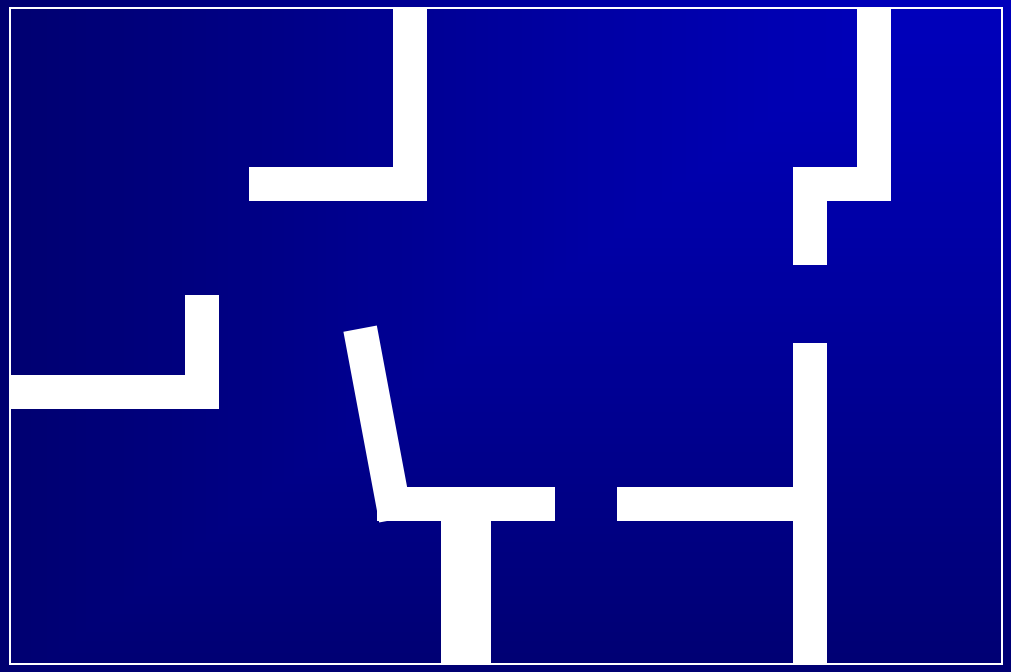
Guarding and Triangulation

Outline

I. Properties of polygon triangulation

II. Solution to the art gallery probelm

The Art Gallery Problem



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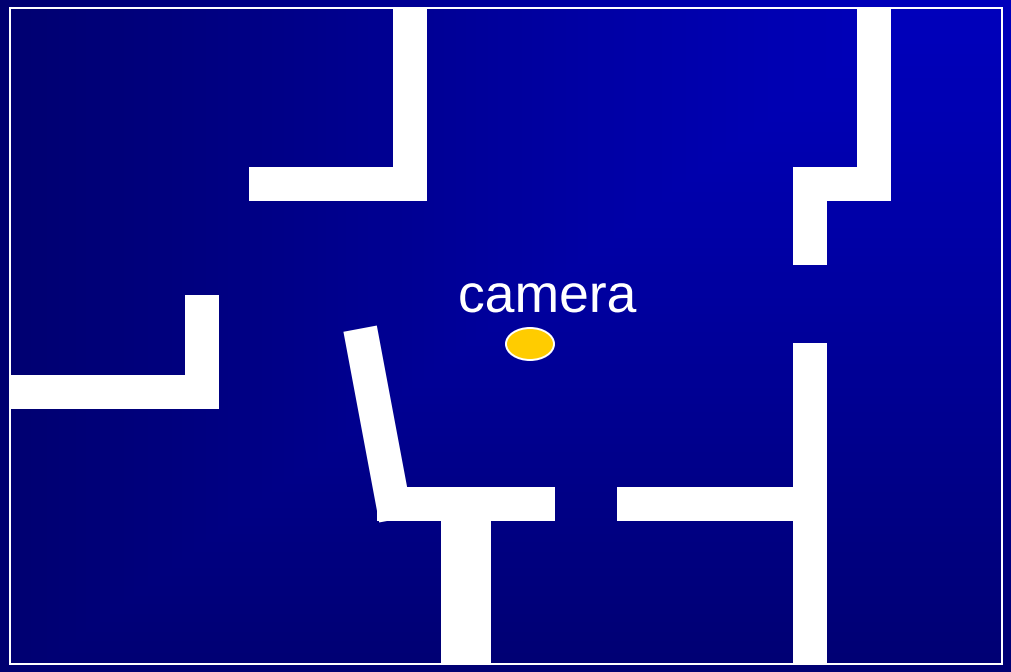

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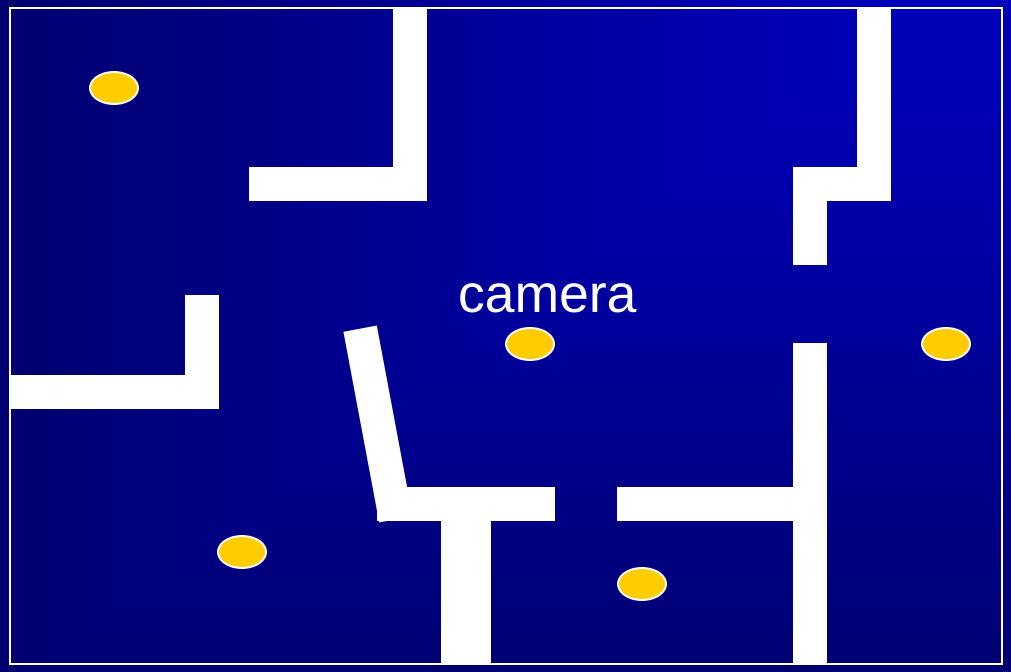

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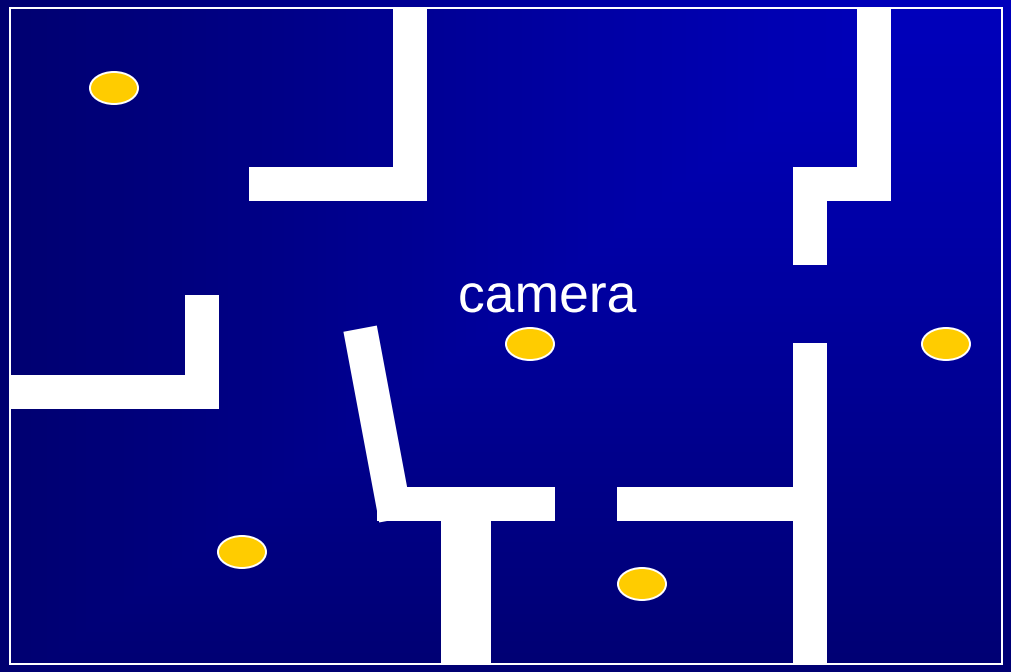

How many cameras are needed to guard a gallery?
Where should they be placed?

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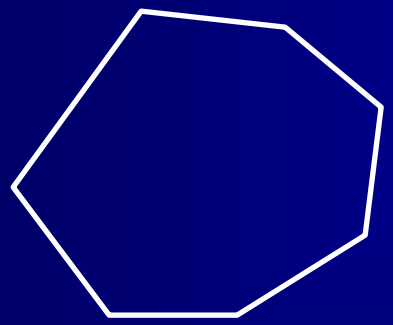

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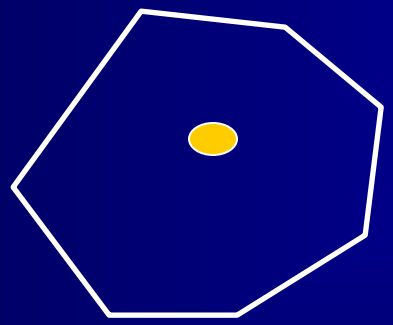

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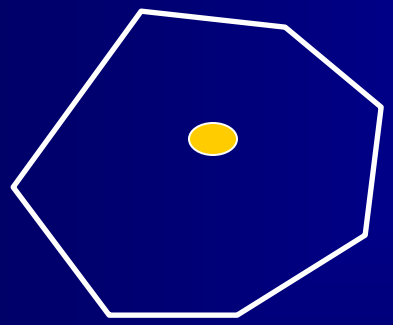

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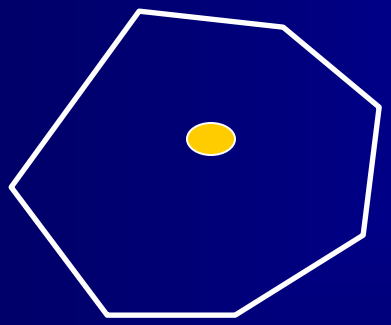

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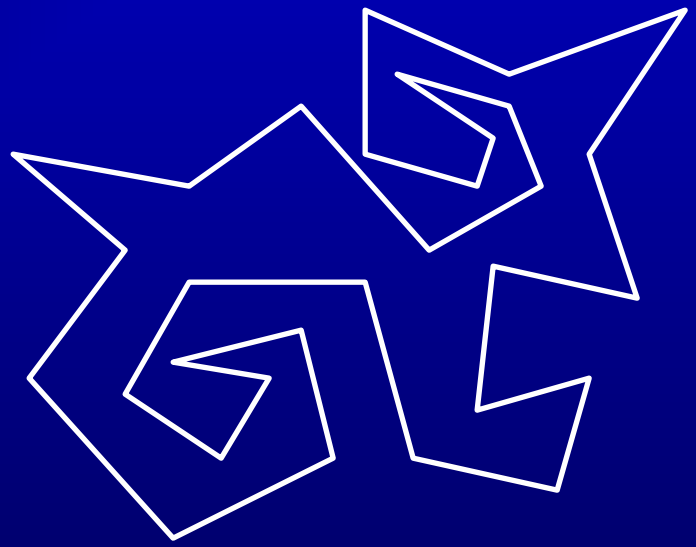

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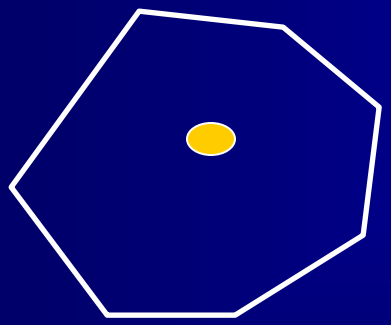

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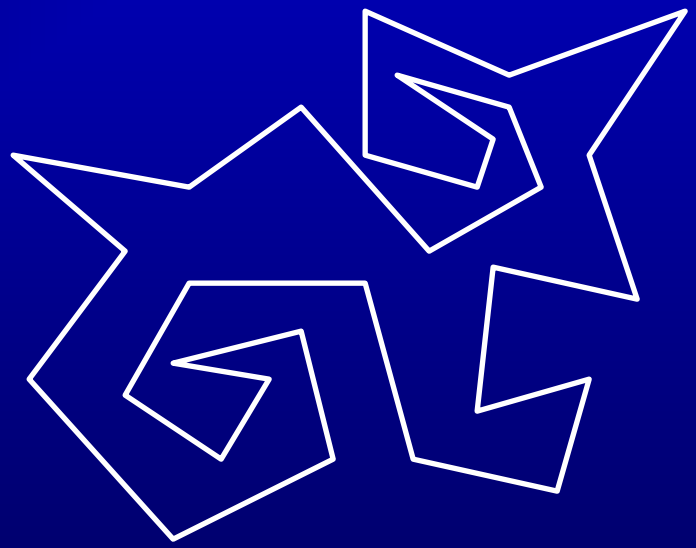

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Bad news: finding the minimum number of cameras for a given polygon is NP-hard (exponential time).

Triangulation

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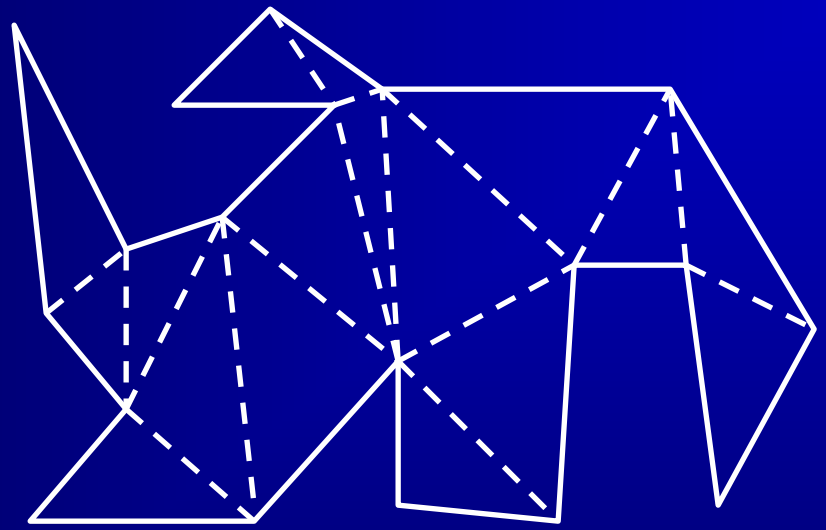

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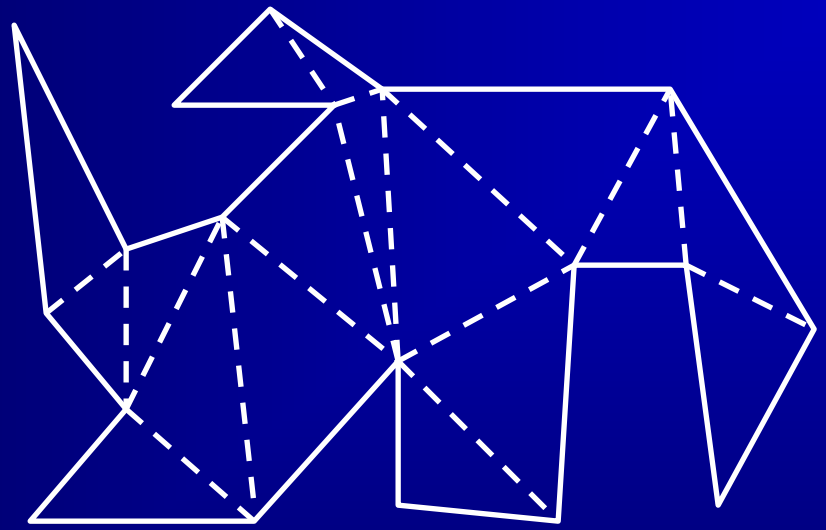

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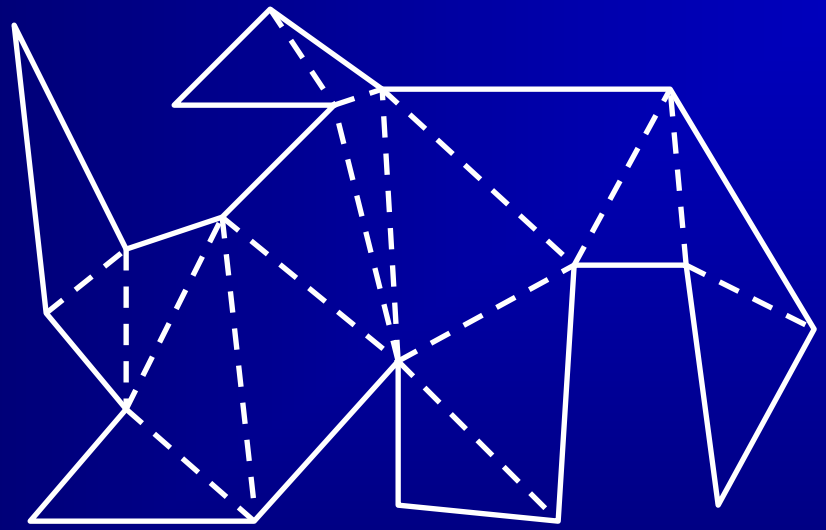

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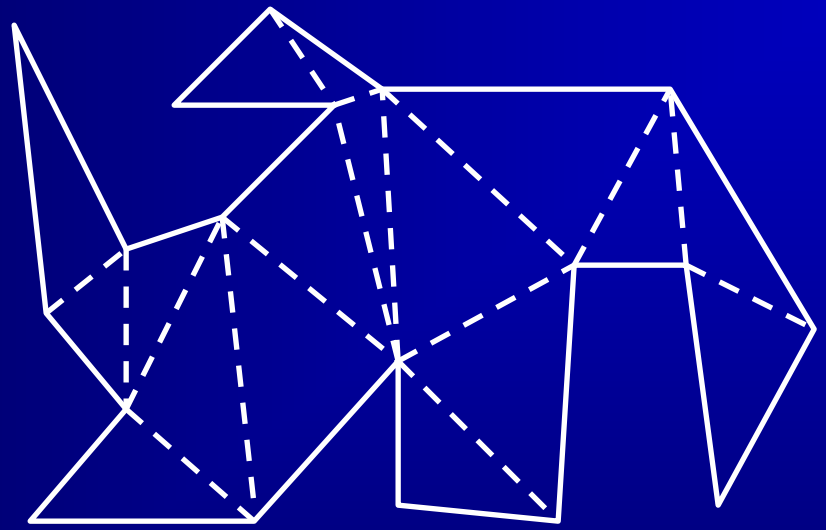

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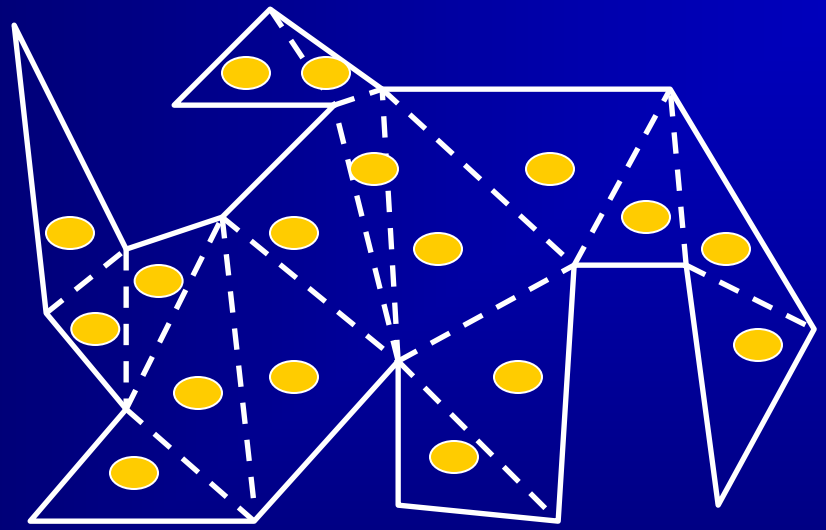

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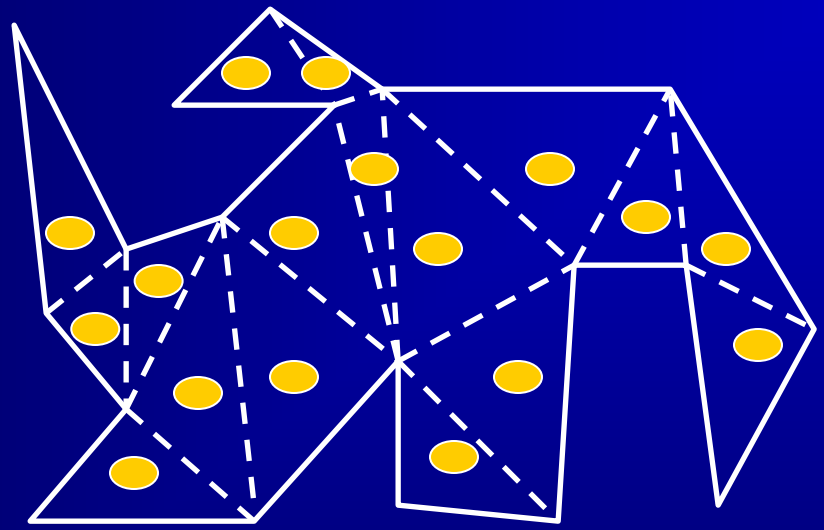

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Triangulation: decomposition of a polygon into triangles by a maximal set of non-intersecting diagonals.

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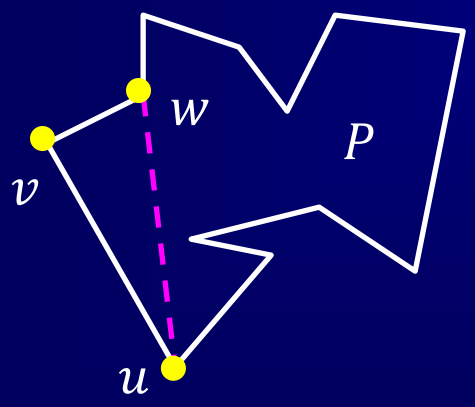

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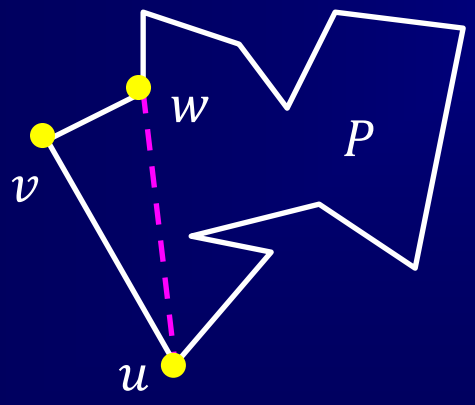

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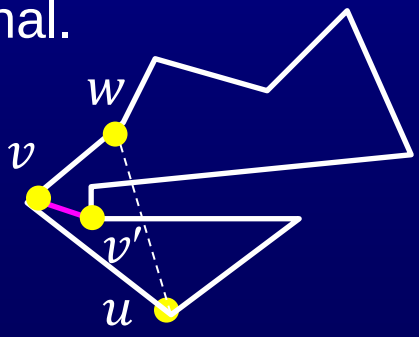
- Otherwise, the triangle determined by u, v, w contains at least one vertex. Let v' be the one closest to v. Then $\overline{vv'}$ is a diagonal.



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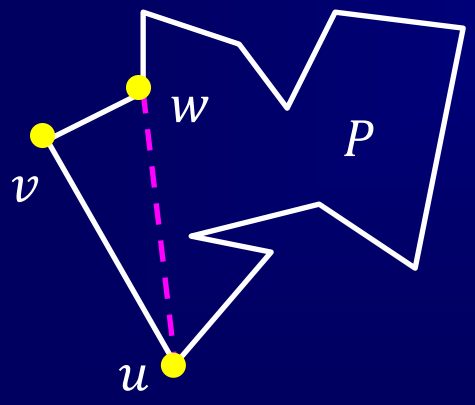

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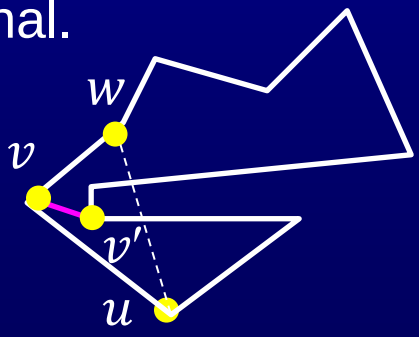

The diagonal splits the polygon into two (which by induction can be triangulated).

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  By induction, the triangulation of P has $(k-2) + (m-2) = k + m - 4 = n + 2 - 4 = n - 2$ triangles.

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A vertex is adjacent to many triangles.
So placing cameras at vertices can do even better …

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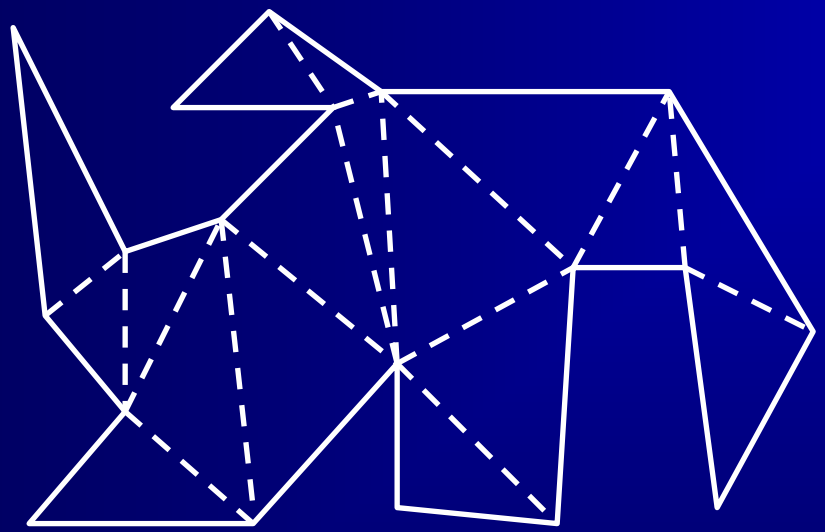
Idea: Select a set of vertices, such that any triangle has at least one selected vertex and place cameras at selected vertices.



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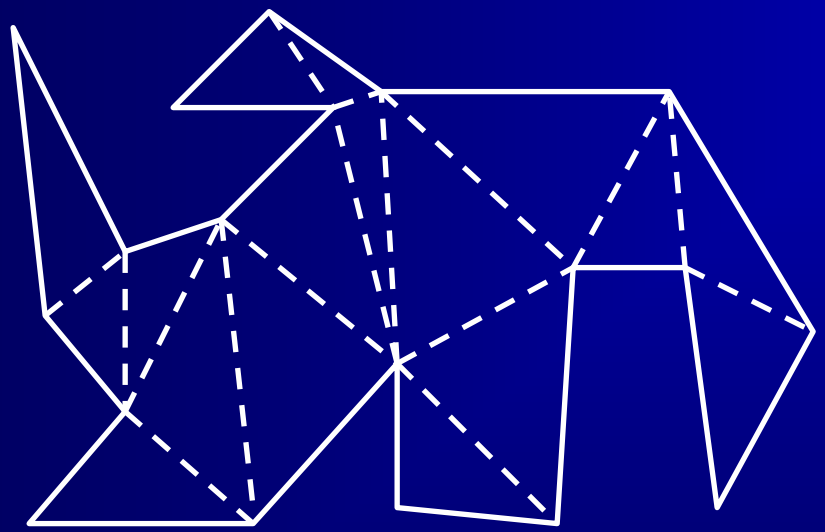

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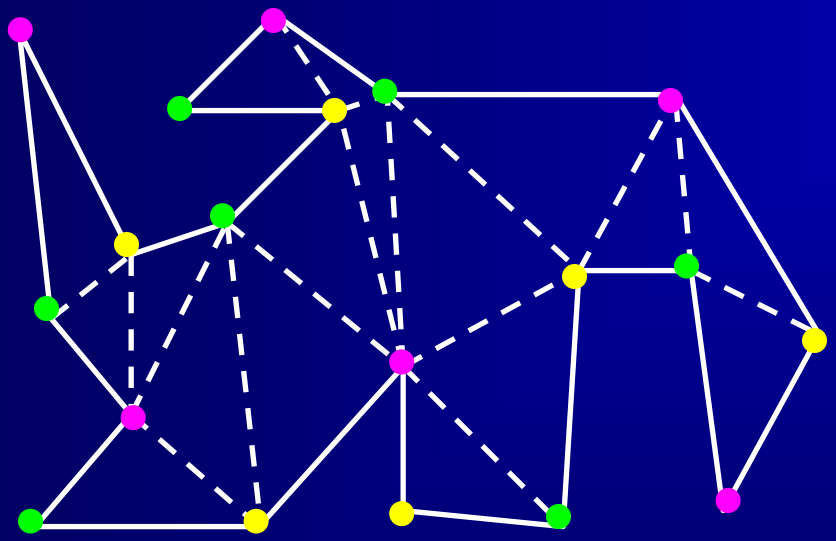

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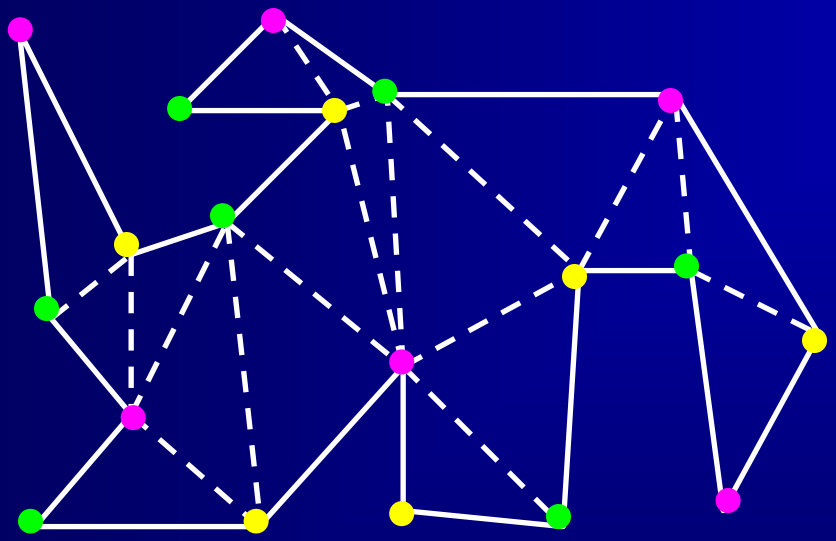

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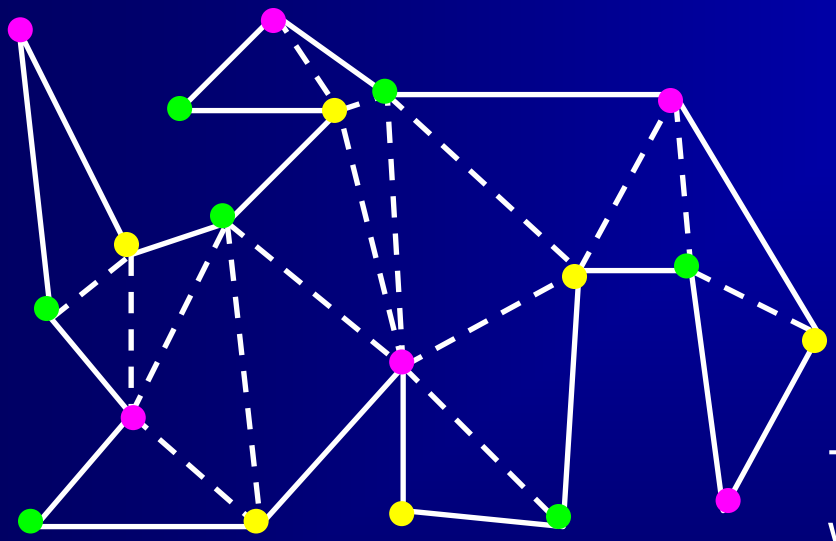

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Thus, the vertices of every triangle will be in three different colors.

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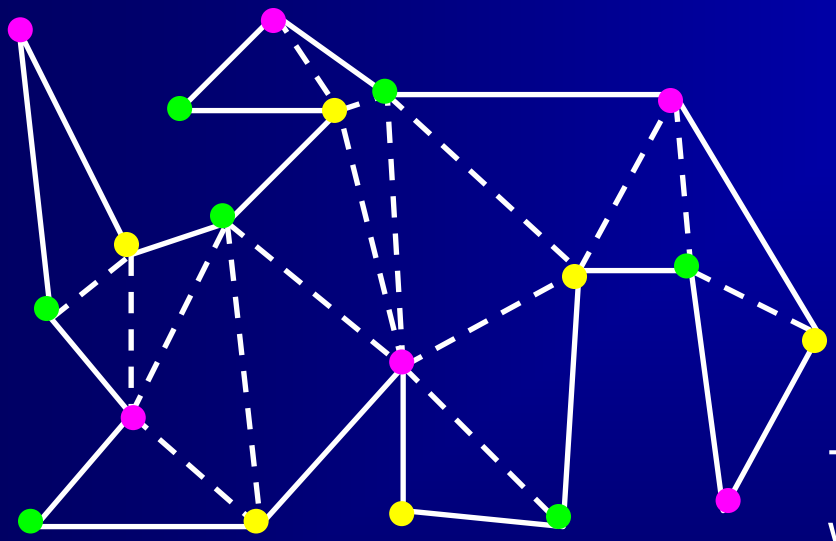

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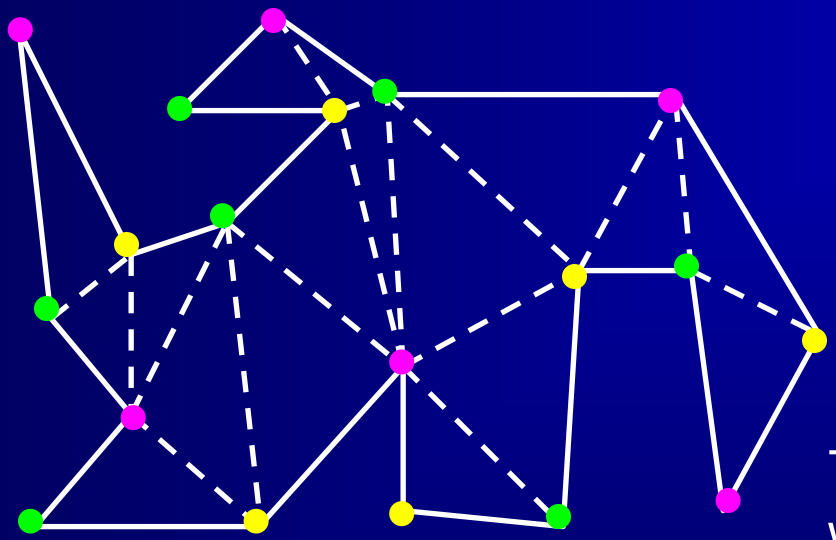

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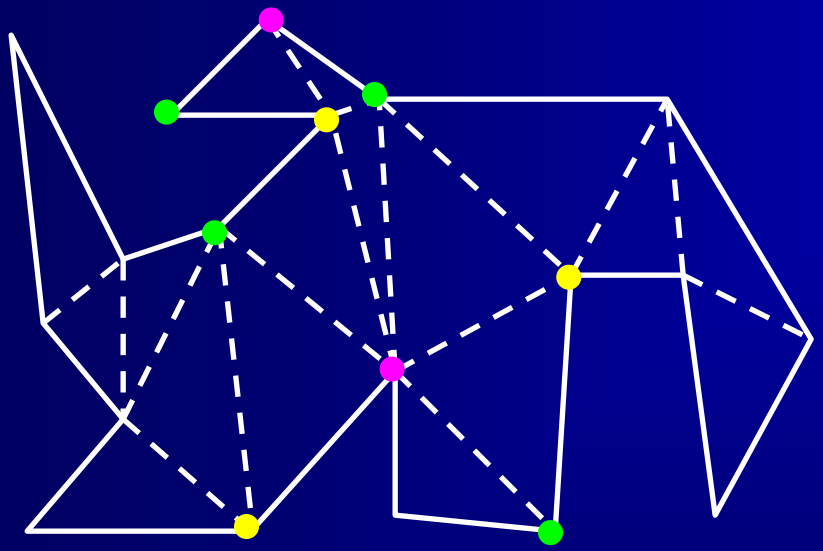
If 3-coloring exists, place cameras at all vertices of the same color.

Choose the smallest color class to place the cameras.

$\Rightarrow \lfloor n/3 \rfloor$ cameras.

The Dual Graph

Dual graph G has a node inside every triangle and an edge between every pair of nodes whose corresponding triangles share a diagonal.



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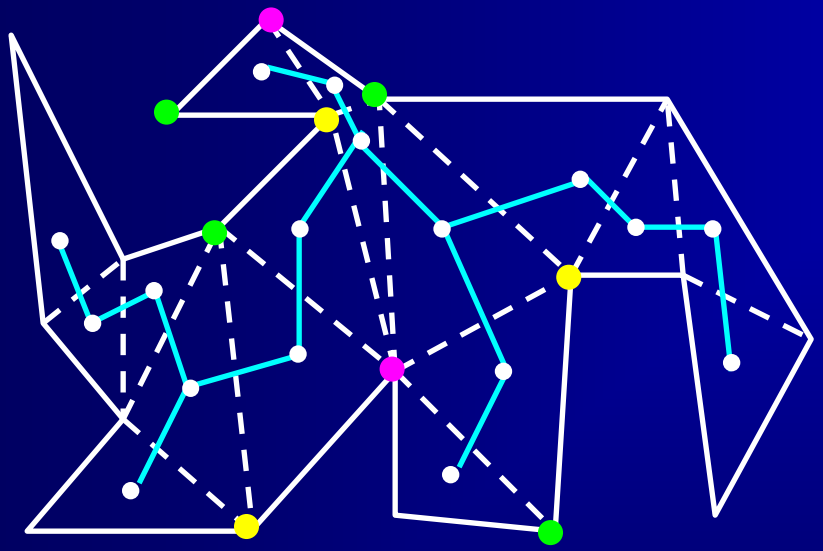

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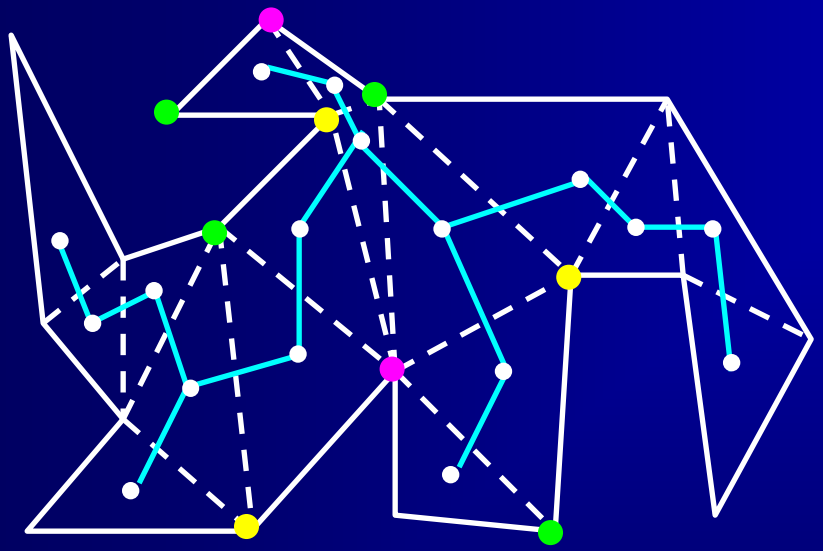

G is connected.

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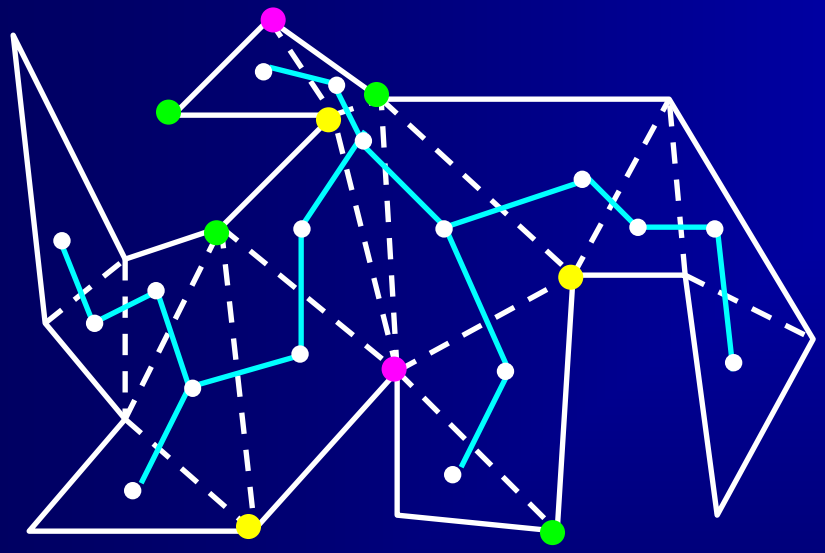

G is connected.

Any diagonal cuts the polygon into two.

Every diagonal corresponds to an edge in the dual graph.

Removal of any edge from the dual graph disconnects it.

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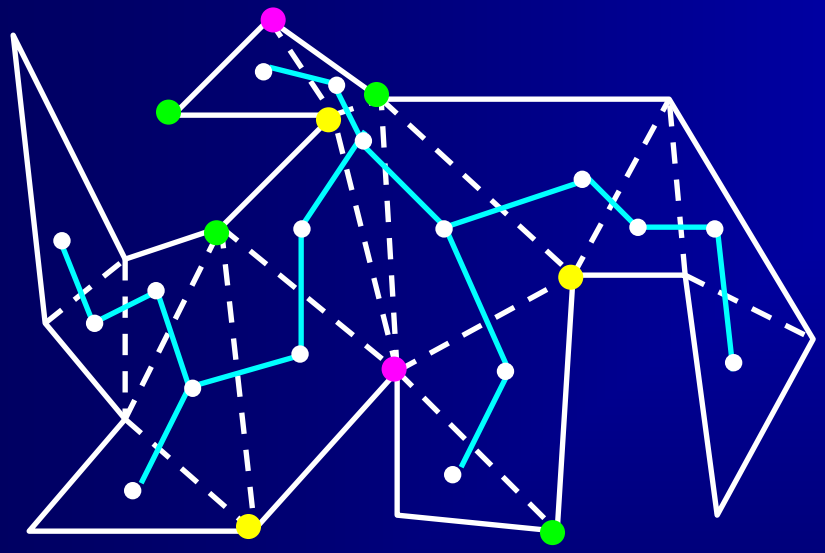

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Thus, the dual graph is a tree.

A 3-Coloring Algorithm

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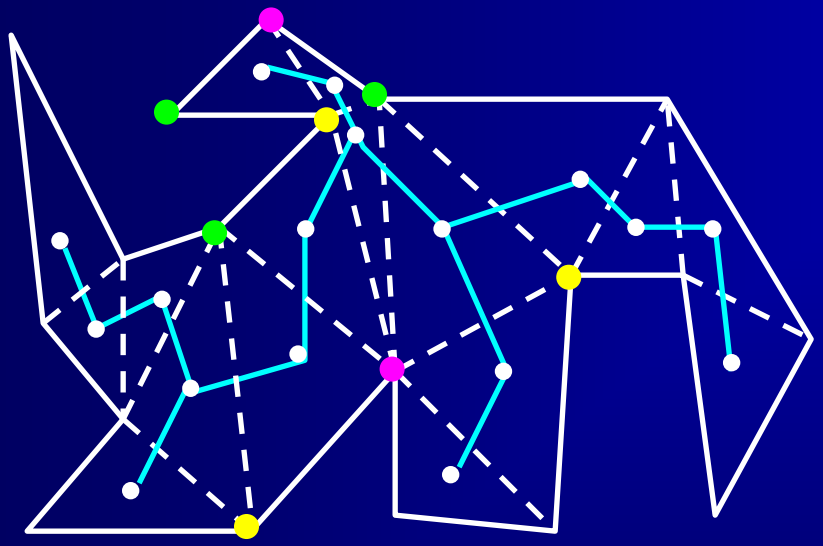

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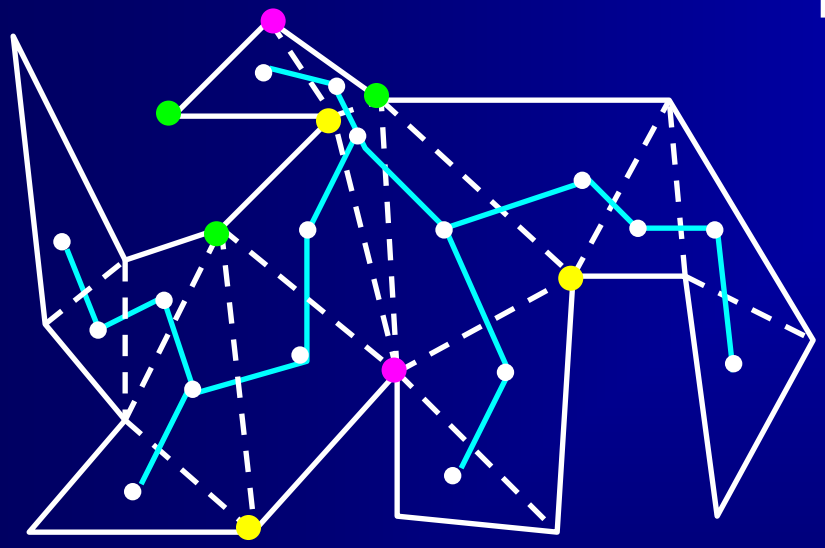

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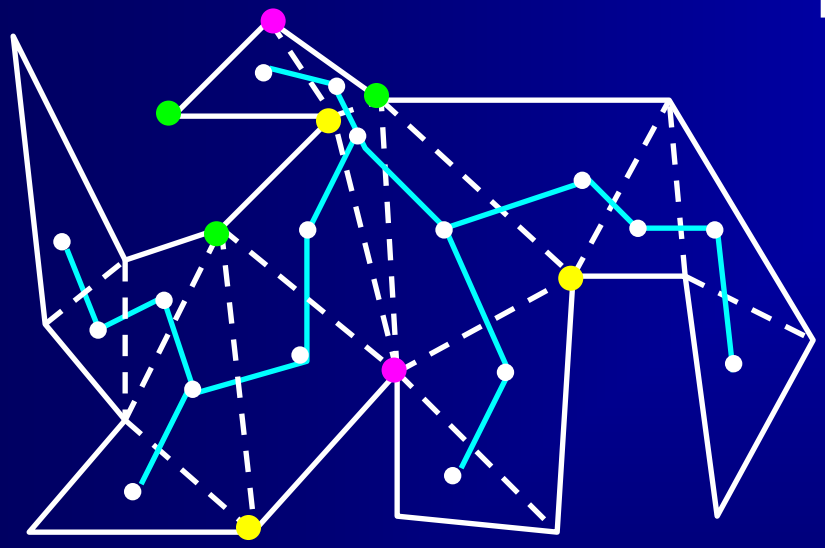

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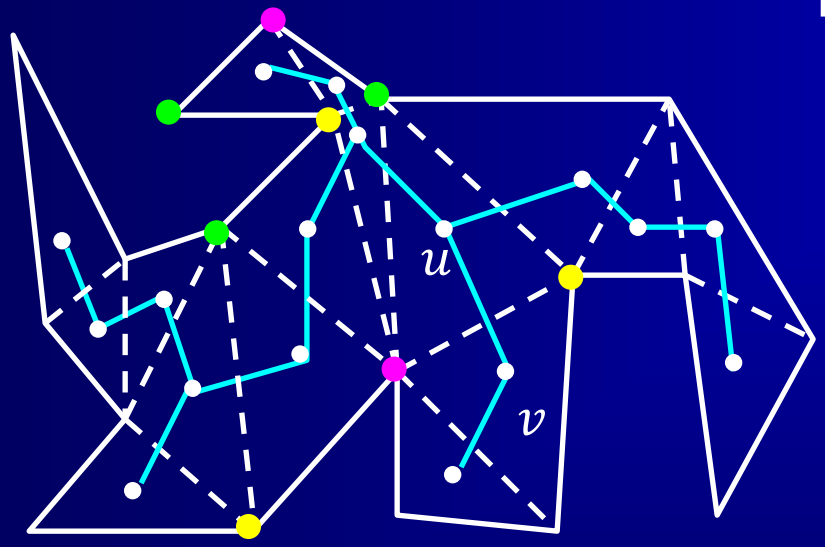

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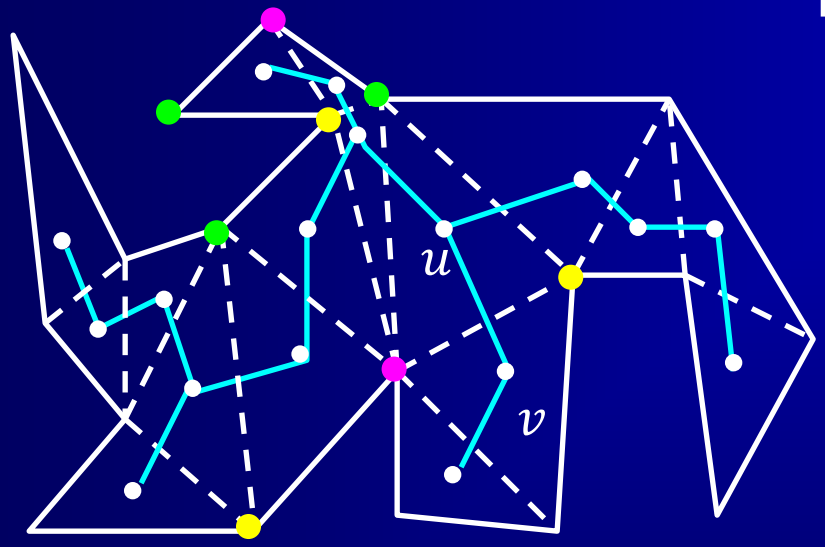

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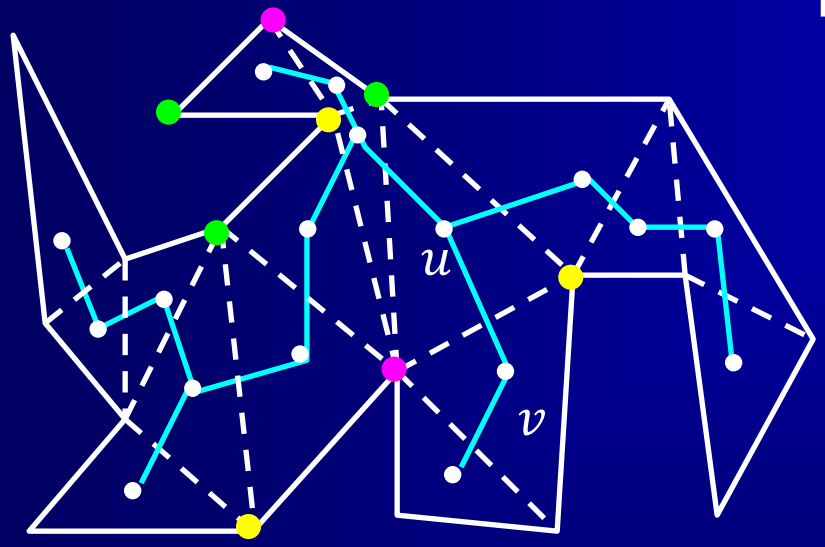

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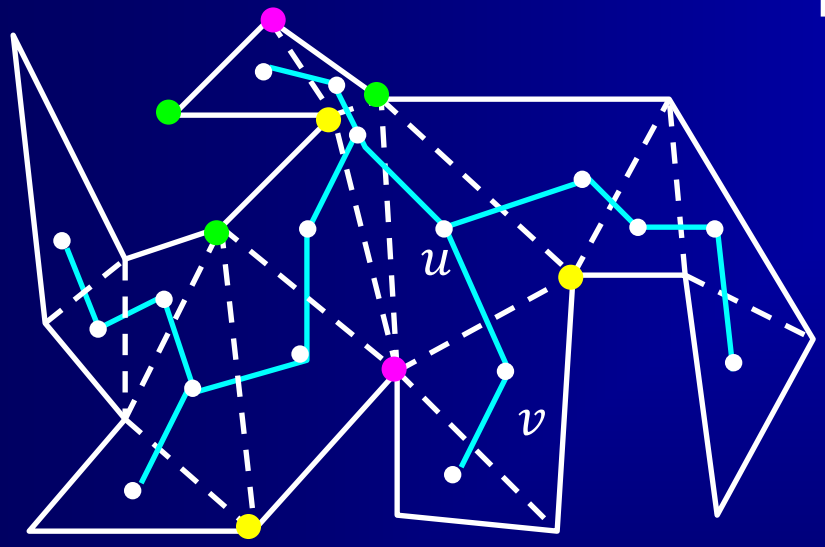

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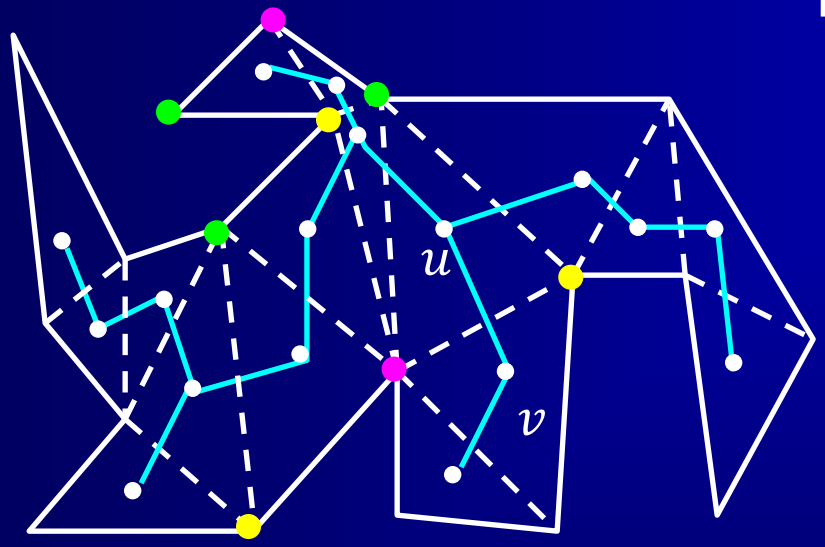

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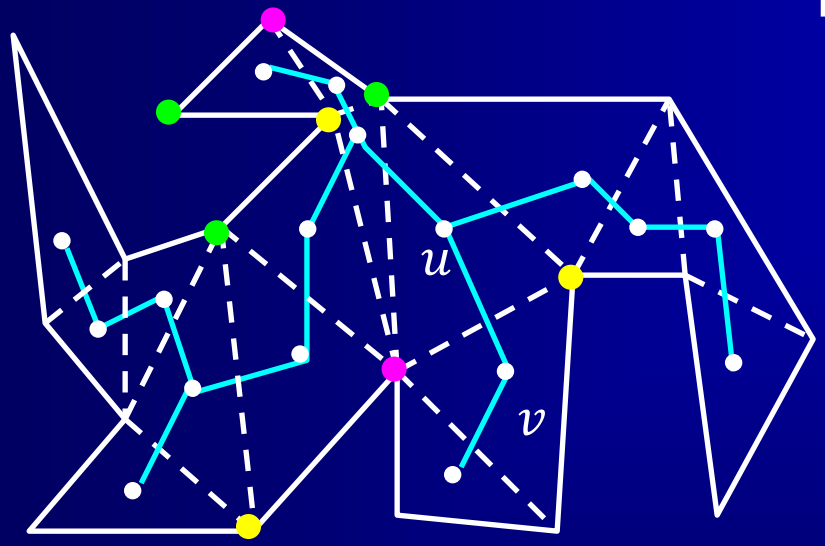

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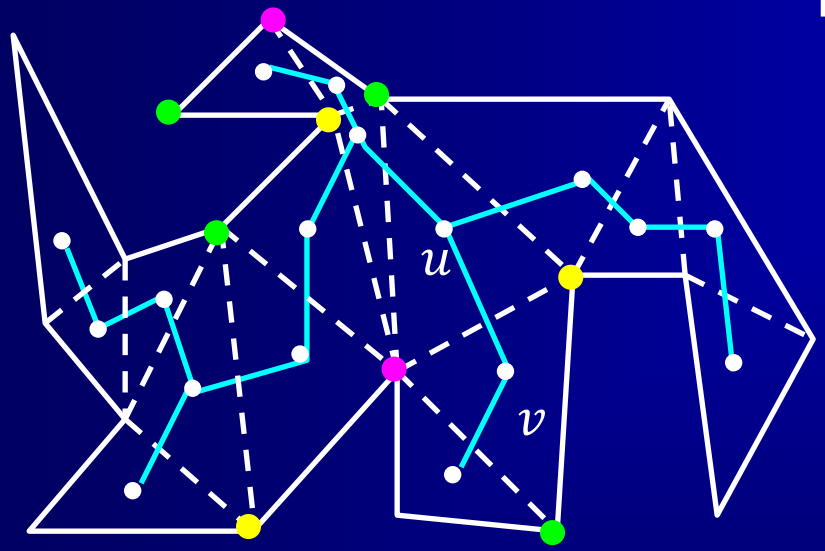

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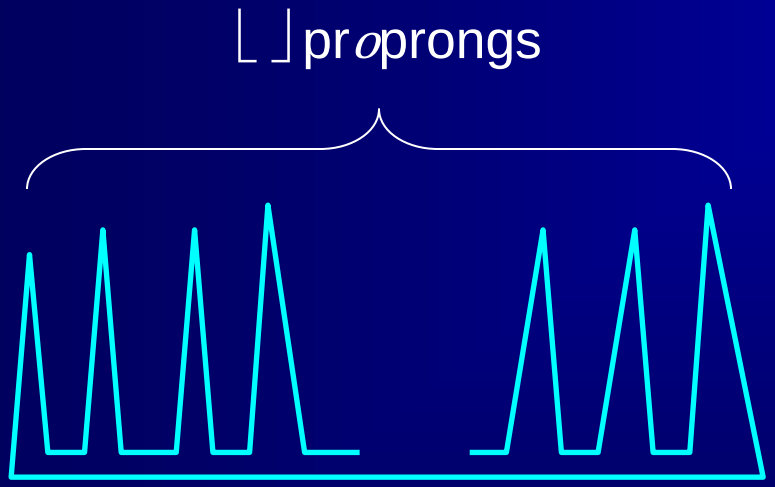

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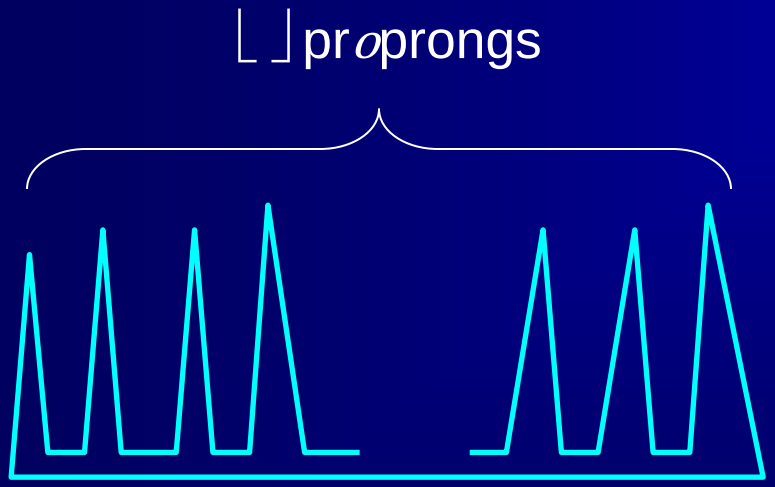

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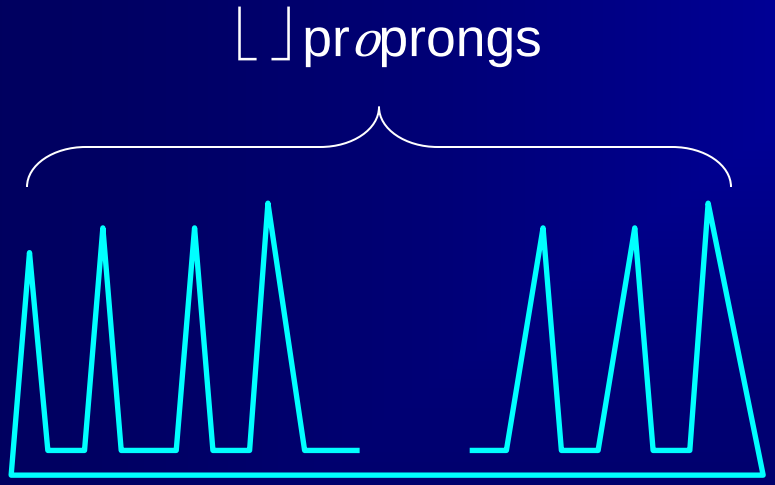

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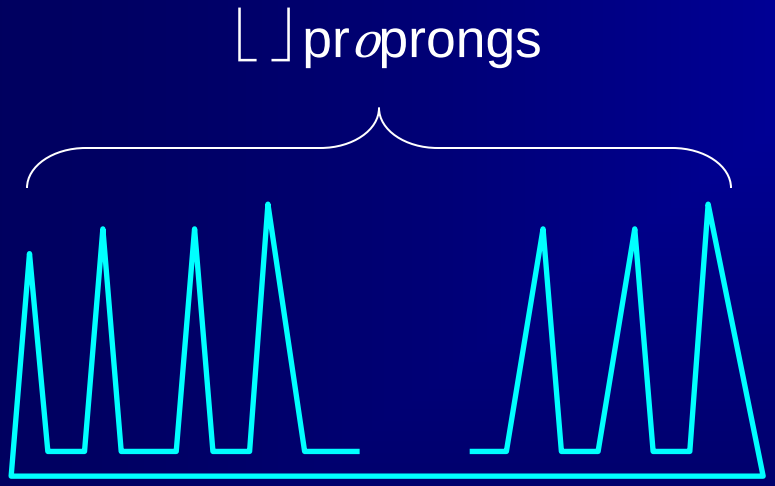

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The 3-coloring approach is optimal in the worst case.

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Solution to the Art Gallery Problem

1. Triangulate a simple polygon with a fast algorithm.

   DCEL representation for the simple polygon so we can visit a neighbor from a triangle in constant time.

2. Generate a 3-coloring by DFS (as presented earlier).

3. Take the smallest color class to place the cameras.