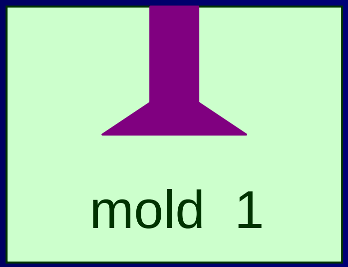
Outline

I. The molding problem

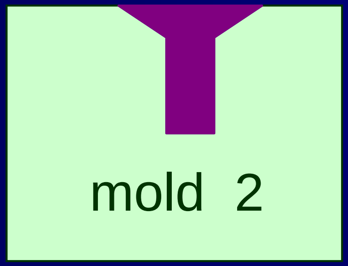
II. Problem transformation

III. Intersection of half-planes

I. The Problem of Molding



Does a given object have a mold from which it can be removed?



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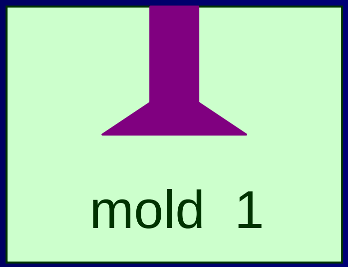
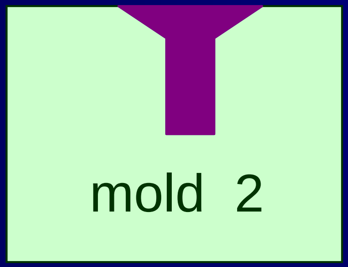


object not removable



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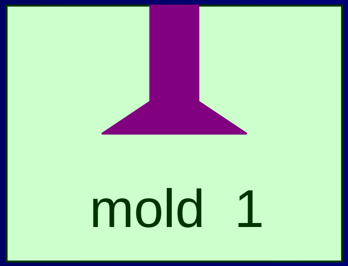

object not
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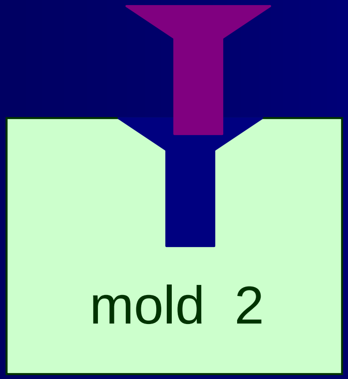

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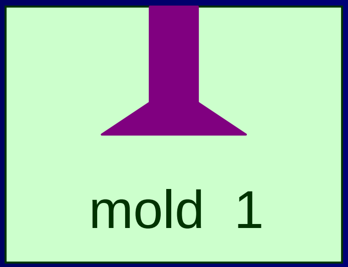

object not
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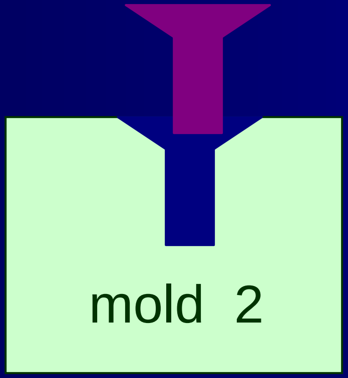

object
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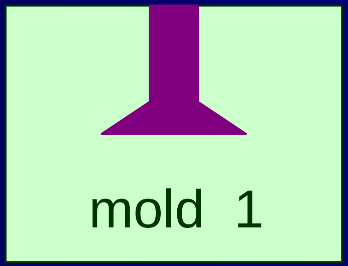
Does a given object have a mold from which it can be removed?

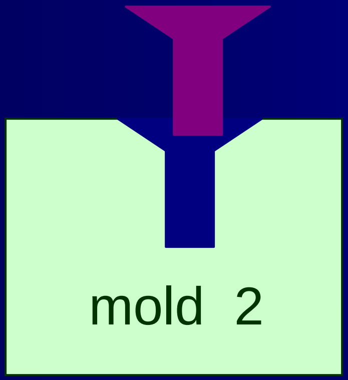
Assumptions

- The object is polyhedral.

I. The Problem of Molding



object not removable



object removable

Does a given object have a mold from which it can be removed?

Assumptions

- The object is polyhedral.
- The mold has only one piece.

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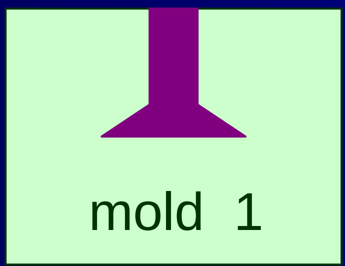
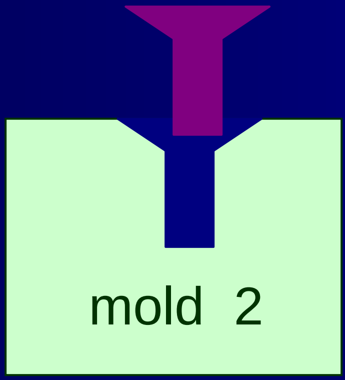


object not removable



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Spherical objects cannot be manufactured using a mold of one piece.

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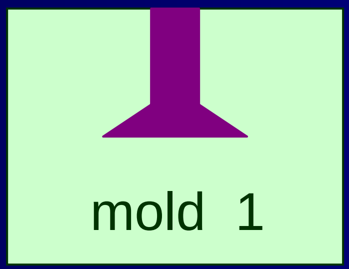

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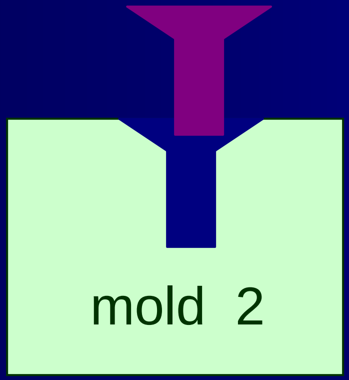

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- The object should be removed by only a single translation.

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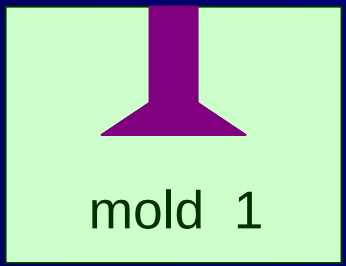

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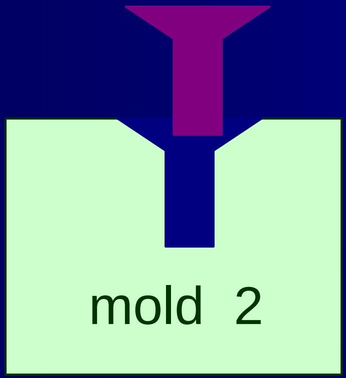

object removable

Assumptions

- The object is polyhedral.
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  Spherical objects cannot be manufactured using a mold of one piece.

- The object should be removed by only a single translation.

  Real screws cannot be removed by just a translation.

Castability

How to choose the orientation?

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How to determine that the object is castable?

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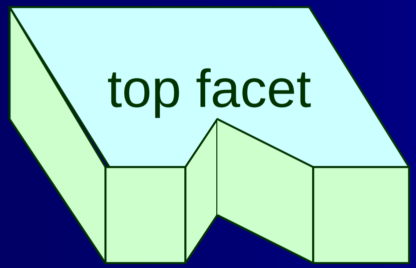
How to determine that the object is castable?

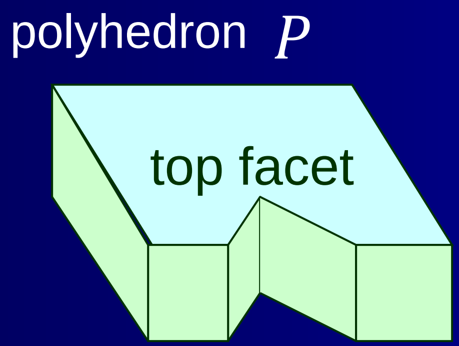
For each potential orientation, determine whether there exists a direction along which the object can be removed from the mold.

Making Things More Precise

polyhedron P



Making Things More Precise



- **The mold is a rectangular block with a concavity that exactly matches the polyhedron.**

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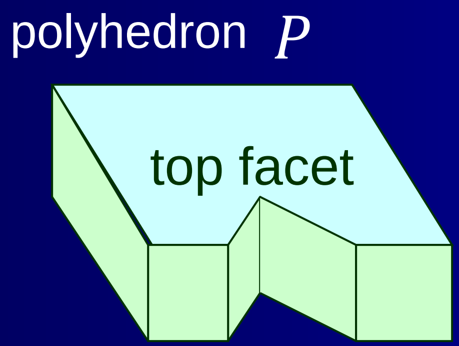

- **The mold is a rectangular block with a concavity that exactly matches the polyhedron.**
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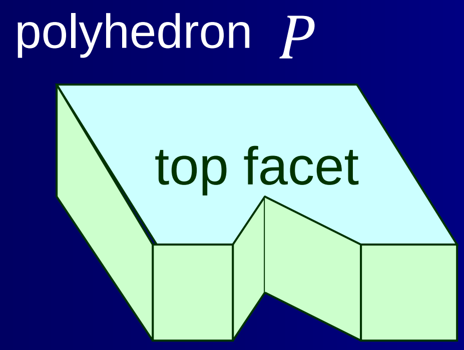

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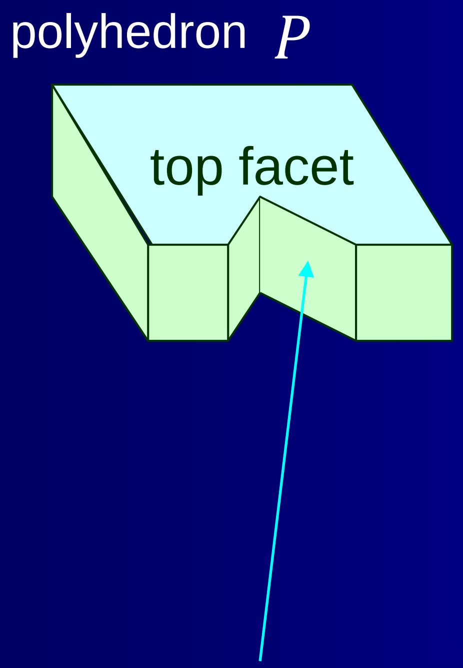

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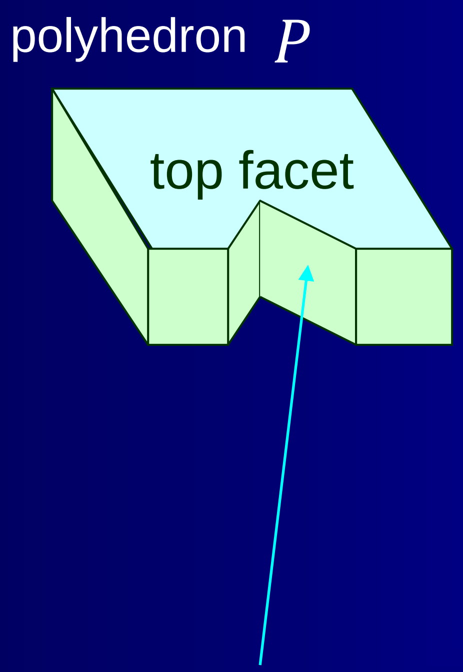
Making Things More Precise



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F : the facet in the mold that corresponds to f.

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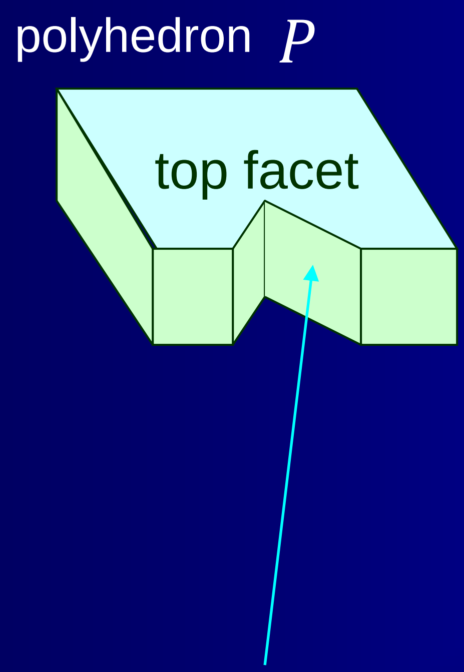

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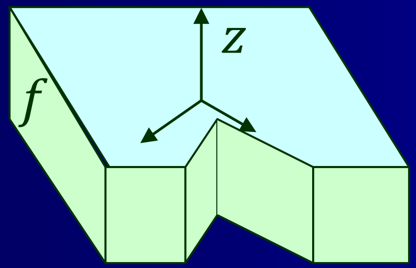
Problem **Decide whether a direction d exists such that P can be translated to infinity without colliding with the mold.**

Necessary Condition for Removal

polyhedron P



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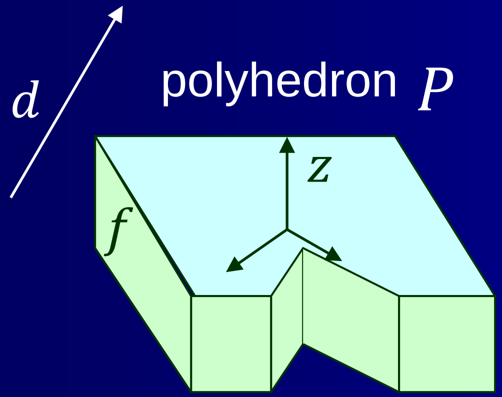

d has a positive z component.

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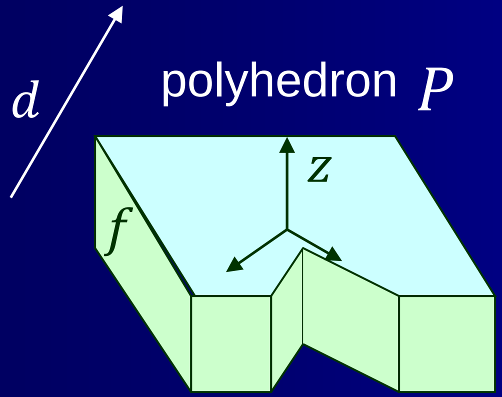
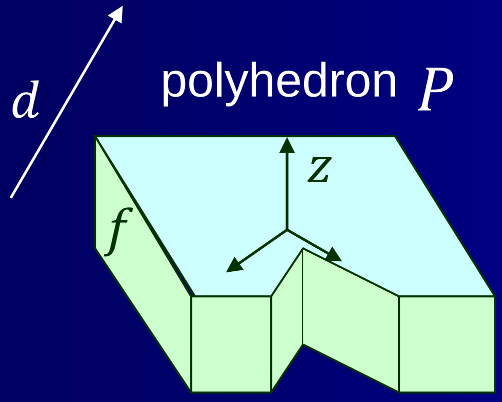


d has a positive *z* component.

The translation of a facet *f* cannot penetrate into the corresponding facet *F* of the mold.

Necessary Condition for Removal



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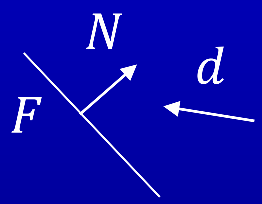

F blocks the translation if d forms an angle $> \pi/2$ with its outward normal N.

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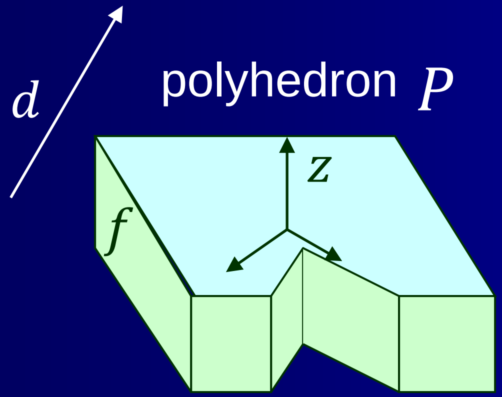

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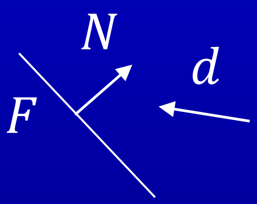
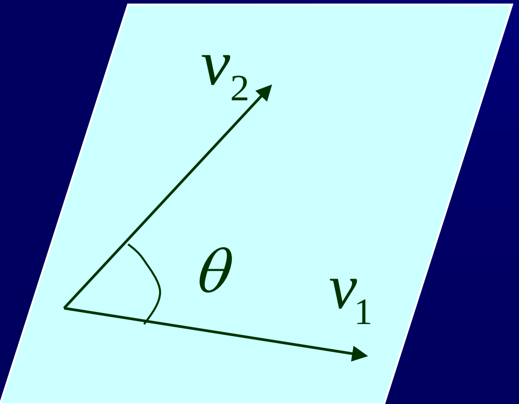


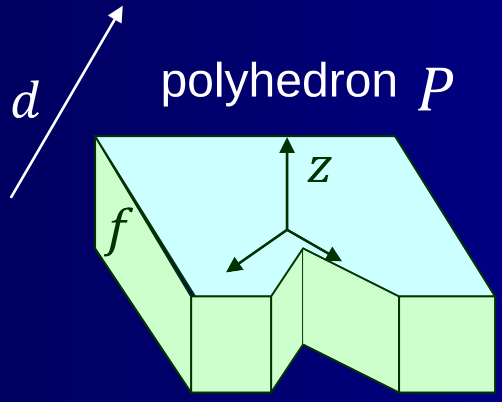


angle between two vectors

$$\cos\theta = v_1 \bullet v_2$$

θ chosen to be in $[0, \pi]$.

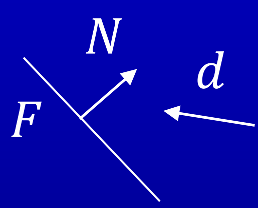
Necessary Condition for Removal



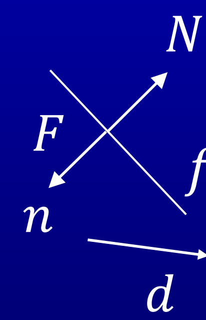
d has a positive z component.

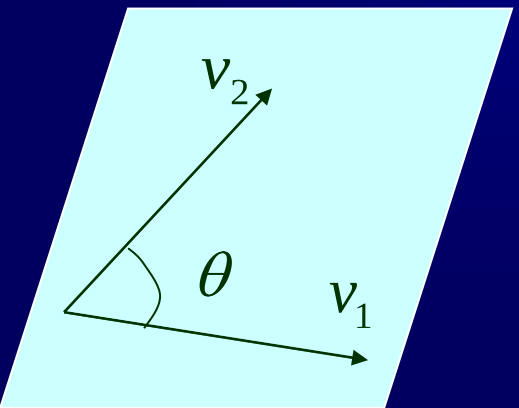
The translation of a facet f cannot penetrate into the corresponding facet F of the mold.

F blocks the translation if d forms an angle $> \pi/2$ with its outward normal N.



F does not block the translation if d forms an angle $\leq \pi/2$ with its outward normal N.





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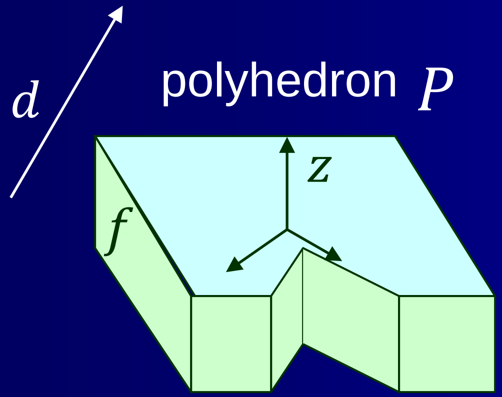

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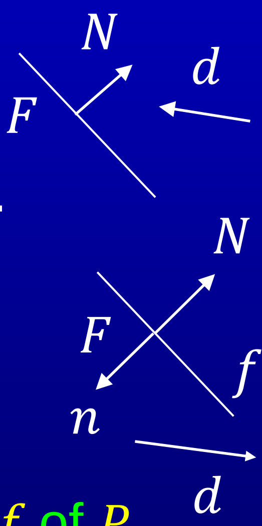
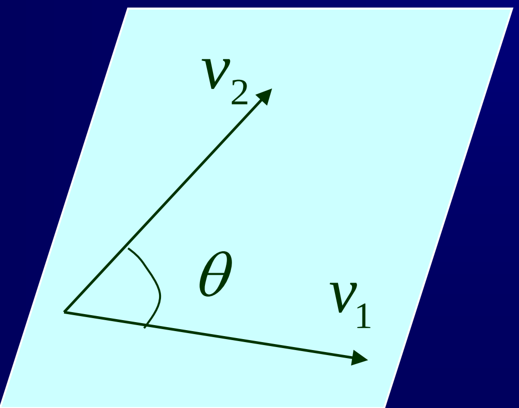


⇒ F blocks the translation if d forms an angle $> \pi/2$ with its outward normal N.

F does not block the translation if d forms an angle $\leq \pi/2$ with its outward normal N.

⇒ d must make an angle $\geq \pi/2$ with the outward normal $n = -N$ of every facet f of P.



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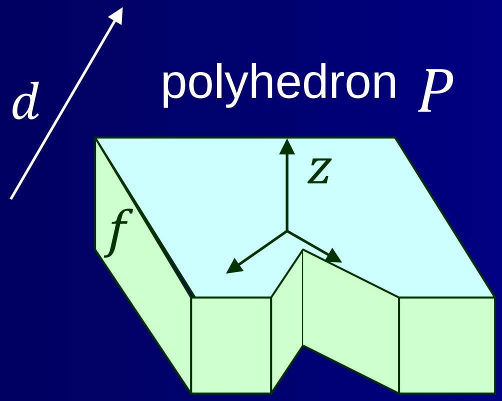

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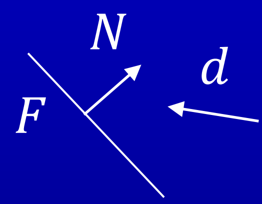

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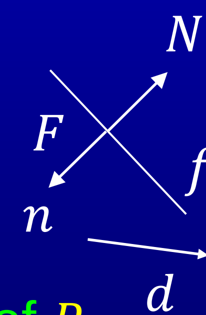

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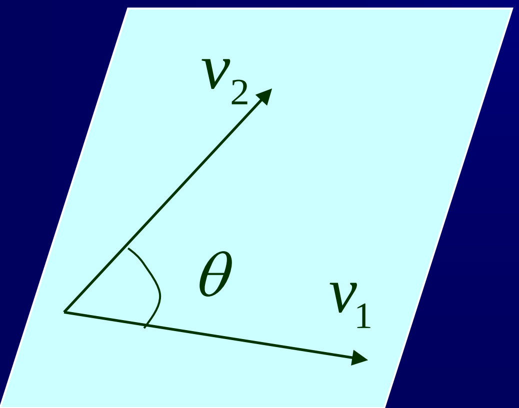
(necessary condition)



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Also a Sufficient Condition

Theorem The polyhedron can translate out of the mold in a direction d if and only if d makes an angle $\geq \pi/2$ with the outward normal of every facet of the polyhedron except the top one.

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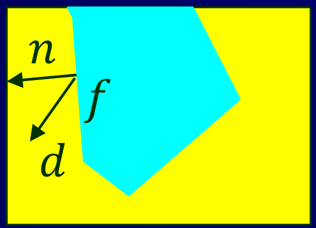

Suppose d makes an angle $< \pi/2$ with the outward normal n of some facet f.

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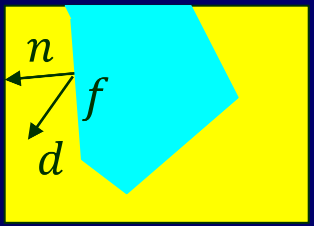

Then any interior point of f collides with the mold in the translation.

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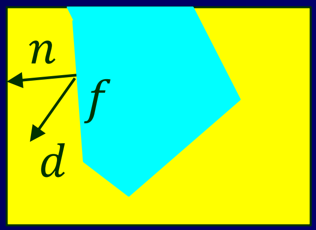

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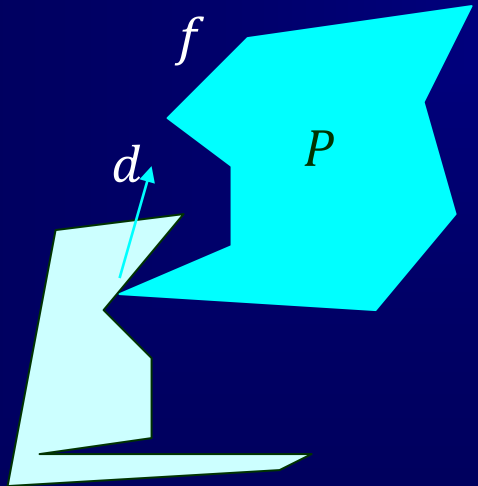

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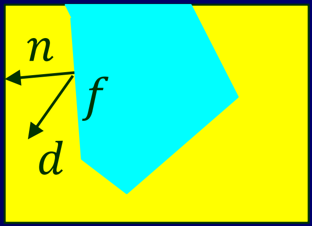

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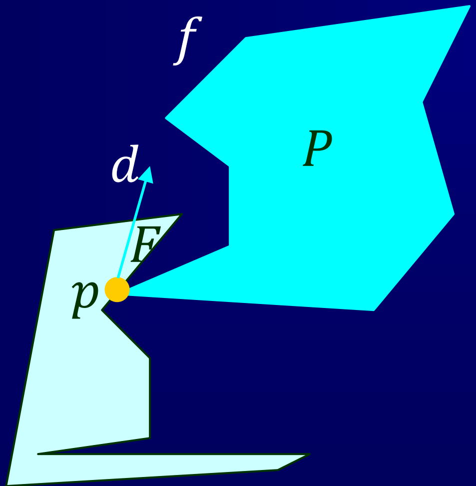

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Suppose P collides with the mold translating in direction d.

Let p be the point on P that collides with a facet F of the mold. p is about to move into the interior.

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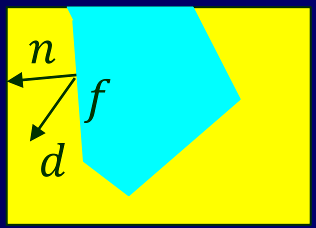

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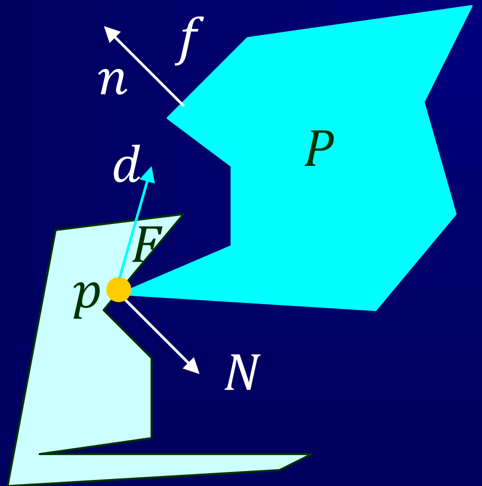

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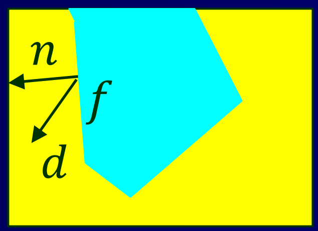

Suppose d makes an angle $< \pi/2$ with the outward normal n of some facet f.

Then any interior point of f collides with the mold in the translation.

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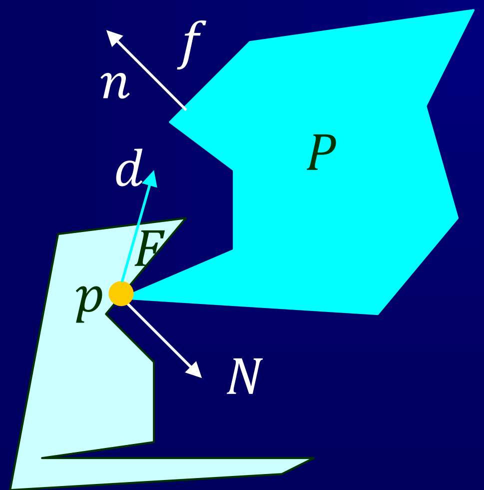

Suppose P collides with the mold translating in direction d.

Let p be the point on P that collides with a facet F of the mold. p is about to move into the interior.

The outward normal N of F makes an angle $> \pi/2$ with d.

Also a Sufficient Condition

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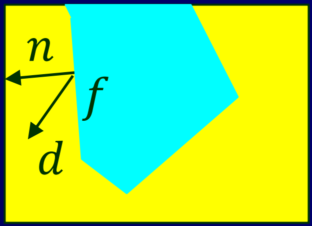

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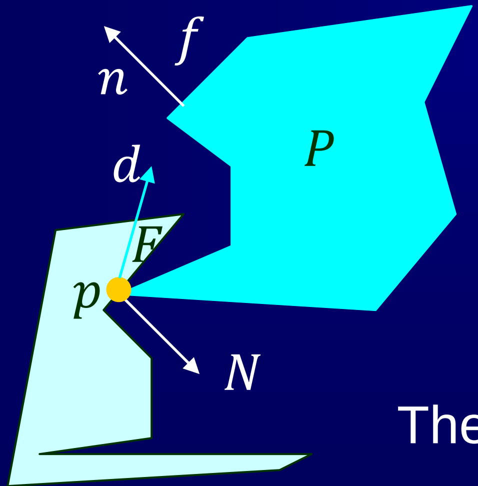

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One Translation vs. Multiple Translations

Polyhedron removable by a sequence of translations.

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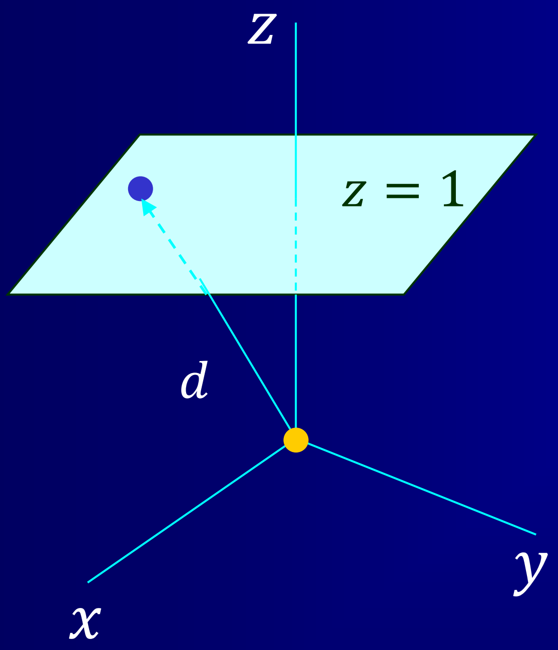
Allowing for multiple translations does not help.

II. Representing a Direction

d to make an angle $\geq \pi/2$ with the normal of every facet.



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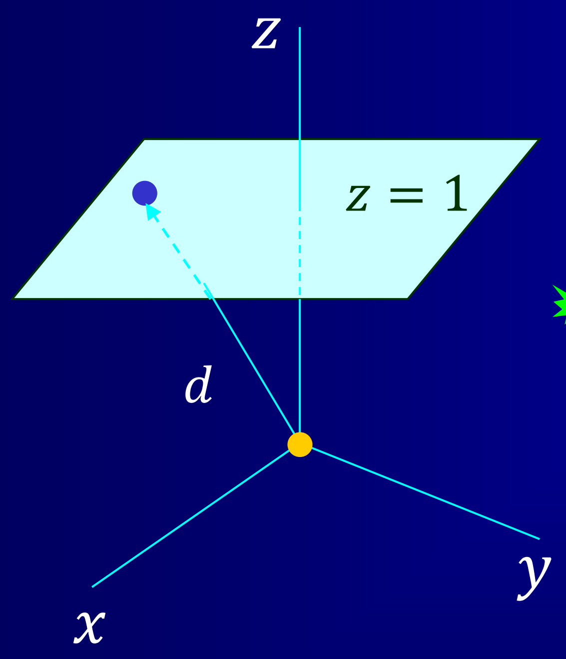

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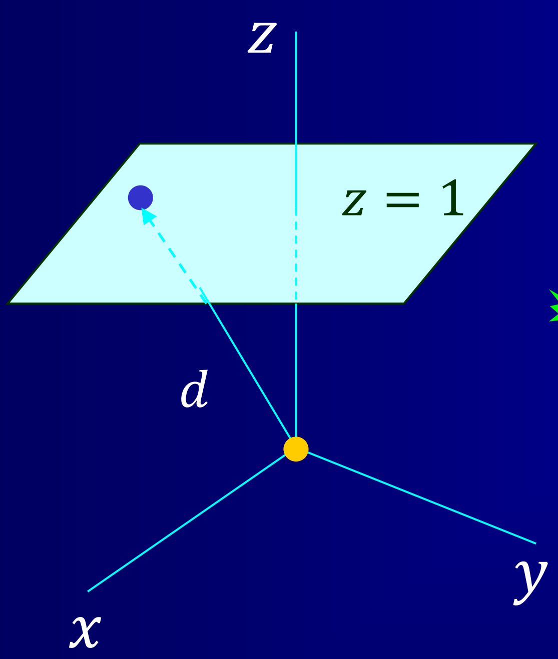
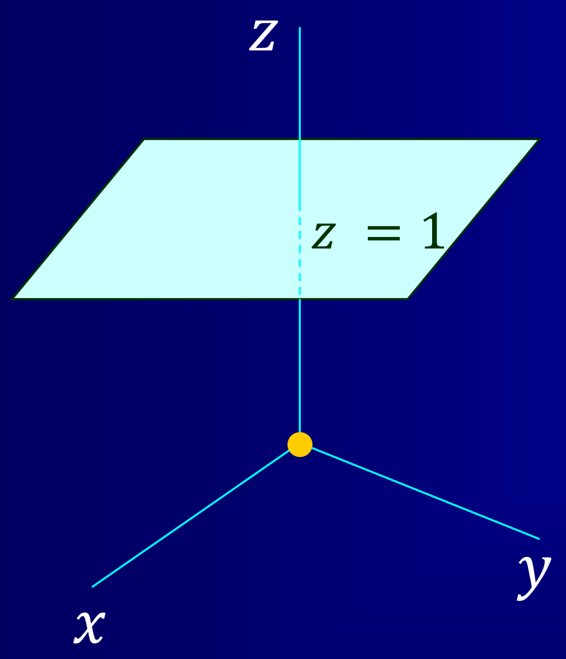


- Every point $(x, y, 1)$ represents a direction.

- The set of all directions with a positive z component is represented by the plane $z = 1$.

Geometric Interpretation

Let $d = (d_x, d_y, 1)$.



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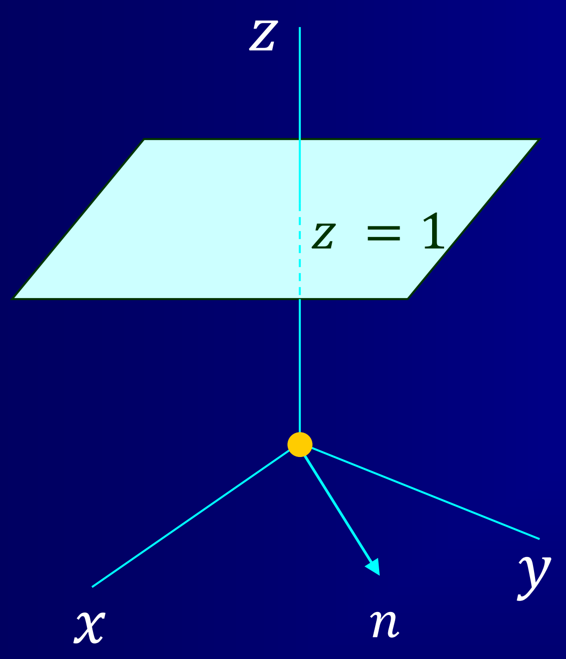

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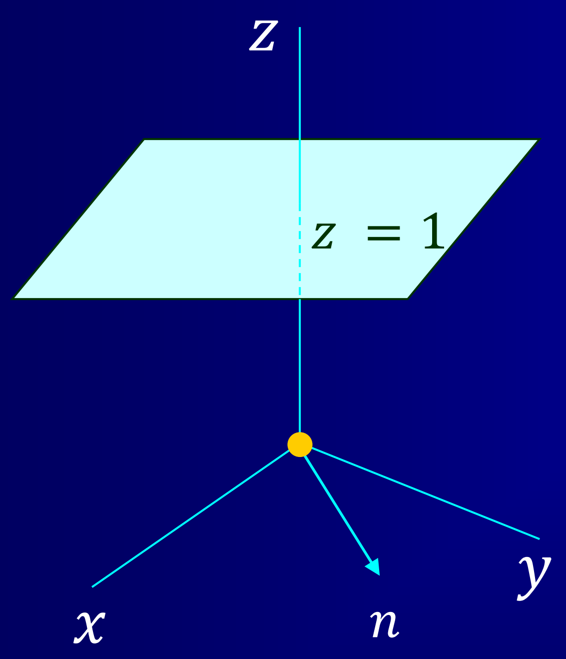
Geometric Interpretation



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$$n \cdot d = n_x d_x + n_y d_y + n_z \leq 0$$

variables

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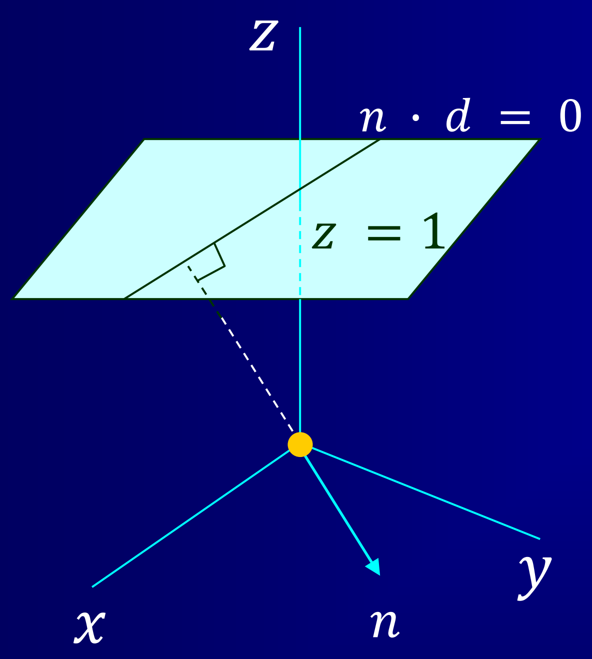

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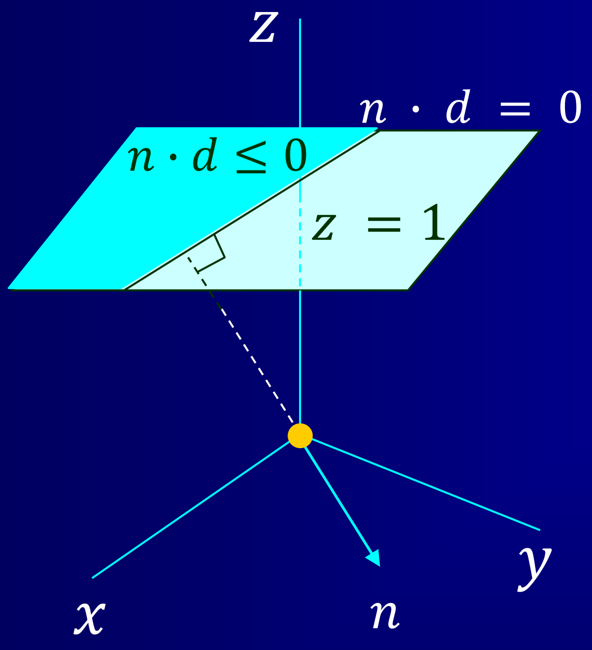

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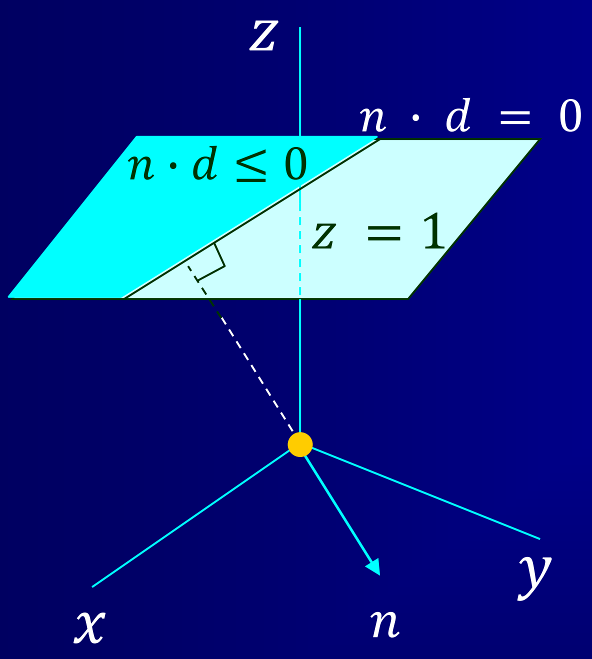

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- An area to one side of the line $n \cdot d = 0$ on the plane $z = 1$ (the d_x-d_y plane)

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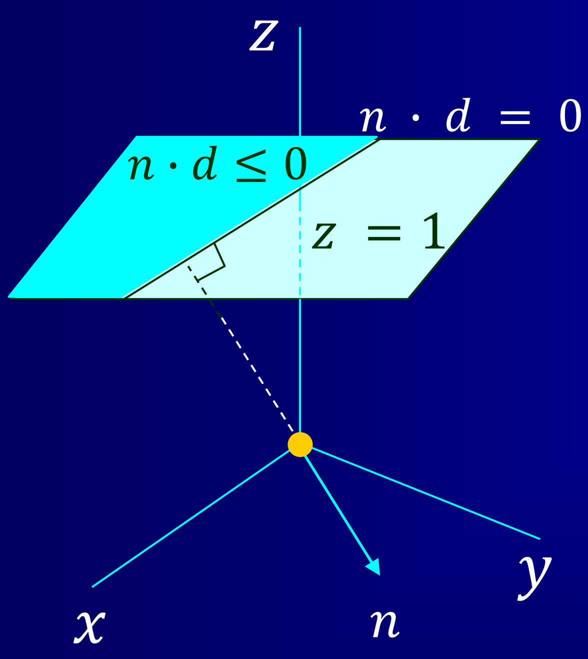

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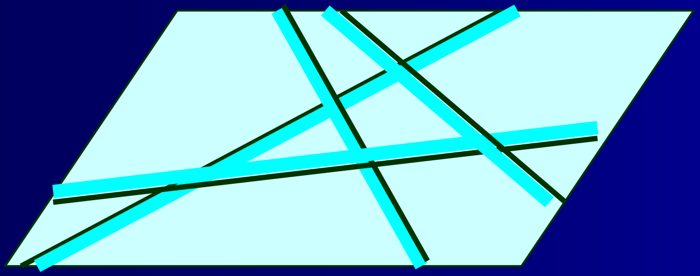
- An area to one side of the line $n \cdot d = 0$ on the plane $z = 1$ (the d_x-d_y plane)

- When the facet is horizontal ($n_x = n_y = 0$), the constraint is either true for all d or false for all d depending on n_z (easy to verify).

Geometric Formulation



Every non-horizontal facet defines a closed half-plane of $z = 1$.

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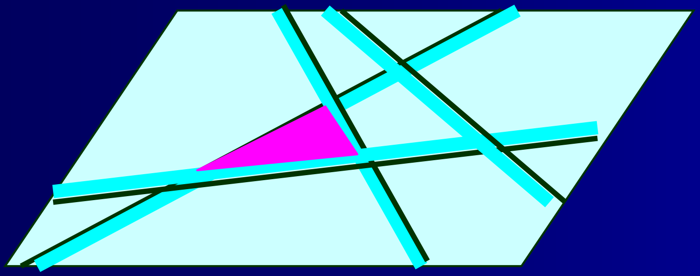

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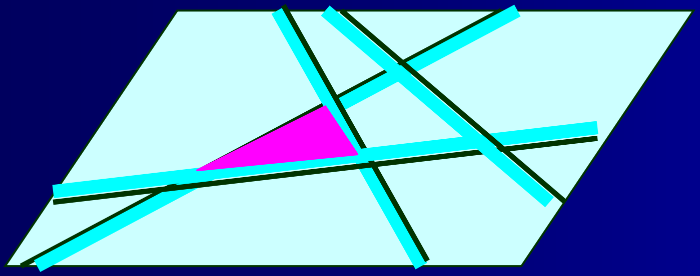

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The ***intersection*** of all such half-planes is the set of points that correspond to a direction in which the polygon can be removed.

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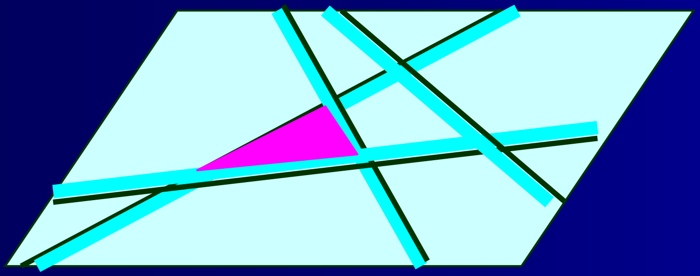

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Given a set of half-planes, compute their common intersection.

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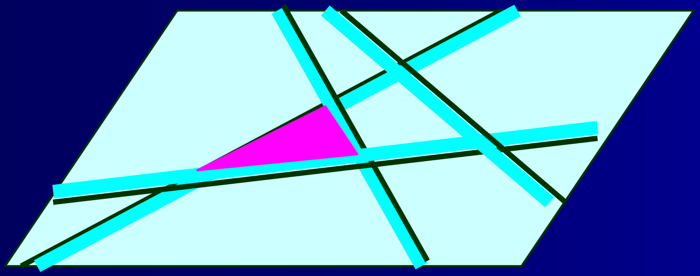

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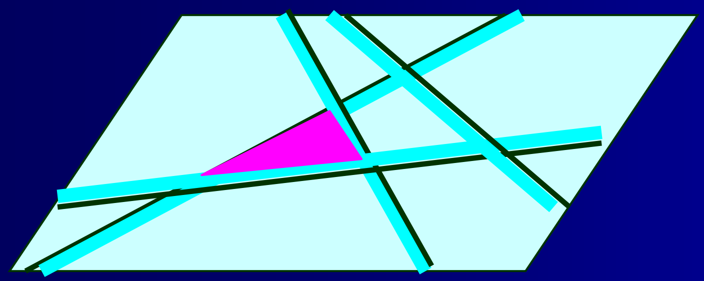

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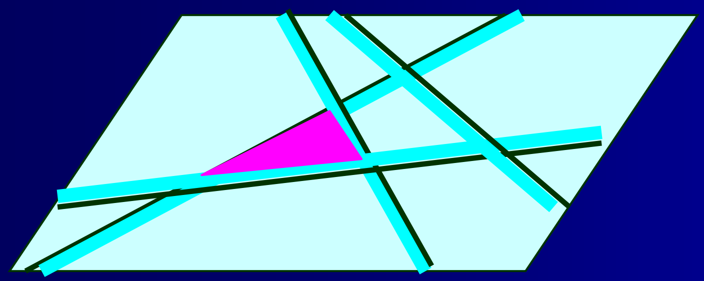

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- ★ This can be done in expected time $O(n^2)$ and $O(n)$ storage.
- ★ If P is castable, a mold and a removal direction can be computed within the same time bound.

III. Intersection of Half-Planes

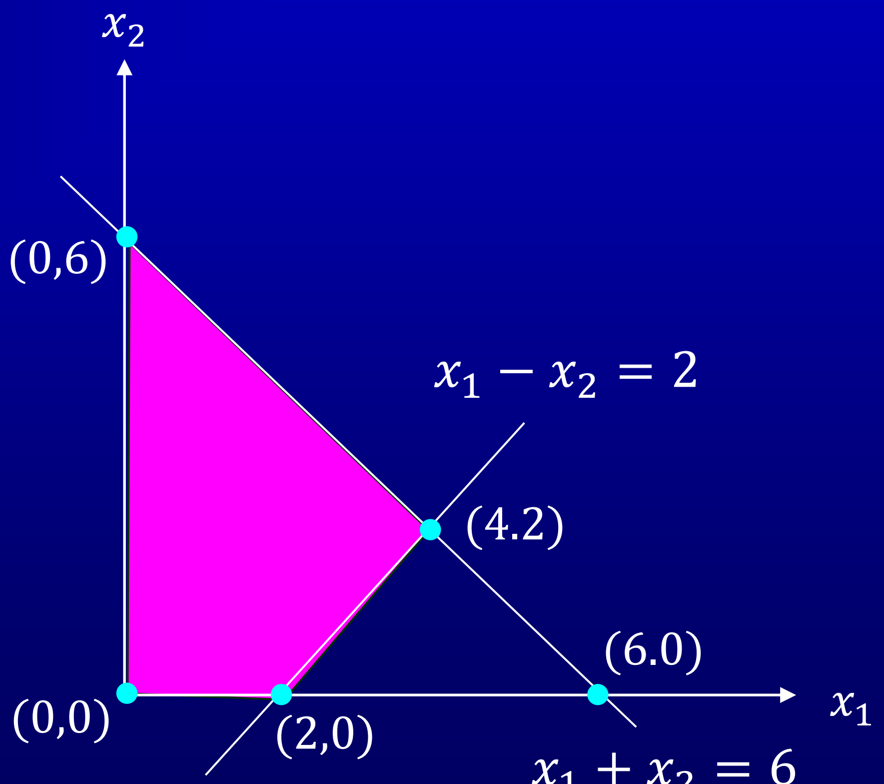
4 half-planes:

$$x_1 - x_2 \leq 2$$

$$x_1 + x_2 \leq 6$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$



General Problem

n half-planes:

$$h_i : \; a_i x + b_i y \le c_i \qquad 1 \le i \le n$$

each a convex set!

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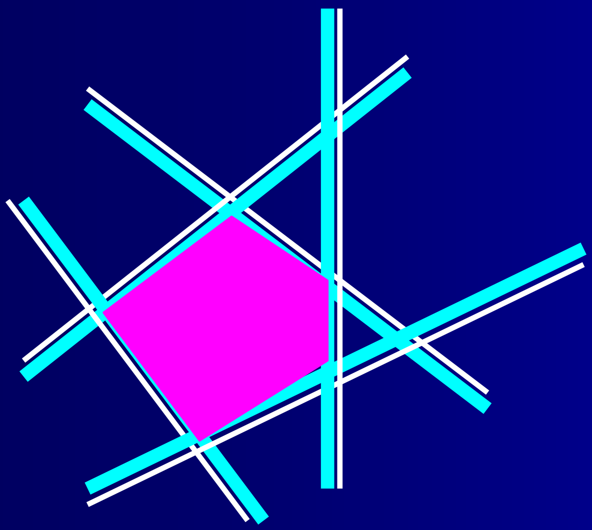
Their intersection must be a convex set:

- a convex polygonal region
- $\leq n$ edges
- possibly unbounded
- possibly degenerating into a line, segment, a point, or an empty set

Some Possible Cases



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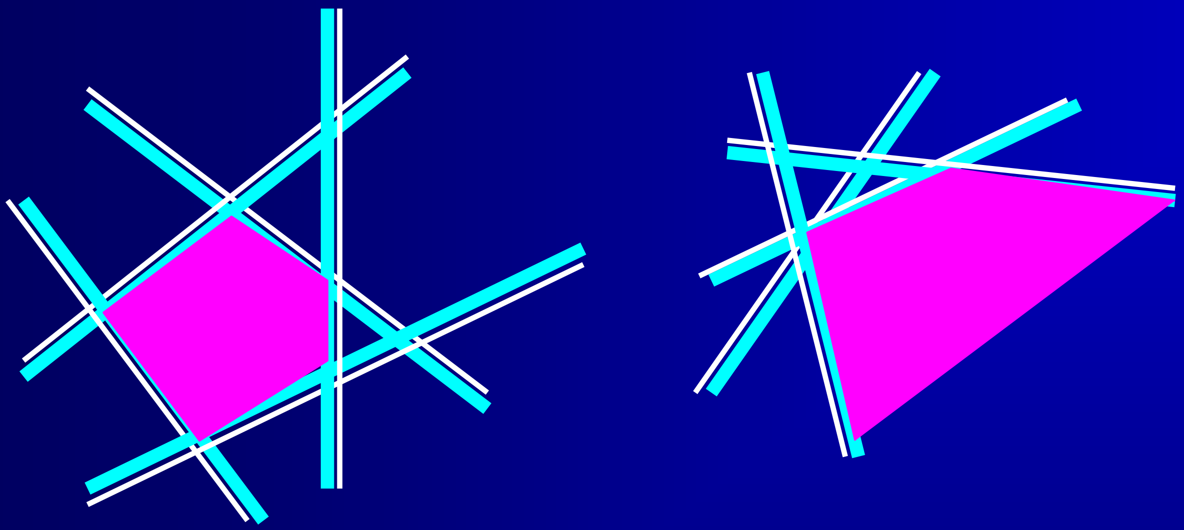

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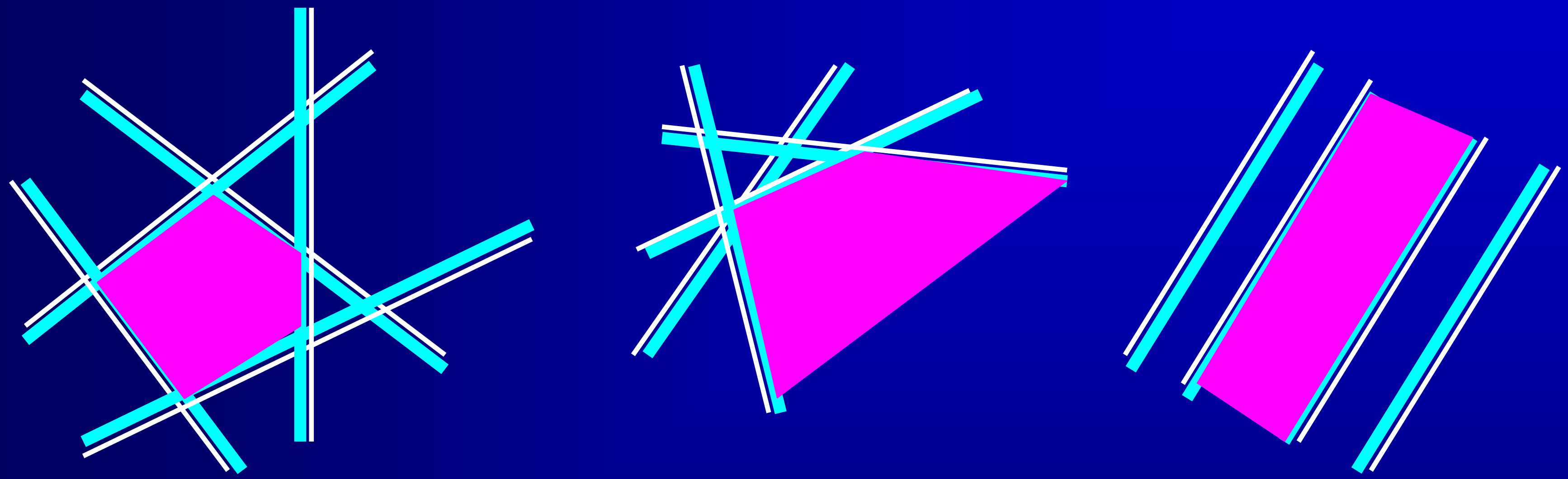

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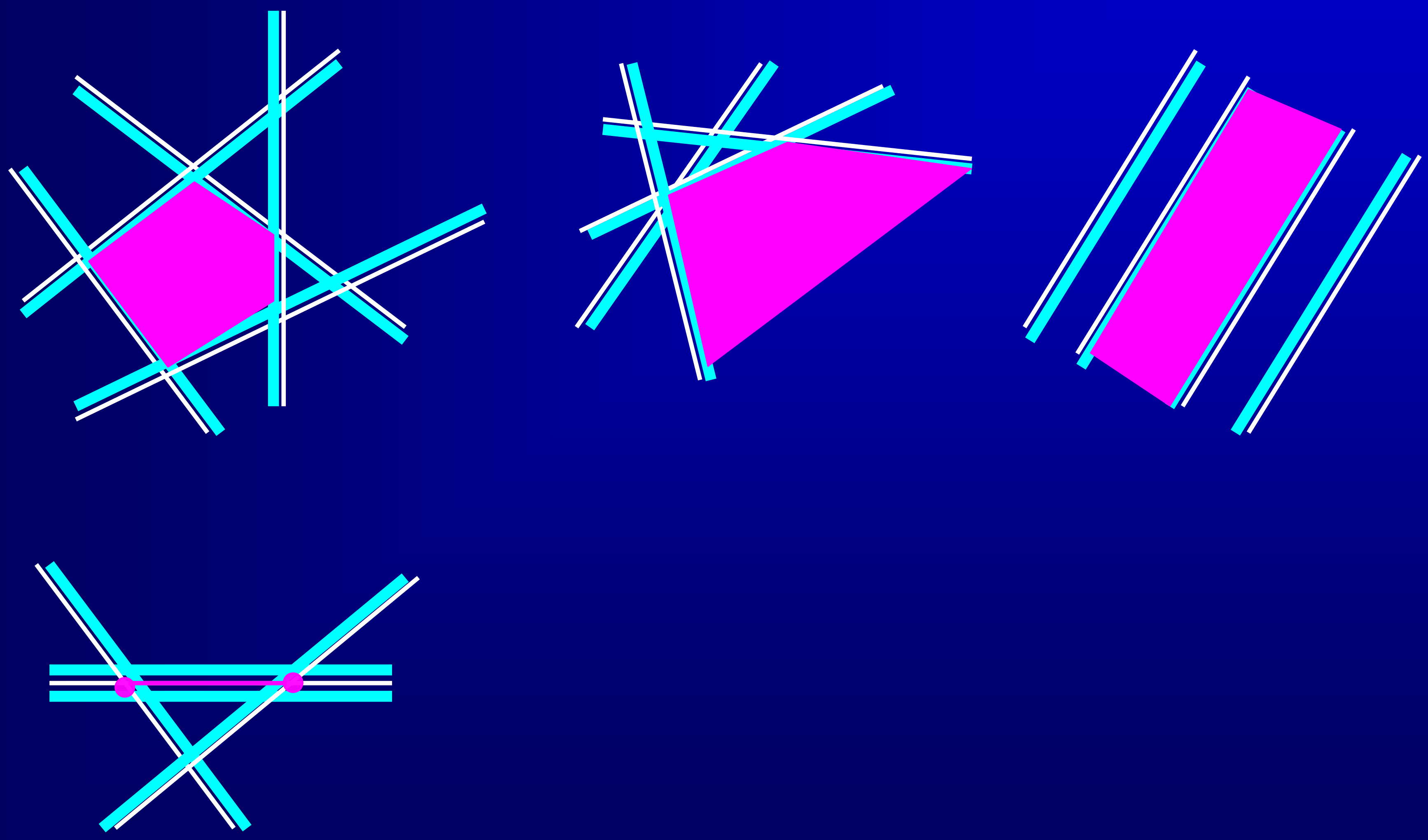

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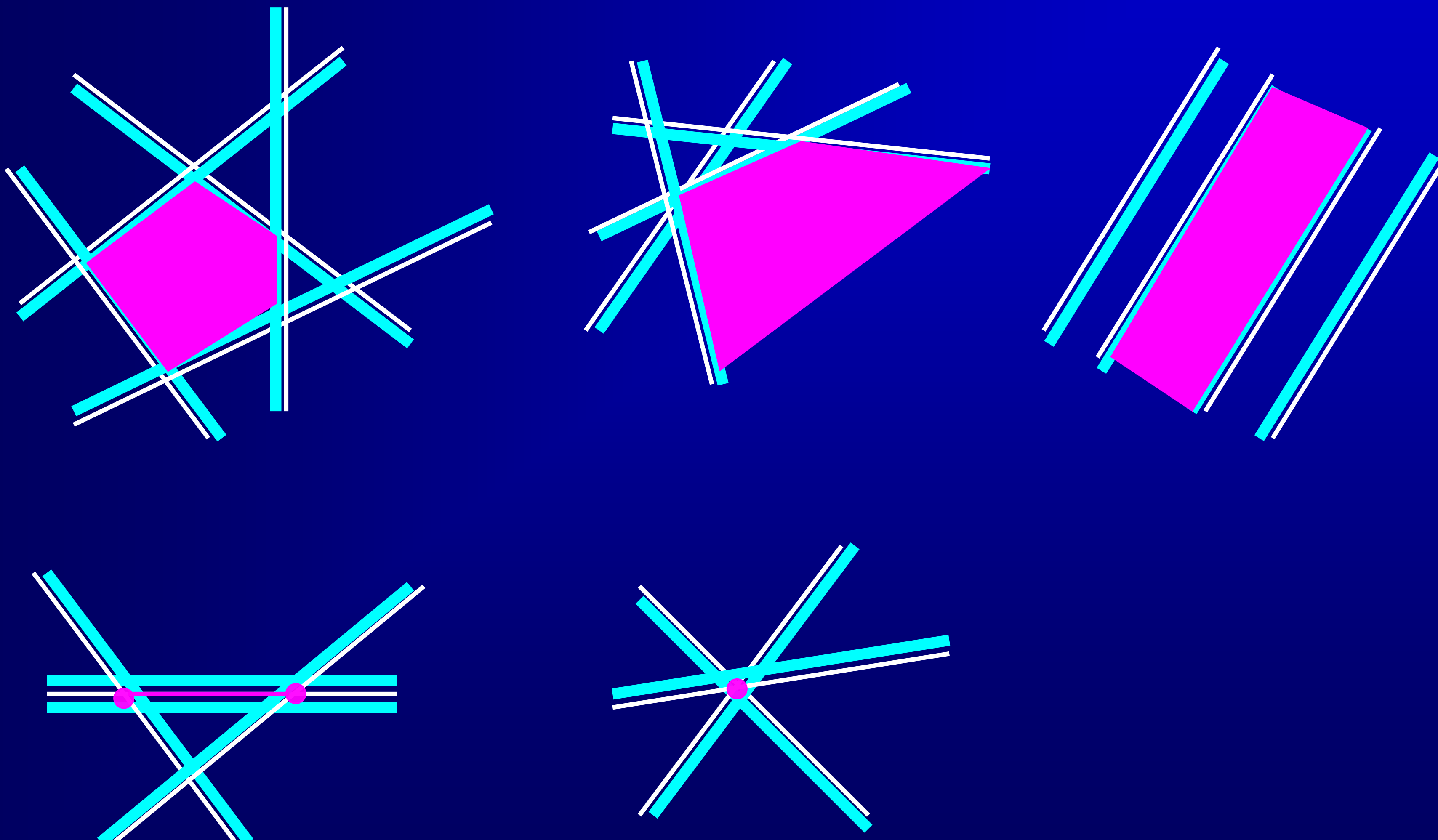

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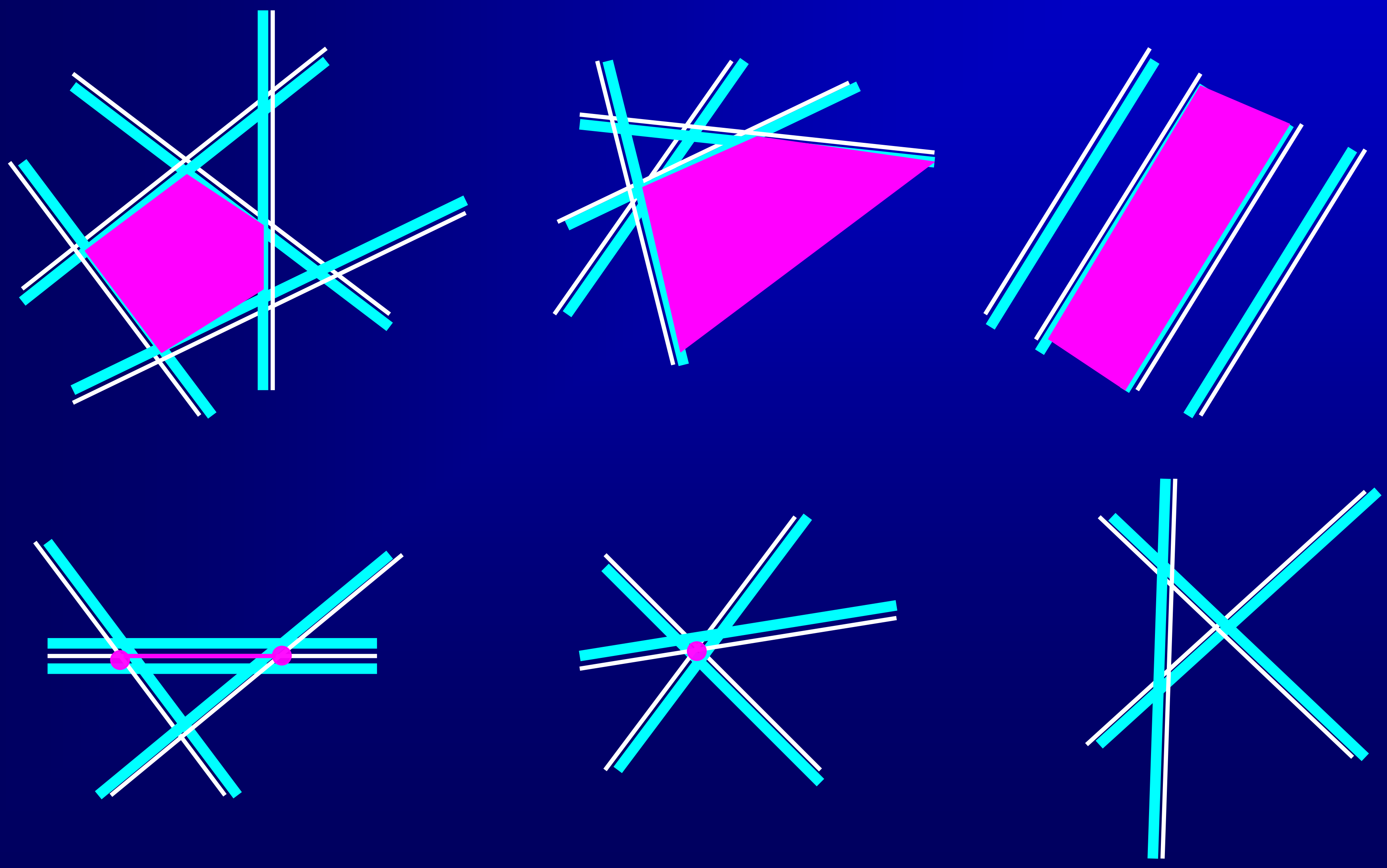

A Divide-and-Conquer Algorithm

```
IntersectHalfplane(H)
Input: A set H of n half-planes in the plane
Output: The convex polygon region C = ∩_{h∈H} h
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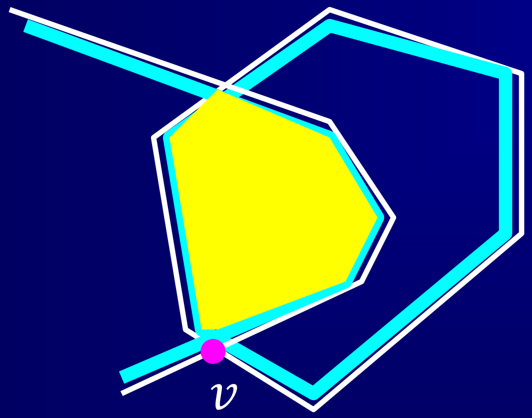

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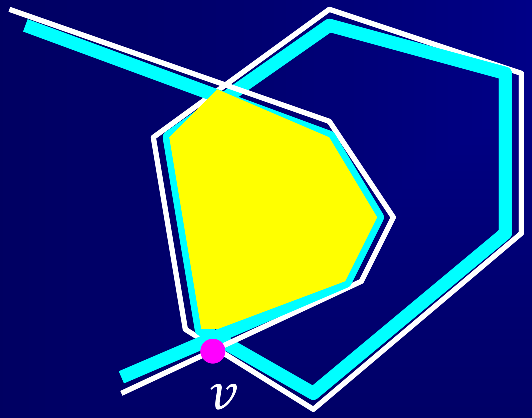

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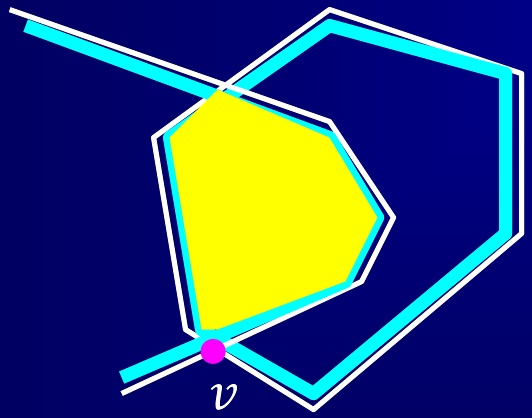

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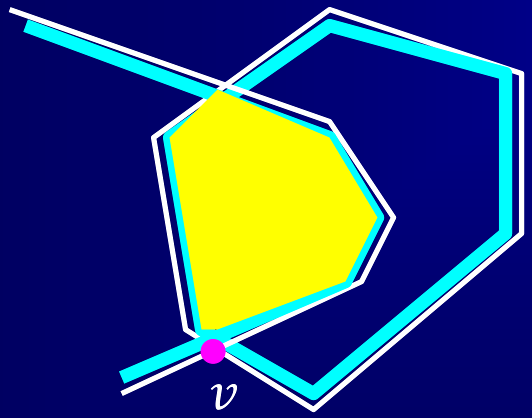

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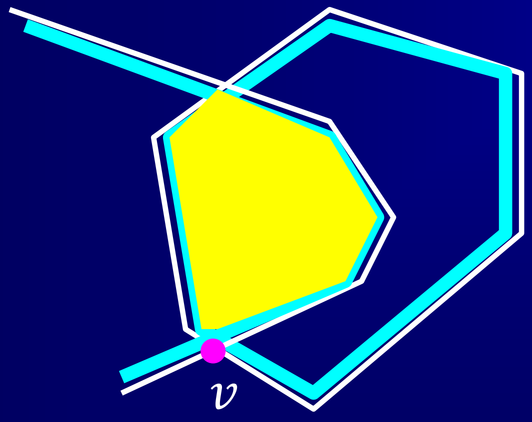

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$\Rightarrow$ IntersectConvexRegion takes time $O(n \log n)$.

The Recurrence

Let $T(n)$ be the running time.

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$\Rightarrow$ $T(n) = O(n\log^2 n)$

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Can we do better?

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- The subroutine for intersection of convex regions is a transplant from that for intersecting two simple polygons.

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Yes!

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- We made use of the *convexity* in our analysis.
- But we haven’t taken full advantage of convexity yet …

Yes!

Assumption (non-degeneracy):

The regions to be intersected are 2-dimensional.

Improvement

Can we do better?

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- We made use of the *convexity* in our analysis.
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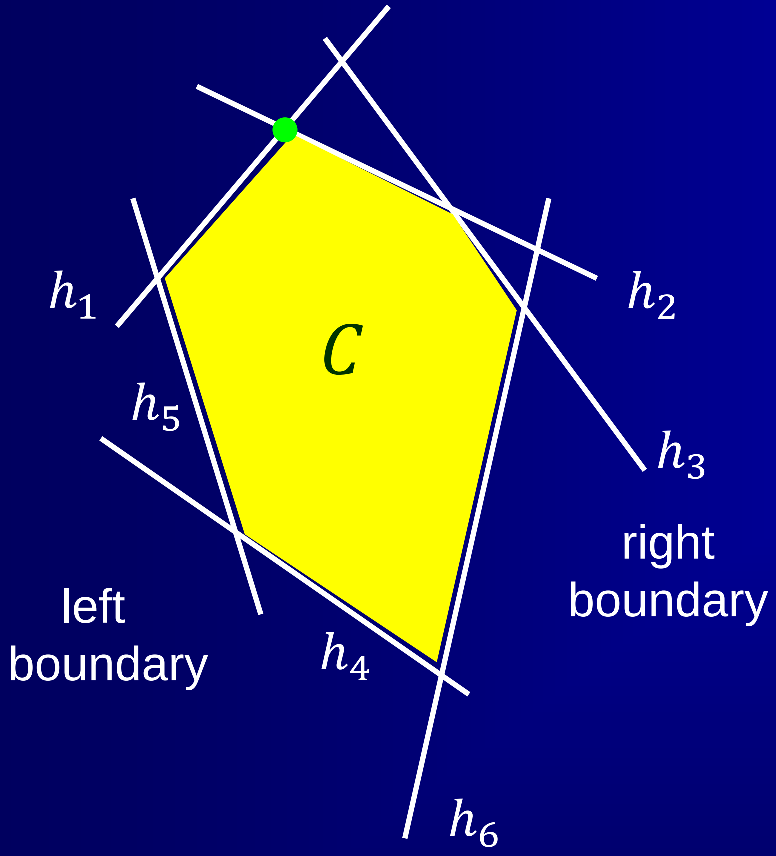
Yes!

Assumption (non-degeneracy):

The regions to be intersected are 2-dimensional.

(The degenerate cases are easier.)

Representing a Convex Region



Left and right boundaries as *sorted* lists of half-planes during traversals from top to bottom.

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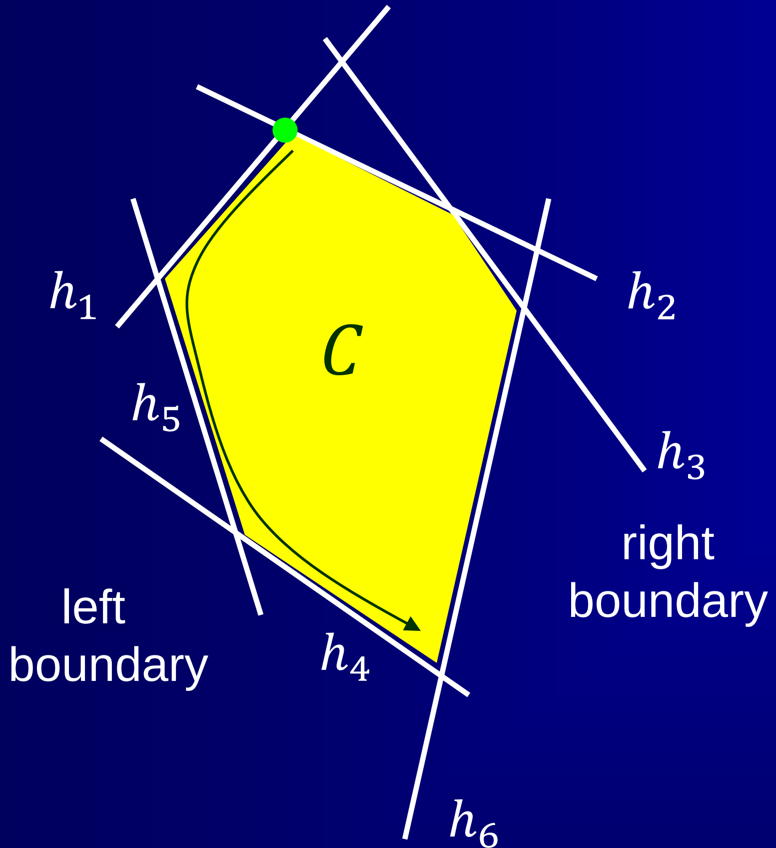

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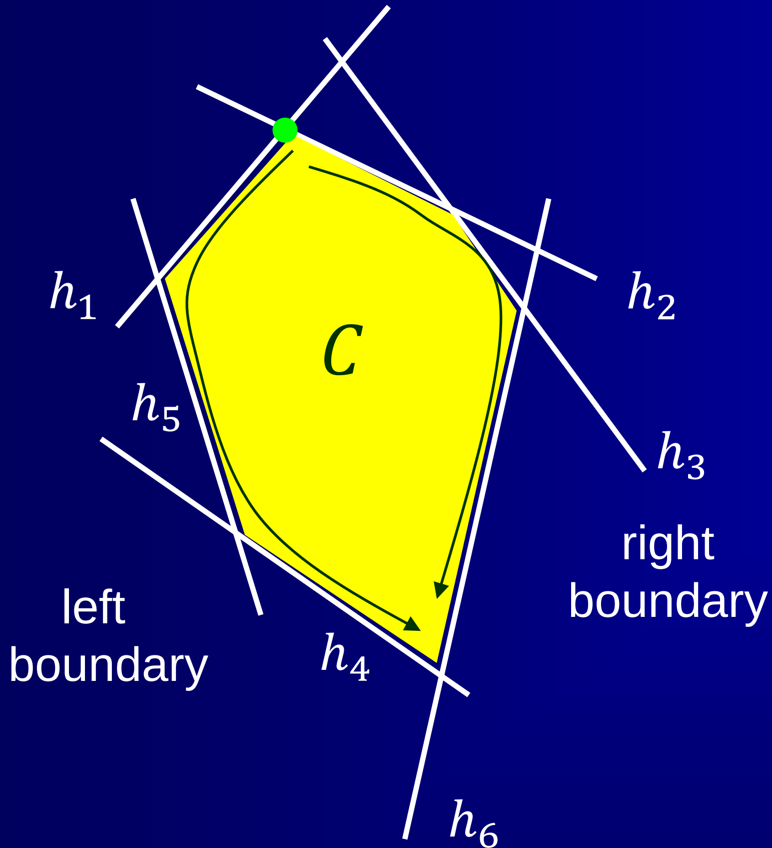

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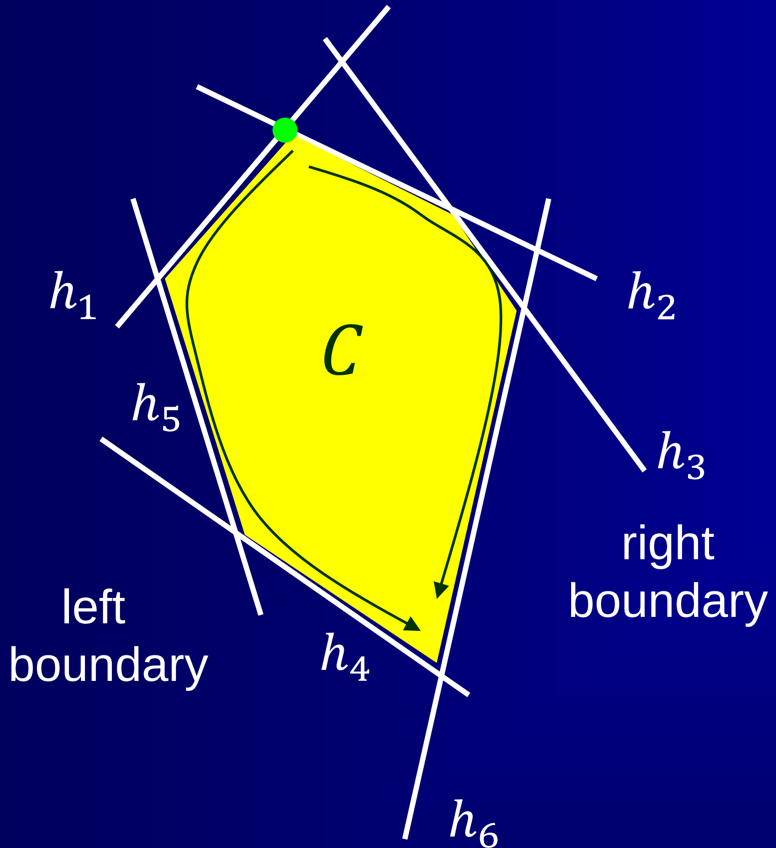

Left and right boundaries as *sorted* lists of half-planes during traversals from top to bottom.

Denote the two lists by L and R.

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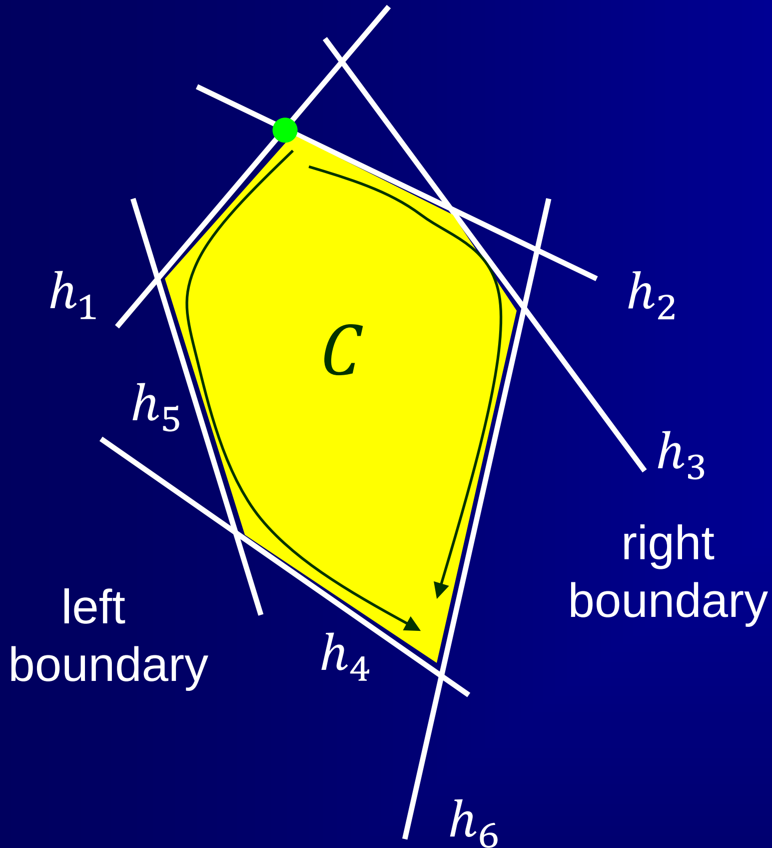

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$L(C)$: h_1, h_5, h_4

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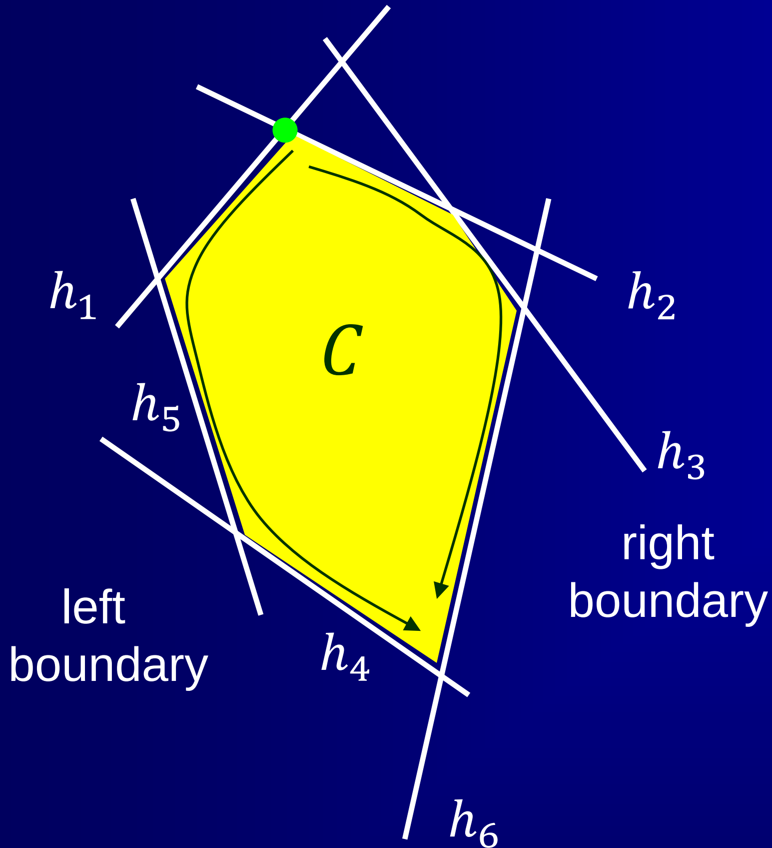

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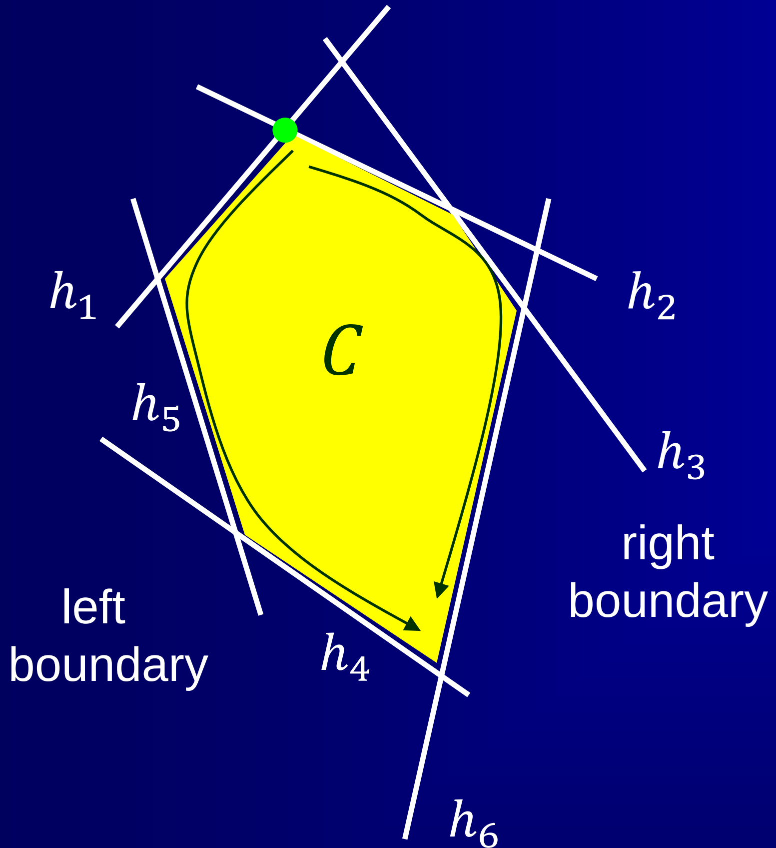

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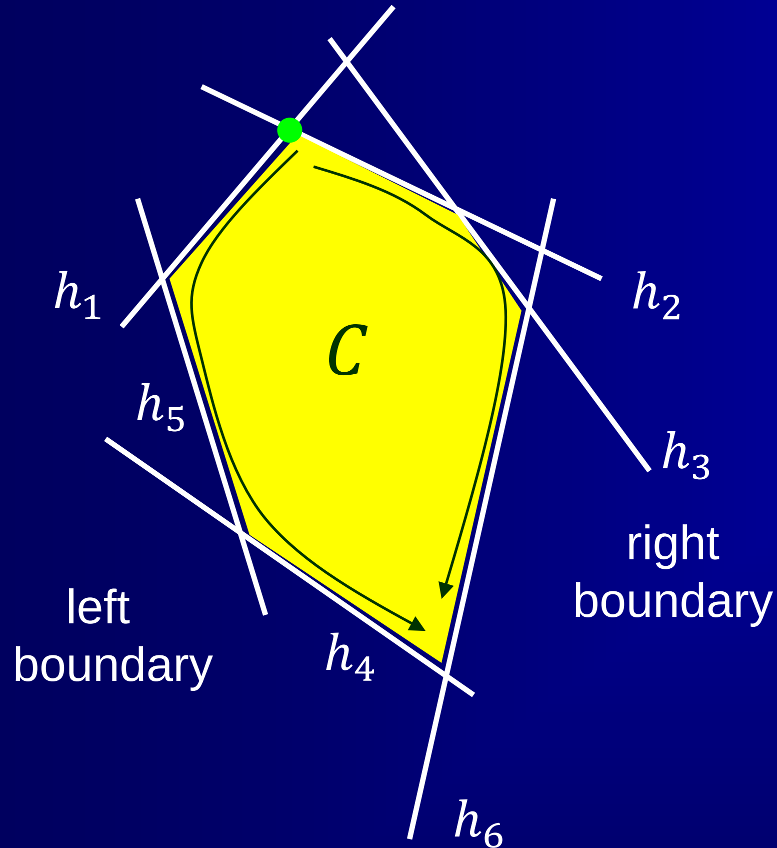
So they are not stored explicitly.

Representing a Convex Region



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A horizontal edge, if exists, belongs to the left boundary if bounding C from above and to the right boundary otherwise.

Plane Sweep Again

Assumption: no horizontal edge (easy to dealt with if not true).

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Sweep downward to merge two convex regions C_1 and C_2.

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- At most four edges intersecting the sweep line.

  *l_e_C*1, *r_e_C*1, *l_e_C*2, *r_e_C*2

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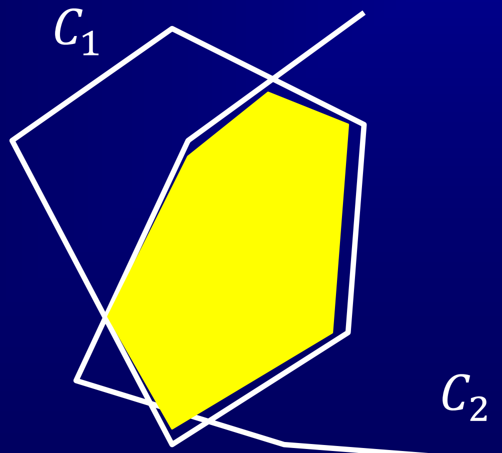

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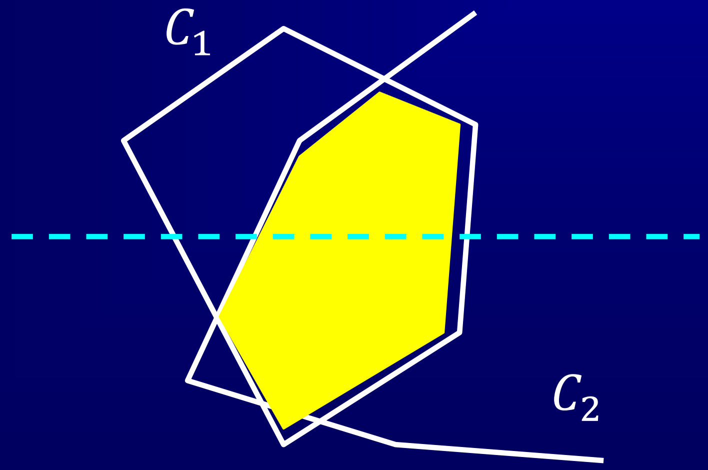

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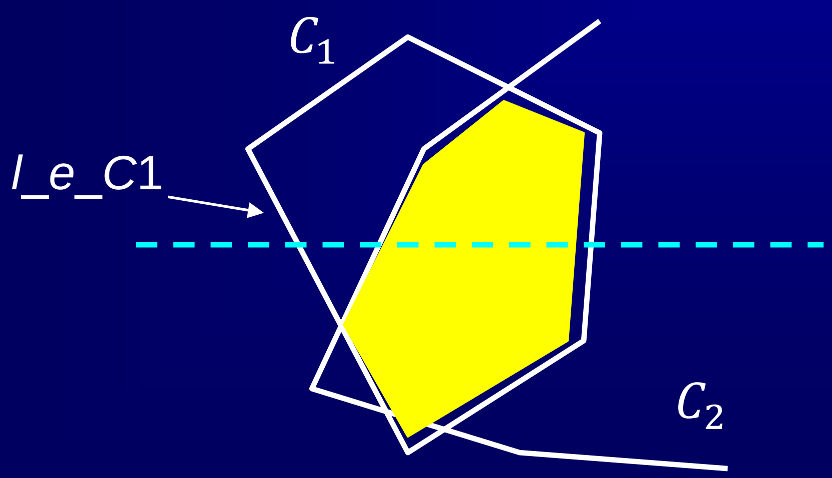

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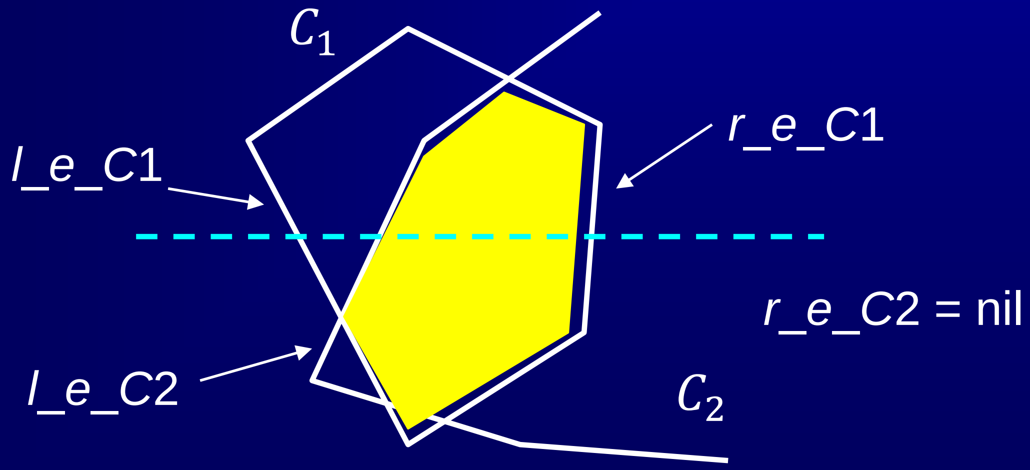

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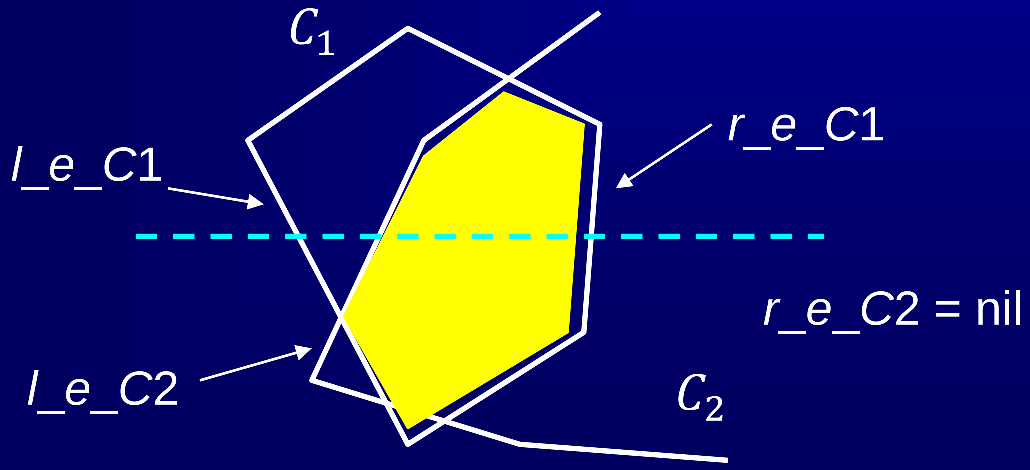
  *l_e_C*1, *r_e_C*1, *l_e_C*2, *r_e_C*2

- Corresponding pointer is set to nil if no intersection.



No Event Queue

Start at

the y-coordinate of the highest vertex of the two chains, or ∞ if one chain has one edge extending upward.

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Highest of the *lower* endpoints of the four edges that intersect the sweep line.

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The new edge e is one of the following:

1. part of C_1 and on the left chain
2. part of C_1 and on the right chain
3. part of C_2 and on the left chain
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Left Boundary of Chain 1

p: upper endpoint of e.

Three possible cases involving e and p in the intersection C:

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Three possible cases involving e and p in the intersection C:

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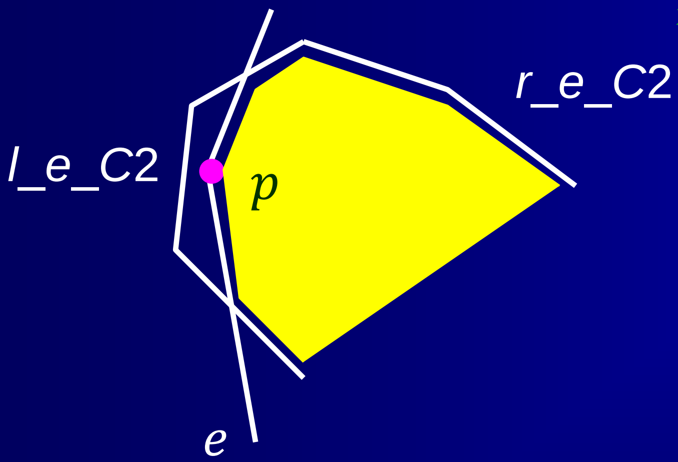

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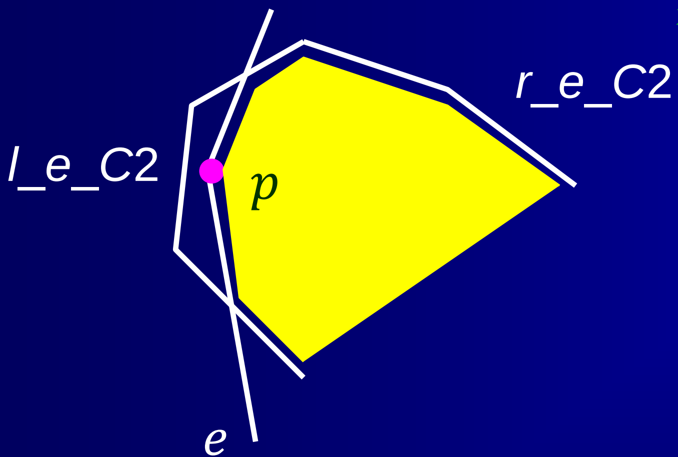

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  - ⇒ C has an edge with p as the upper endpoint.

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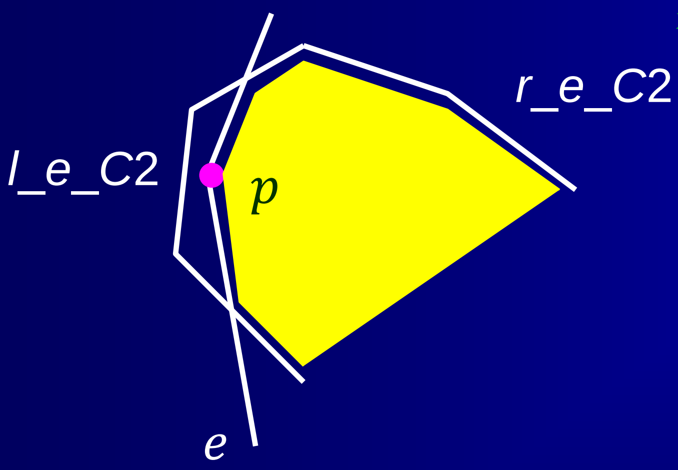

- p lies inside C_2

→ C has an edge with p as the upper endpoint.

This can be determined by checking whether p is between *l_e_C2* and *r_e_C2*.

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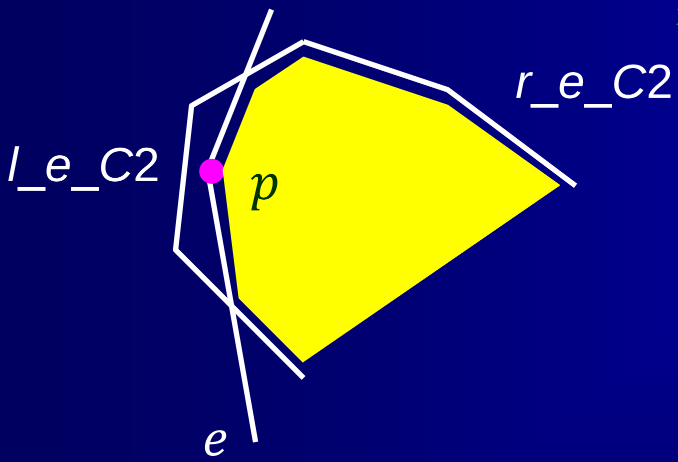

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➡ C has an edge with p as the upper endpoint.

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Add the half-plane with e part of its boundary to the list L.

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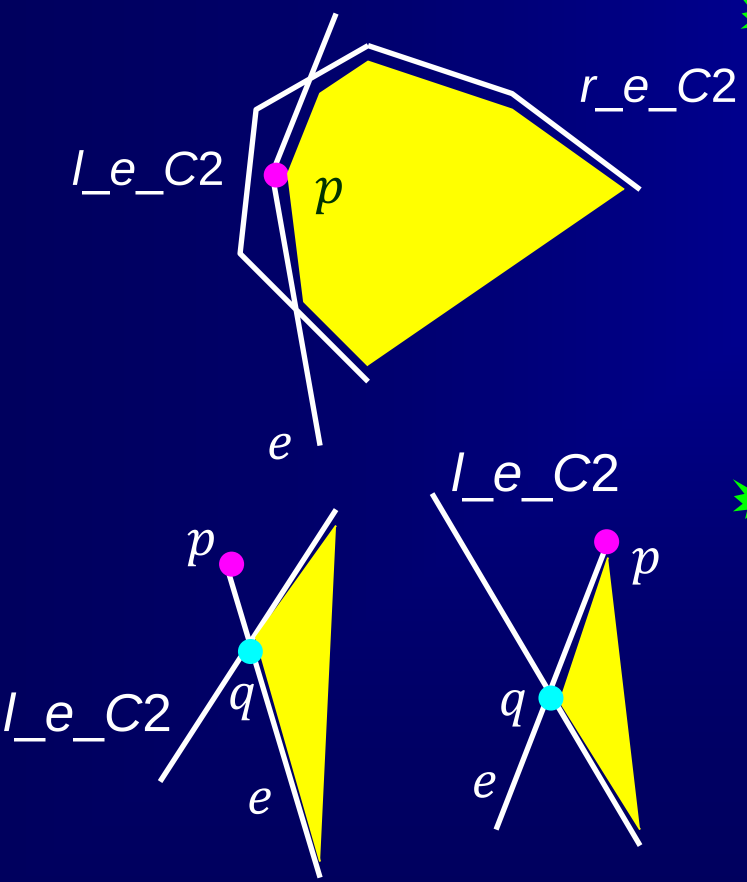

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- e intersects *l_e_C2*.

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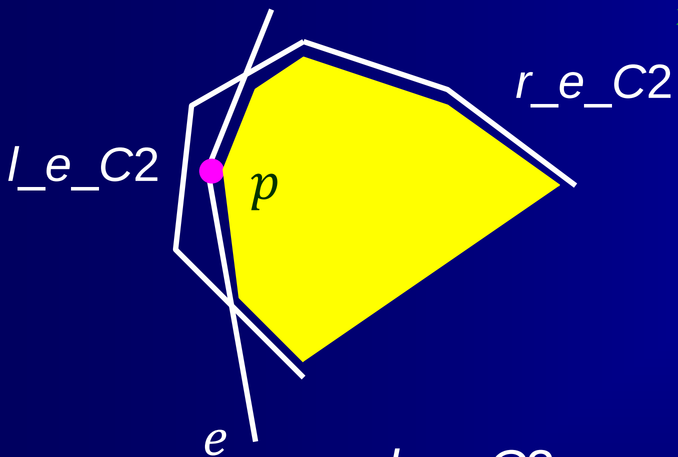

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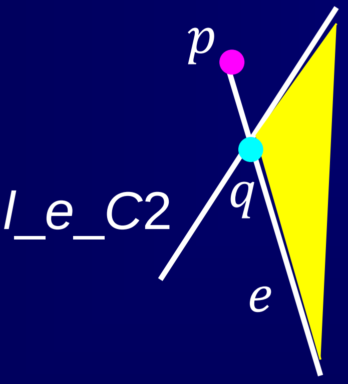

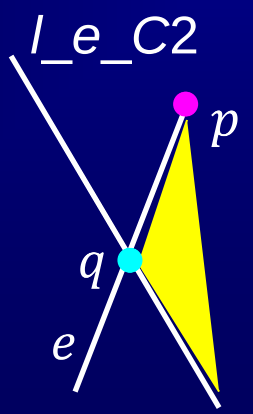

- e intersects l_e_C2.

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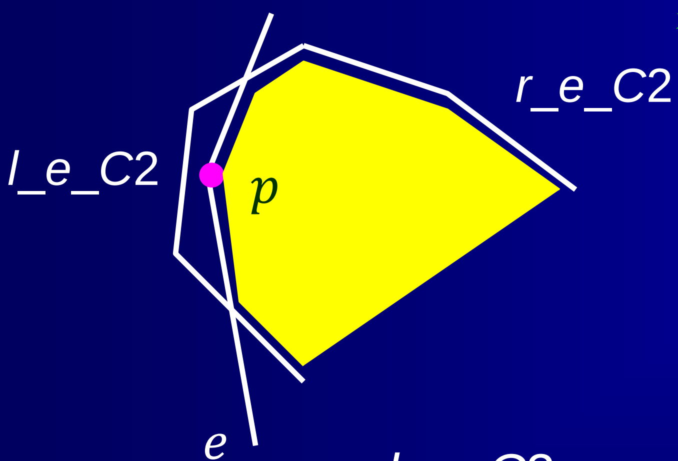

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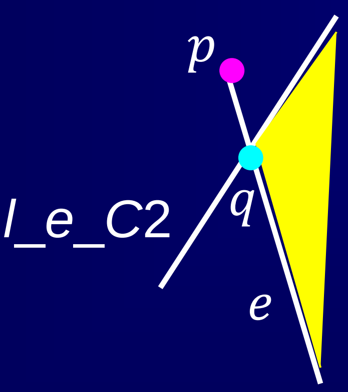

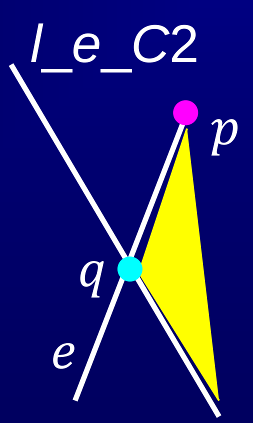

- e intersects *l_e_C2*.
  - ➡ The intersection q is a vertex of C.
  - ➡ The edge of C starting at q is part of either e (p outside of C_2) or *l_e_C2* .

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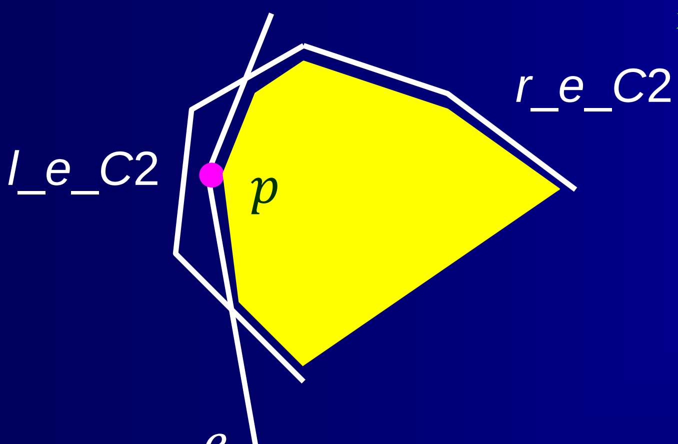

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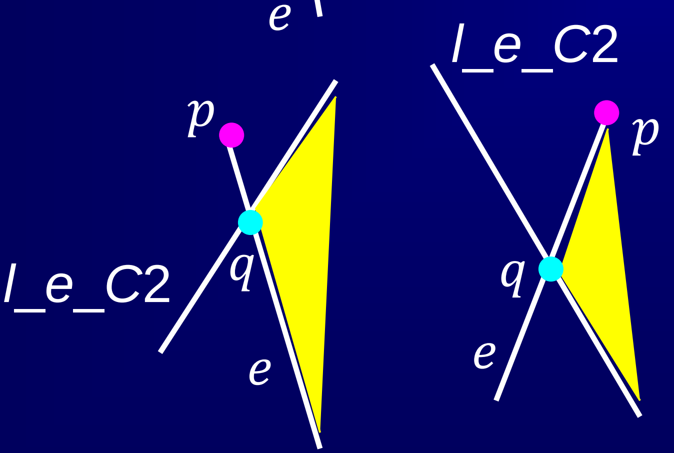
    Add the half-plane with e part of its boundary to the list $\boldsymbol{L}$.



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  Add the appropriate edge(s) to the list $\boldsymbol{L}$.

cont'd

- e intersects *r_e_C*2.

cont'd

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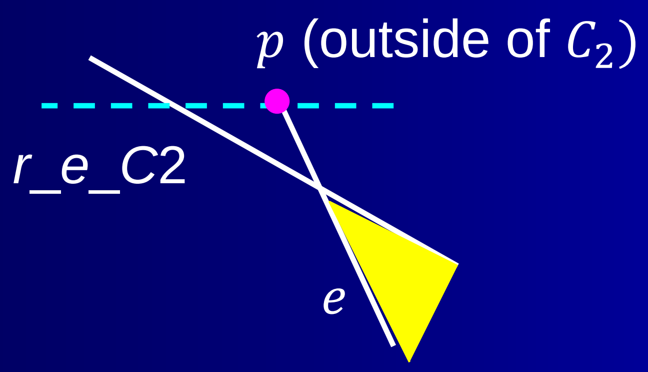

cont'd

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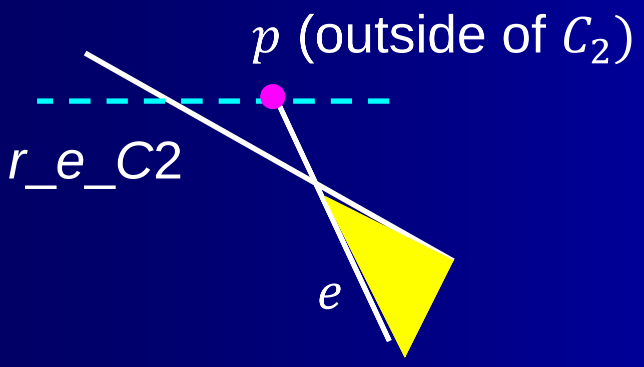

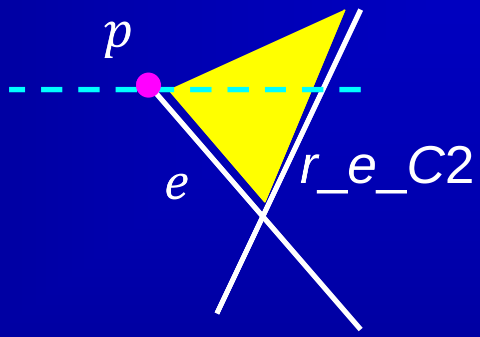

cont'd

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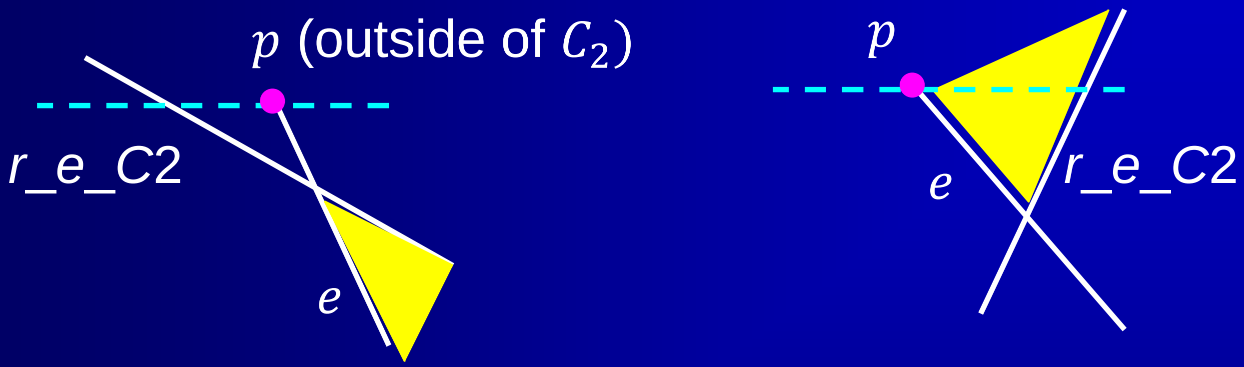

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cont'd

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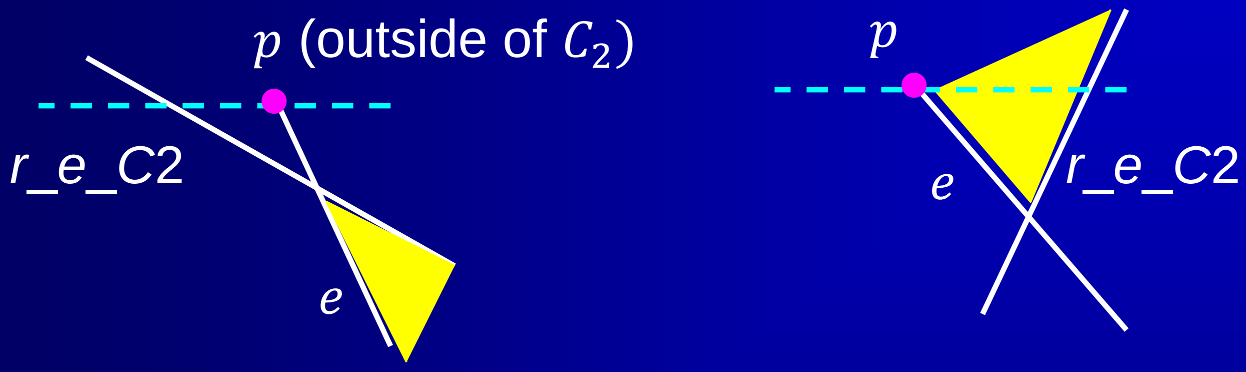

→ Each of e and *r_e_C2* contributes an edge to C at the intersection.

Case 1. The new edges start at the intersection.

cont'd

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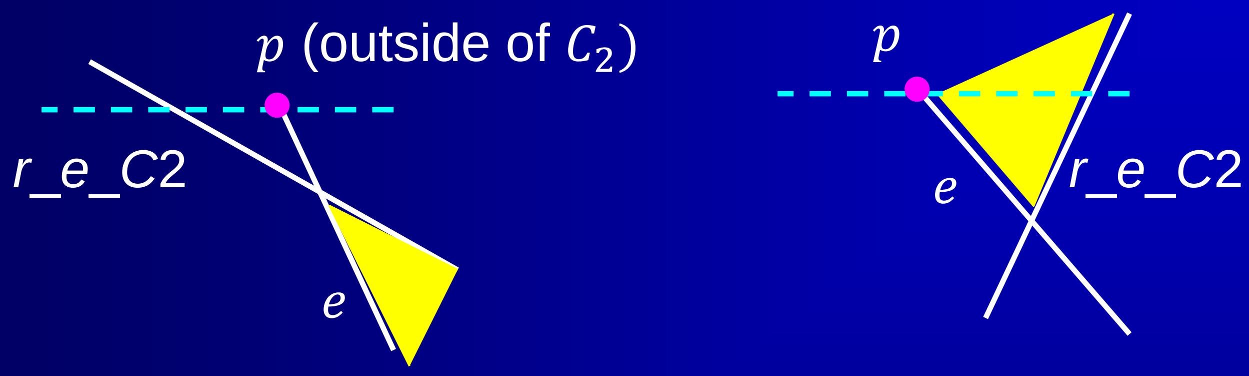

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Case 1. The new edges start at the intersection.
Add the half-plane defining e to L and
the one defining *r_e_C*2 to R.

cont'd

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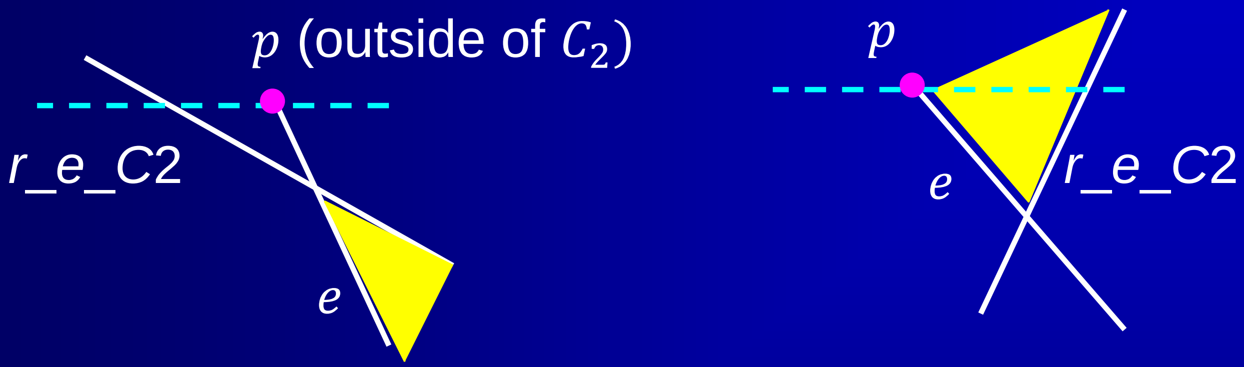

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Case 2. The new edges end at the intersection.

cont'd

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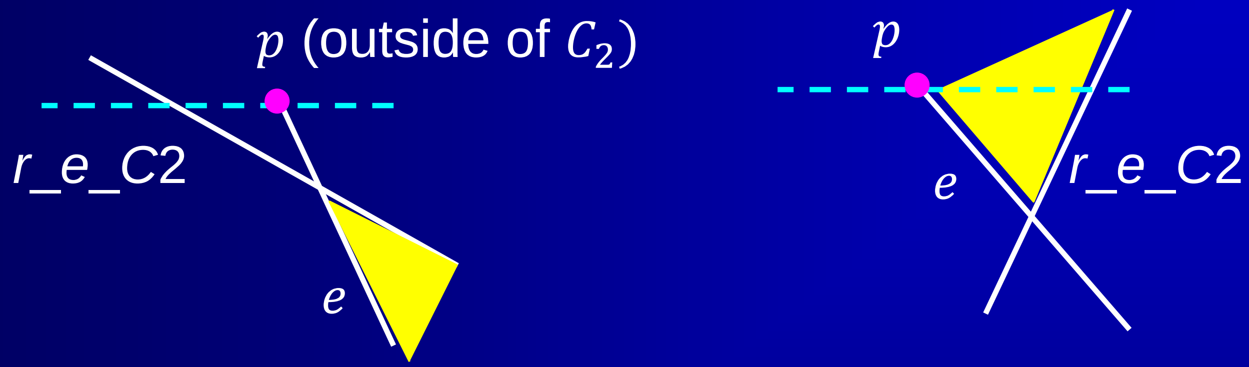

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Add the half-plane defining e to $\boldsymbol{L}$ and the one defining *r_e_C*2 to $\boldsymbol{R}$.

Case 2. The new edges end at the intersection.

Do nothing because these edges have been discovered.

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Theorem The common intersection of n half-planes in the plane can be computed in $O(n \log n)$ time and $O(n)$ storage.