Guarding and Triangulation

Outline

I. Properties of polygon triangulation

II. Solution to the art gallery problem
The Art Gallery Problem
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How many cameras are needed to guard a gallery? Where should they be placed?
I. Simple Polygon Model

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- Convex polygon
  - one camera

- An arbitrary $n$-gon ($n$ vertices)
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Bad news: finding the minimum number of cameras for a given polygon is NP-hard (exponential time).
Triangulation

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Triangulation: decomposition of a polygon into triangles by a maximal set of non-intersecting diagonals.

Draw diagonals between pair of vertices. an open line segment that connects two vertices and lie in the interior of the polygon.

Guard the polygon by placing a camera in every triangle …
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- Otherwise, the triangle determined by \( u, v, w \) contains at least one vertex. Let \( v' \) be the one closest to \( v \). Then \( vv' \) is a diagonal.
### Theorem 1
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**Proof**
By induction.

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- Assume true for all $m < n$.

**Existence**
Let $v$ be the leftmost vertex and $u$ and $w$ its two neighbors.
- $uw$ in the interior of $P \Rightarrow$ it is a diagonal.
- Otherwise, the triangle determined by $u, v, w$ contains at least one vertex. Let $v'$ be the one closest to $v$. Then $vv'$ is a diagonal.

The diagonal splits the polygon into two (which by induction can be triangulated).
Proof (cont’d)
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- Other vertices of $P$ each occurs in exactly one subpolygon.
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By induction, the triangulation of $P$ has

$$(k - 2) + (m - 2) = k + m - 4 = n + 2 - 4 = n - 2$$

triangles.
Theorem 1 $\Rightarrow$ $n - 2$ cameras can guard the simple polygon.
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$\Rightarrow$ # cameras can be reduced to roughly $n/2$.

A vertex is adjacent to many triangles.
So placing cameras at vertices can do even better …
Idea: Select a set of vertices, such that any triangle has at least one selected vertex and place cameras at selected vertices.
3-Coloring

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If 3-coloring exists, place cameras at all vertices of the same color.

Choose the smallest color class to place the cameras.

⇒ $\lfloor n/3 \rfloor$ cameras.
The Dual Graph

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$G$ is connected.
The Dual Graph

**Dual graph** $G$ has a node inside every triangle and an edge between every pair of nodes whose corresponding triangles share a diagonal.

Any diagonal cuts the polygon into two.
Every diagonal corresponds to an edge in the dual graph.
Removal of any edge from the dual graph disconnects it.
Dual graph $G$ has a node inside every triangle and an edge between every pair of nodes whose corresponding triangles share a diagonal. Any diagonal cuts the polygon into two. Every diagonal corresponds to an edge in the dual graph.

Removal of any edge from the dual graph disconnects it. Thus, the dual graph is a tree.
A 3-Coloring Algorithm

A 3-coloring can be found through a graph traversal (such as DFS).
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During DFS, maintain the invariant:

All polygon vertices of encountered triangles have been colored such that no adjacent two have the same color.
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Start DFS at any node of $G$. Color the three vertices of the corresponding triangle.
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Suppose node $v$ is visited from $u$. Their triangles $T(v)$ and $T(u)$ are adjacent.
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Only one vertex of $T(v)$ is not colored.
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Suppose node $v$ is visited from $u$. Their triangles $T(v)$ and $T(u)$ are adjacent. Only one vertex of $T(v)$ is not colored. Its color is uniquely determined.
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A Worst Case

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\(\lfloor \text{proprongs} \rfloor\)
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Any simple polygon can be guarded with $\lceil n/3 \rceil$ cameras.

There exists no position at which a camera can oversee two prongs.
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Any simple polygon can be guarded with \( \lceil n/3 \rceil \) cameras.

There exists no position at which a camera can oversee two prongs.

\( \lfloor n/3 \rfloor \) cameras are needed.
A triangulated polygon can always be 3-colored.

Any simple polygon can be guarded with $\lceil n/3 \rceil$ cameras.

There exists no position at which a camera can oversee two prongs.

$\lceil n/3 \rceil$ cameras are needed.

The 3-coloring approach is optimal in the worst case.
Art Gallery Theorem

For a simple polygon with \( n \) vertices, \( \lceil n/3 \rceil \) cameras are sufficient to have every interior point visible from at least one of the cameras.
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For a simple polygon with $n$ vertices, $\left\lfloor n/3 \right\rfloor$ cameras are sufficient to have every interior point visible from at least one of the cameras.

Solution to the Art Gallery Problem

1. **Triangulate a simple polygon with a fast algorithm.**
   
   DCEL representation for the simple polygon so we can visit a neighbor from a triangle in constant time.

2. **Generate a 3-coloring by DFS** (as presented earlier).

3. **Take the smallest color class to place the cameras.**