Heuristic Functions

Outline

I. Properties of Heuristics

II. Variations of A* search

III. Generating heuristics

* Figures are from the textbook site (or drawn by the instructor) unless the source is specifically cited.

I. Admissible Heuristic

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- Whether it is optimal depends on the heuristic.

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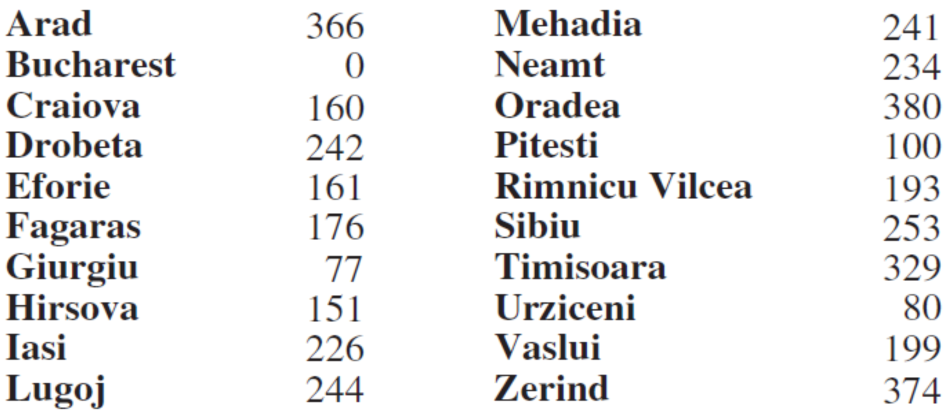

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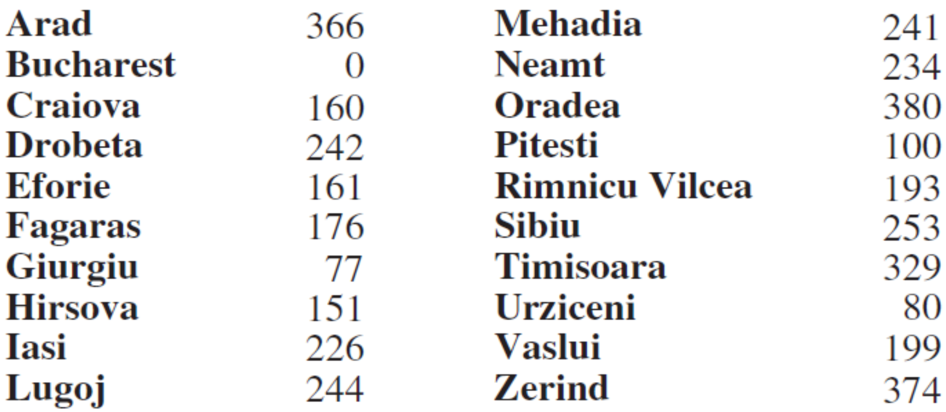

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h_{SLD} is admissible because the actual distance to Bucharest cannot be less than the straight-line distance.

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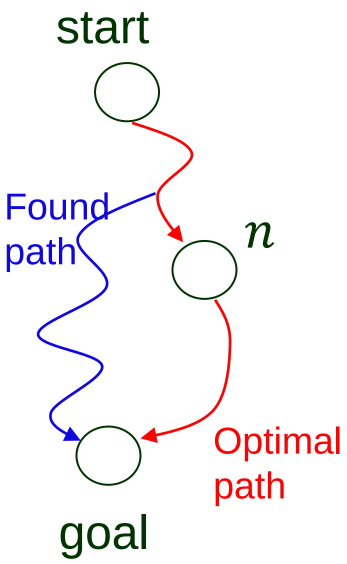

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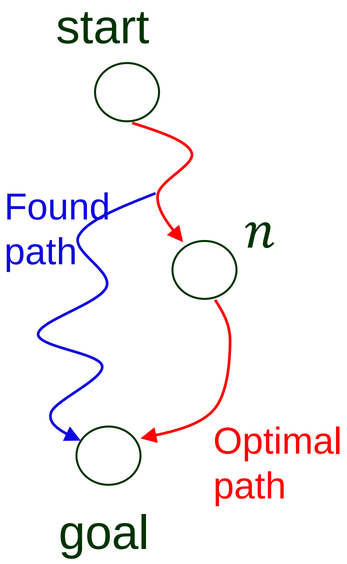

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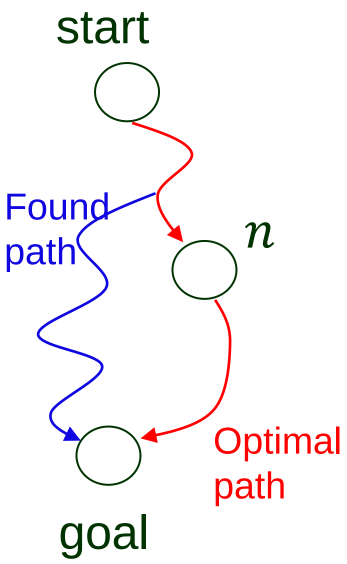

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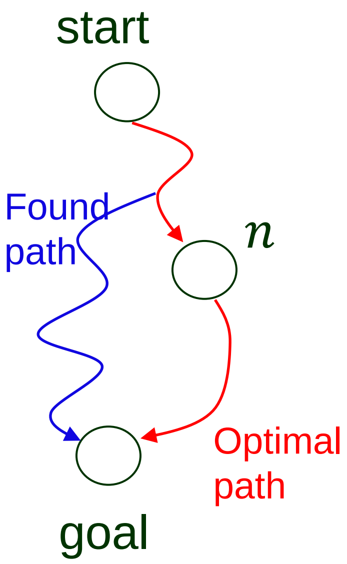

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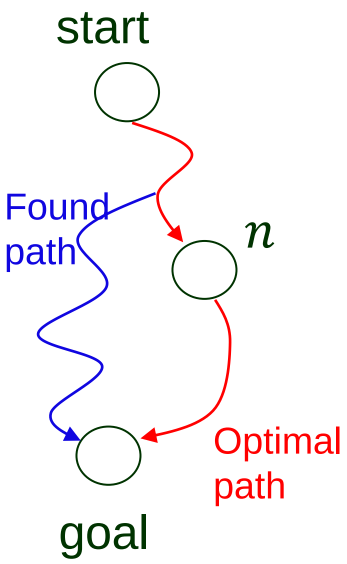

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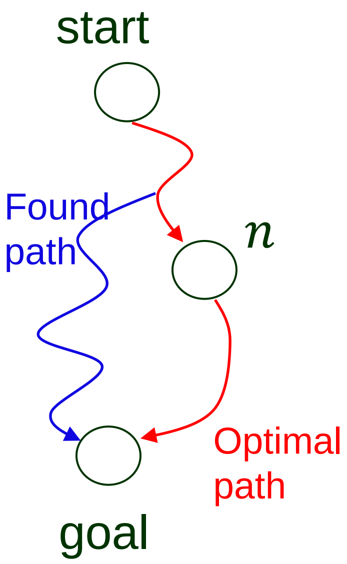

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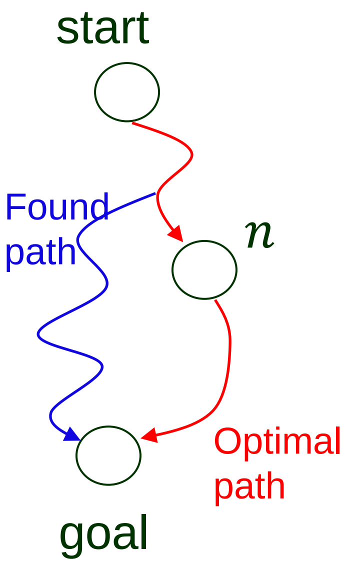

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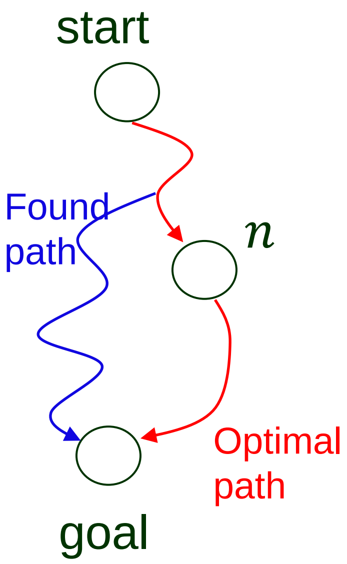

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But $$\begin{aligned} f(n) &= g(n) + h(n) \\ &= g^*(n) + h(n) && (n \text{ on the optimal path}) \\ &\le g^*(n) + h^*(n) && (h(n) \le h^*(n) \text{ due to admissibility}) \\ &= C^* \end{aligned}$$

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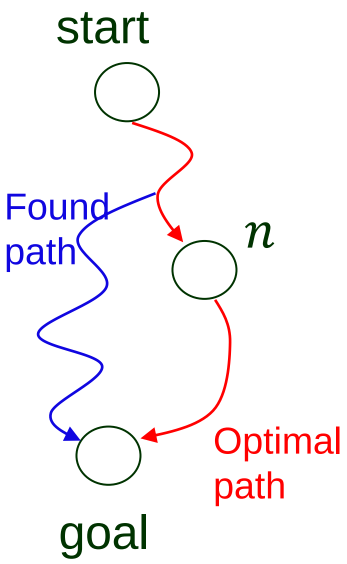

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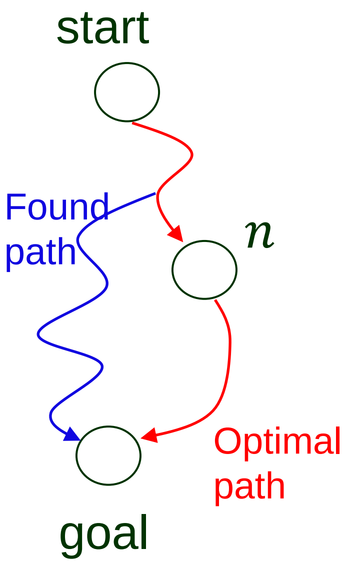

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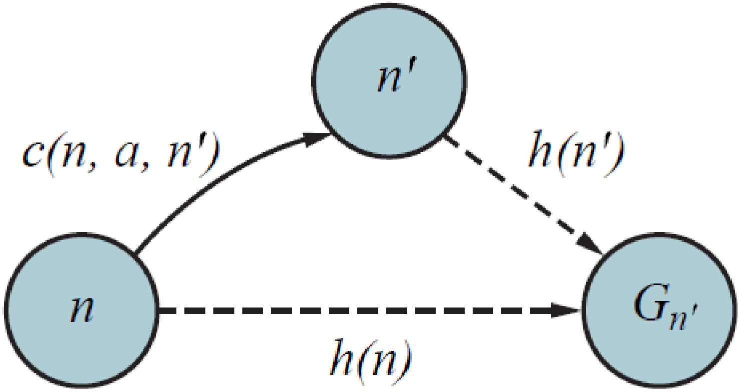
Consistent Heuristic

A heuristic h is *consistent* if for every two nodes n and n' such that n' is generated from n by some action a, the following inequality holds:

$$h(n) \leq c(n, a, n') + h(n')$$

(*triangle inequality*)



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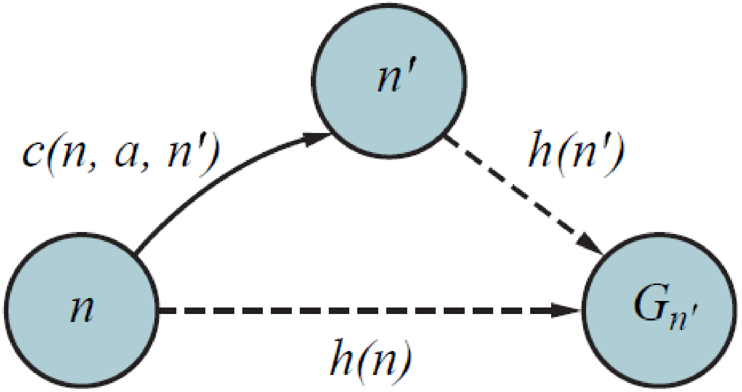

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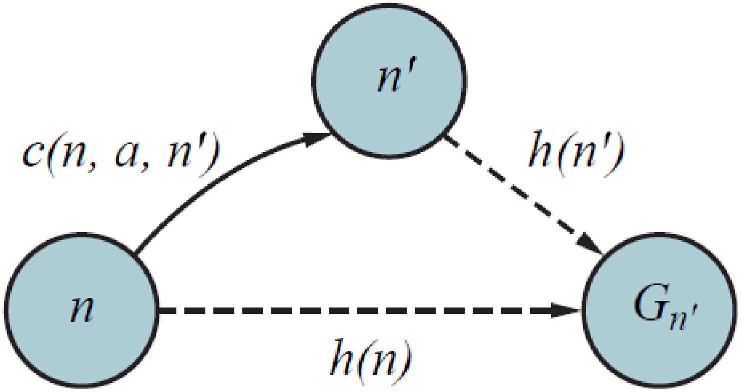


- Every consistent heuristic is admissible.
- A* with a consistent heuristic is cost-optimal.

What If the Heuristic Is Inadmissible?

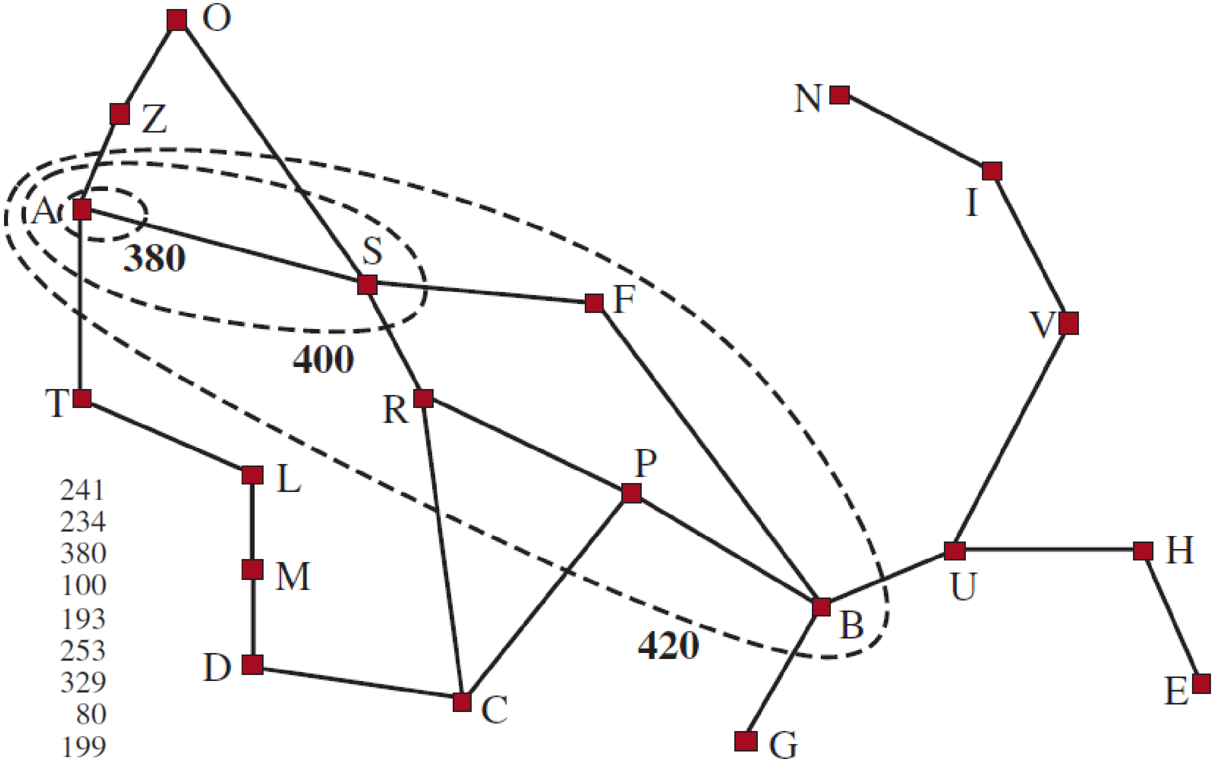
In such a situation, A* may or may not be cost-optimal.

Two of the cases where A* finds an optimal path:

- h is admissible for all the nodes on one optimal path.
- $h(n)$ does not overestimate the cost on each node n by more than the difference between the costs of the optimal and the second-best paths.

Search Contours

A *contour* labeled by a cost c encloses all the nodes n with $f(n) = g(n) + h(n) \leq c$.



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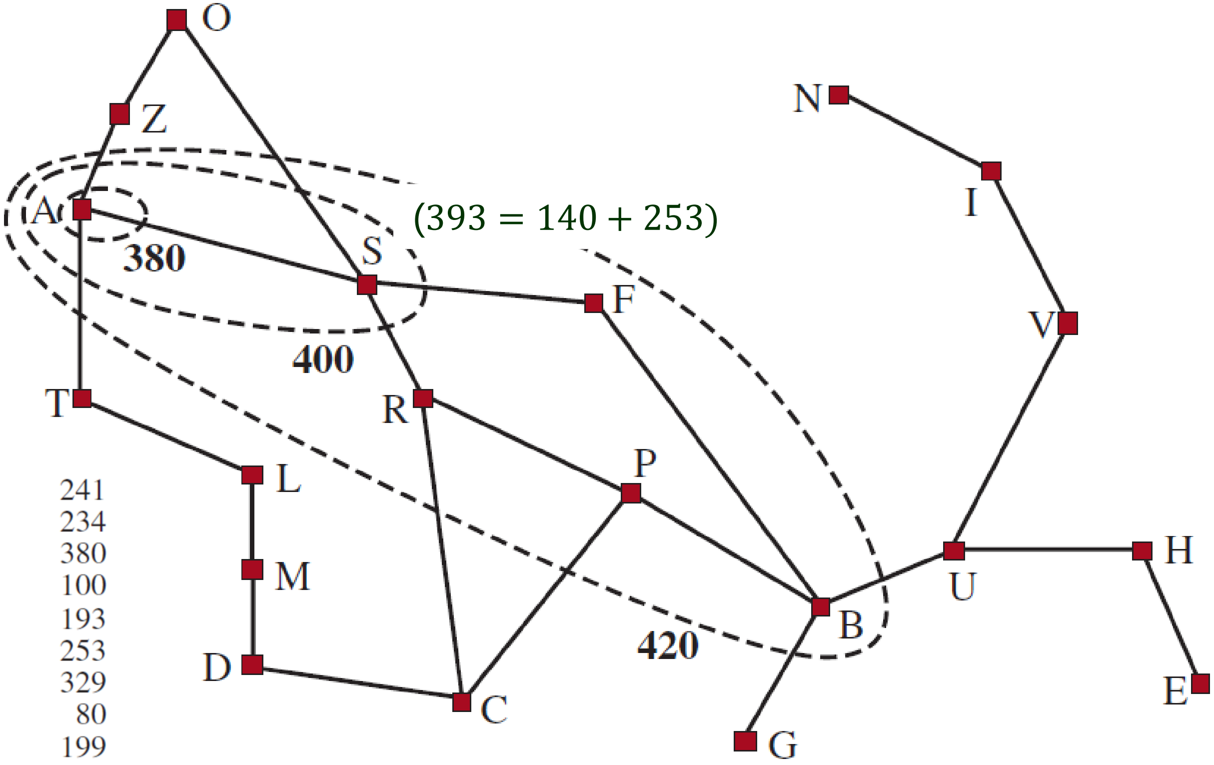

$(393 = 140 + 253)$

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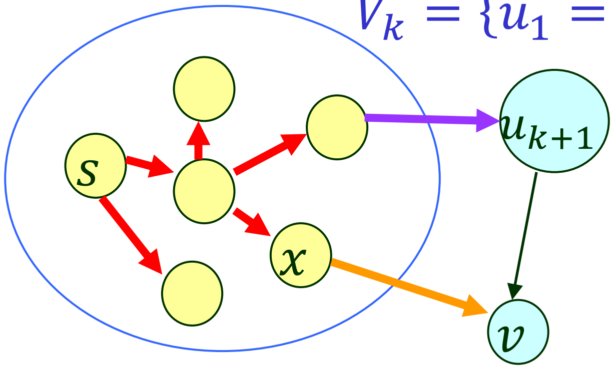

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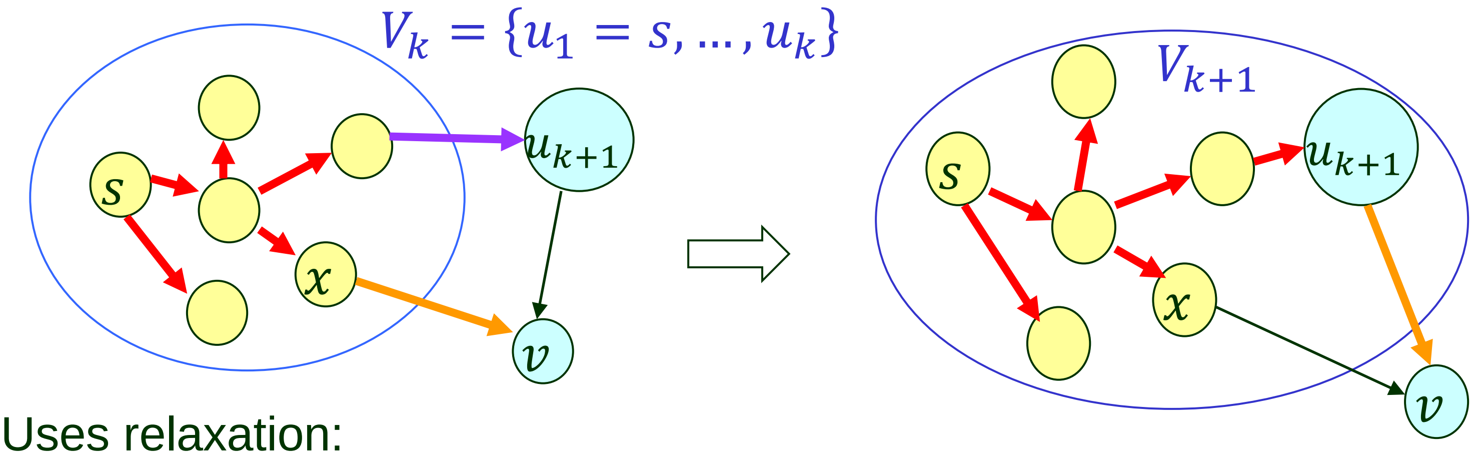

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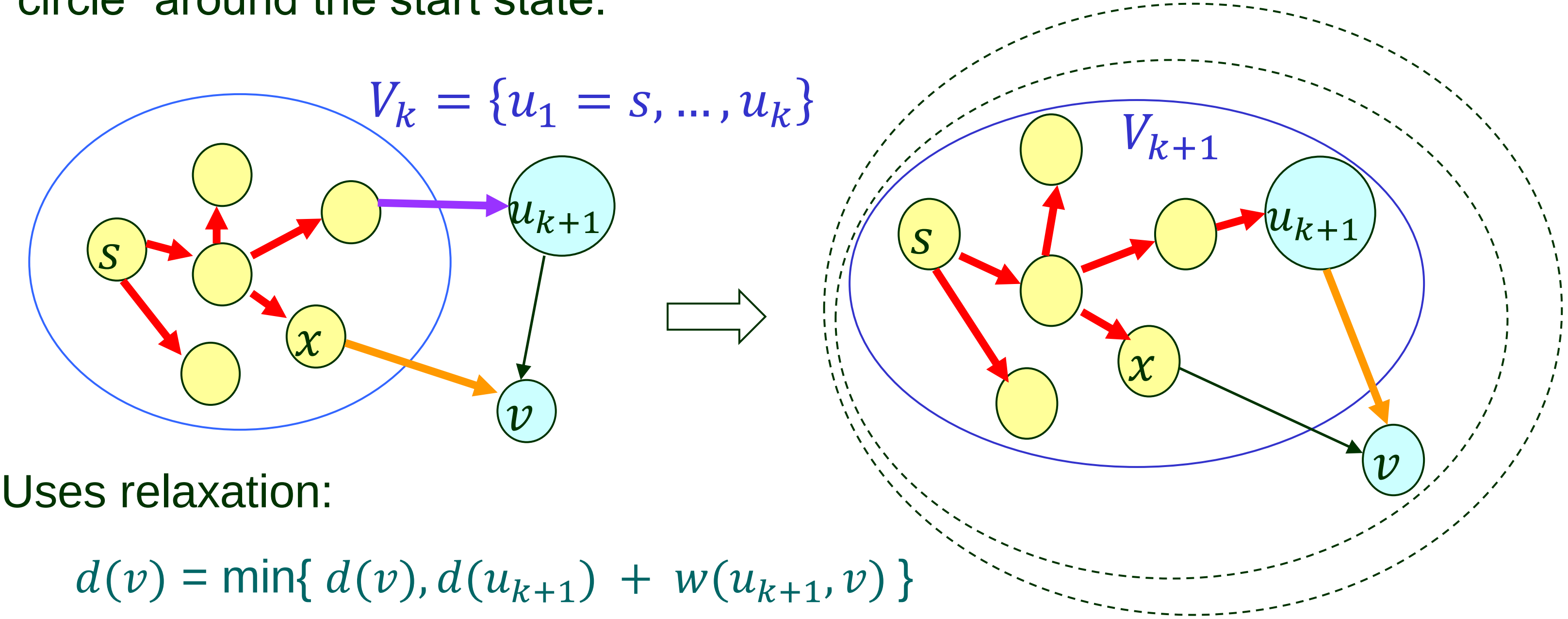

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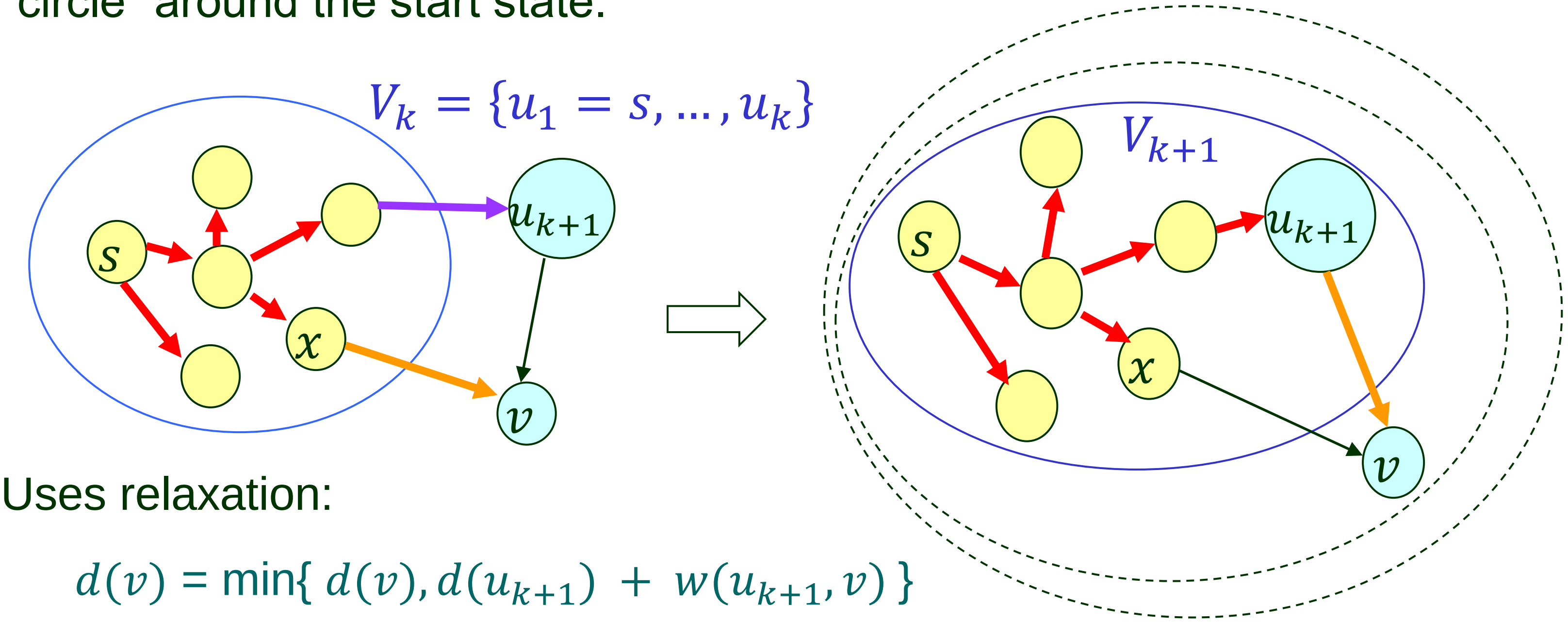

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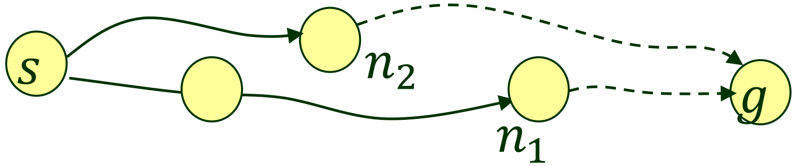
- With a good h, the $g + h$ bands will stretch toward a goal state.



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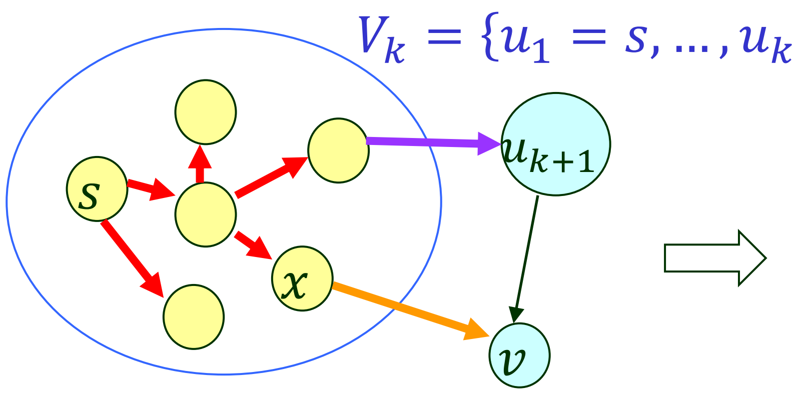

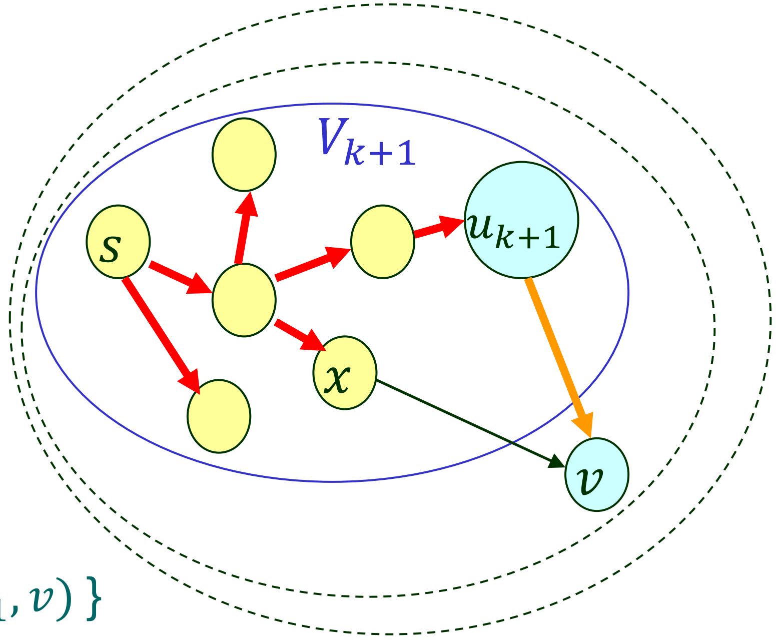

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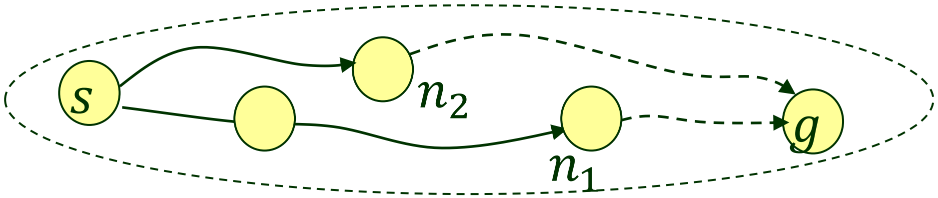
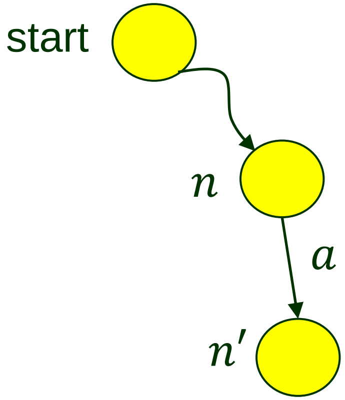


Monotonicity?

- The g cost increases along a path because action costs are positive.



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Cost at its successor n': $\underbrace{g(n) + c(n, a, n')}_{= g(n')} + h(n')$

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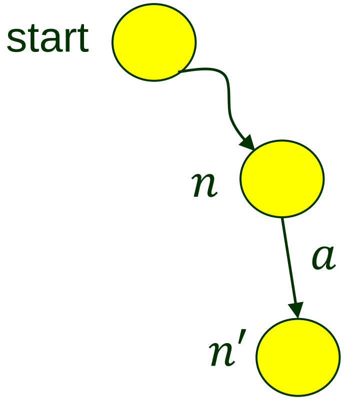

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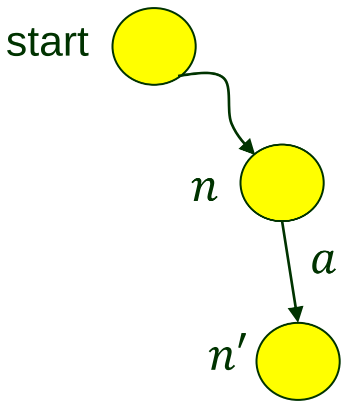

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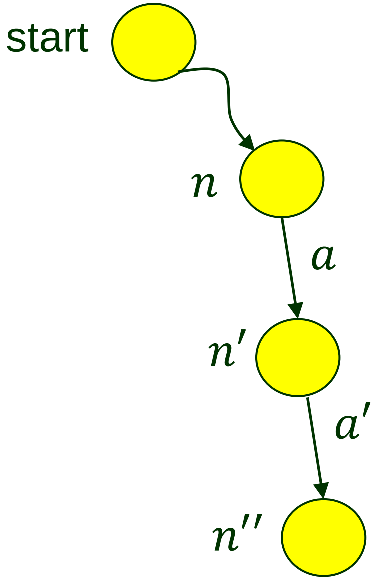
$$\Updownarrow$$

$$h(n) \le c(n, a, n') + h(n')$$

(consistent heuristic)

Consecutive Nodes Scored the Same



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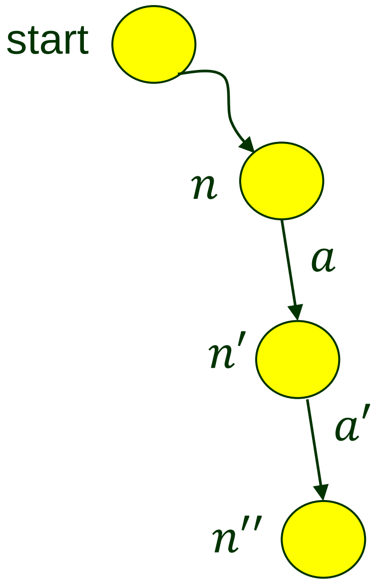

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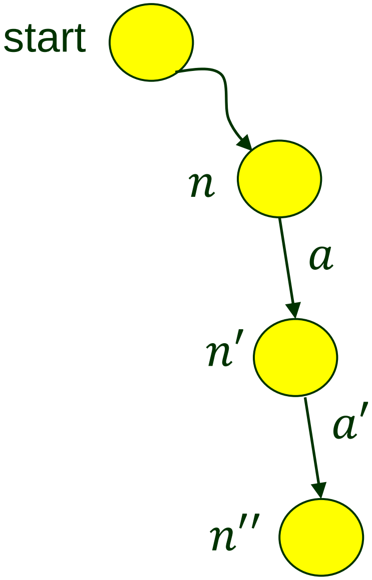

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$$\Updownarrow$$

$$h(n) - h(n') = g(n') - g(n)$$
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h decreases as much as g increases after an action.

Efficiency of A*

h: admissible

C^*: cost of the optimal solution path

- A* will expand every node reachable via a sequence of nodes that have costs $< C^*$.
- A* will not expand any node n with $f(n) > C^*$.
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A* prunes away nodes unnecessary for finding an optimal solution.

- ♣ A* may take exponential time with a poor heuristic function.

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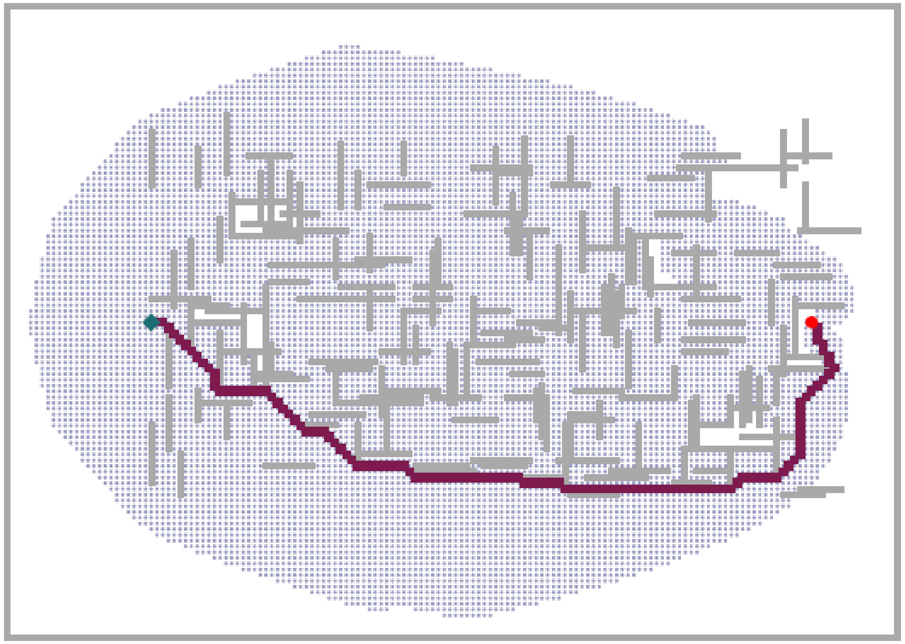
Use an inadmissible heuristic.

Idea of *detour index*: multiplier applied to the straight-line distance.

e.g., a detour index of 1.3 implies a good estimate of 13 miles between two cities that are 10 miles apart.

Weighted A*

Evaluation function: $f(n) = g(n) + W \times h(n)$ for some $W > 1$.



A* search

Gray bars: obstacles
Dots: reached states

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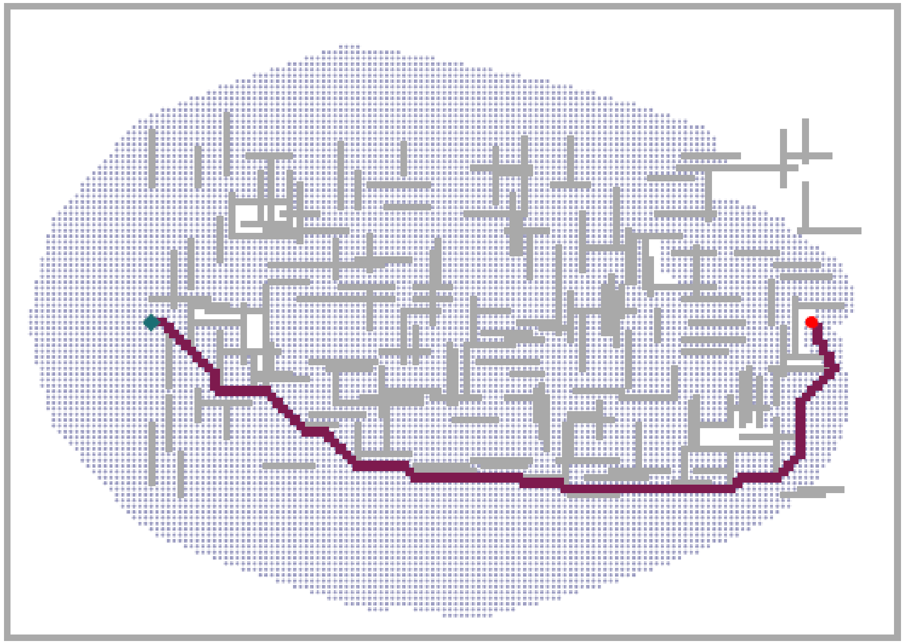

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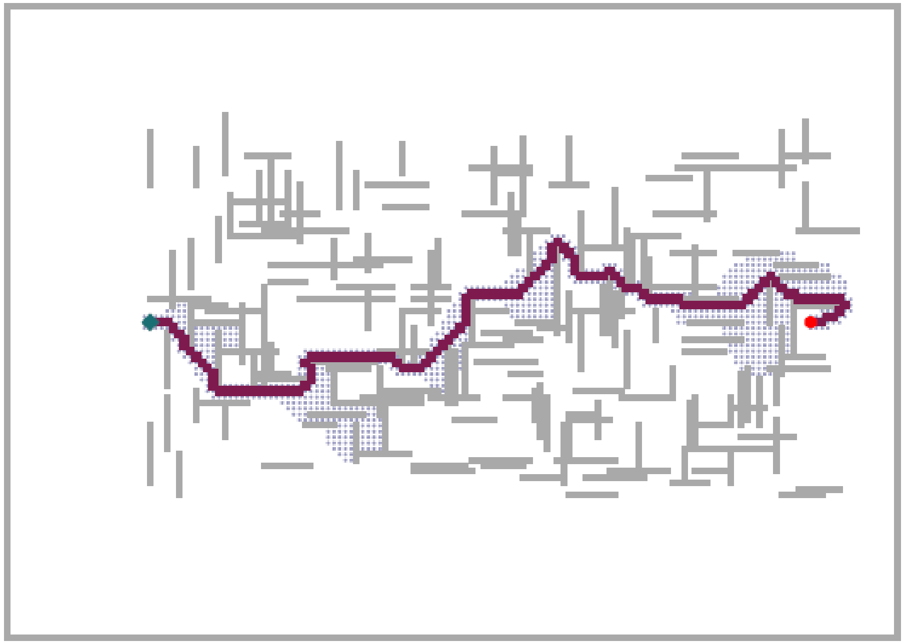

Weighted A* search
($W = 2$ on the same grid)

Weighted A* As a Generalization

- Weighted A* finds a solution with cost between C^* and $W \times C^*$.
- Cost is usually much closer to C^* in practice.

Weighted A* search $g(n) + W \times h(n) \quad (1 \leq W < \infty)$

Weighted A* As a Generalization

- Weighted A* finds a solution with cost between C^* and $W \times C^*$.
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A* search	$g(n) + h(n)$	$(W = 1)$
Uniform-cost search	$g(n)$	$(W = 0)$
Best-cost search	$h(n)$	$(W = \infty)$
Weighted A* search	$g(n) + W \times h(n)$	$(1 \le W < \infty)$

Memory-Bounded Search

- Beam search keeps the k nodes with the best f scores.
  - ♦ Less memory and faster execution.
  - ♣ Incomplete and suboptimal.

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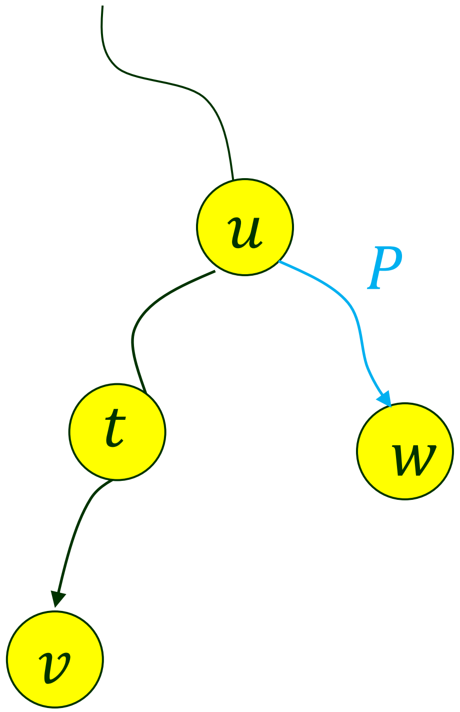
  - ♣ In the worst case, #iterations = #states

Recursive Best-First Search (RBFS)



$f_{limit}(v)$: f-value of the *best alternative path* from any *ancestor* of the node v.

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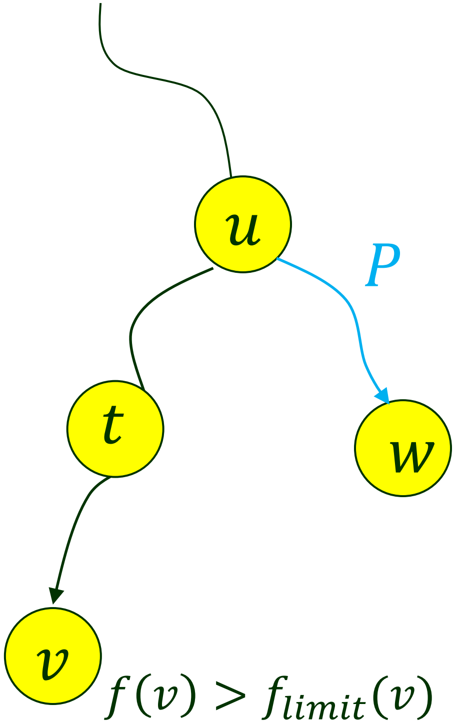

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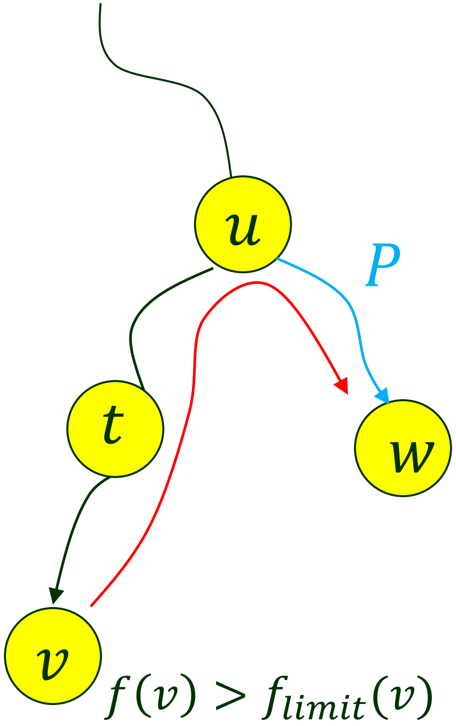

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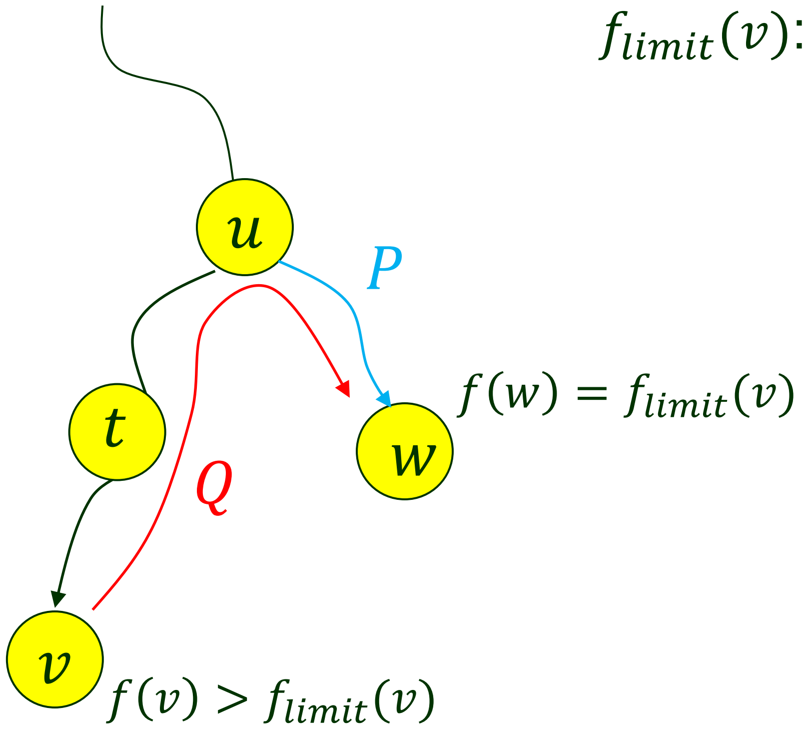

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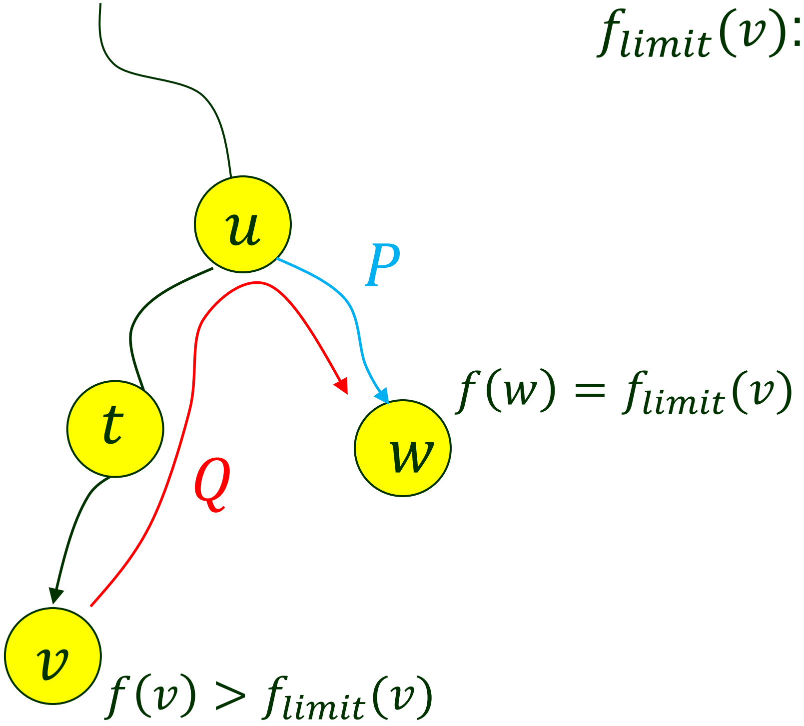

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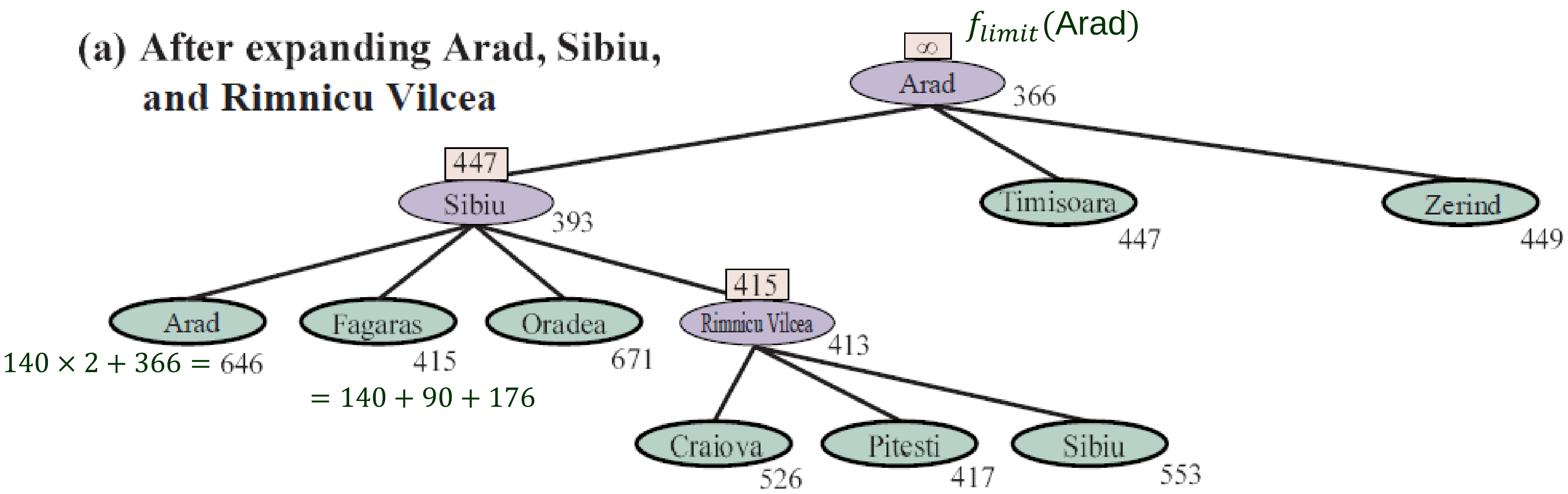
  - update the f-value of every node along the path Q with the best f- value of its children.

RBFS Example



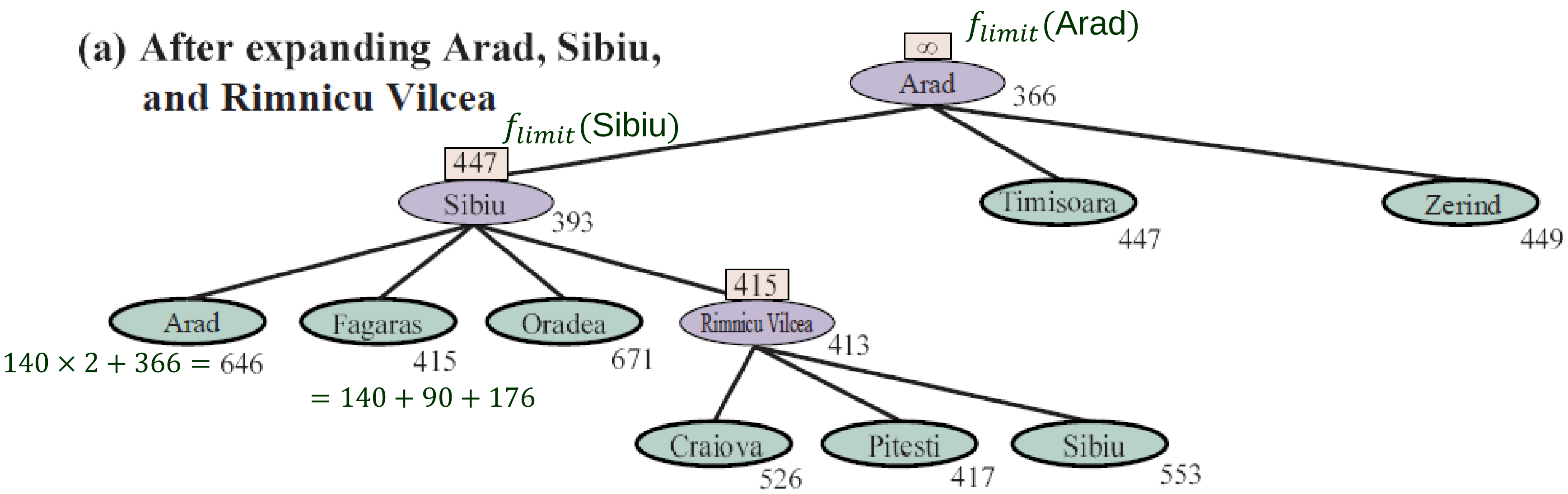
(a) After expanding Arad, Sibiu, and Rimnicu Vilcea

f_{limit}(Arad)

$140 \times 2 + 366 = 646$

$= 140 + 90 + 176$

RBFS Example



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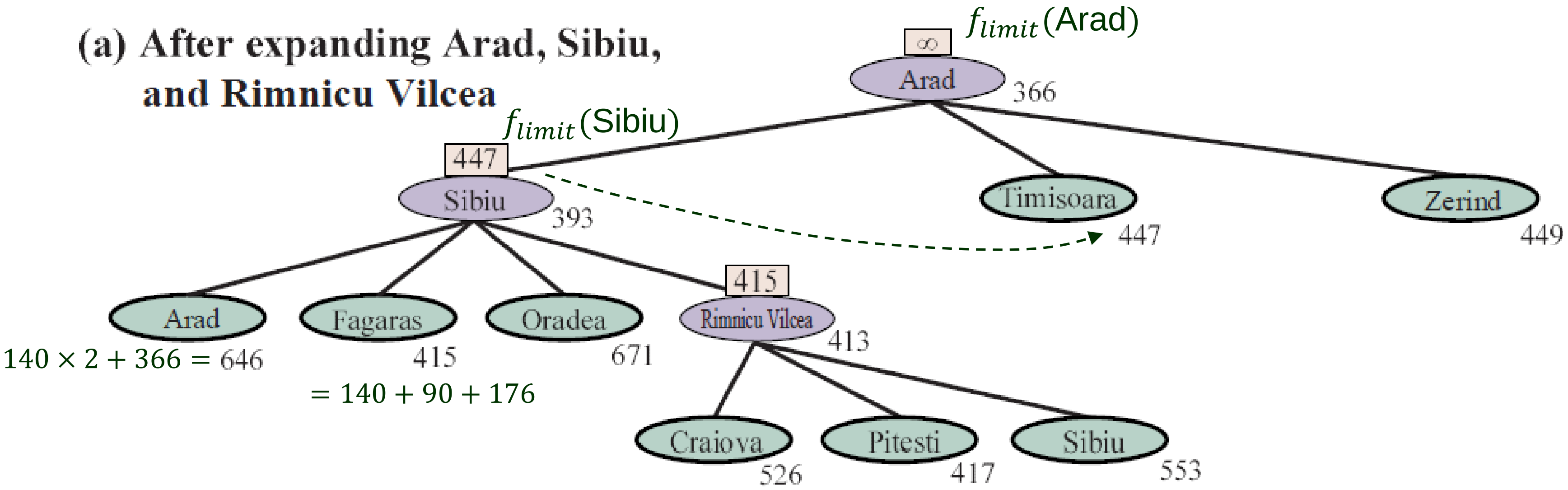

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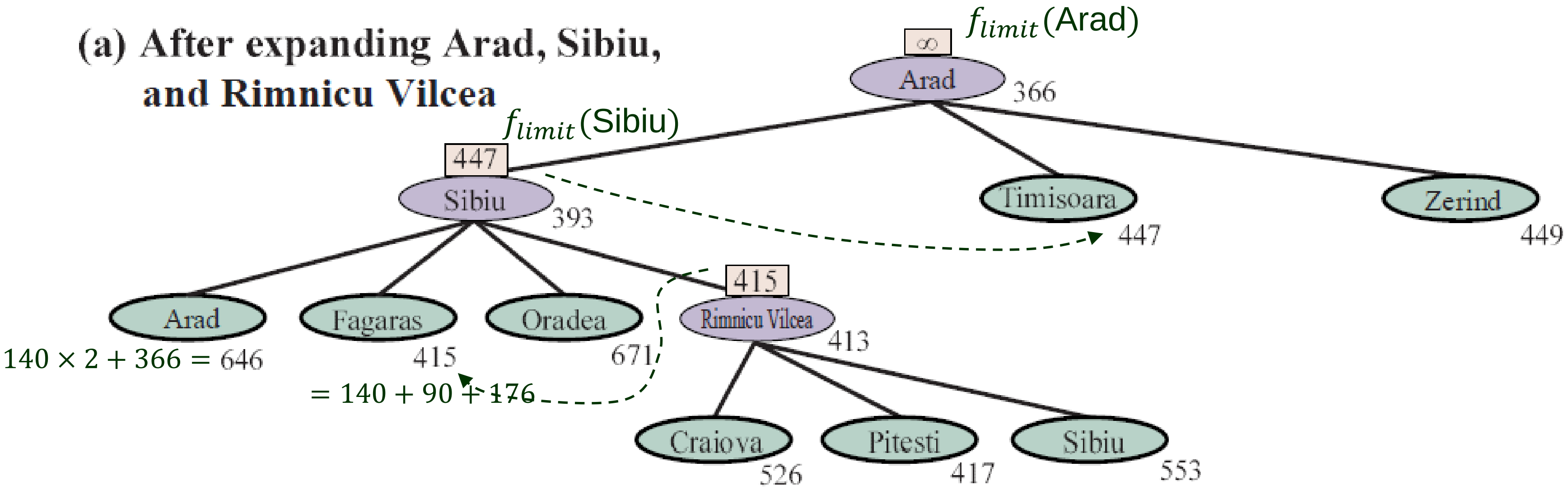

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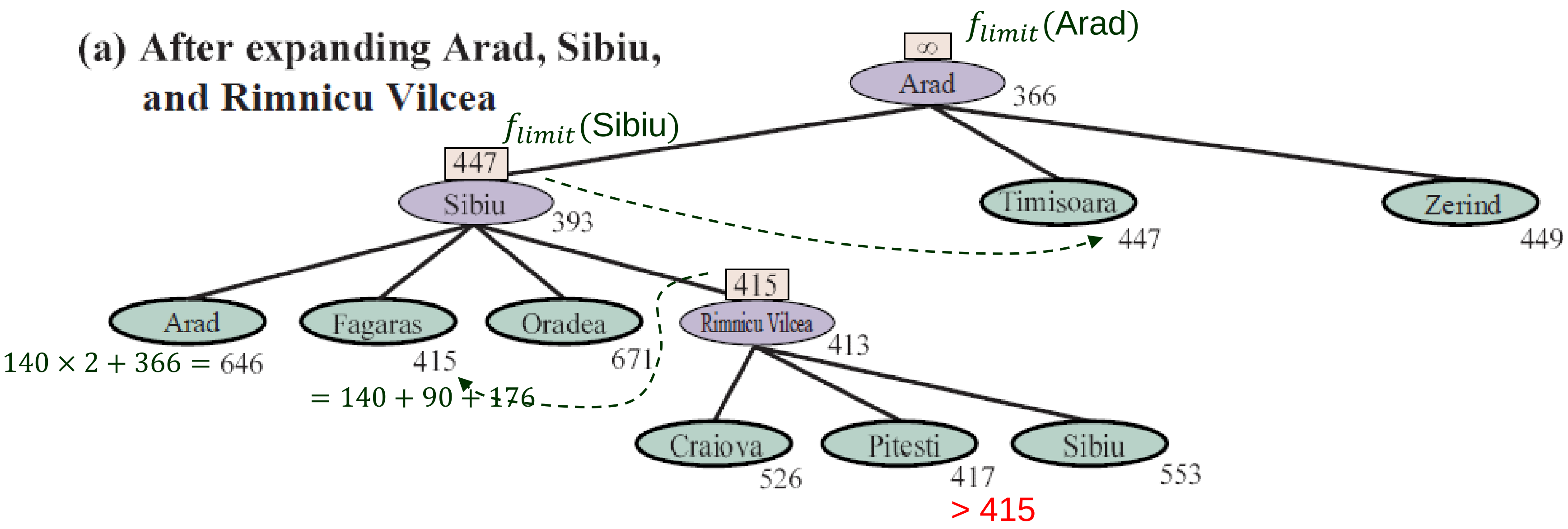

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> 415

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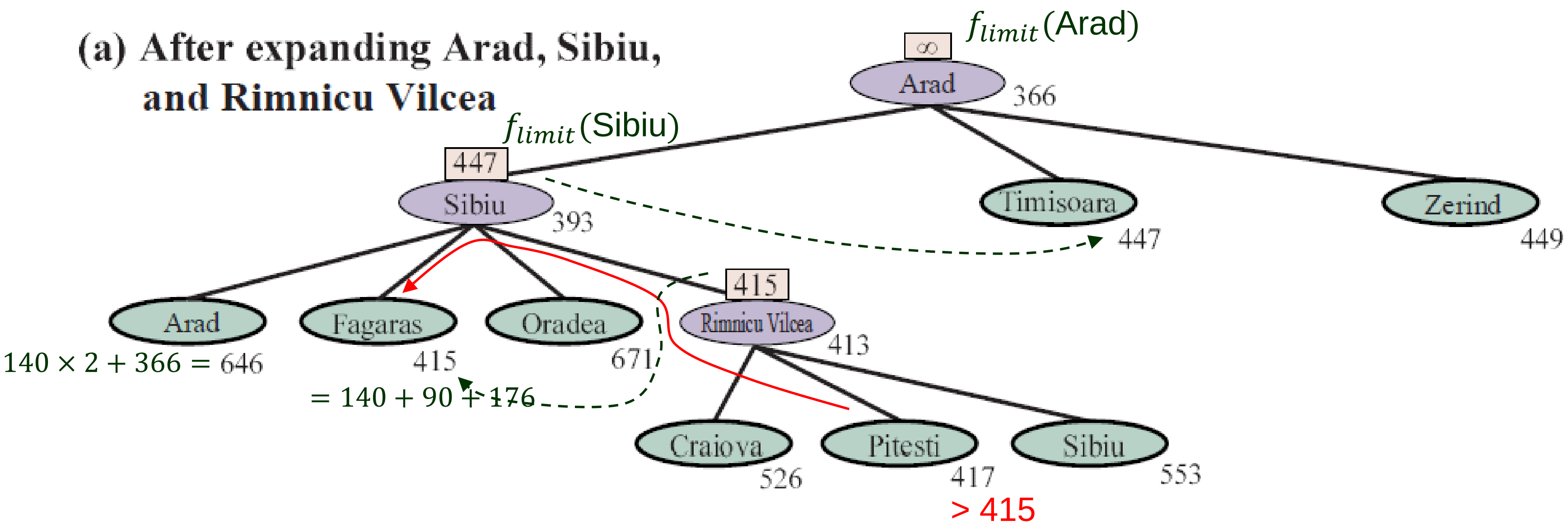
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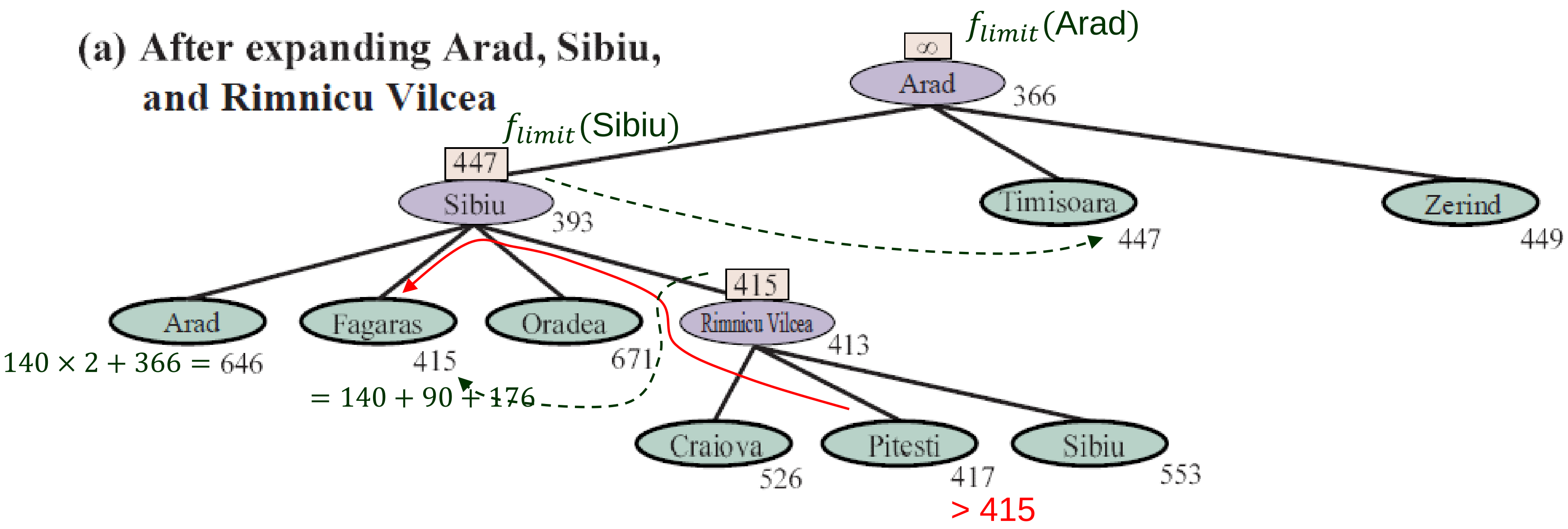
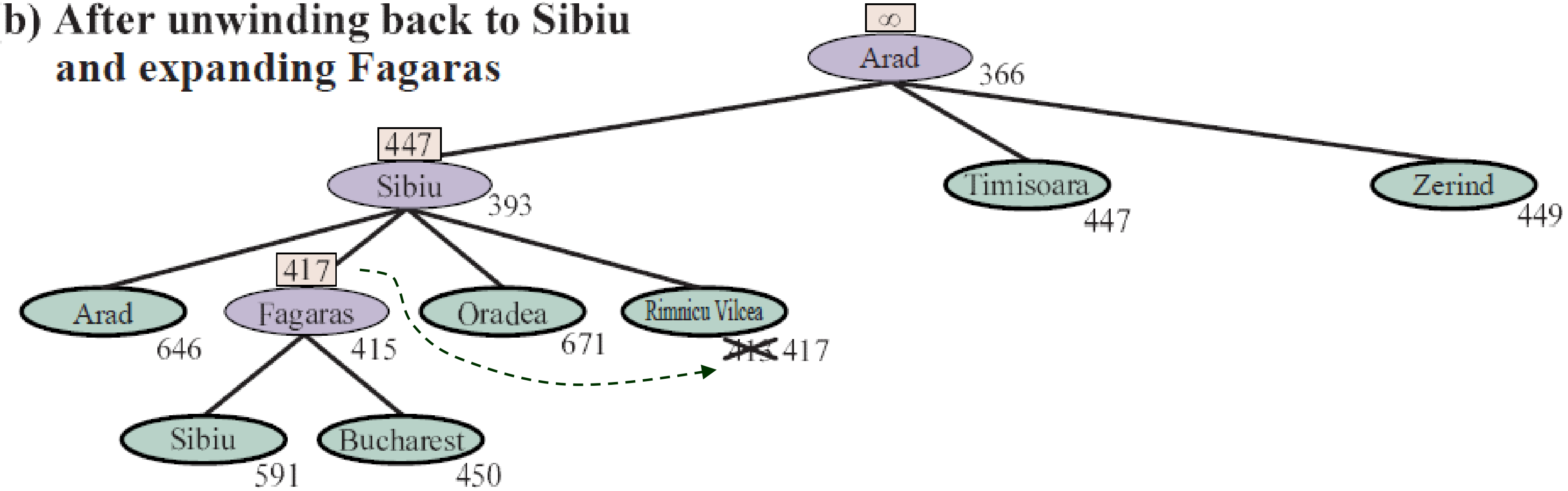


(b) After unwinding back to Sibiu and expanding Fagaras



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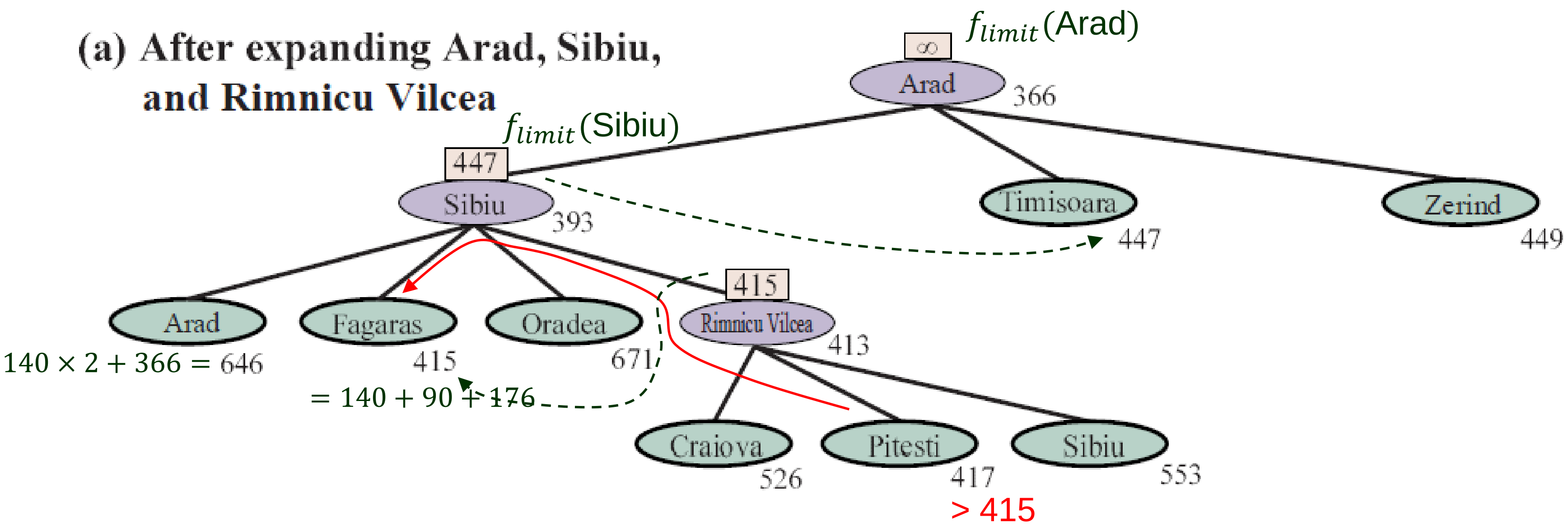

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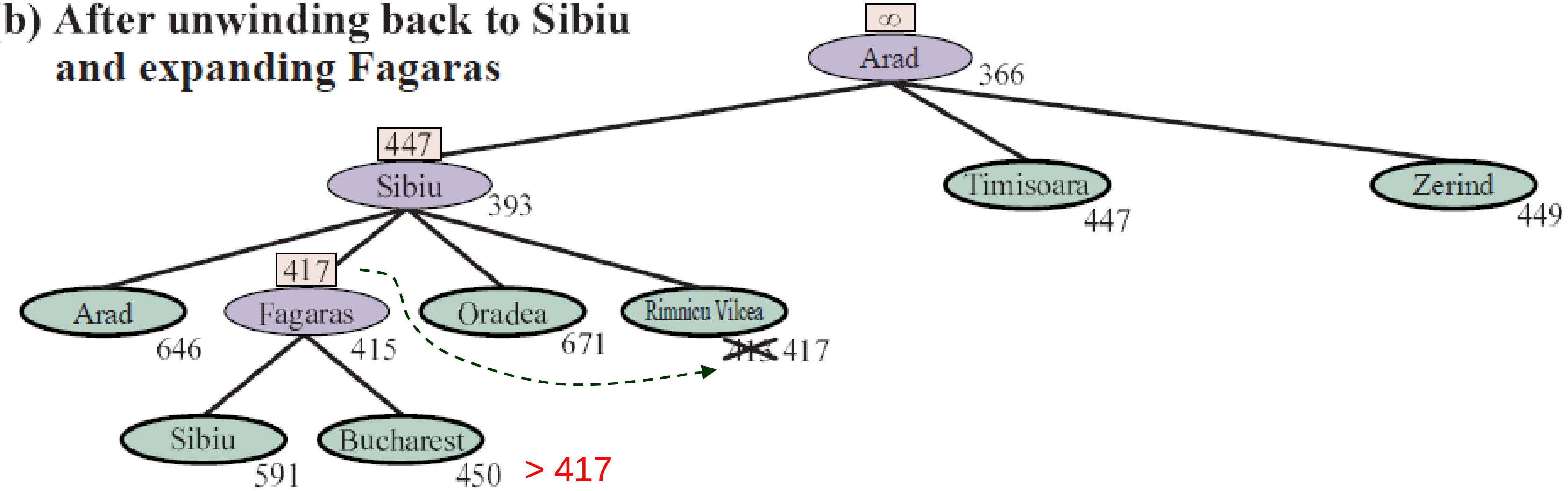

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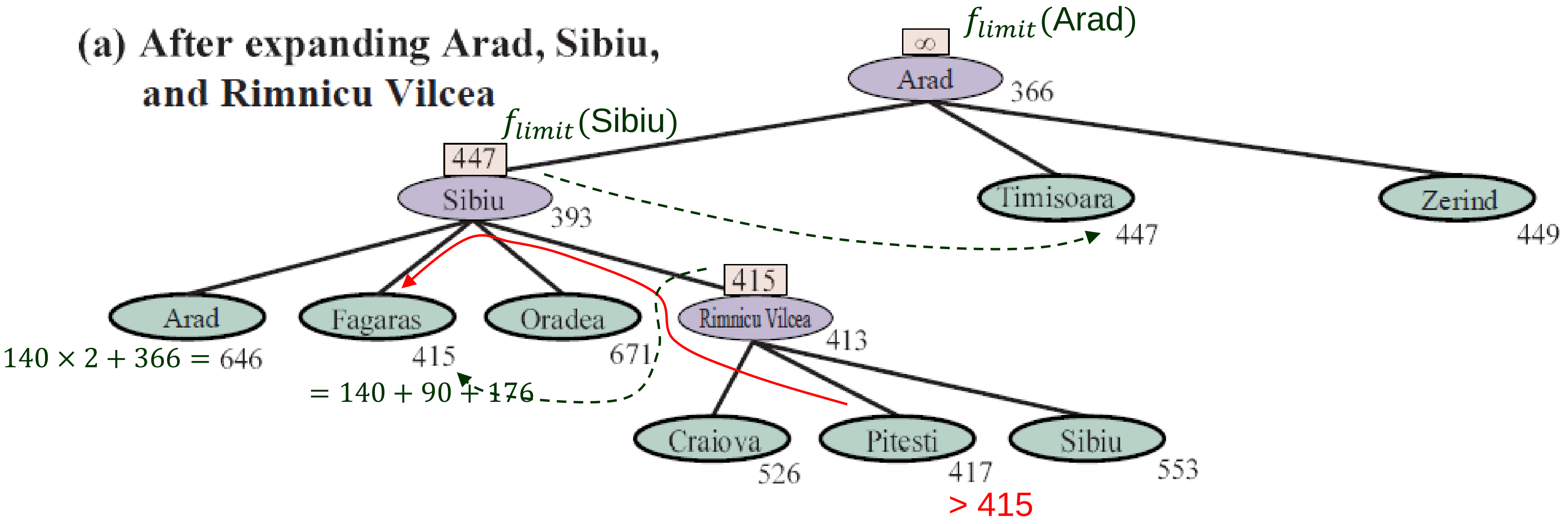

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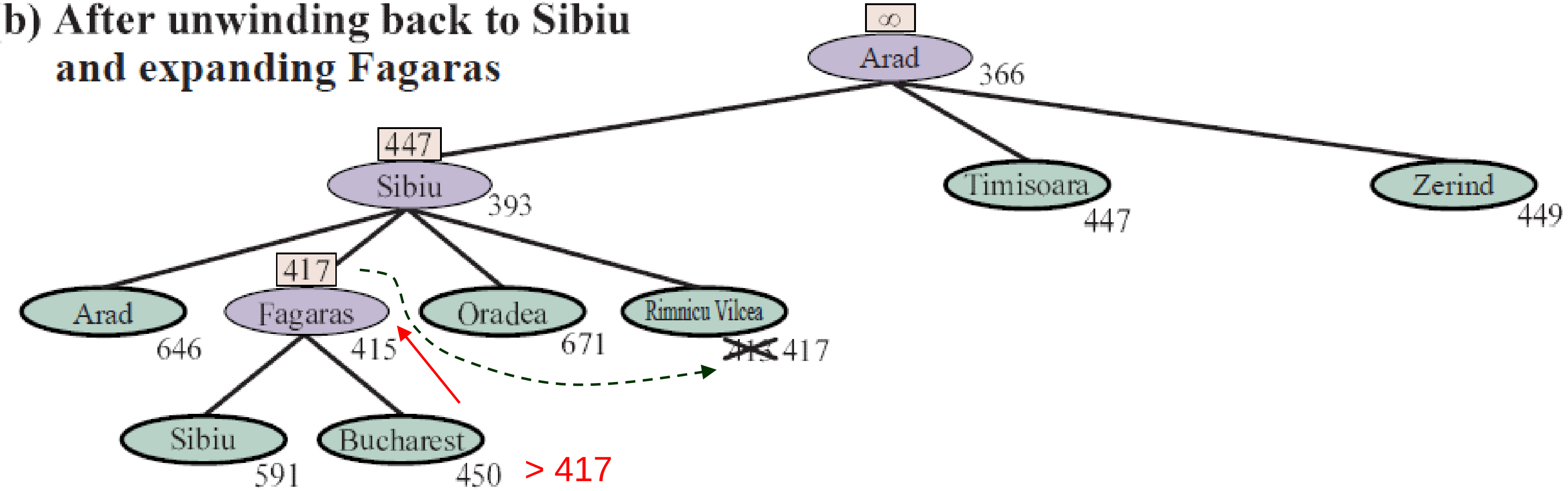

RBFS Example (cont'd)

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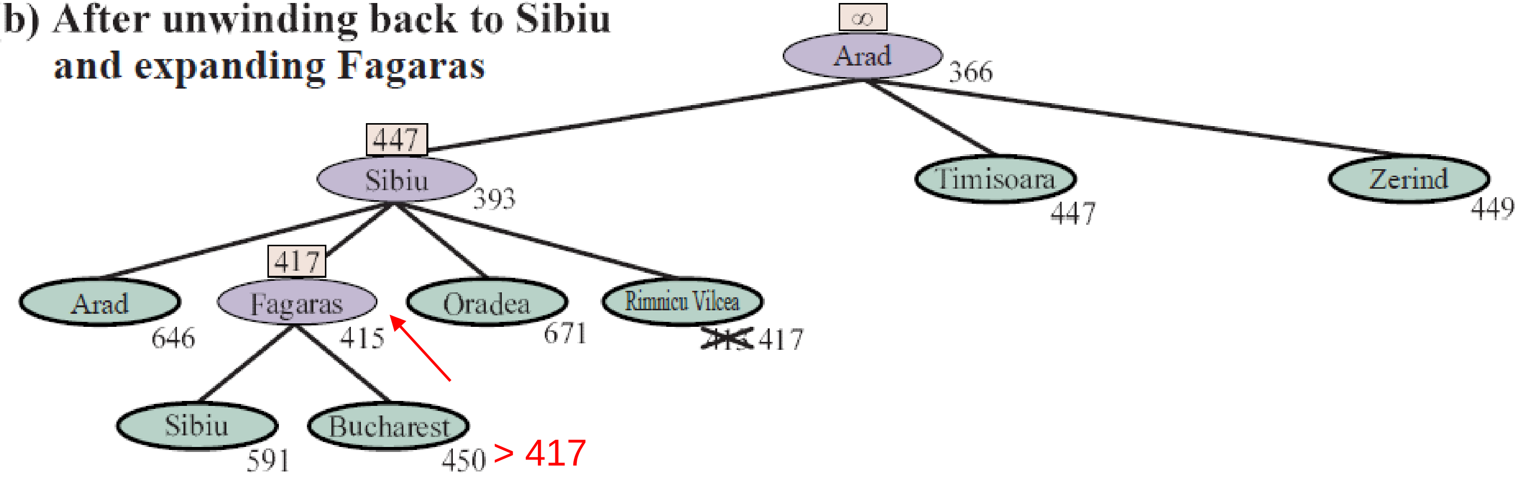

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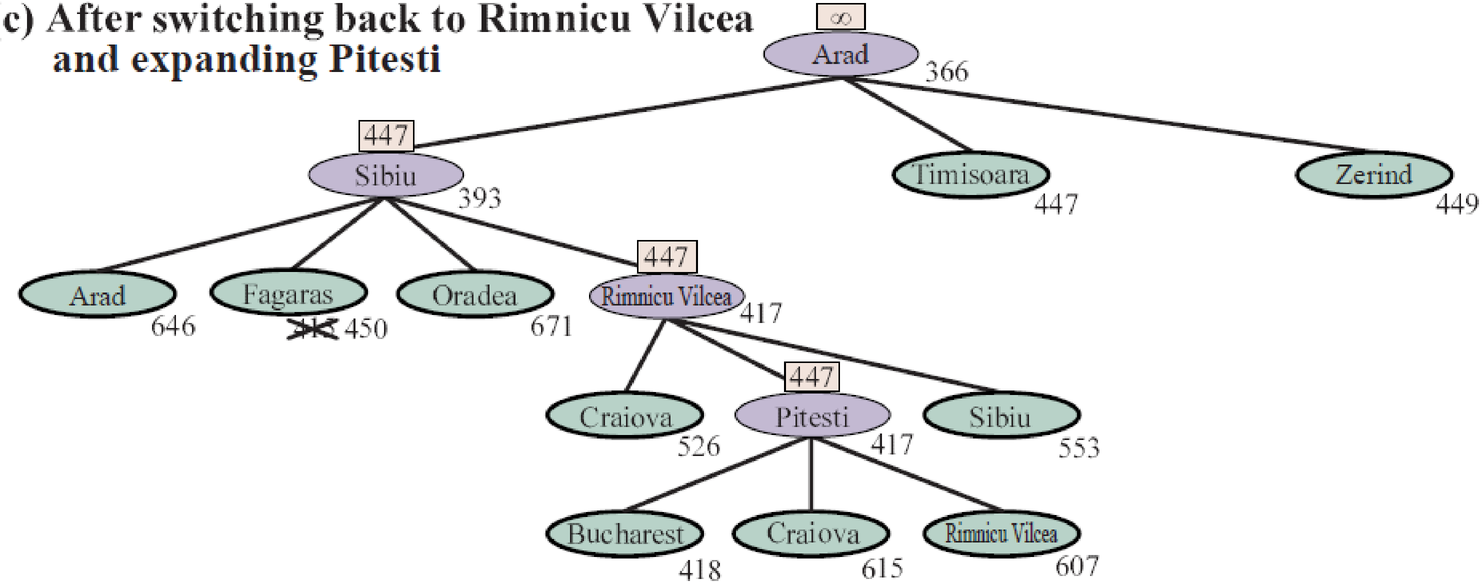

RBFS Example (cont’d)

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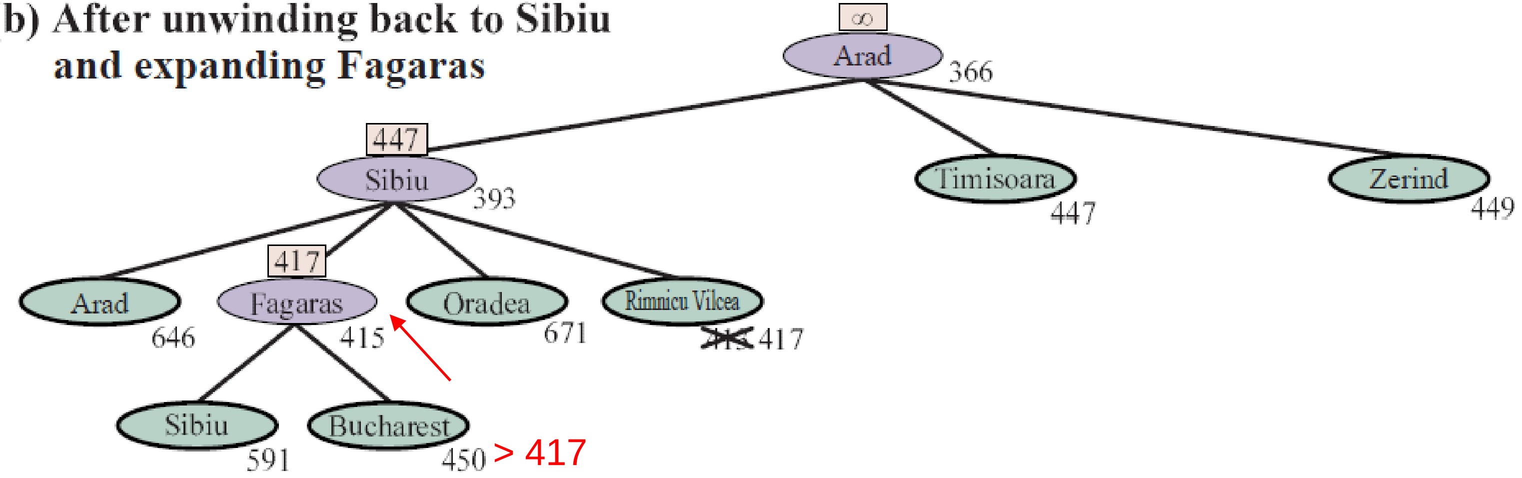

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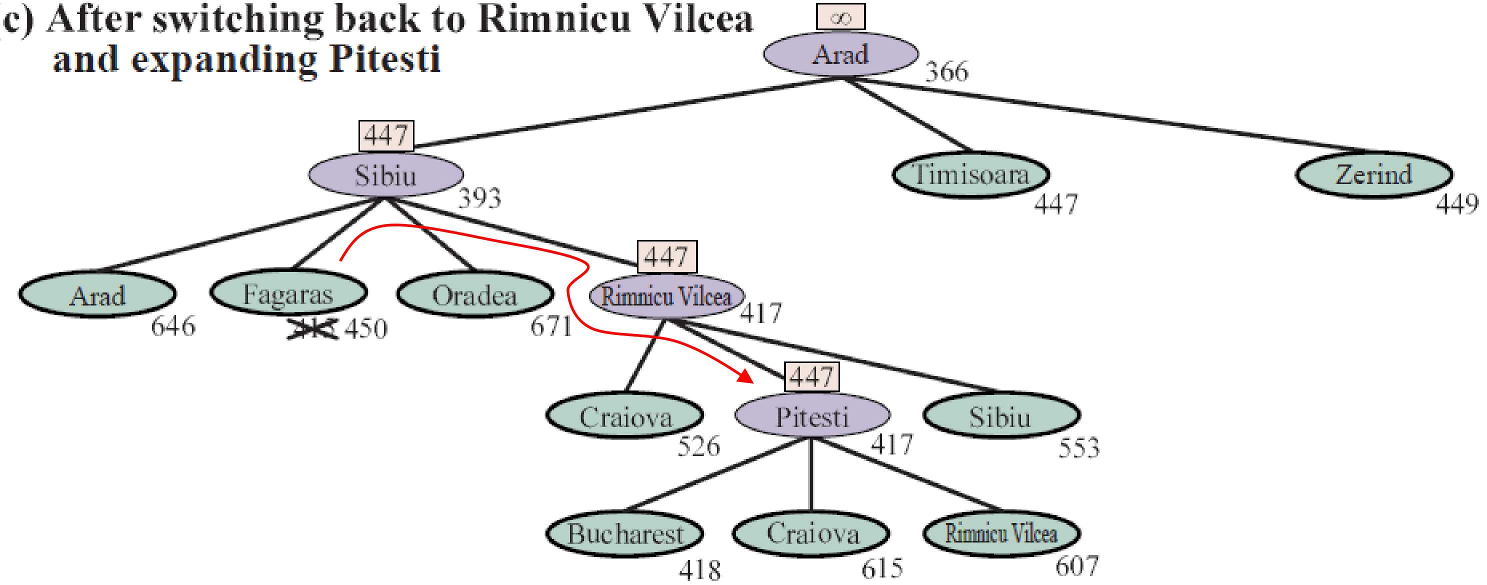

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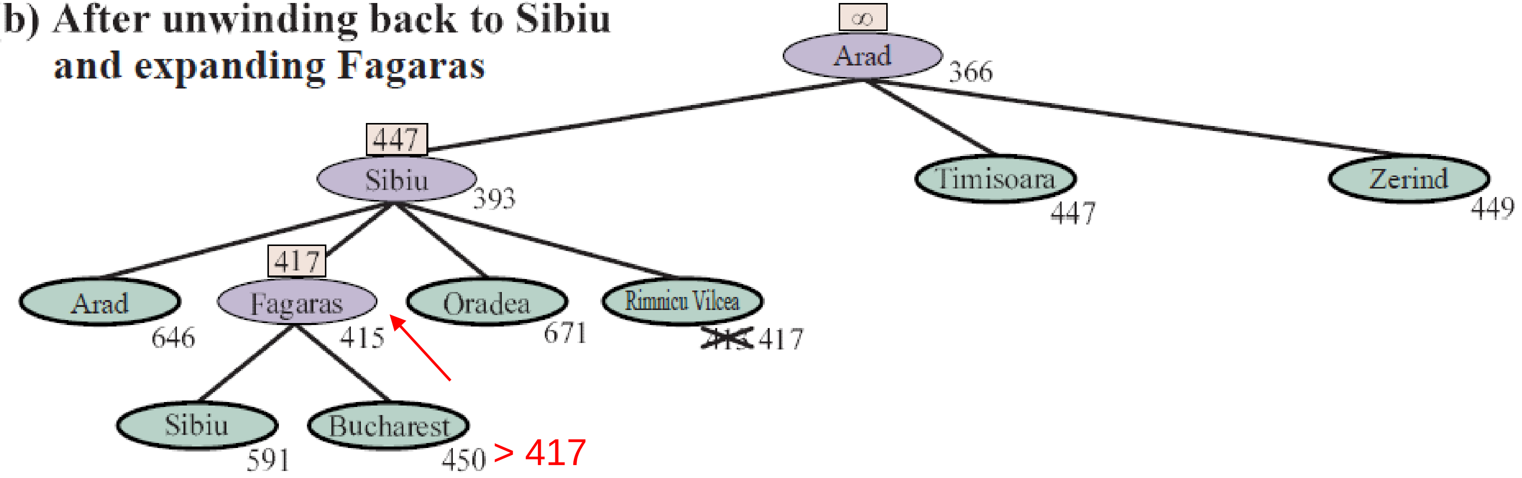

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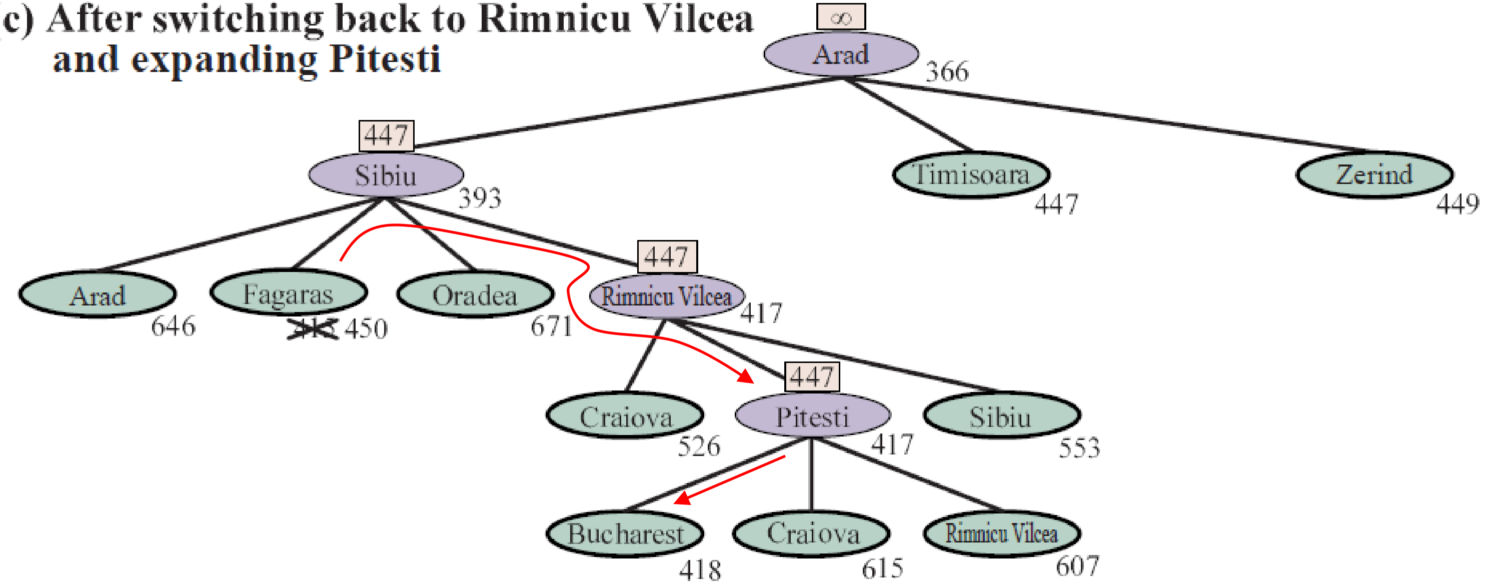
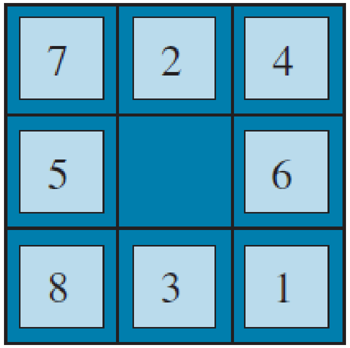


RBFS Summary

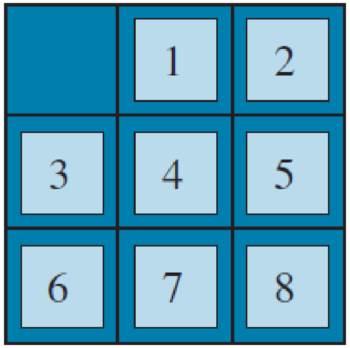
- Optimal with admissible heuristic function $h(n)$.
- Space complexity $O(\underset{\uparrow\ \text{branching factor}}{b}\underset{\uparrow\ \text{depth}}{d})$.
- Time complexity difficult to analyze, depending on
  - accuracy of $h(n)$
  - how often the best path changes
- Slightly more efficient than IDA*.
- Both IDA* and RBFS suffering from using too little memory and may explore the same state multiple times.

III. 8-Puzzle: Large Search Space



7	2	4
5		6
8	3	1

Start State



	1	2
3	4	5
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Goal State

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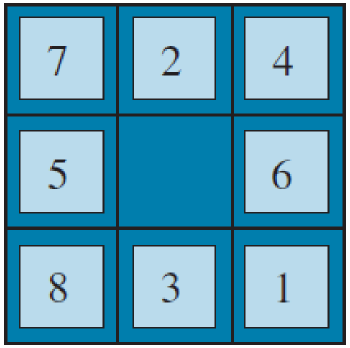

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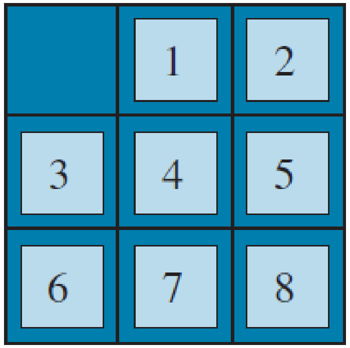

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$\frac{9!}{2} = 181{,}400$ reachable states from start.

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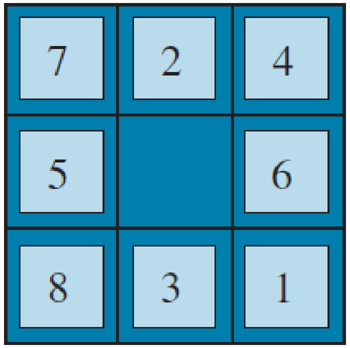

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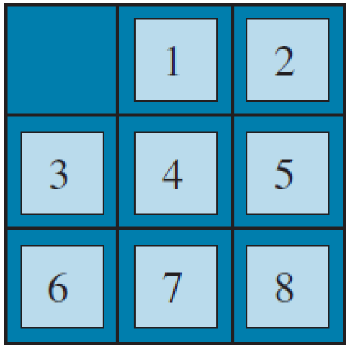

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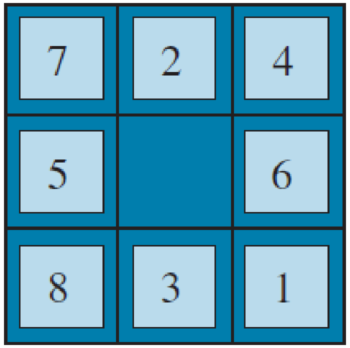
$\frac{9!}{2} = 181{,}400$ reachable states from start.

$\frac{16!}{2} > 10^{13}$ reachable states for the 15-puzzle!

Two Heuristics



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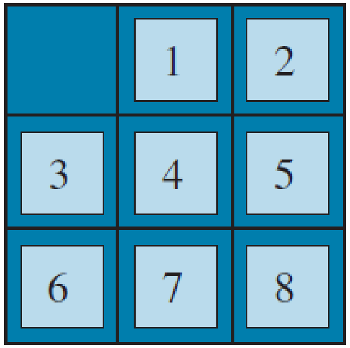

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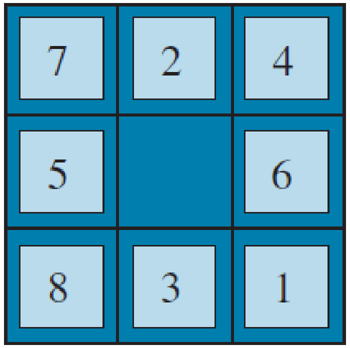
Goal State

Heuristics are needed for searching the vast state space.

Two Heuristics



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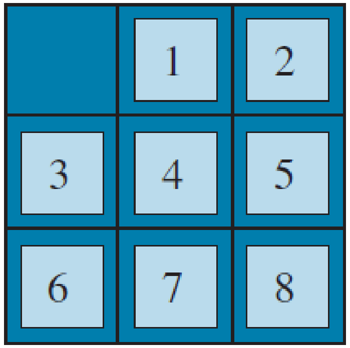

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Goal State

Heuristics are needed for searching the vast state space.

- h_1 = # tiles misplaced

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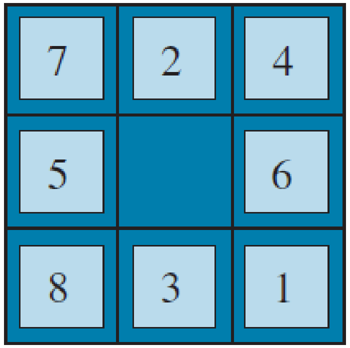

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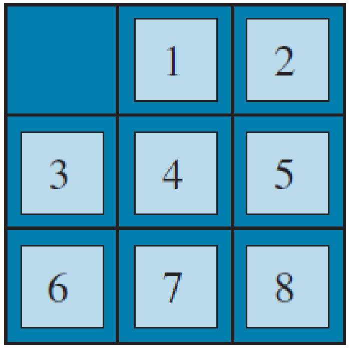

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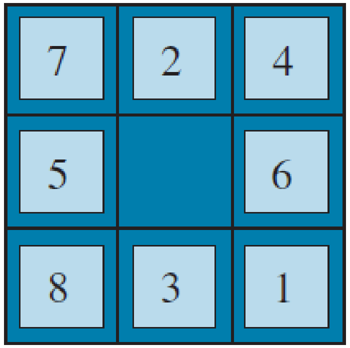

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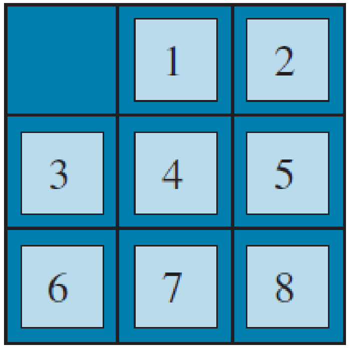

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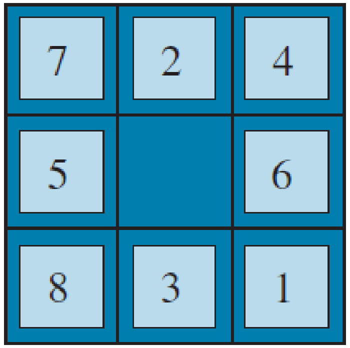
Two Heuristics



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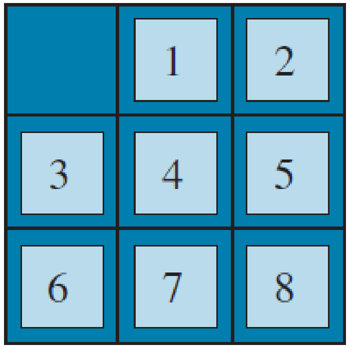

Goal State

Heuristics are needed for searching the vast state space.

♦ h_1 = # tiles misplaced

Admissible: any tile out of place will require ≥ 1 move to fix.

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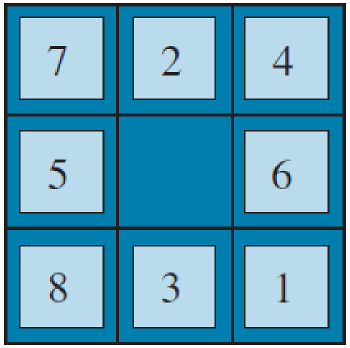
Two Heuristics



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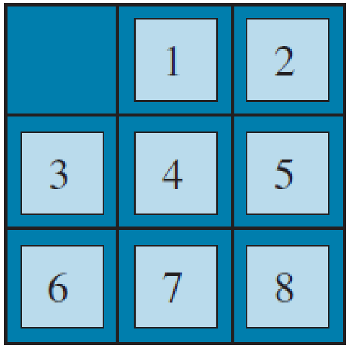

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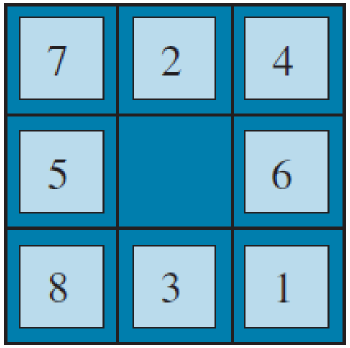
Heuristics are needed for searching the vast state space.

◆ h_1 = # tiles misplaced

Admissible: any tile out of place will require ≥ 1 move to fix.

◆ h_2 = sum of Manhattan distances of the tiles from their goal positions

Two Heuristics



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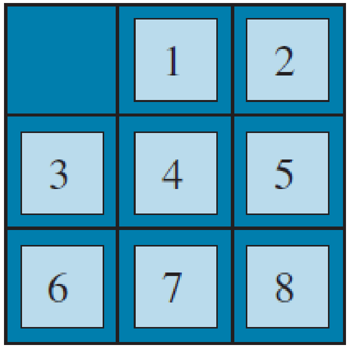


Goal State

Heuristics are needed for searching the vast state space.

- h_1 = # tiles misplaced

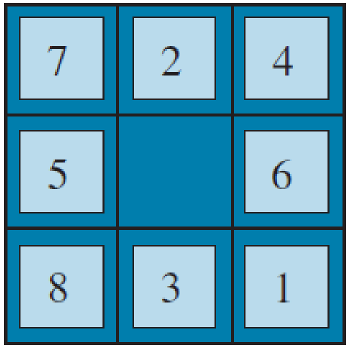
  Admissible: any tile out of place will require $\geq$ 1 move to fix.

Misplaced tiles:
1, 2, 3, 4, 5, 6, 7, 8 ⇨ $h_1 = 8$

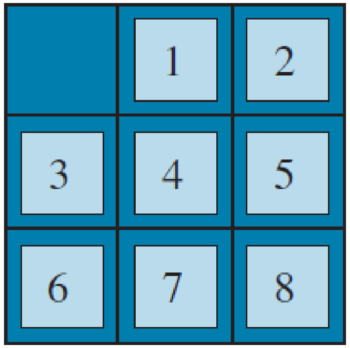
- h_2 = sum of Manhattan distances of the tiles from their goal positions

Tile	1	2	3	4	5	6	7	8
Manhattan distance	3	1	2	2	2	3	3	2

Two Heuristics



Start State



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$h_2 = 18$

Heuristics are needed for searching the vast state space.

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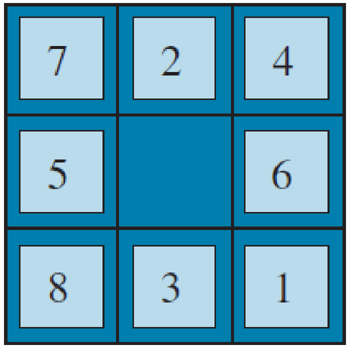
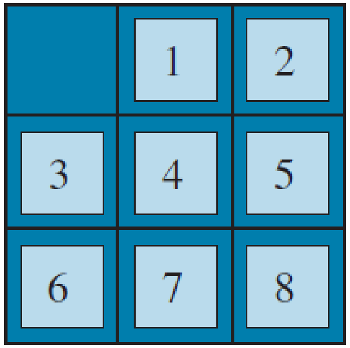


7	2	4
5		6
8	3	1

Start State



	1	2
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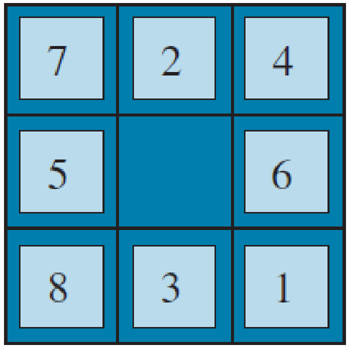

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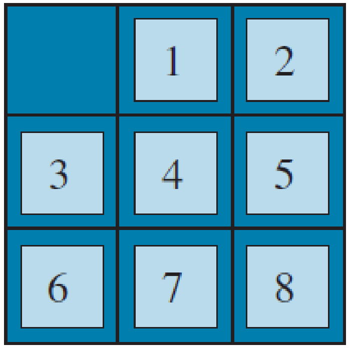

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Neither heuristic overestimates the shortest solution (26 actions for the problem instance).

Heuristic Accuracy on Performance

Quality of a heuristic is often measured by the effective branching factor.

If A* generates N nodes to find a solution at depth d, then its *effective branching factor* b^* is the root of the following equation:

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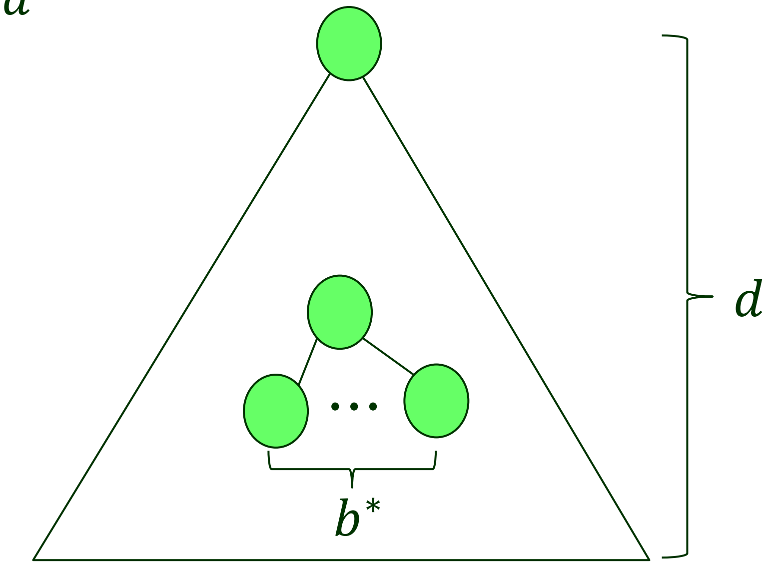

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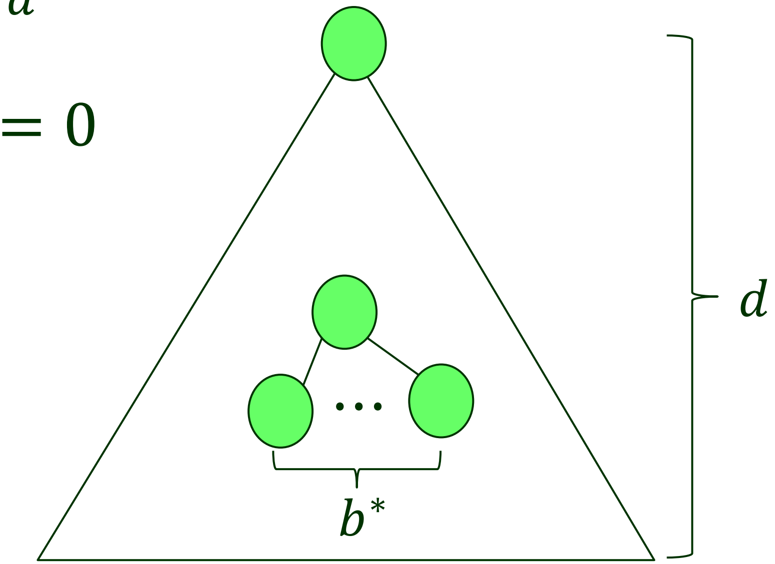

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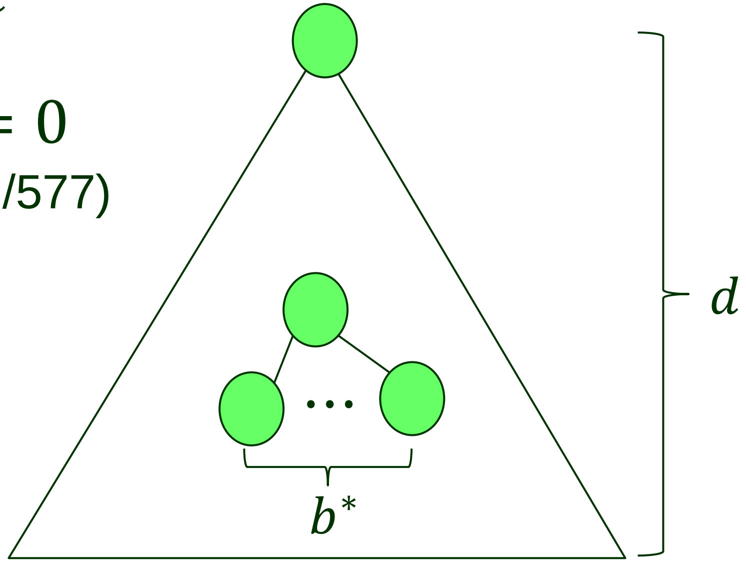

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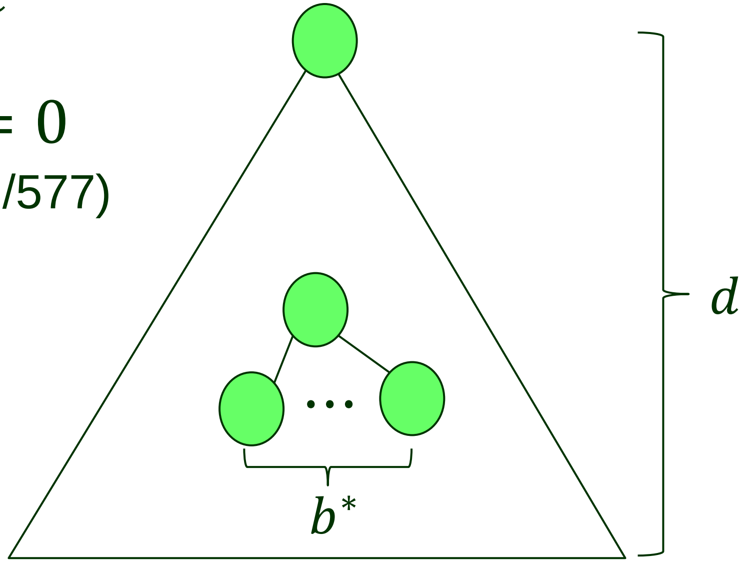
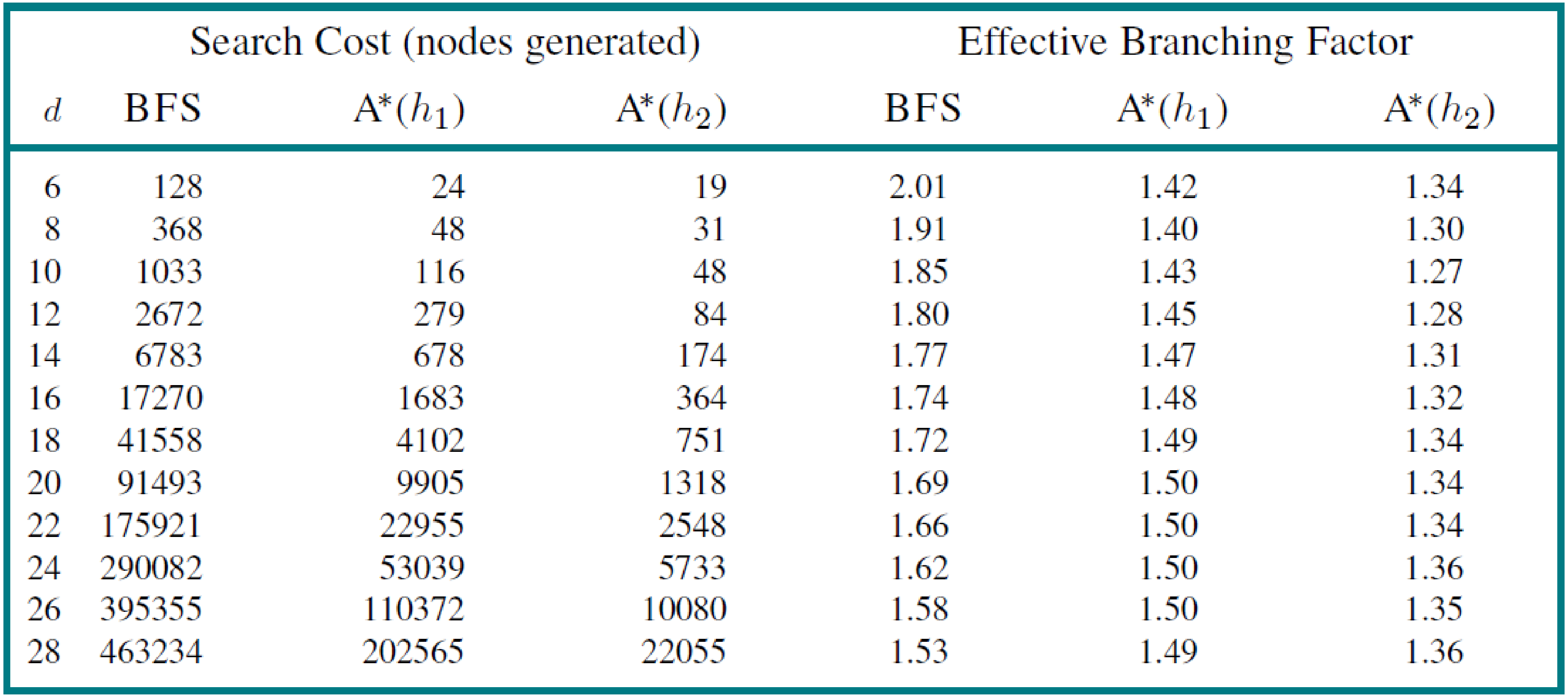


Intuitively, the $N + 1$ nodes handled by A* would fill a tree of height d in which every node at depth $< d$ has exactly b^*children.

e.g. A* finds a solution at depth 5 using 52 nodes has $b^* = 1.92$.

Performance Comparison on 8-Puzzle



	Search Cost (nodes generated)			Effective Branching Factor		
d	BFS	A*(h_1)	A*(h_2)	BFS	A*(h_1)	A*(h_2)
6	128	24	19	2.01	1.42	1.34
8	368	48	31	1.91	1.40	1.30
10	1033	116	48	1.85	1.43	1.27
12	2672	279	84	1.80	1.45	1.28
14	6783	678	174	1.77	1.47	1.31
16	17270	1683	364	1.74	1.48	1.32
18	41558	4102	751	1.72	1.49	1.34
20	91493	9905	1318	1.69	1.50	1.34
22	175921	22955	2548	1.66	1.50	1.34
24	290082	53039	5733	1.62	1.50	1.36
26	395355	110372	10080	1.58	1.50	1.35
28	463234	202565	22055	1.53	1.49	1.36

Figure 3.26 Comparison of the search costs and effective branching factors for 8-puzzle problems using breadth-first search, A* with h_1 (misplaced tiles), and A* with h_2 (Manhattan distance). Data are averaged over 100 puzzles for each solution length d from 6 to 28.

#Misplaced Tiles vs Manhattan Distance

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$$h\,1\;\;n \leq \;\; h\,2\;\;n < \;\; C * \;- g\;n$$

$$* \;\; 2\;2\;2\;1\;1\;1\;(n) \leq h_2(n) < C^* - g(n)$$

$$\Downarrow$$

The node n will be expanded by A* with h_1.

- ♣ h_1 might cause other nodes to be expanded as well.

Generating Heuristics by Relaxation

- An admissible heuristic can be derived from exact solution cost of a *relaxed problem*.

with fewer restrictions on the actions

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Admissibility & Consistency

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The heuristic is consistent.

Heuristics from Formal Specification

Formal specification of a problem (8-puzzle):

> A tile can move from square X to square Y if X is adjacent to Y and Y is blank.

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- Program ABSOLVER generates heuristics automatically from problem definitions, including the best one for the 8-puzzle and the first one for the Rubik's Cube puzzle.

Multiple Heuristics Available

Admissible heuristics $h_1, h_2, \cdots, h_k$ are available but none is clearly better than the others.

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