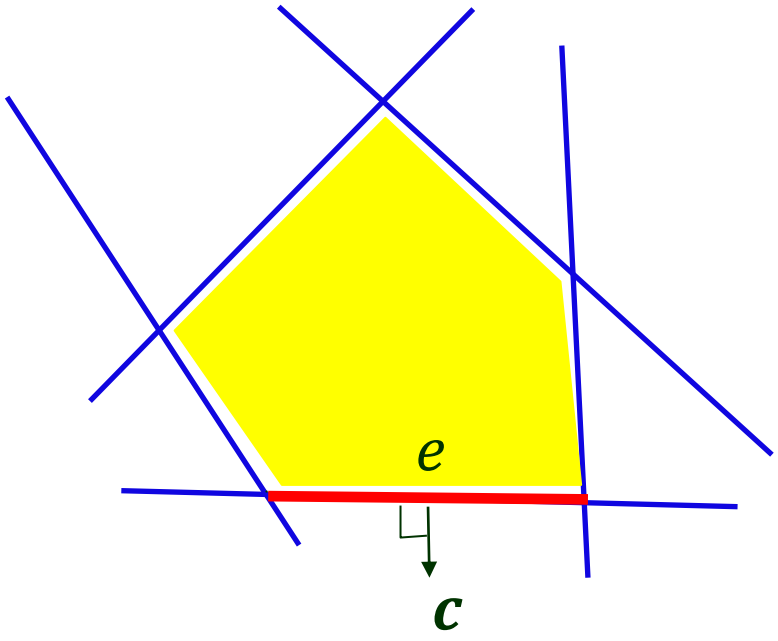


$e \perp \boldsymbol{c}$

(c) a continuum of solutions
(every point on the edge e is optimal)

For uniqueness, choose lexicographically the smallest vertex.

Two More Possibilities



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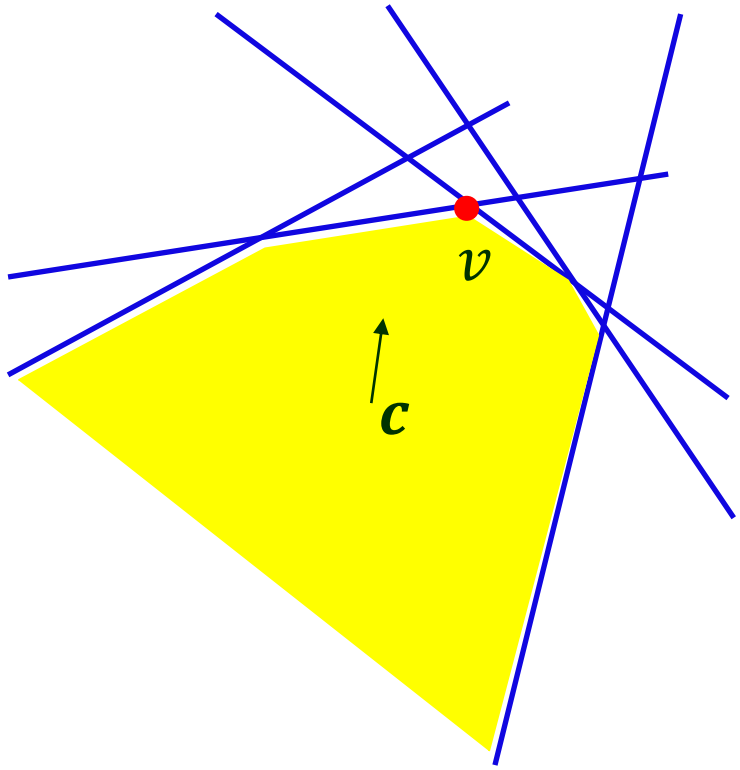

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Incremental Construction

Subroutine *UNBOUNDEDLP*$(H, \boldsymbol{c})$ outputs

- a ray completely contained in R if unbounded;
- two half-planes $h_1, h_2 \in H$ such that the LP $(\{h_1, h_2\}, c)$ is bounded otherwise.

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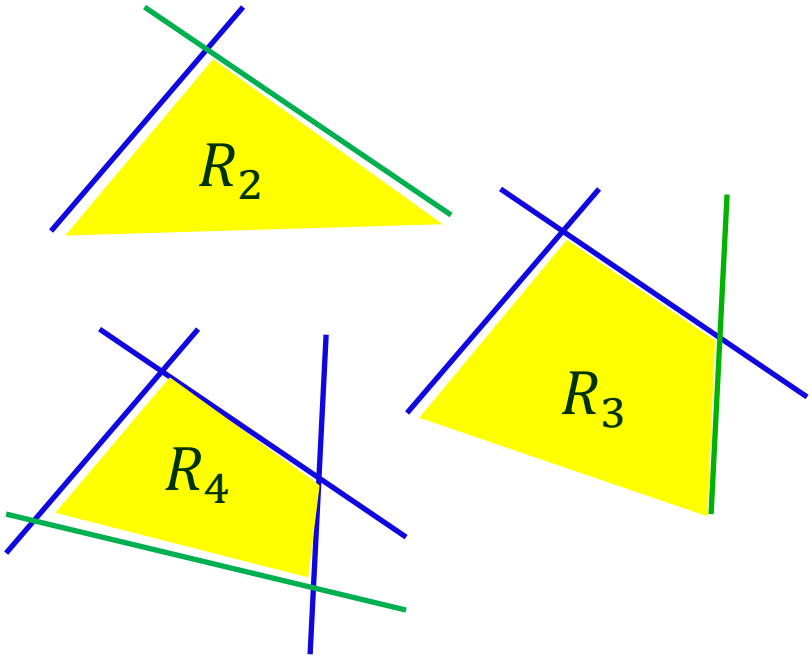

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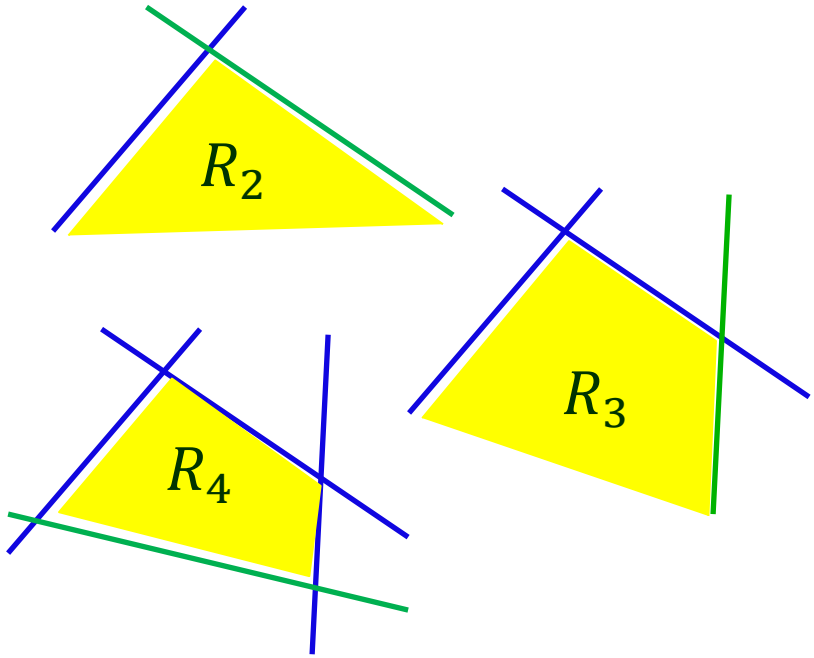

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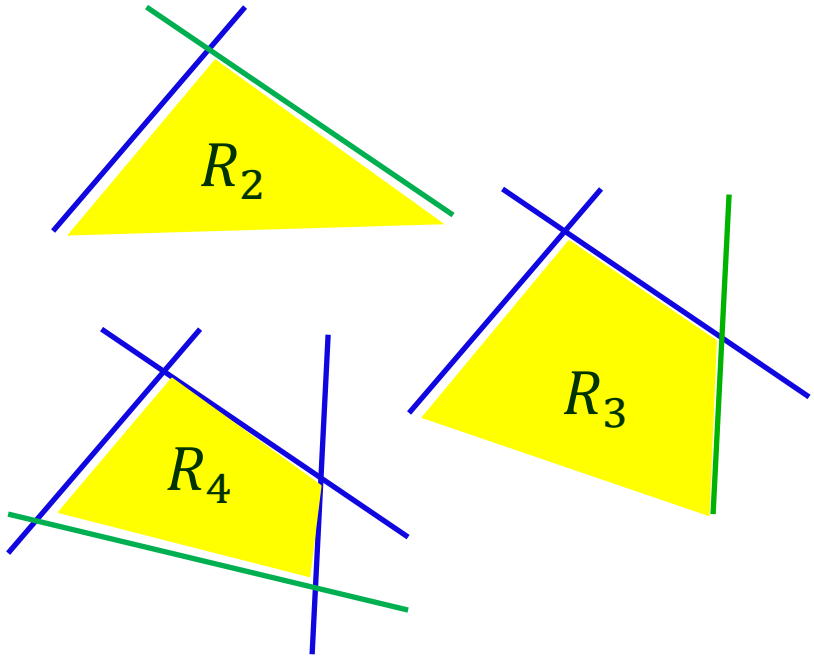
$$R_2 \supseteq R_3 \supseteq \cdots \supseteq R_n$$

If $R_i = \emptyset$ then $R_j = \emptyset$ for all $j > i$, infeasible!



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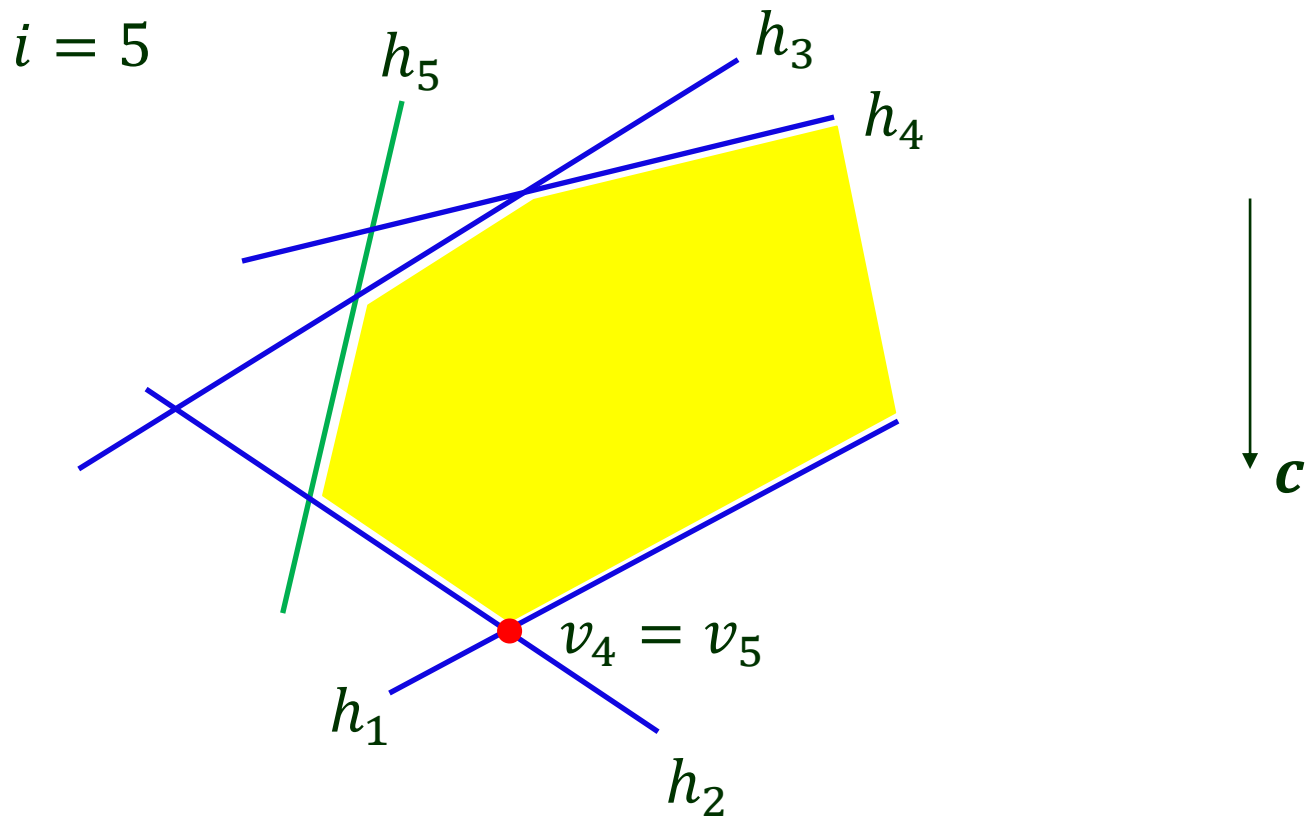
v_i: optimal vertex of R_i .



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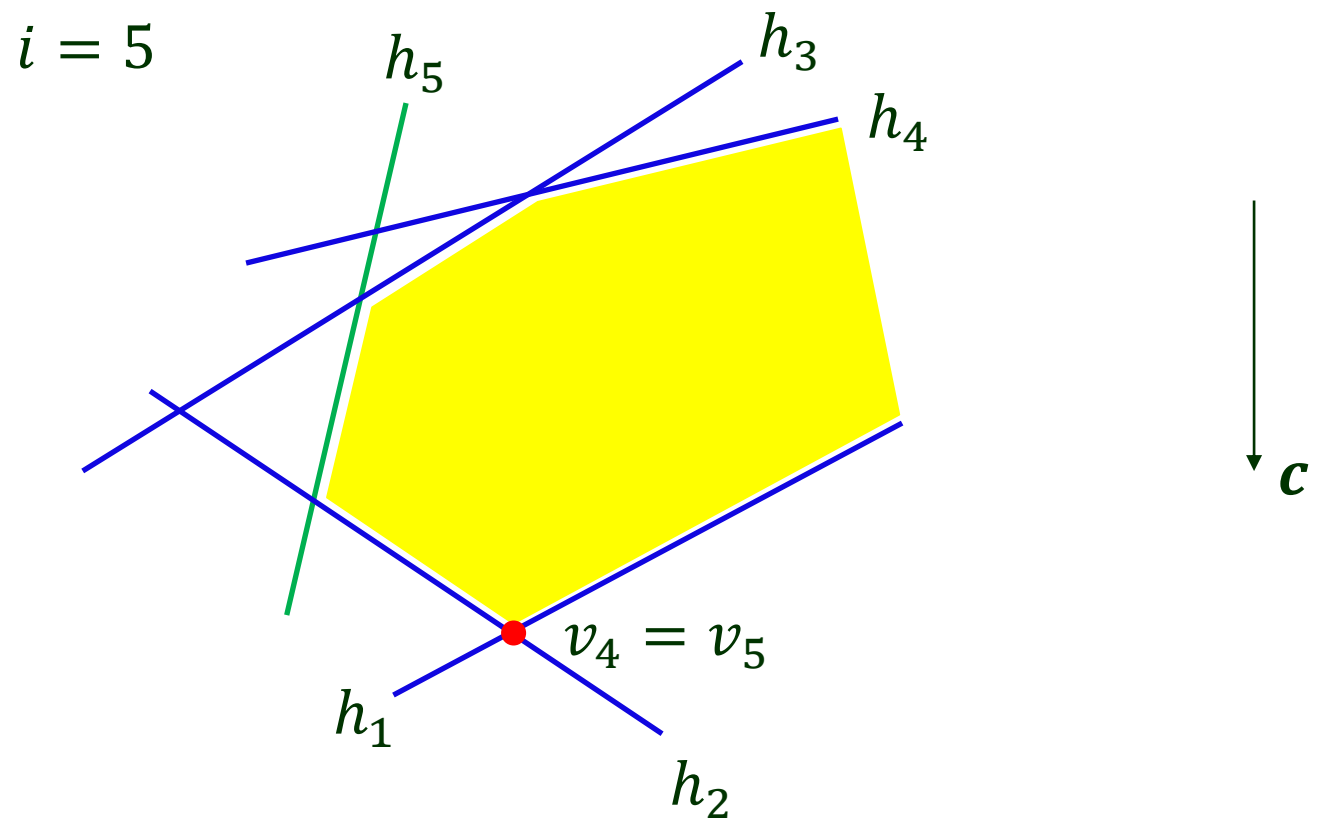

$v_4 \in h_5 \Rightarrow$ No change

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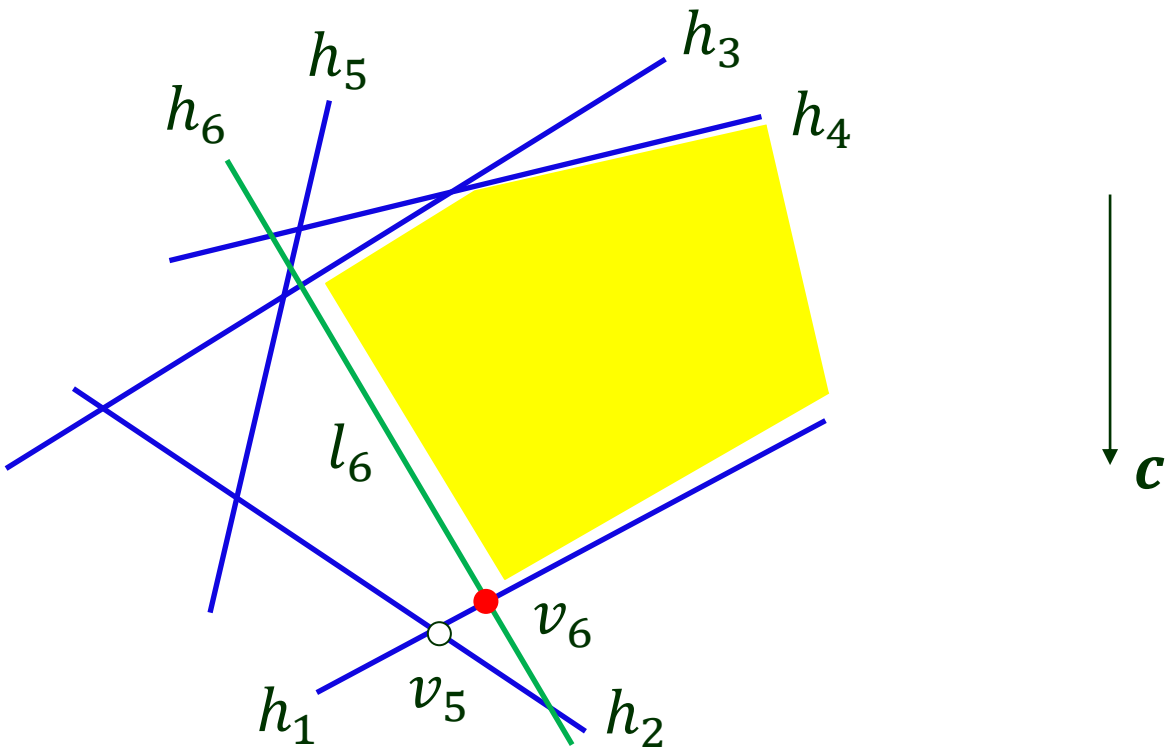
$i = 6$



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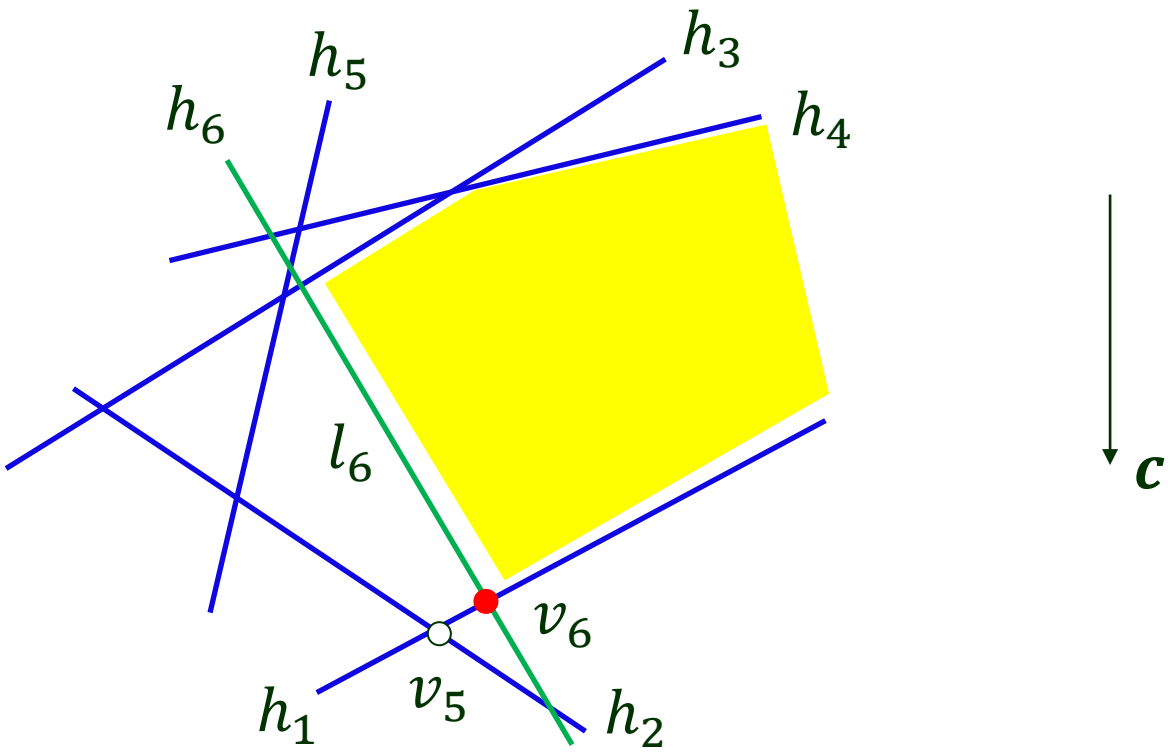

$v_5 \notin h_6 \Rightarrow$ new optimal vertex $v_6 \in l_6$ (bounding line of h_6)

Evolution of the Optimal Vertex

Lemma i) $v_{i-1} \in h_i \Rightarrow v_i = v_{i-1}$

ii) $v_{i-1} \notin h_i \Rightarrow$ either $R_i = \varnothing$ or $v_i \in l_i$

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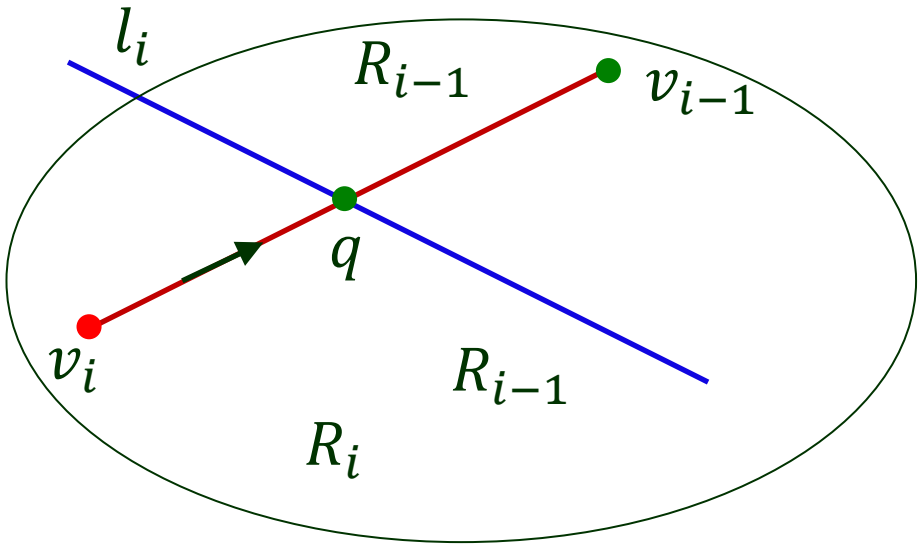

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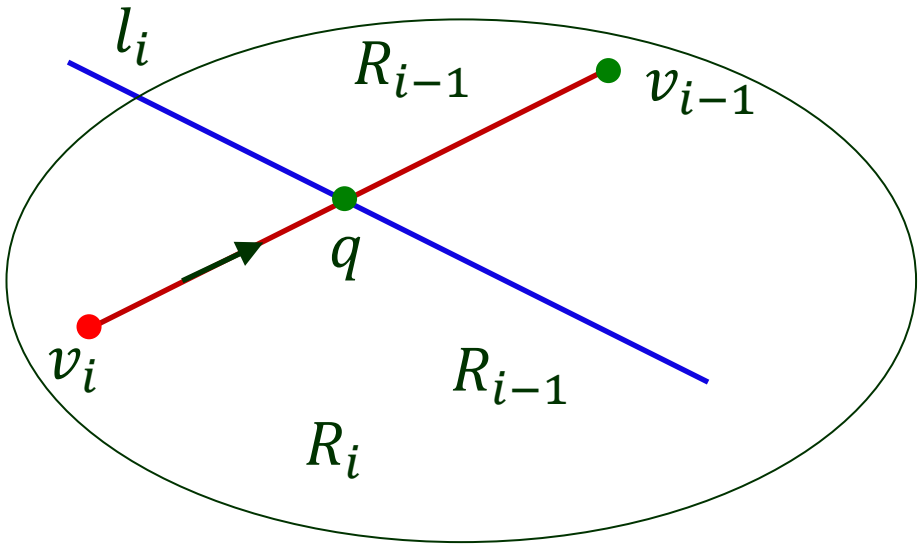

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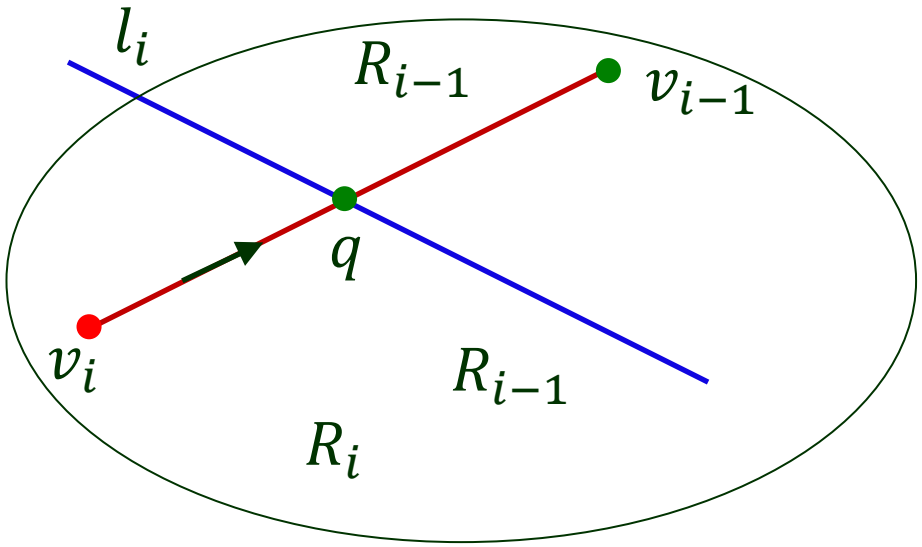

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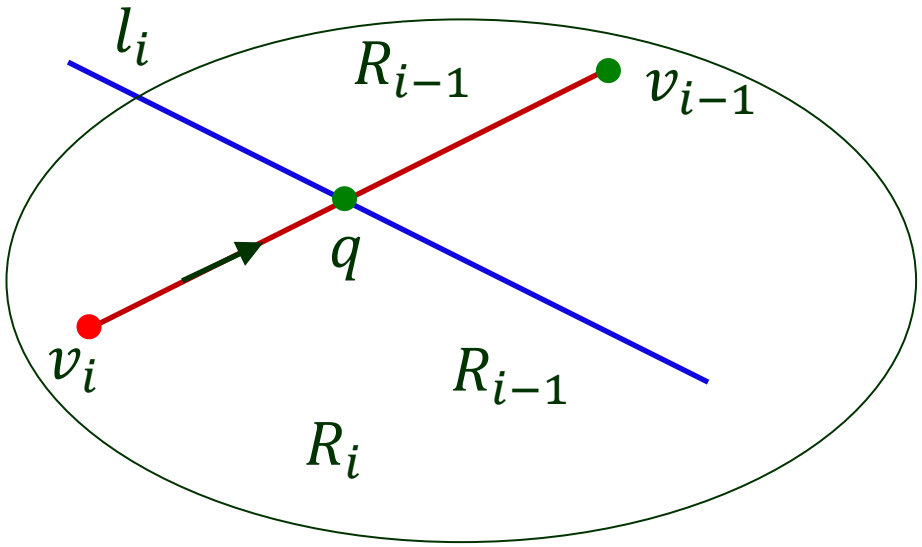

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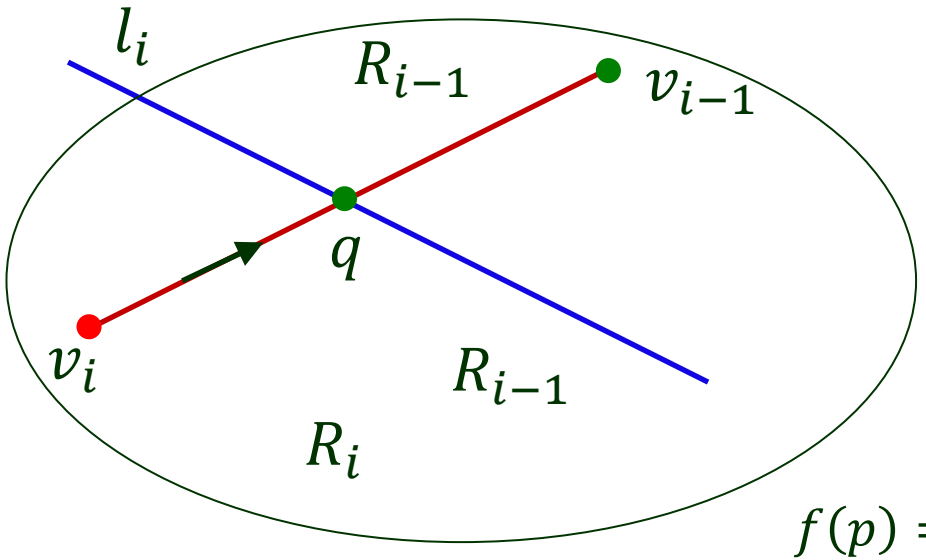

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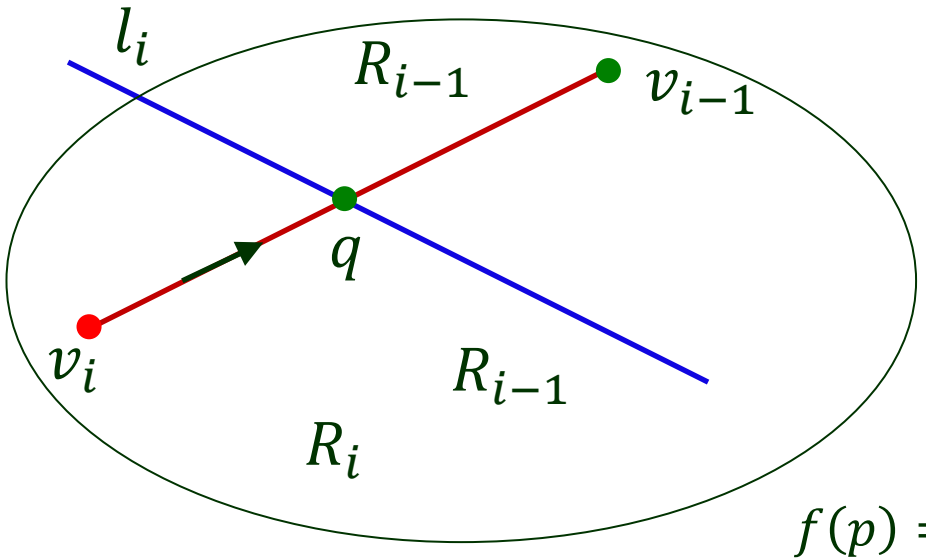

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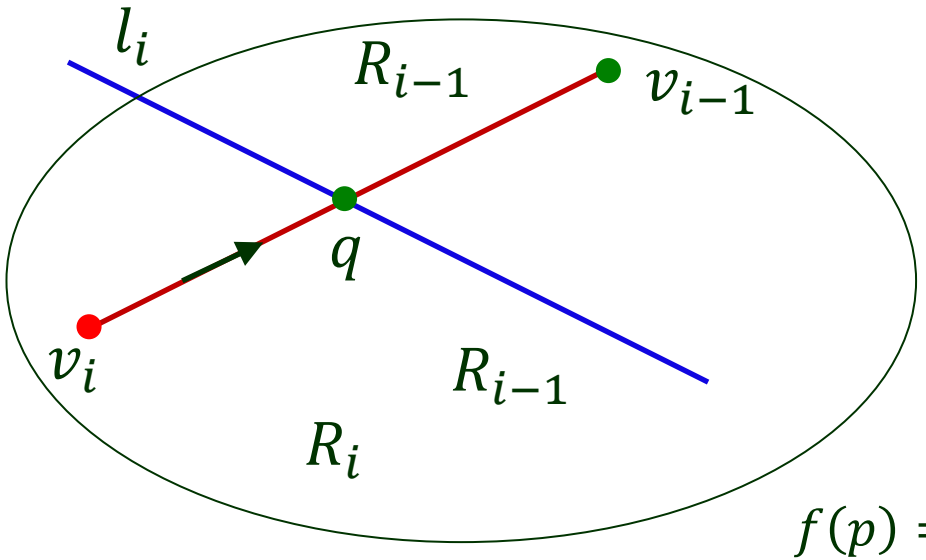

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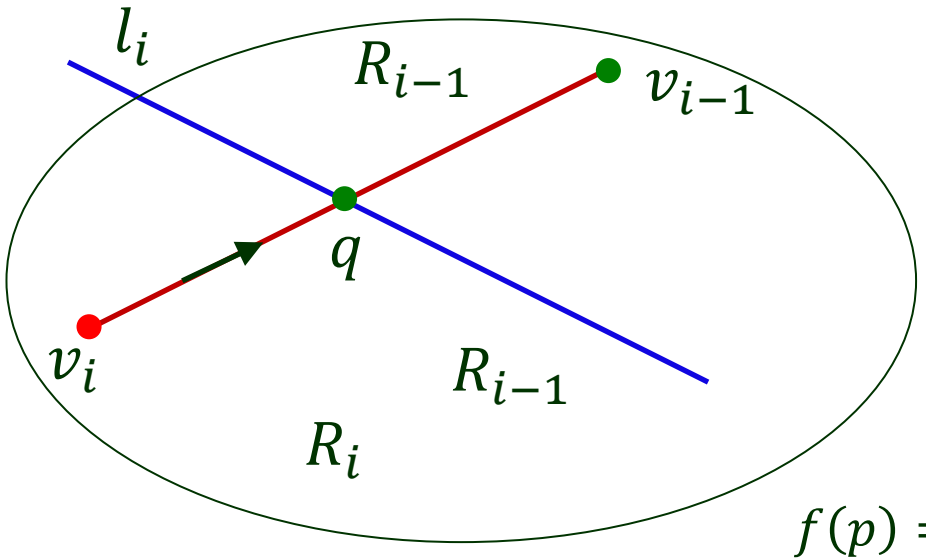

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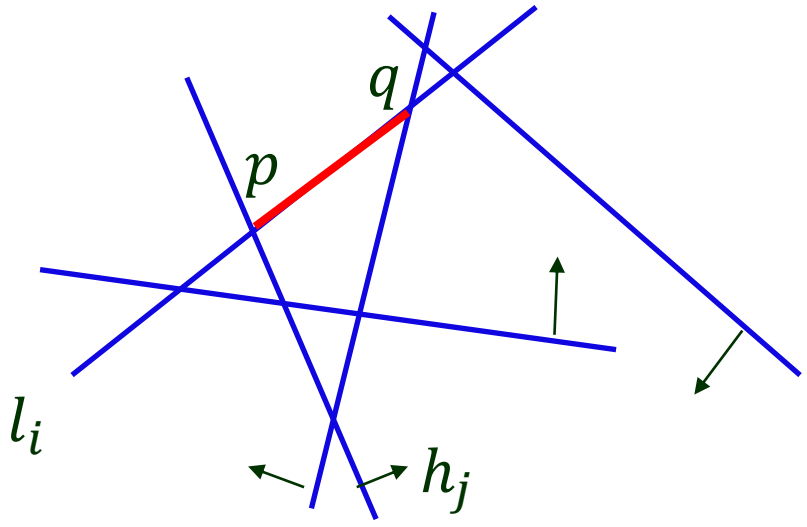
□

Updating the Optimal Vertex

How to find v_i if $v_{i-1} \notin h_i$?



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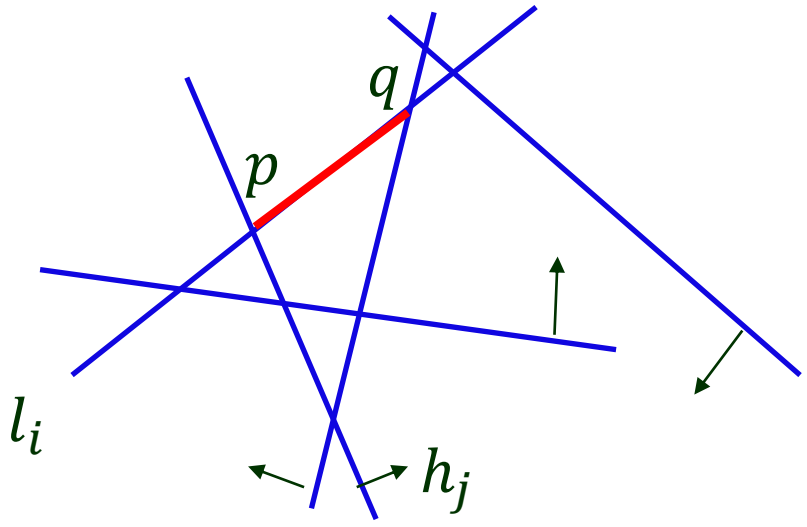

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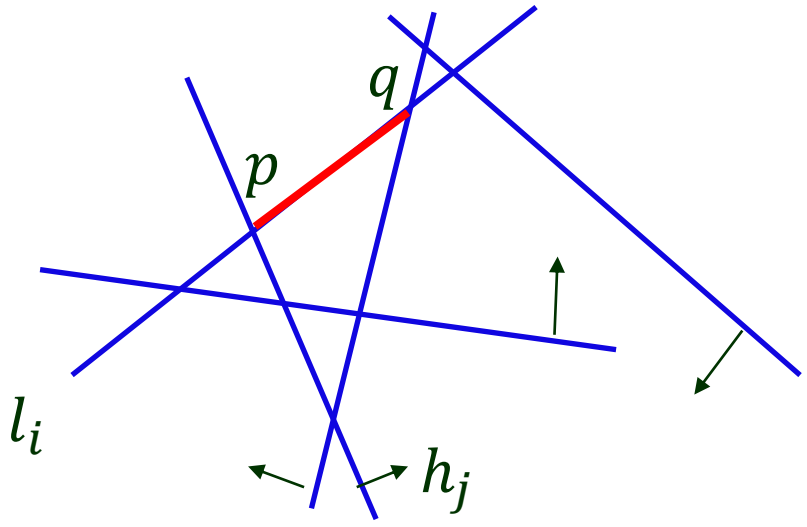

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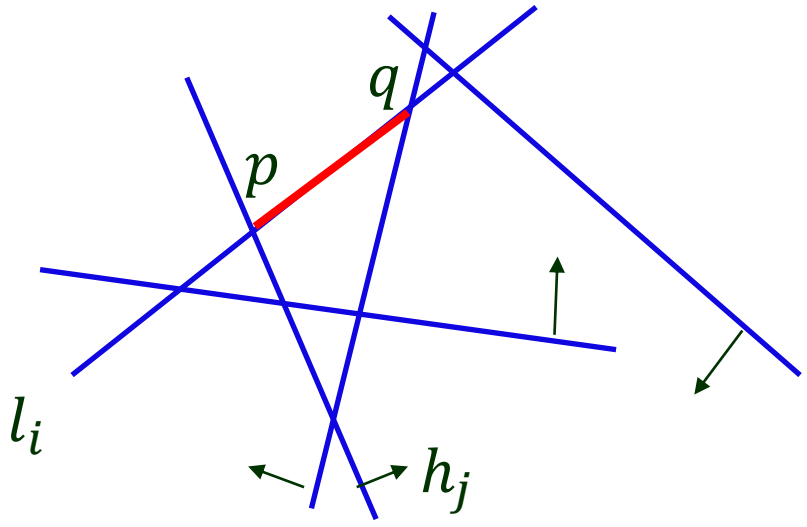

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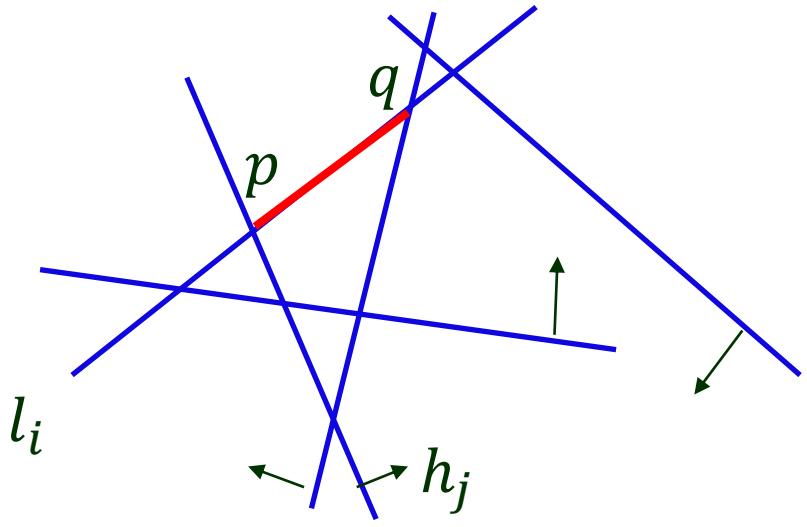

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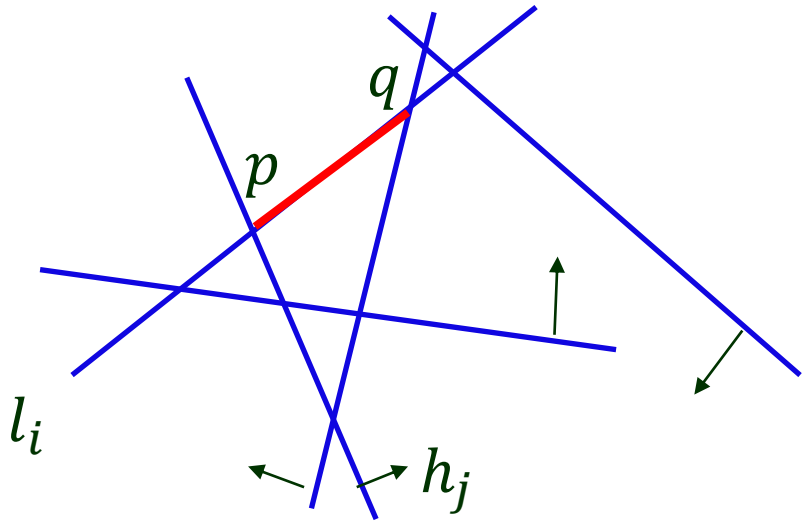

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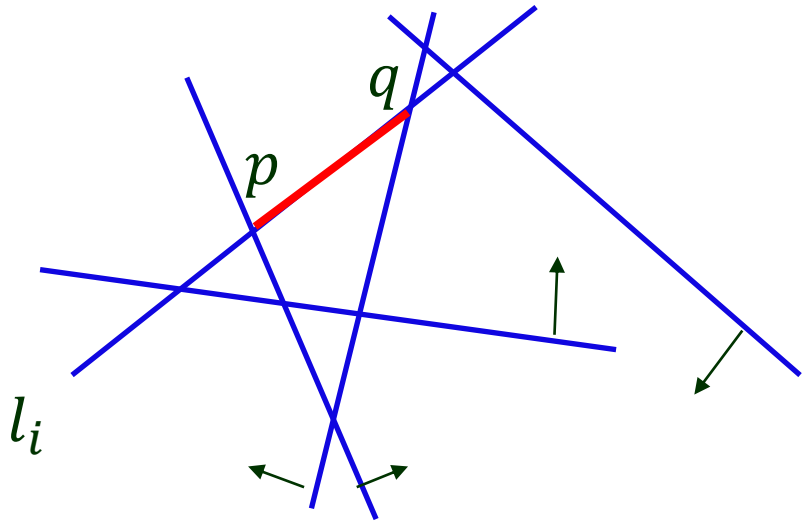

Iterate over half-planes $h_1, h_2, \ldots, h_{i-1}$ to intersect with l_i.

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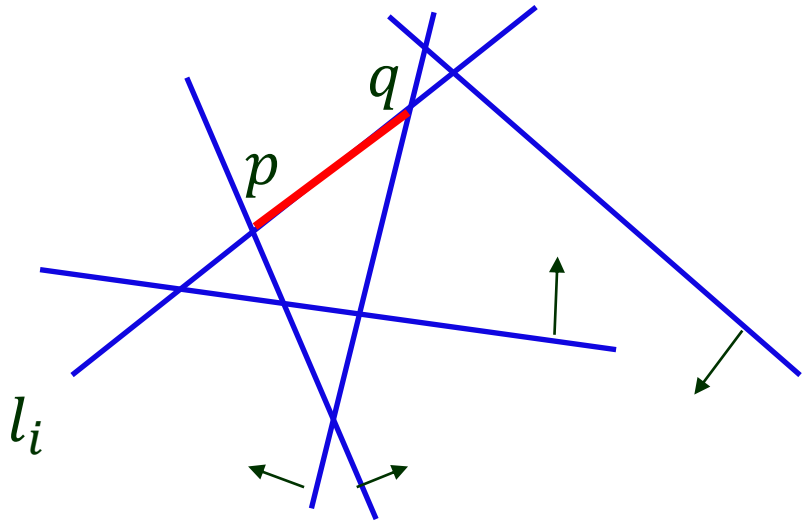

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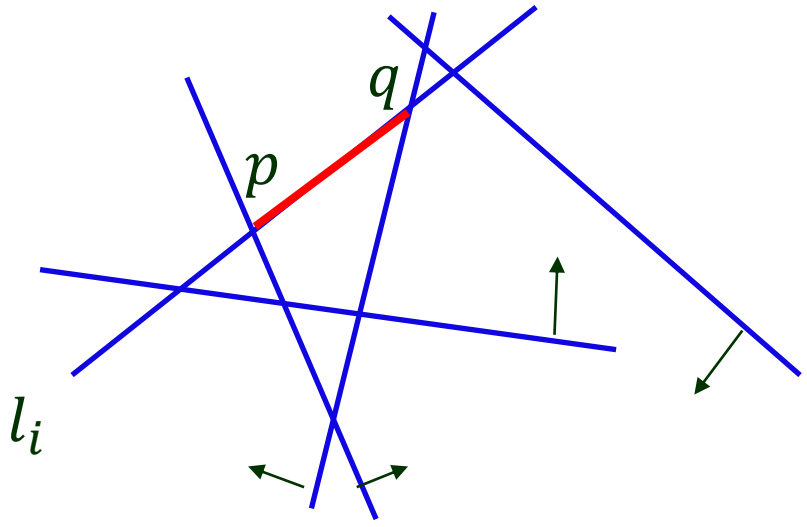

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It may end earlier once the intersection becomes empty.

The Incremental Algorithm

```
if UNBOUNDEDLP reports that LP is infeasible
   then return
   else if LP is unbounded  // to be discussed
           then return a ray along which it is unbounded
           else h_1, h_2 ← certificate half-planes
                v_2 ← l_1 ∩ l_2
                for i ← 3 to n
                   do if v_{i-1} ∈ h_i
                         then v_i ← v_{i-1}
                         else v_i ← point p on l_i that maximizes f
                                    subject to h_1, h_2, …, h_{i-1}  // O(i)
                              if p does not exist
                                 then LP is infeasible
                return v_n // optimal solution
```

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Theorem The algorithm solves an LP with n constraints and 2 variables in $O(n^2)$ time and $O(n)$ storage.

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Total running time $\sum_{i=3}^{n} O(i) = O(n^2)$

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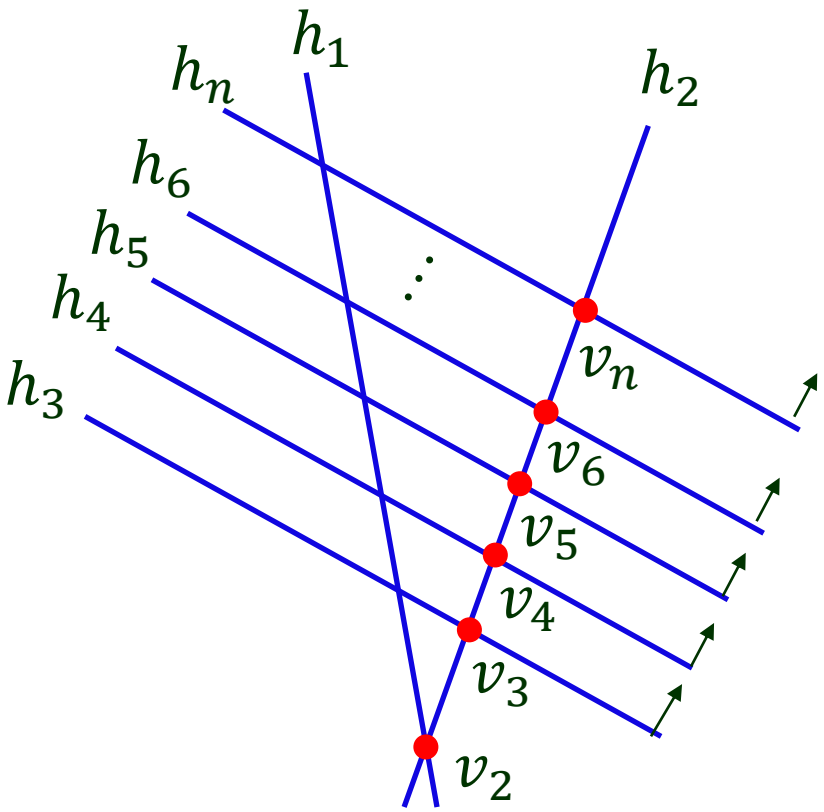

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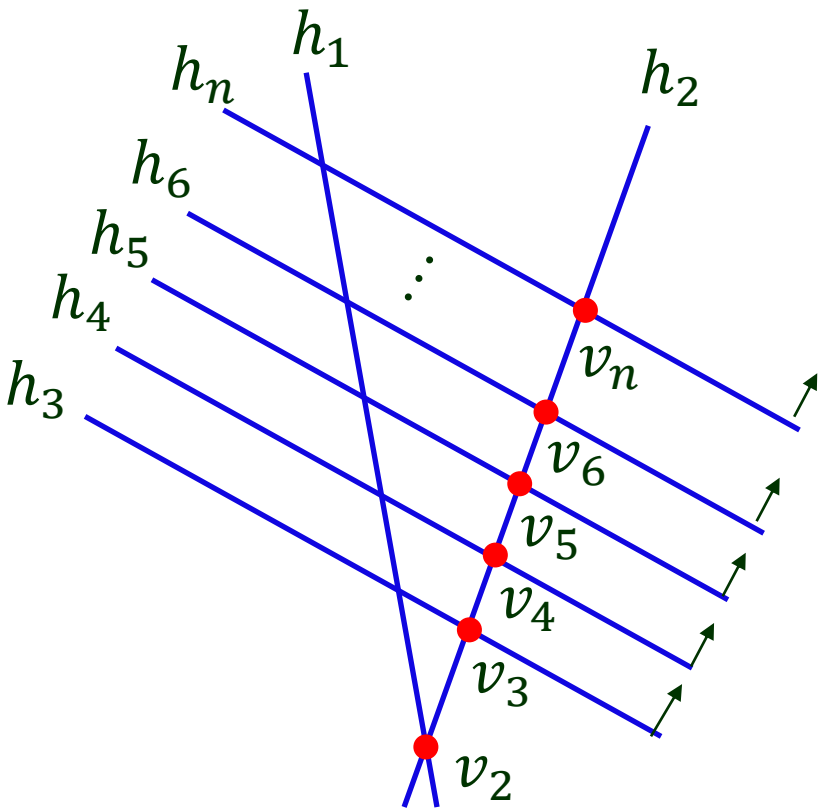

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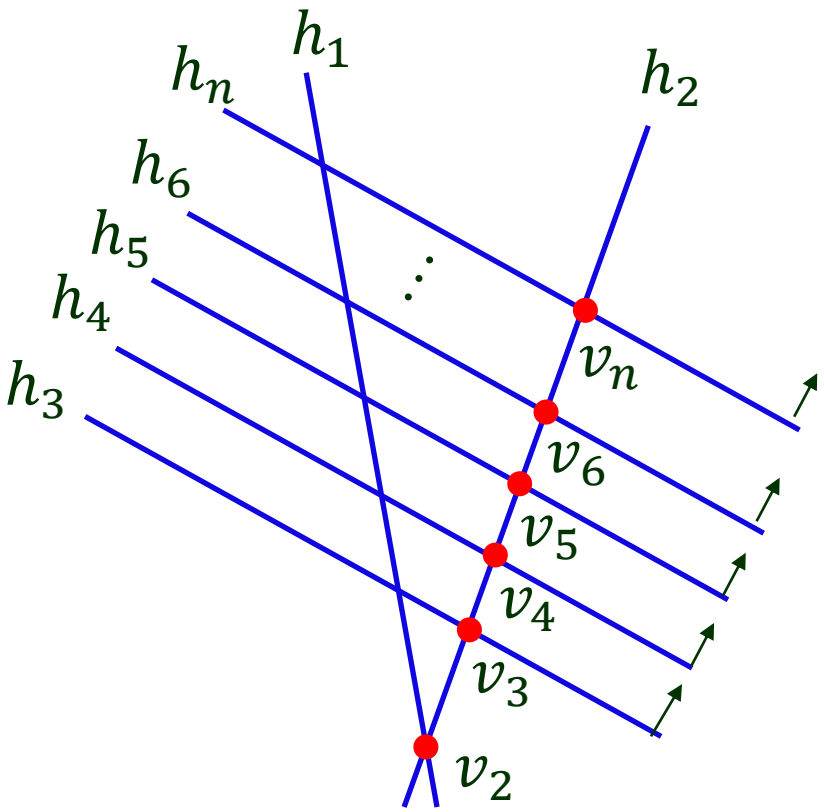

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$\Theta(n^2)$ time!

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How about adding them in the order $h_n, h_{n-1}, \dots, h_3$?

Namely, change their indices: $n \to 3, n-1 \to 4, \dots, 3 \to n$.

No change of the optimal vertex!

$$R_3 = R_4 = \cdots = R_n$$

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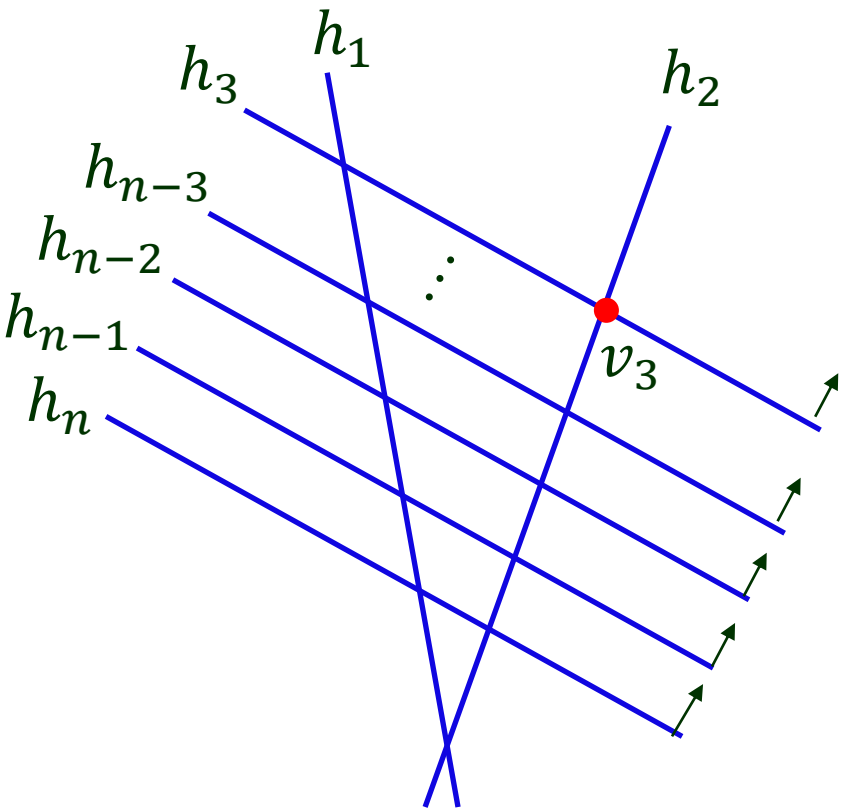

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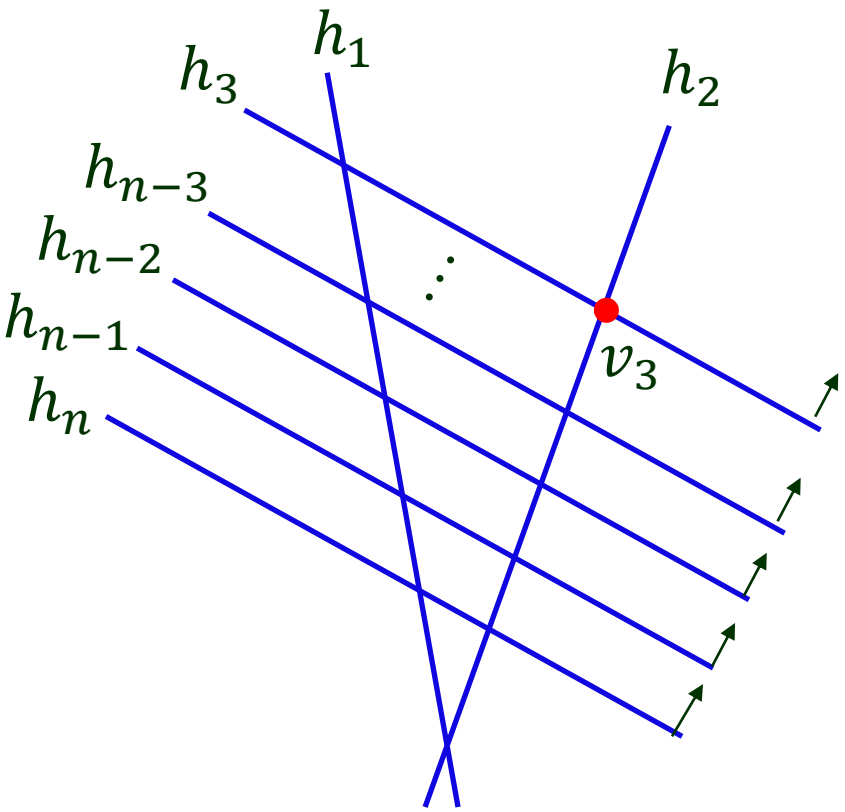

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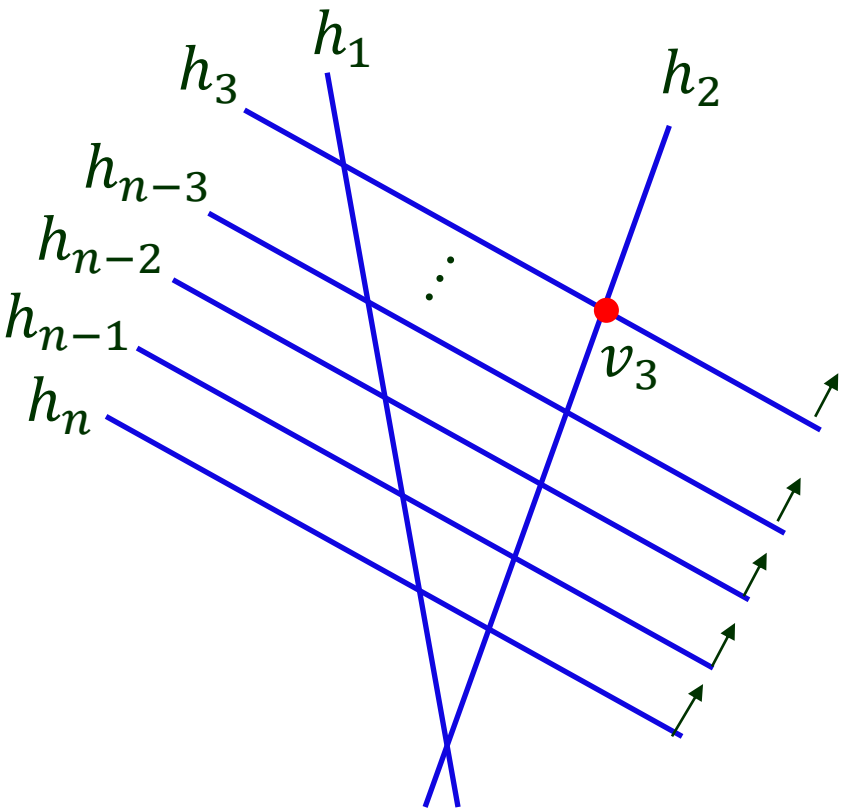

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Any set of half-planes has such a good order, but there is no easy way to find it.

Our approach: randomization.