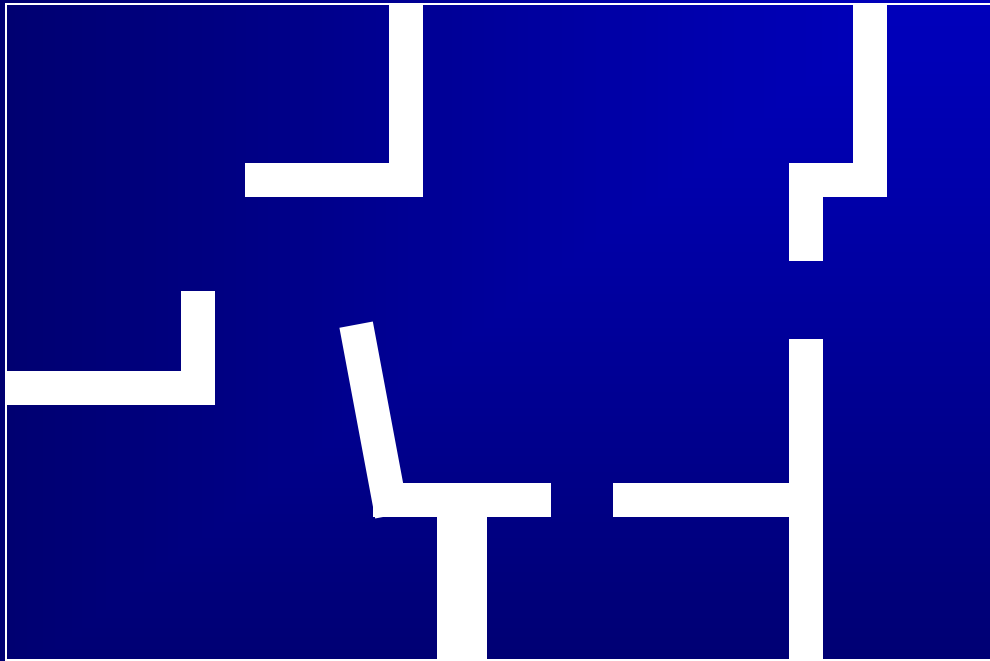


Guarding and Triangulation

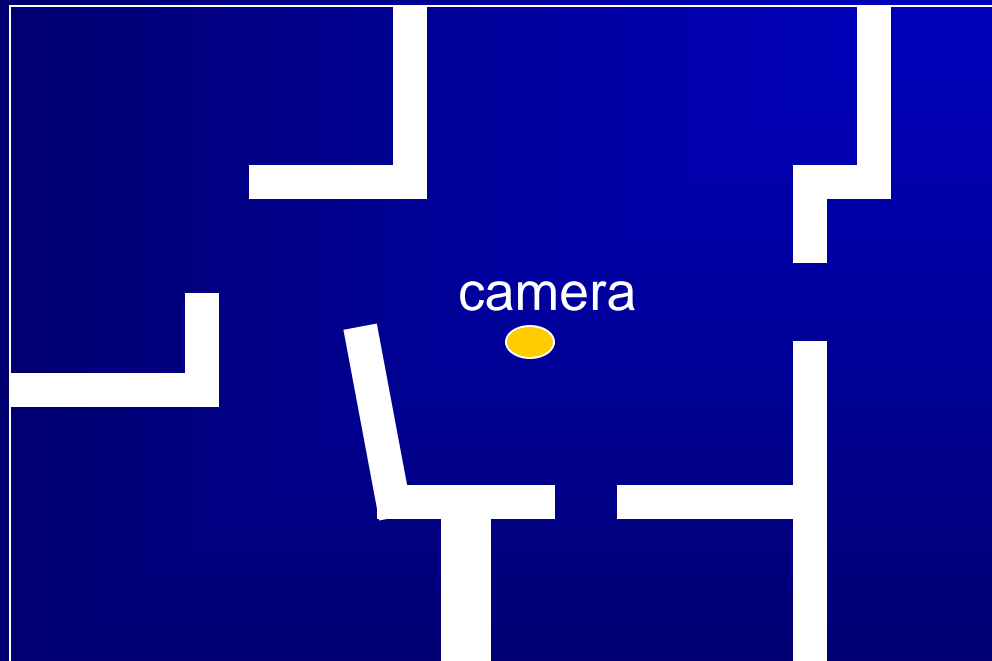
Outline

- I. Properties of polygon triangulation
- II. Solution to the art gallery problem

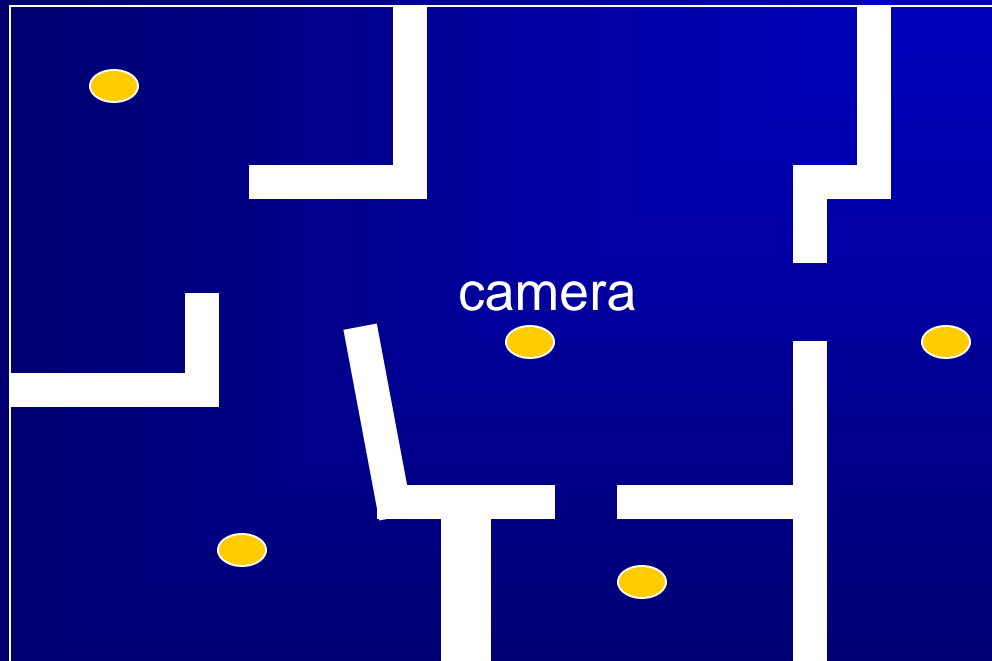
The Art Gallery Problem



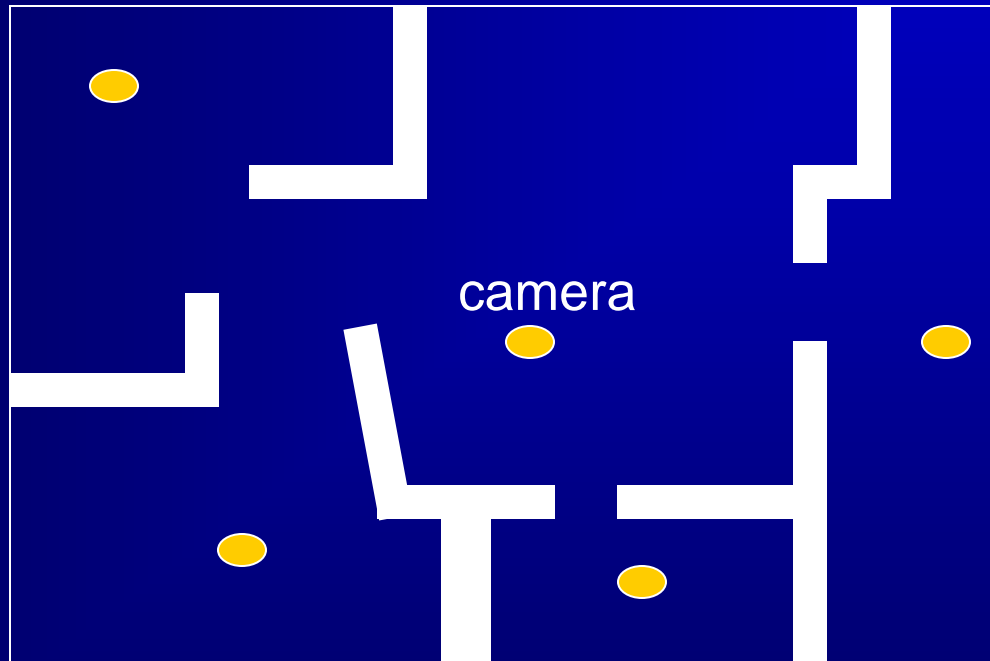
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The Art Gallery Problem



The Art Gallery Problem



How many cameras are needed to guard a gallery?
Where should they be placed?

I. Simple Polygon Model

Model the art gallery as a region bounded by some *simple polygon* (no self-crossing).

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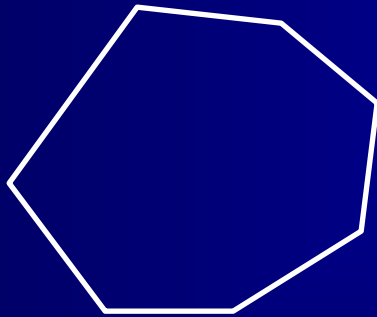
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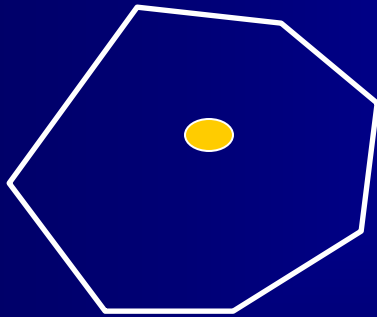


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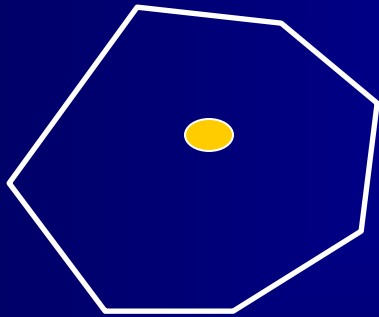


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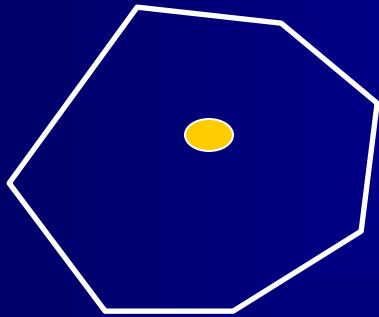


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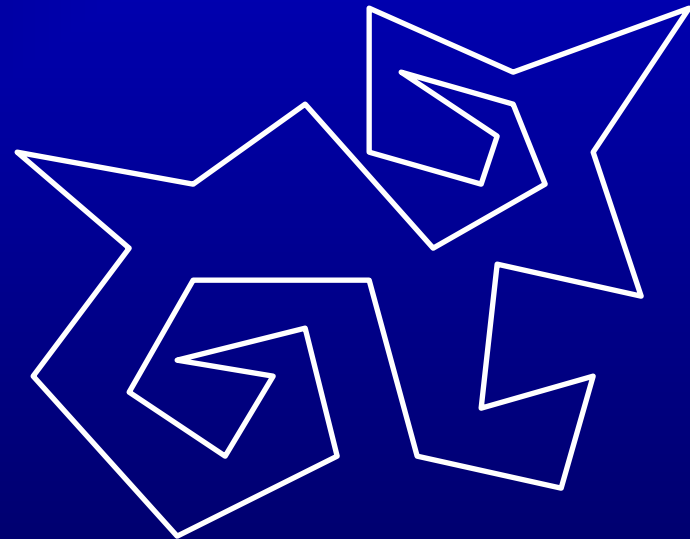
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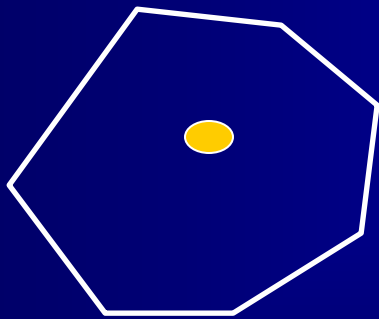


an arbitrary n -gon (n vertices)

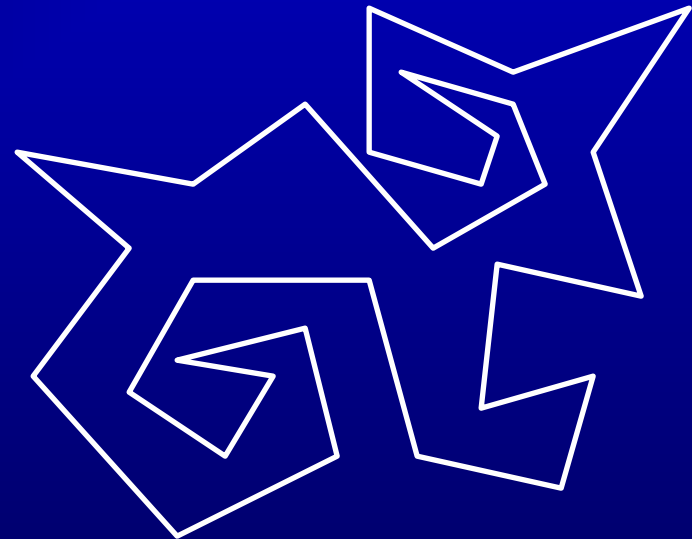
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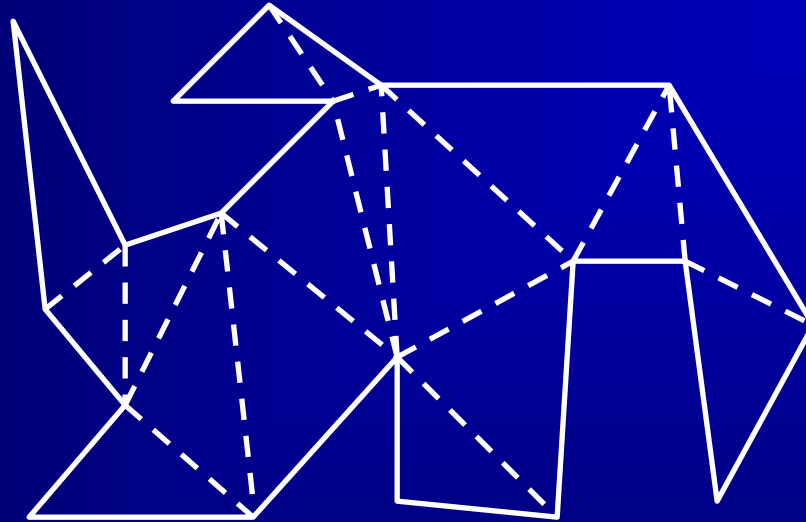
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Bad news: finding the minimum number of cameras for a given polygon is **NP-hard** (exponential time).

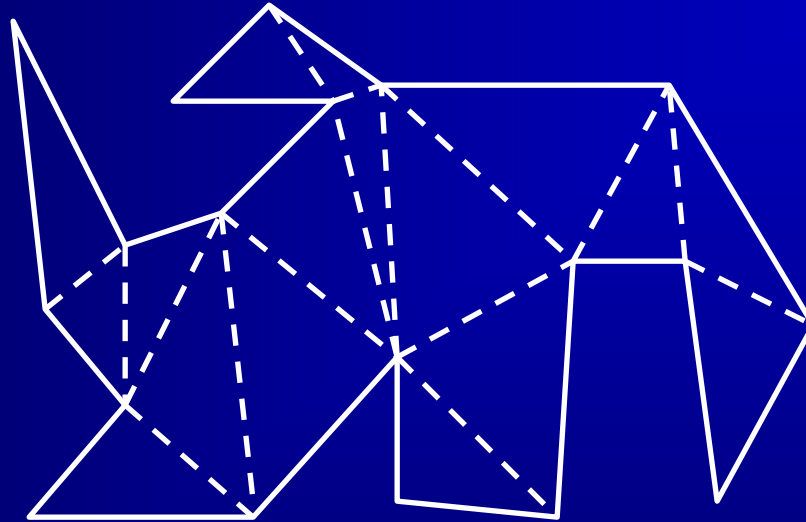
Triangulation

To make things easier, we decompose a polygon into pieces that are easy to guard.



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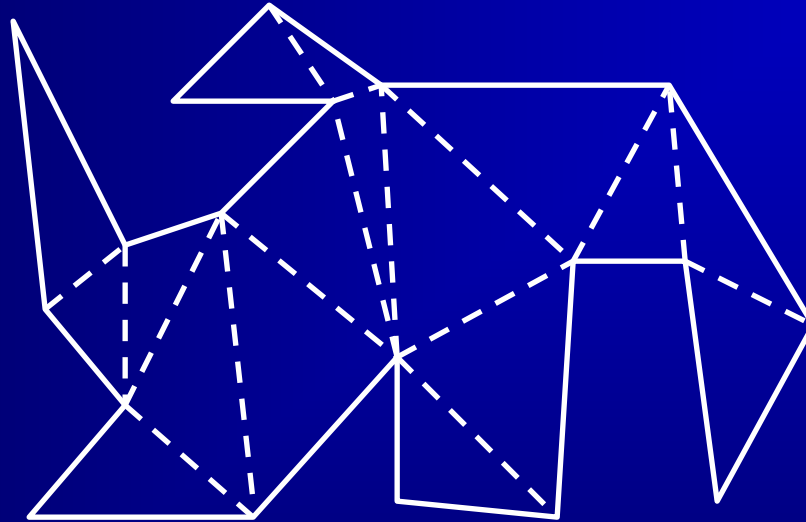
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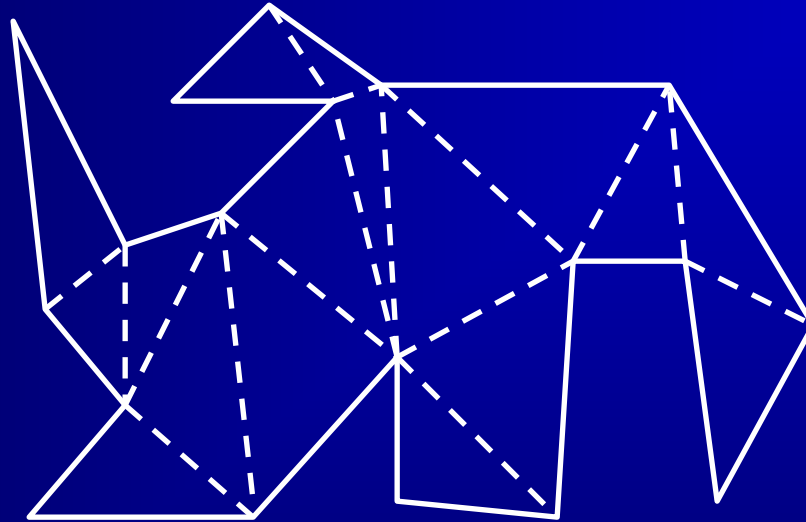


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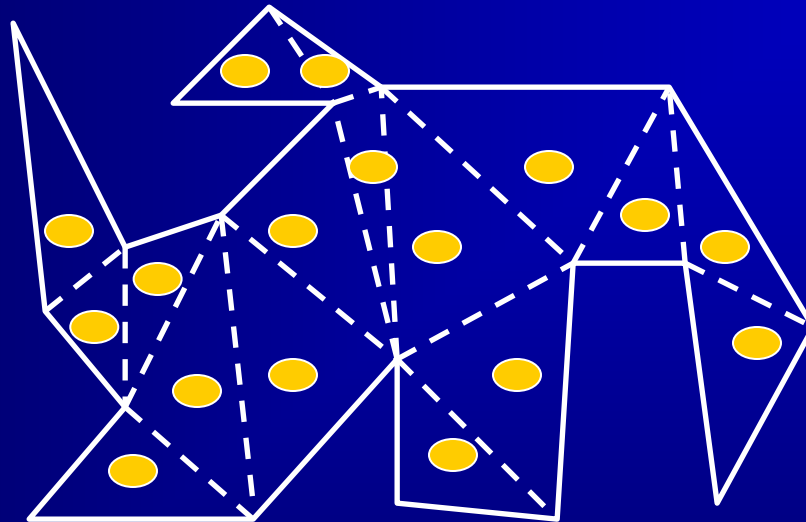
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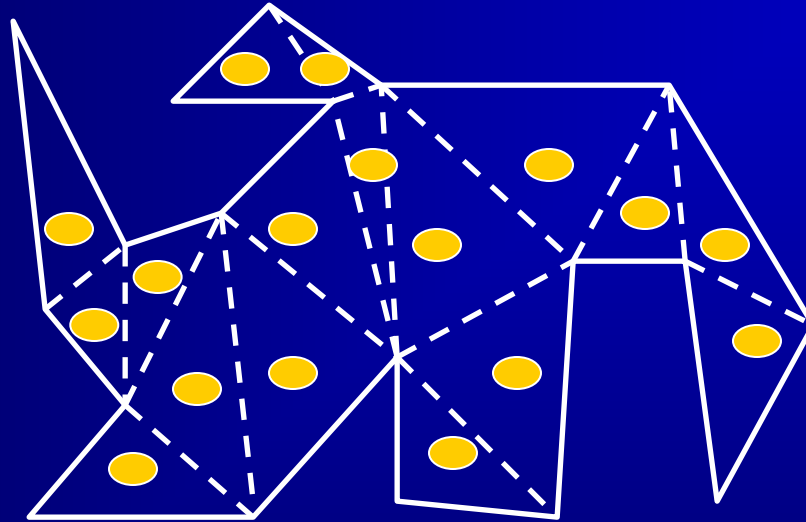
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Triangulation: decomposition of a polygon into triangles by a maximal set of non-intersecting diagonals.

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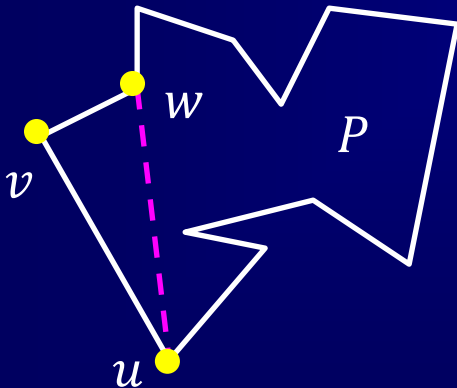
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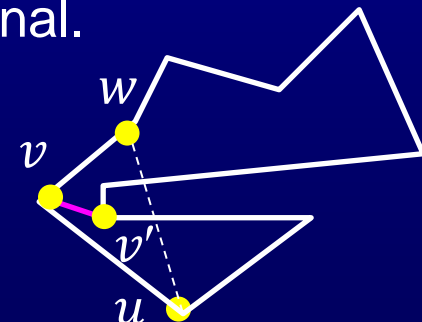
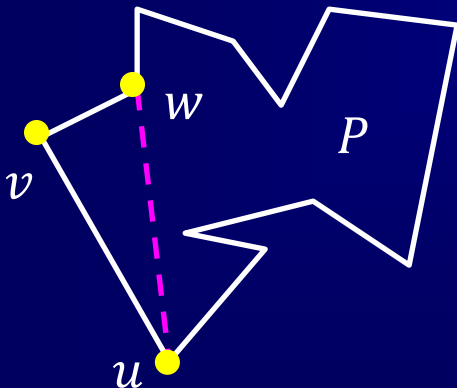
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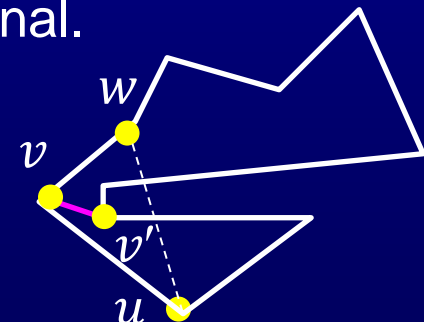
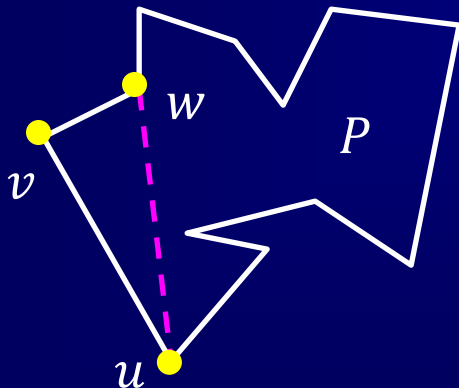
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The diagonal splits the polygon into two (which by induction can be triangulated).

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By induction, the triangulation of P has

$(k - 2) + (m - 2) = k + m - 4 = n + 2 - 4 = n - 2$
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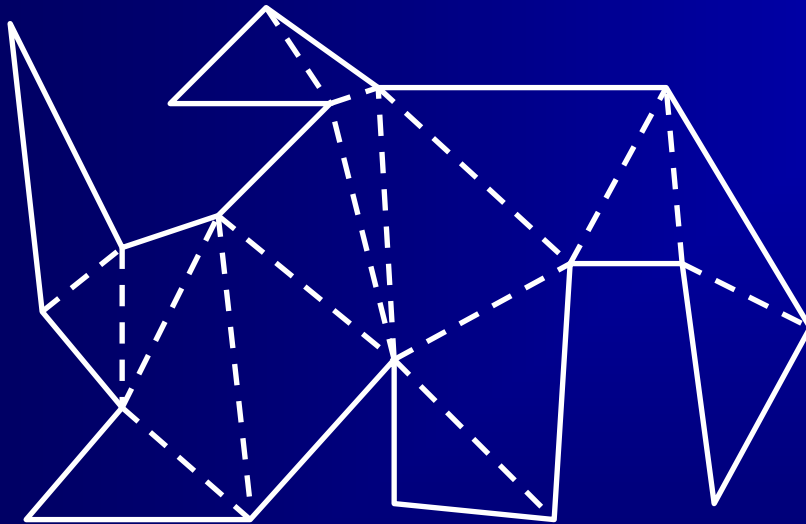
\Rightarrow # cameras can be reduced to roughly $n/2$.

A vertex is adjacent to many triangles.

So placing cameras at vertices can do even better ...

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Idea: Select a set of vertices, such that any triangle has at least one selected vertex and place cameras at selected vertices.

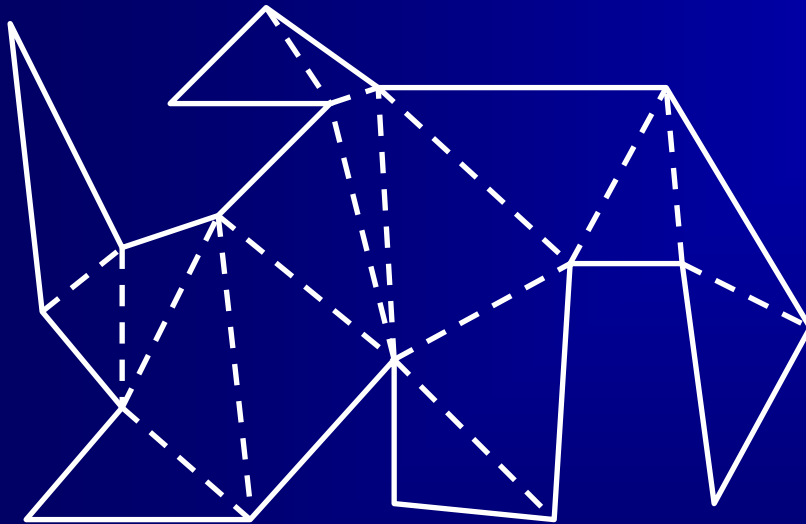


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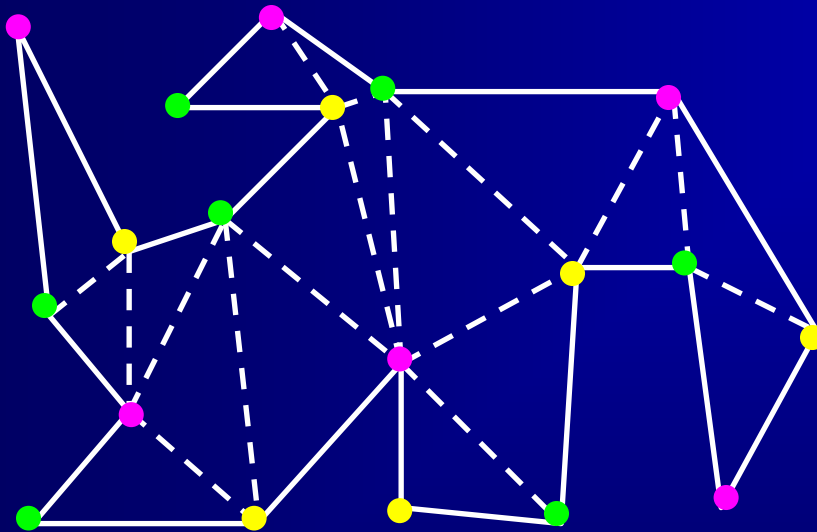
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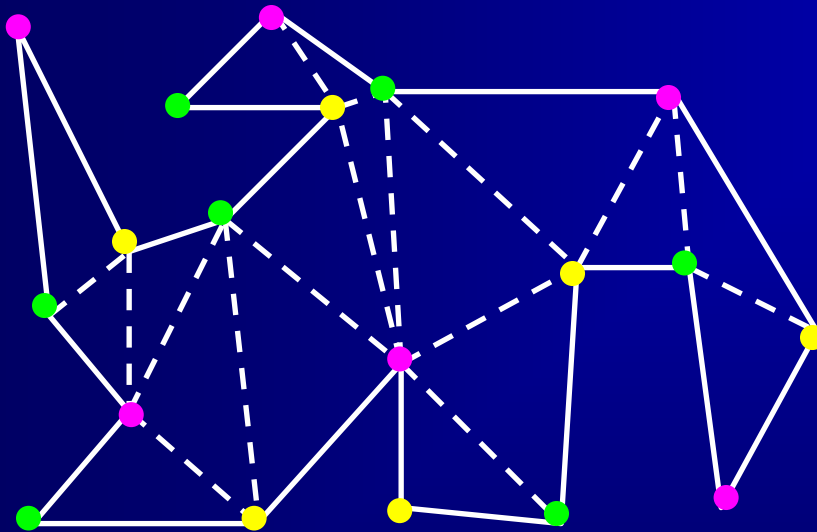


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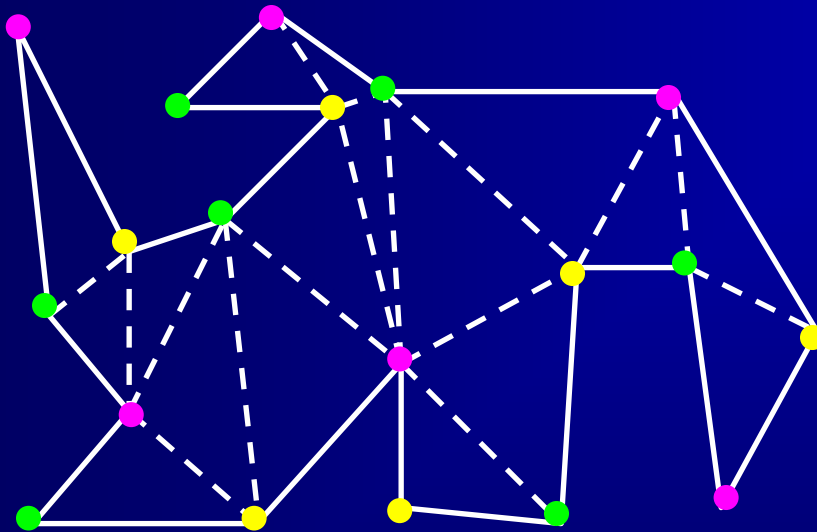
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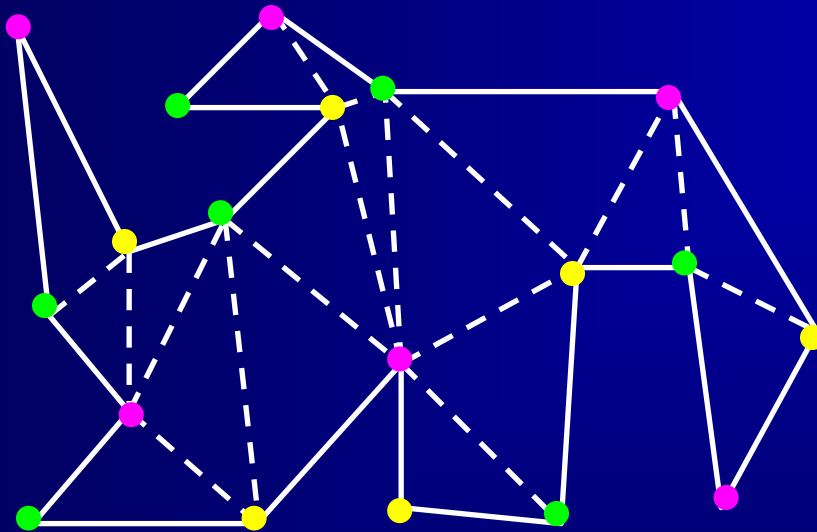
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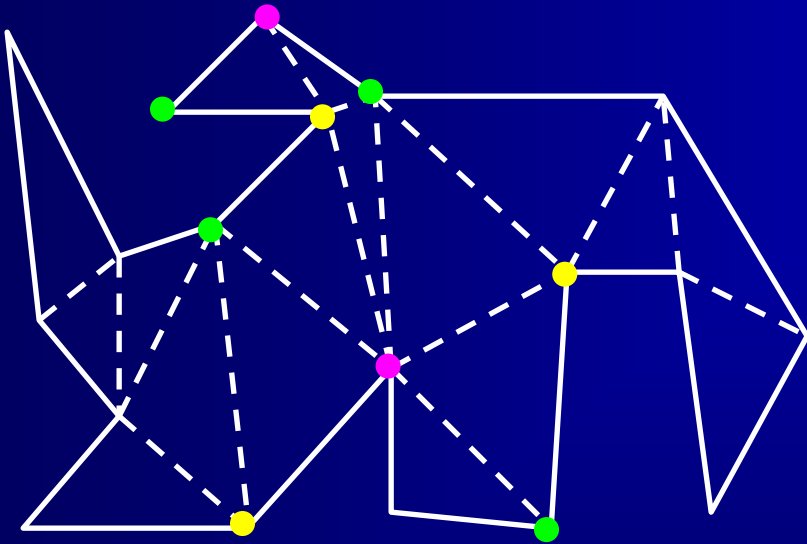
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Choose the smallest color class to place the cameras.

$\Rightarrow \lfloor n/3 \rfloor$ cameras.

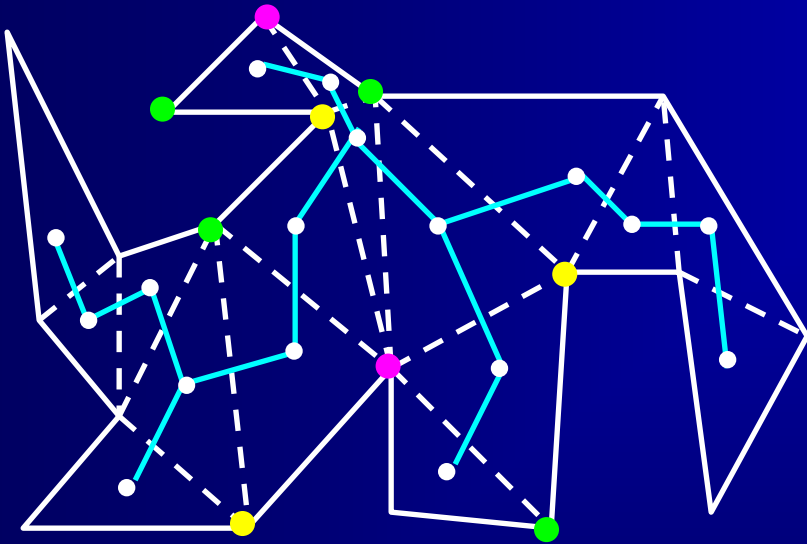
The Dual Graph

Dual graph G has a node inside every triangle and an edge between every pair of nodes whose corresponding triangles share a diagonal.



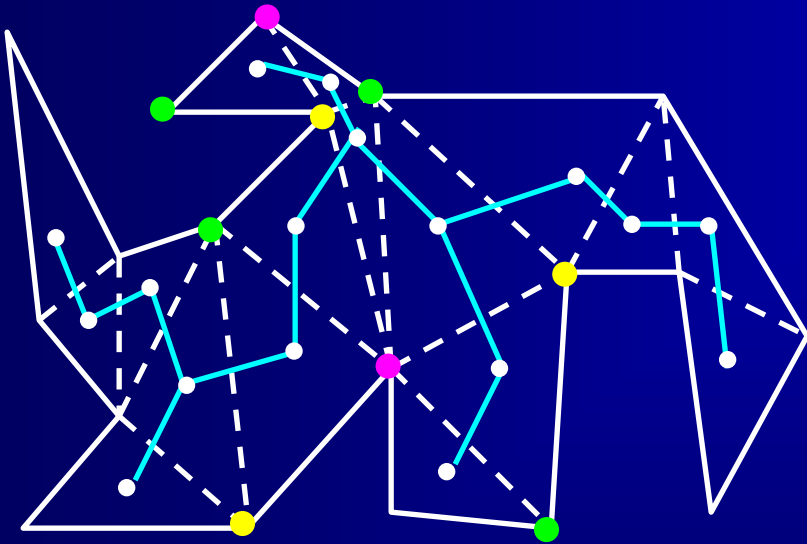
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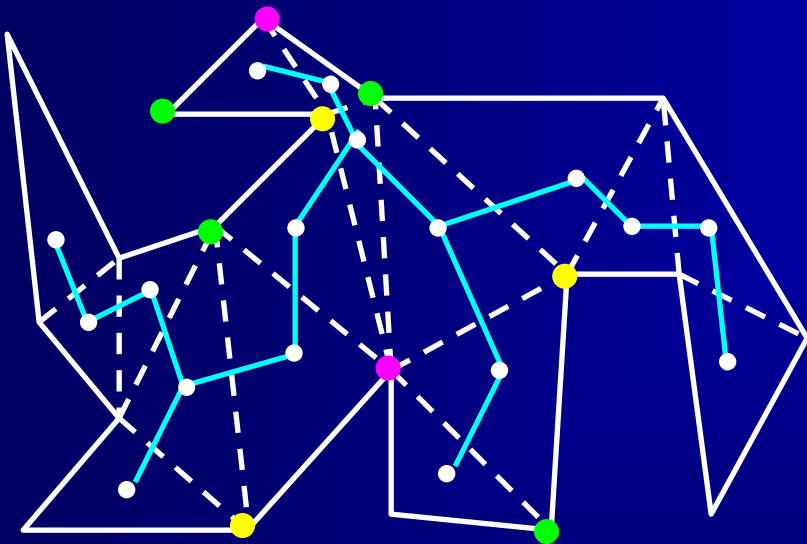


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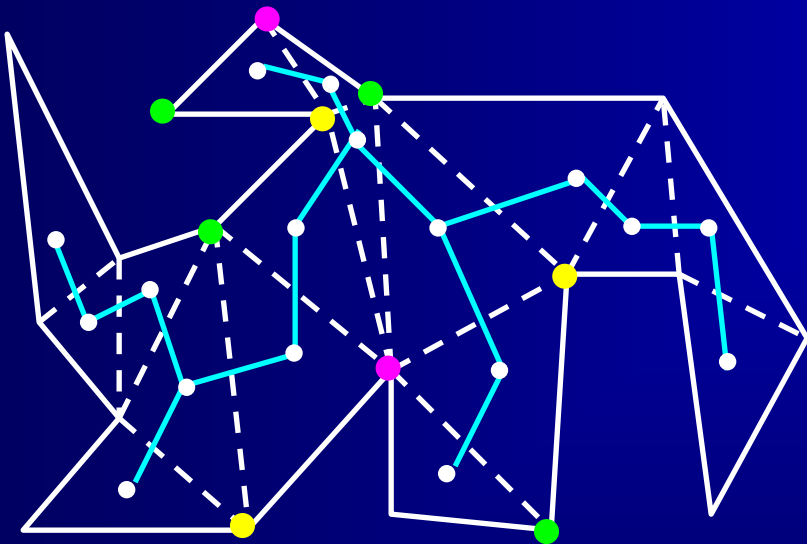
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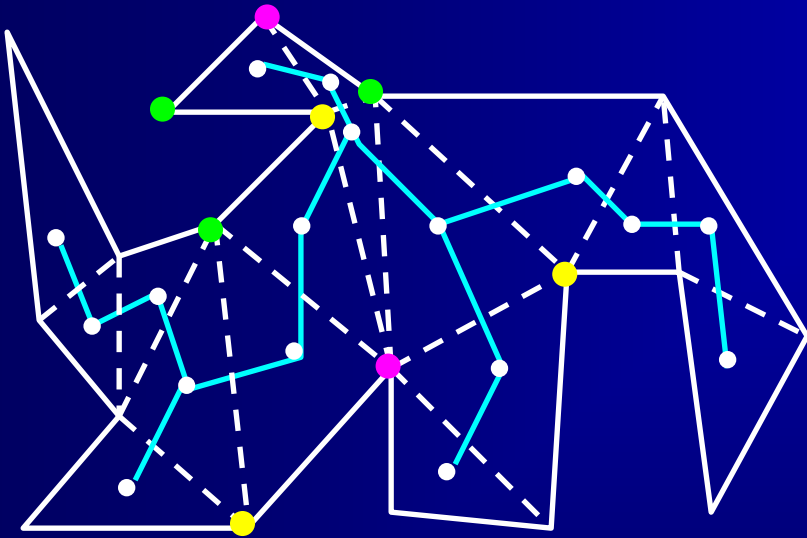
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Thus, the dual graph is a tree.

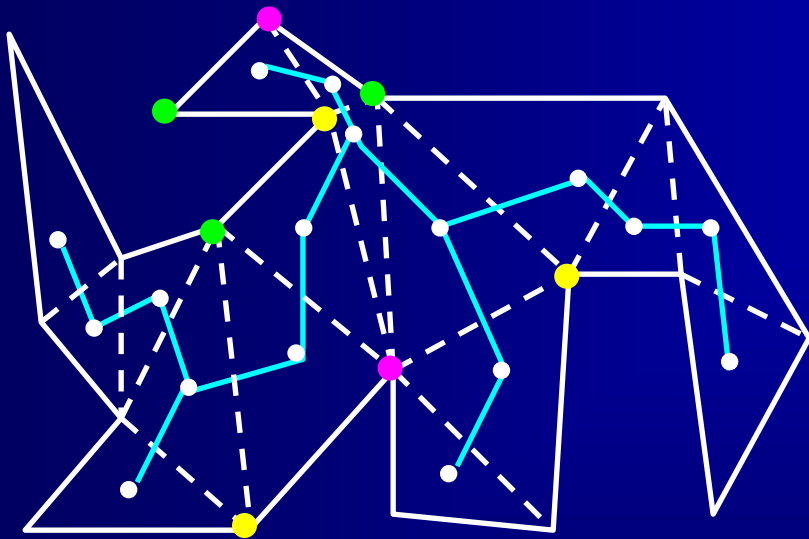
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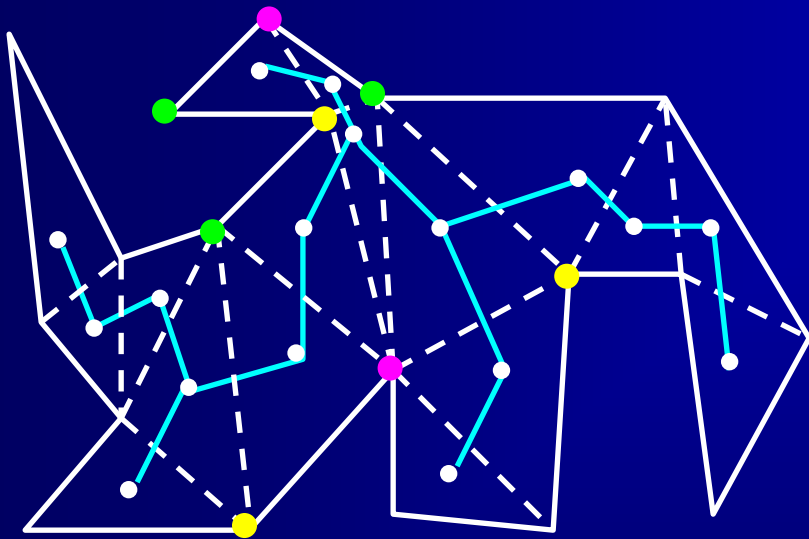
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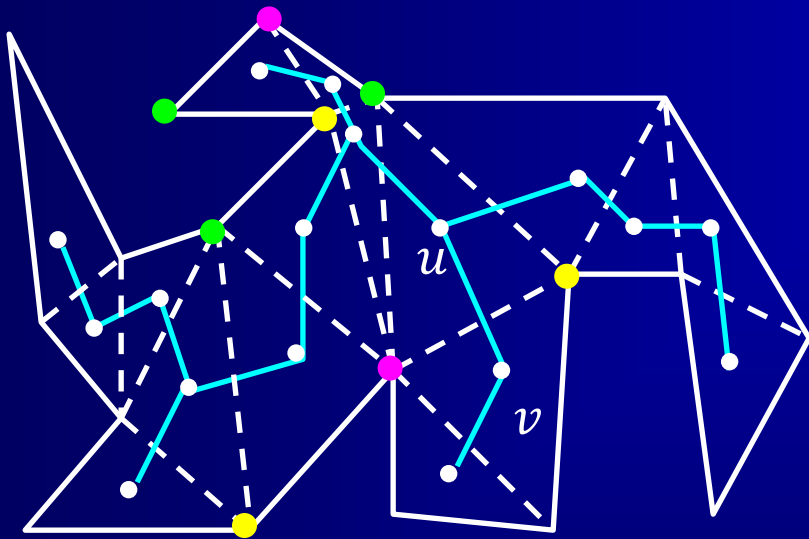
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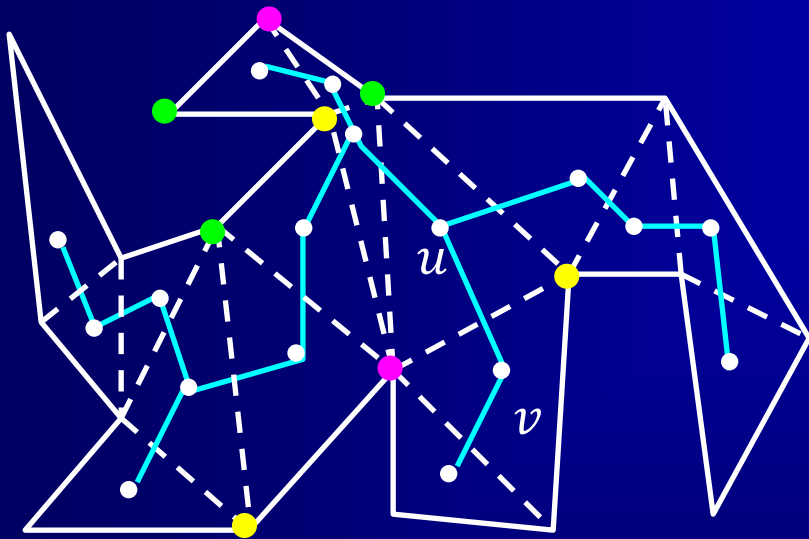
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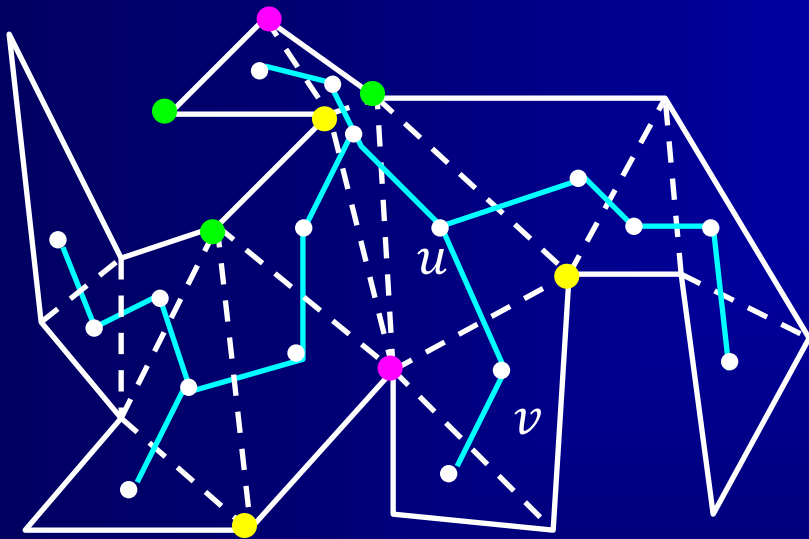
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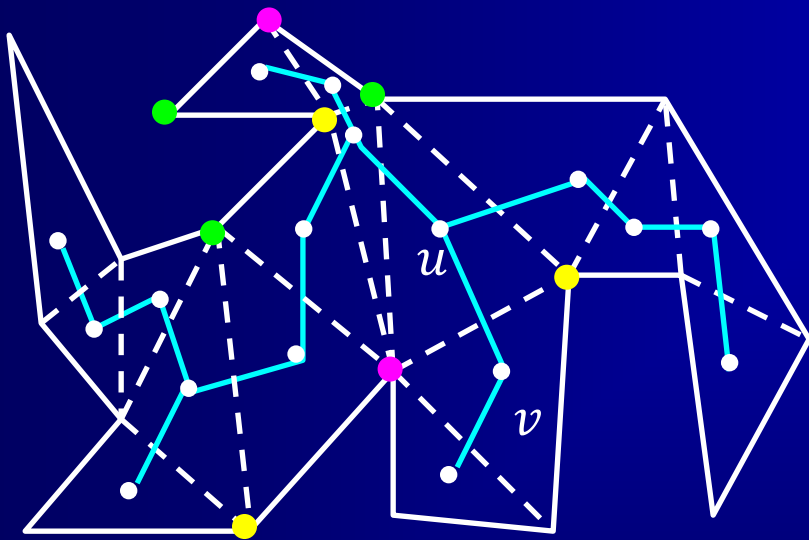
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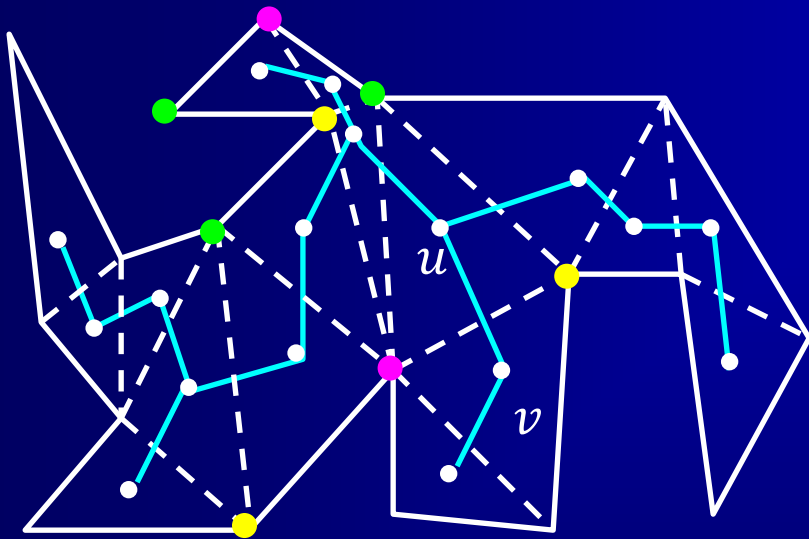
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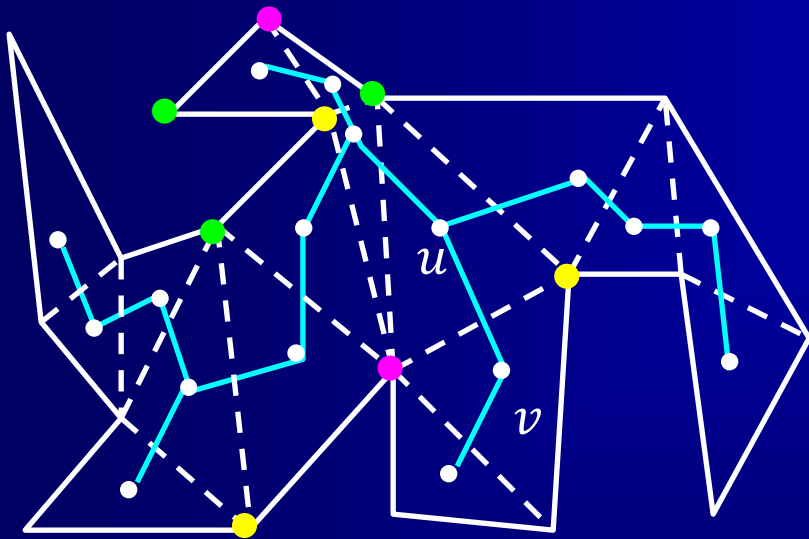
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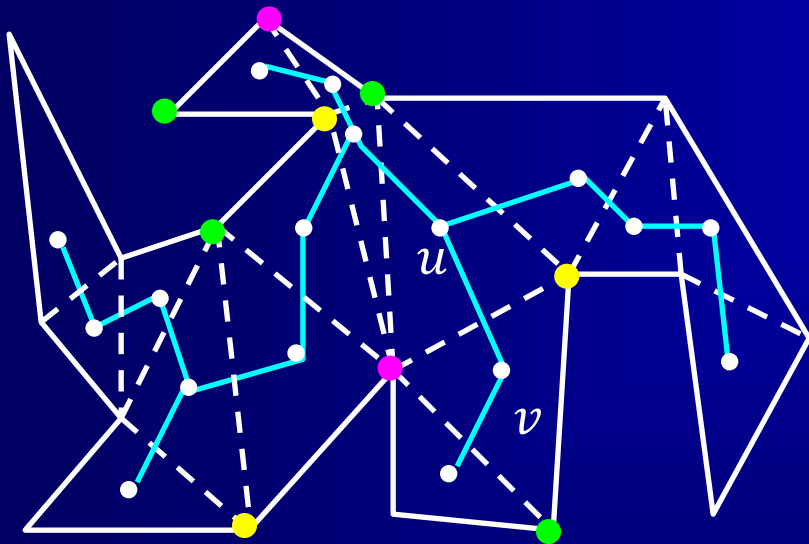
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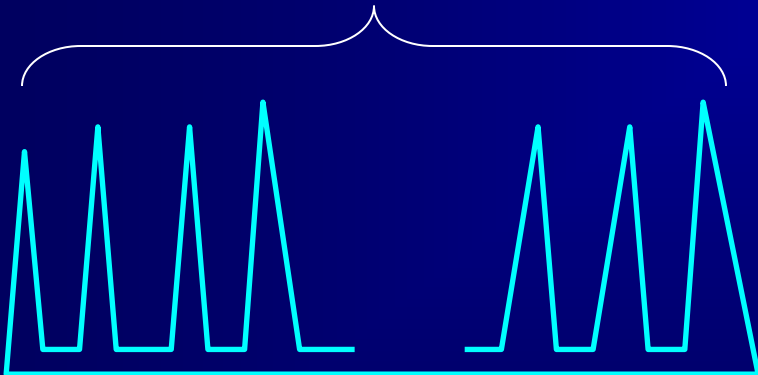
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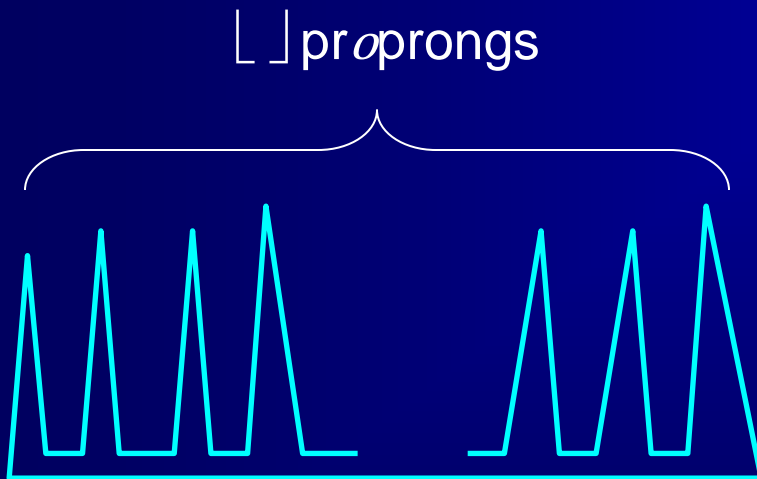
$\lfloor n \rfloor$ prongs



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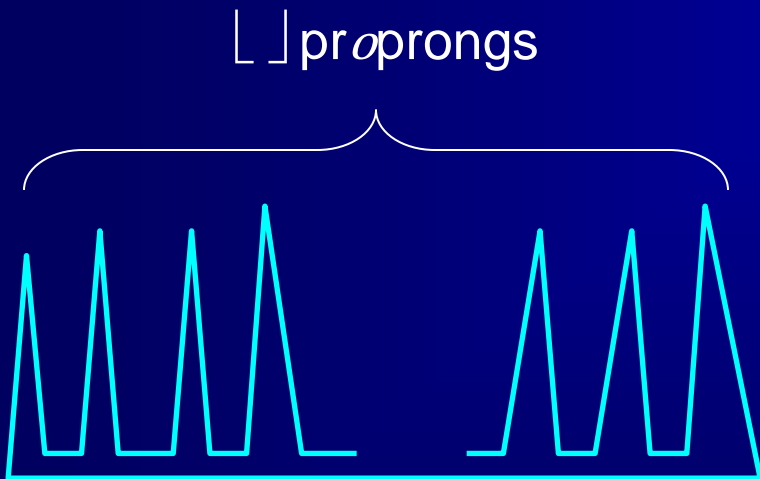


There exists no position at which a camera can oversee two prongs.

A Worst Case

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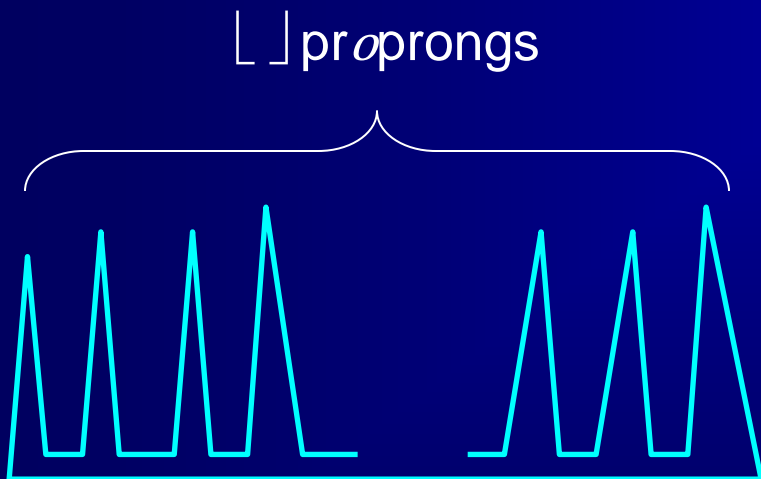
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$\lfloor n/3 \rfloor$ cameras are needed.

The 3-coloring approach is optimal in the worst case.

Art Gallery Theorem

For a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras are sufficient to have every interior point visible from at least one of the cameras.

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Solution to the Art Gallery Problem

1. Triangulate a simple polygon with a fast algorithm.

DCEL representation for the simple polygon so we can visit a neighbor from a triangle in constant time.

2. Generate a 3-coloring by DFS (as presented earlier).
3. Take the smallest color class to place the cameras.