Hidden Markov Models

Outline

I. Simplified matrix algorithms

II. HMM application to robot localization

* Figures/images are from the textbook site or drawn by the instructor unless the source is cited specifically.

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- The state is described by a single, discrete random variable X_t.
- Its values are the possible states of the world.
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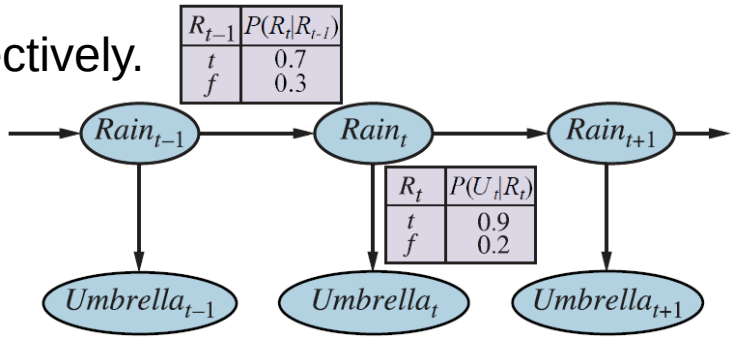

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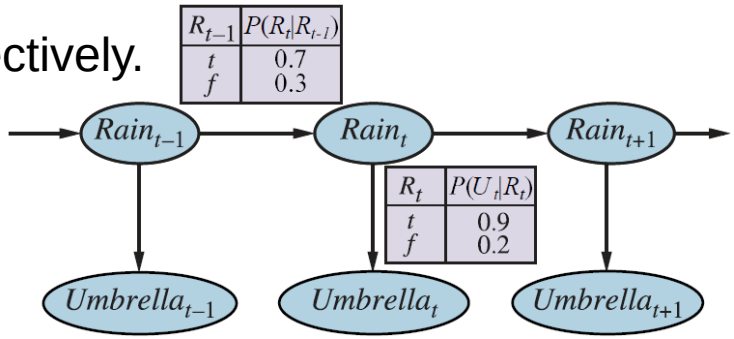

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The value e_t of the evidence variable E_t is known at time t.

For $i = 1, \ldots, n$, the likelihood of the state value c_i causing e_t to appear is

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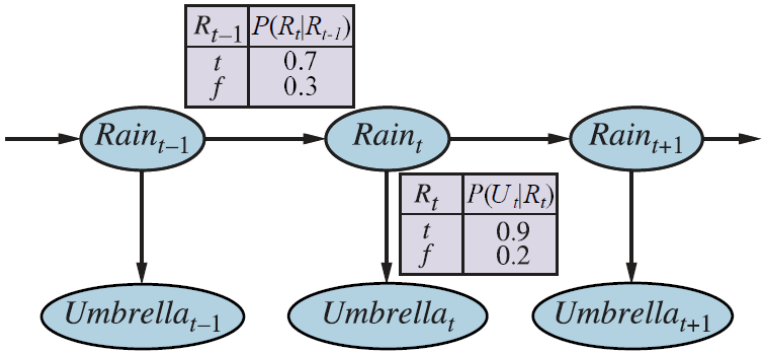

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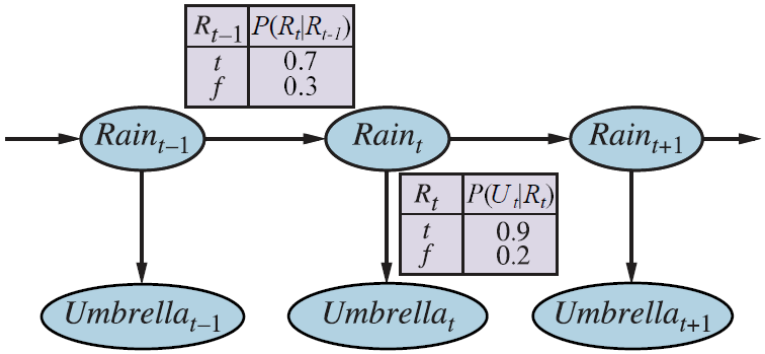
$$O_1 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix} \qquad O_3 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.8 \end{pmatrix}$$



Matrix Formulations of Filtering and Smoothing

Forward message used in filtering: $\boldsymbol{f}_{1:t} \equiv \boldsymbol{P}(\boldsymbol{X}_t \mid \boldsymbol{e}_{1:t}) = \begin{pmatrix} P(c_1 \mid \boldsymbol{e}_{1:t}) \\ \vdots \\ P(c_n \mid \boldsymbol{e}_{1:t}) \end{pmatrix}$

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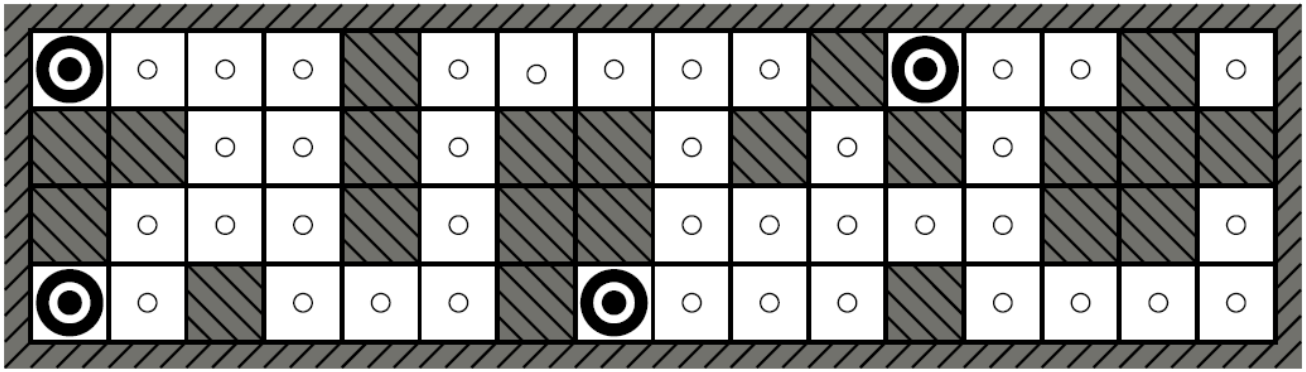
- Allows smoothing to be carried out in constant space, independent of the lengths of the sequence.
- Leads to an algorithm whose time complexity is independent of the length d of the lag.

  Smoothing at time $t - d$, where the current time is t.

II. Revisiting the Localization Task

The robot had a single action Move and a perfect sensor to report whether obstacles are immediately to the north, east, south, and west.



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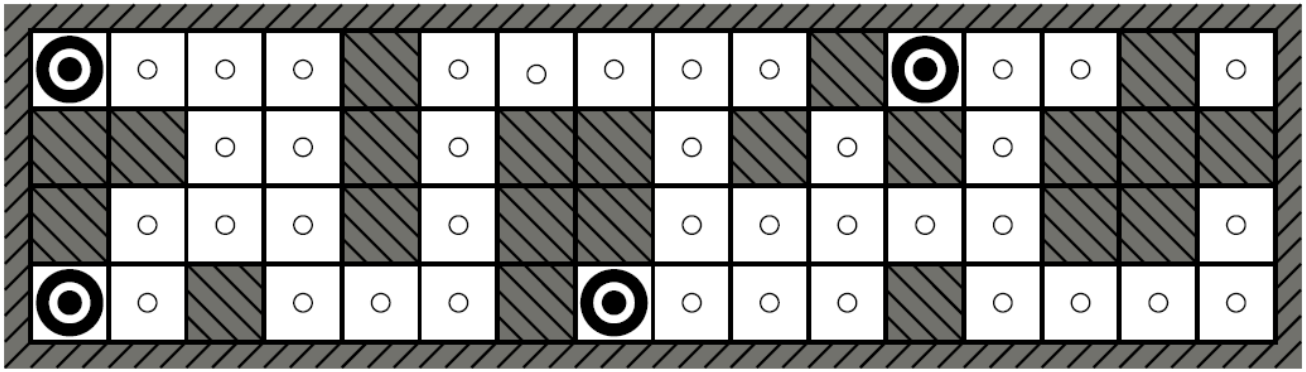

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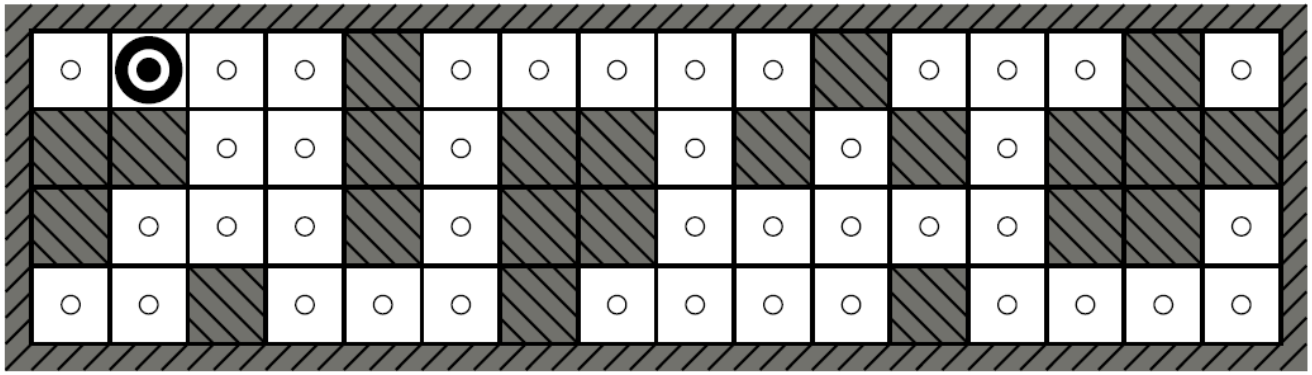

Possible locations after $E_1 = 1011$, $E_2 = 1010$.

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- State X_t: robot location
- $\{1, \ldots, n\}$: set of empty squares (labelled by integers)
- NEIGHBORS(i): set of empty squares that are adjacent to i
- $N(i)$: size of NEIGHBORS(i)

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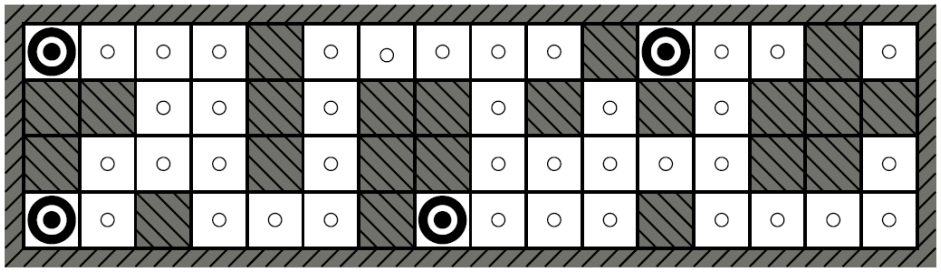
$$P(X_{t+1} = j \mid X_t = i) = \boldsymbol{T}_{ij} = \begin{cases} 1/N(i) & \text{if } j \in \text{NEIGHBORS}(i) \\ 0 & \text{otherwise} \end{cases}$$



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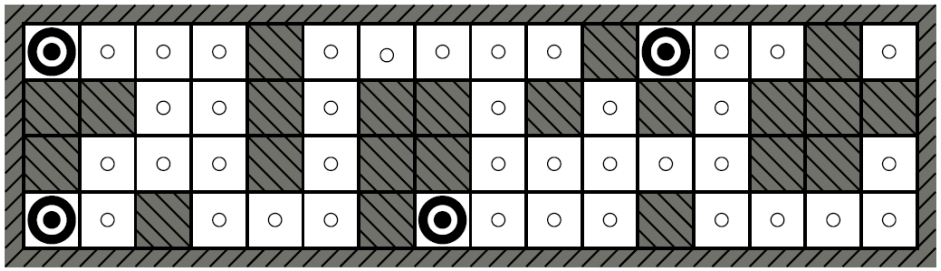

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Assume a uniform distribution of the robot's starting location:

$$P(X_0 = i) = 1/n \quad \text{for } 1 \leq i \leq n$$

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diagonal observation matrix

A square with obstacles to the north and south would produce a sensor reading of 1110 with probability $(1-\varepsilon)^3\varepsilon^1$.

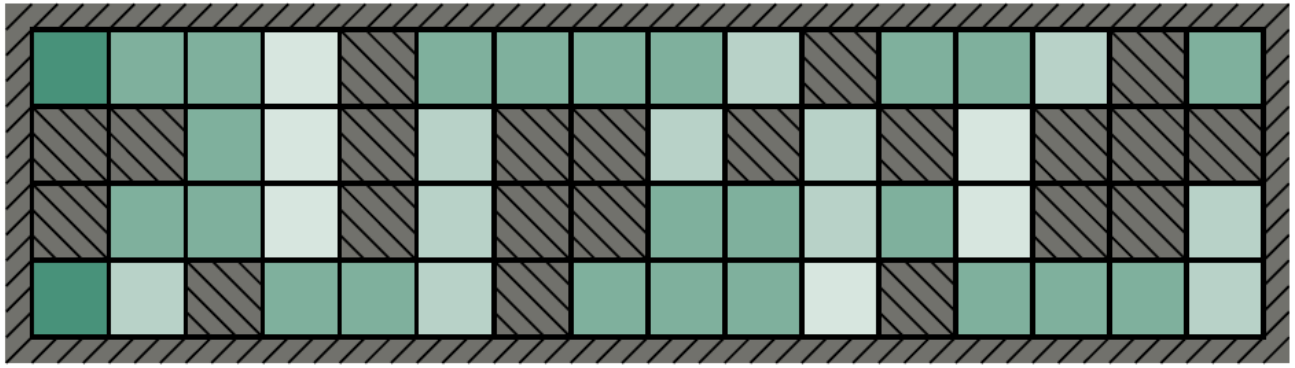
Localization

Posterior distribution over locations using the filtering equation:

$$\boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t+1}) = \alpha \boldsymbol{O}_{t+1} \boldsymbol{T}^T \boldsymbol{P}(\boldsymbol{X}_t \mid \boldsymbol{e}_{1:t})$$

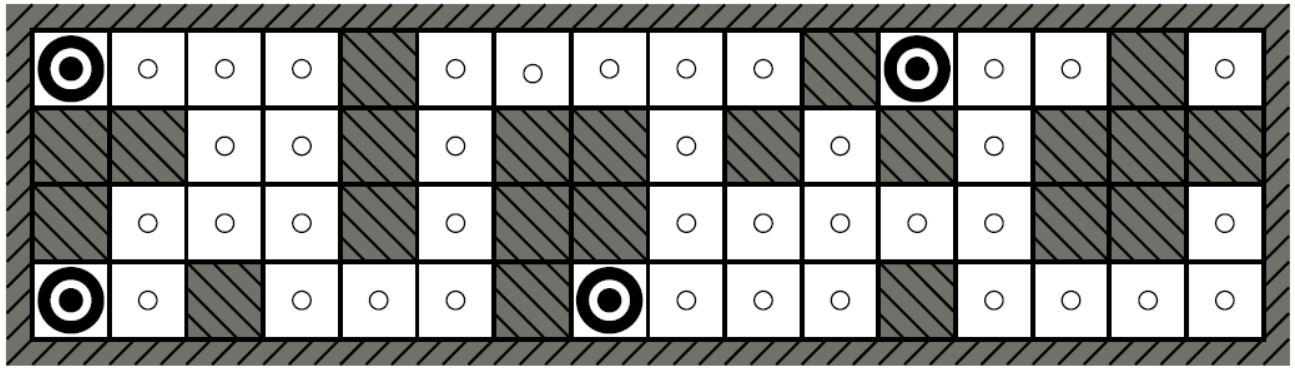
$n \times 1$ column vector $\boldsymbol{f}_{1:t+1}$

$\varepsilon = 0.2$



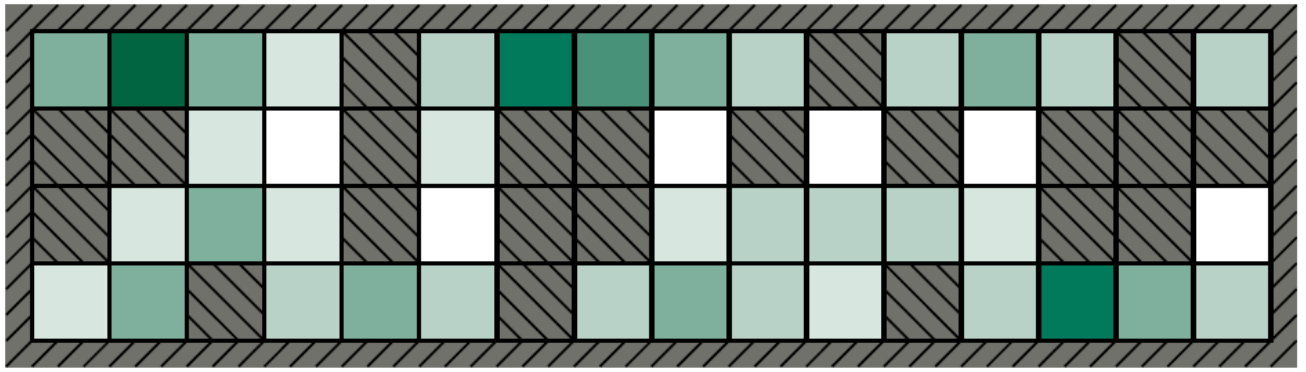
(a) Posterior distribution over robot location after $E_1 = 1011$

Perfect sensing



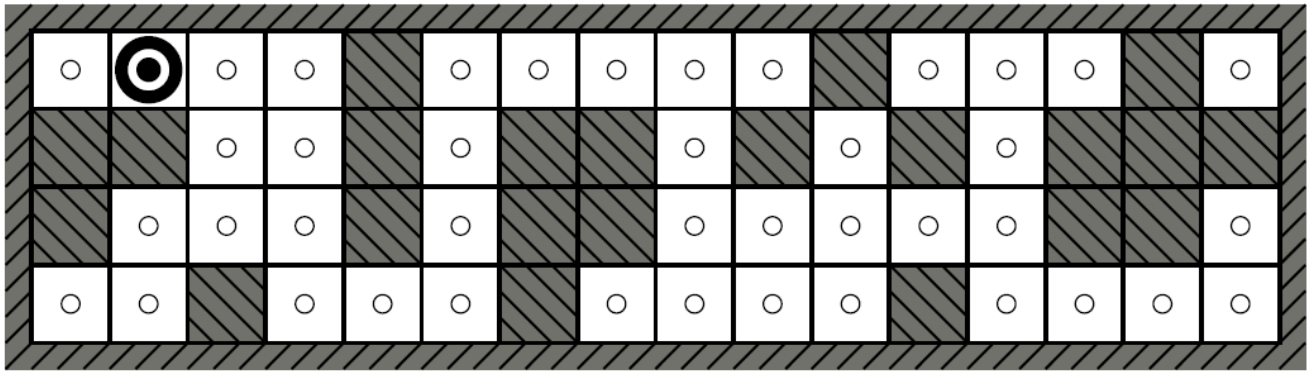
Localization (cont'd)

$\varepsilon = 0.2$



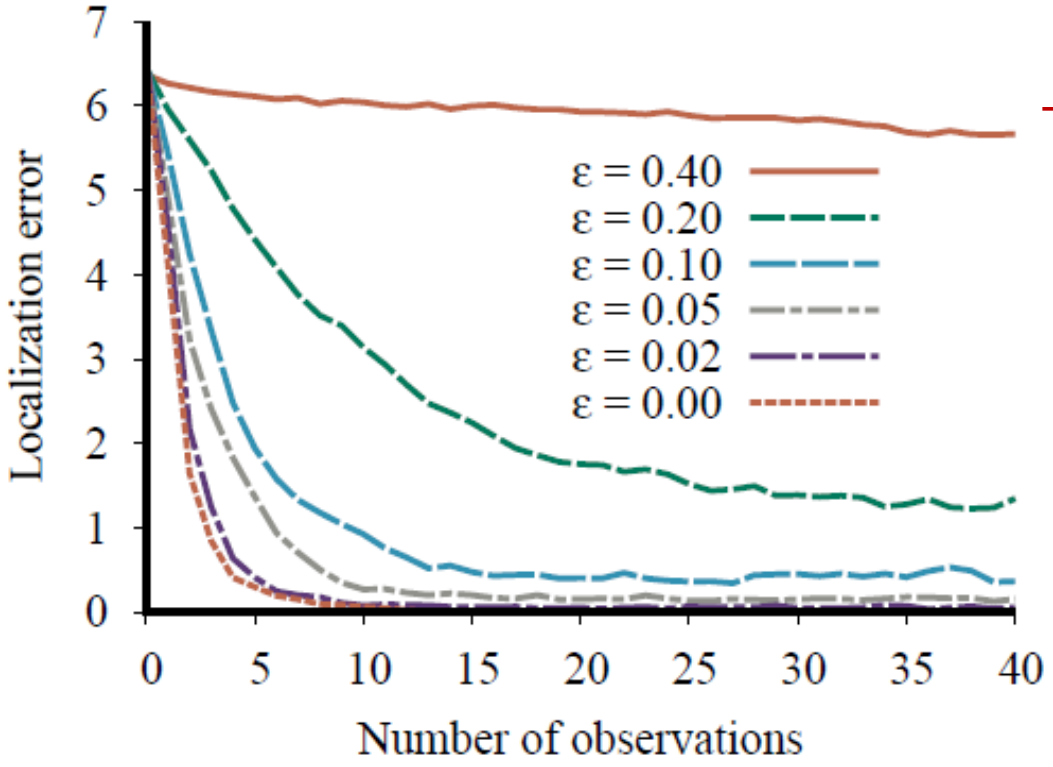
(b) Posterior distribution over robot location after $E_1 = 1011$, $E_2 = 1010$

Perfect sensing



Localization Error

Measured as the Manhattan distance from the true location.

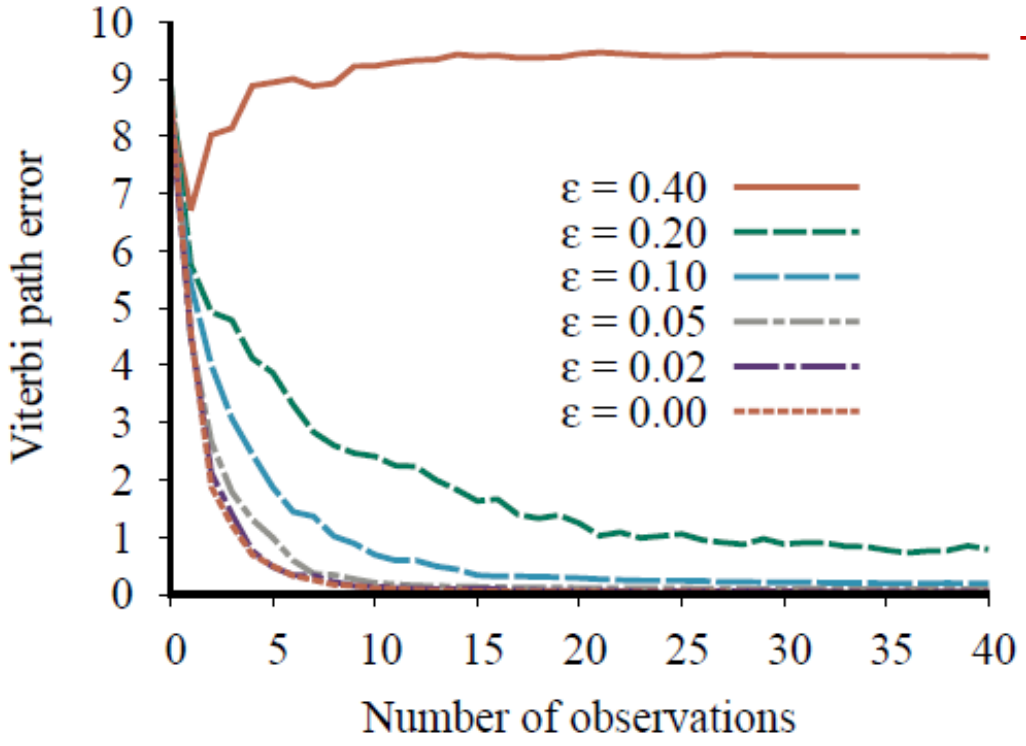


The robot is lost!

The robot can also use smoothing to work out where it was at a given past time.

Viterbi Path Error

Error measured as the average Manhattan distance of states on the Viterbi Path (most likely explanation) from corresponding states on the true path.



The robot is lost!