Lecture notes for *Principles of Artificial Intelligence* (Com S 4720/5720)
Yan-Bin Jia, Iowa State University

Inference in Temporal Models

Outline

I. States and observations

II. Transitions and sensor models

III. Filtering

* Figures/images are from the textbook site or the instructor unless the source is cited specifically.

I. Time and Uncertainty

- Probabilistic reasoning in *static worlds* discussed so far.

Every random variable
has a single fixed value.

- Real situations are *dynamic* with evidence evolving with time and thus actions predicted (and chosen) based on the history of evidence:
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How to model dynamic situations?

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high speed camera

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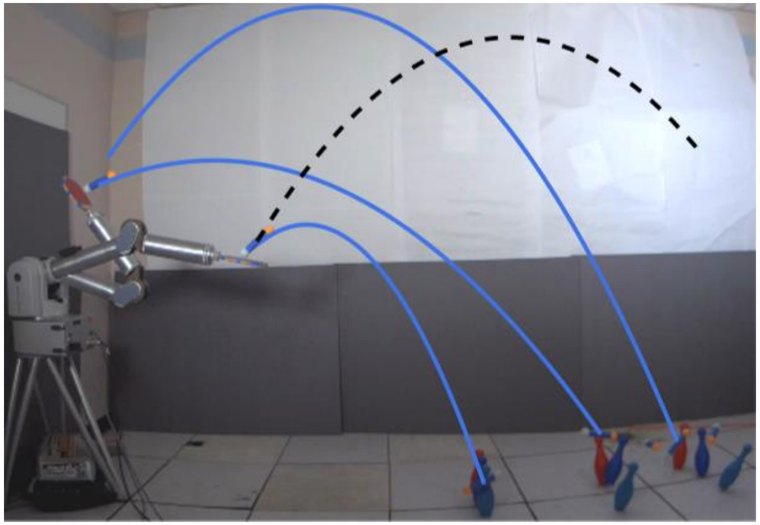

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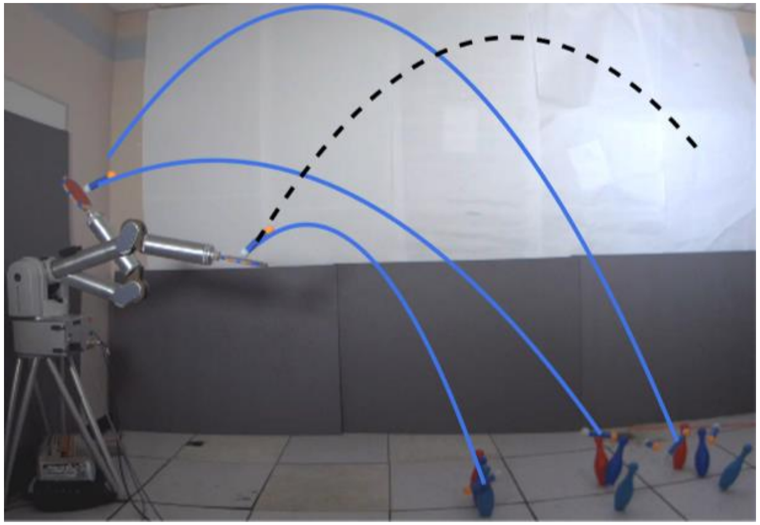

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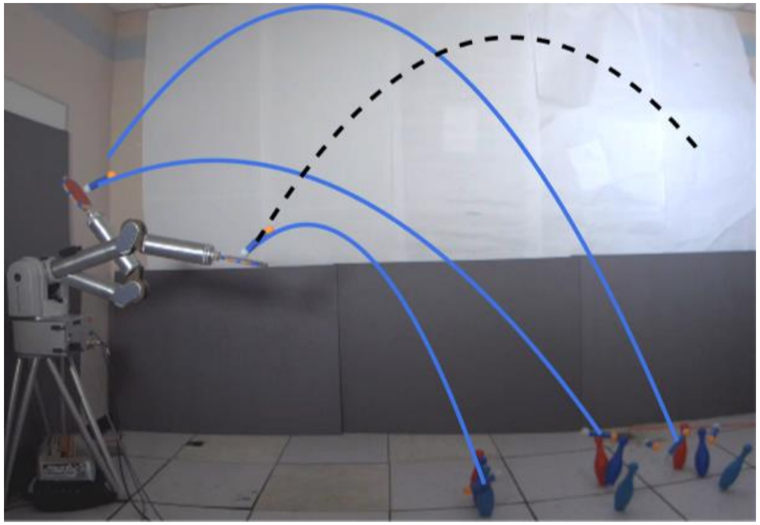

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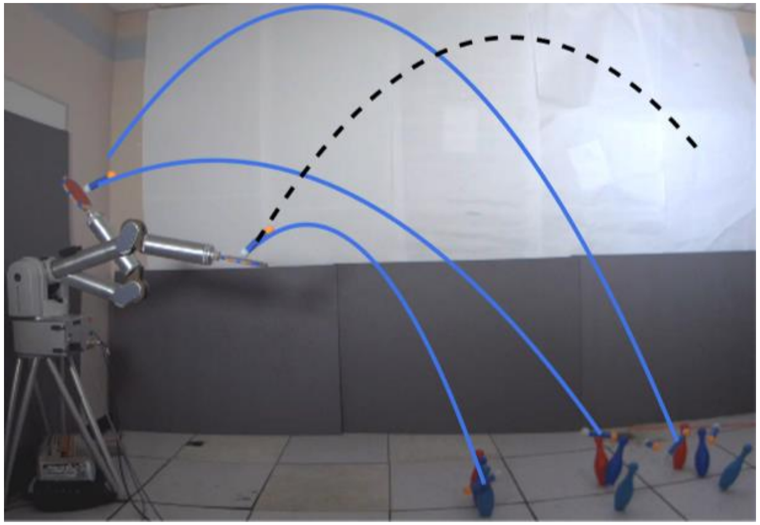

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$$\boldsymbol{E}_t = \boldsymbol{e}_t \text{ for some observed values } \boldsymbol{e}_t$$

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The umbrella world is represented by

state variable sequence: $R_0, R_1, R_2, \dots$

evidence variable sequence: $U_1, U_2, U_3, \dots$

$U_{3:5}$ corresponds to U_3, U_4, U_5.

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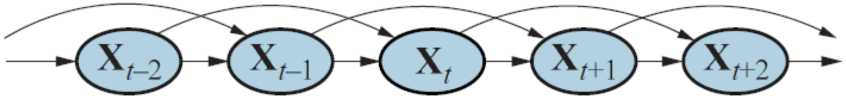

2nd order Markov process
(transition model $\boldsymbol{P}(\boldsymbol{X}_t \mid \boldsymbol{X}_{t-1}, \boldsymbol{X}_{t-2})$)

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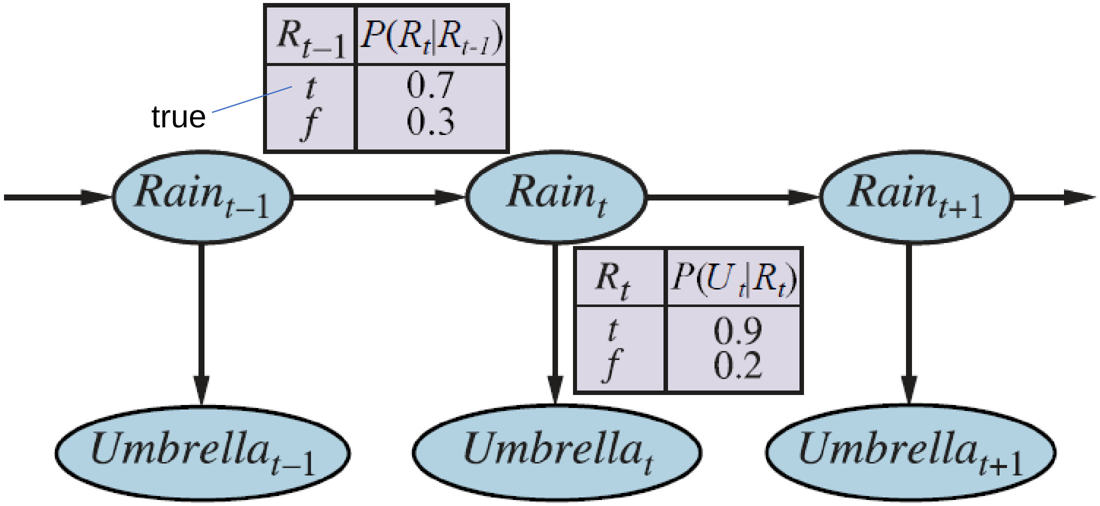
$$\mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_{0:t}, \mathbf{E}_{1:t-1}) = \mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_t)$$

sensor/observation
model

Bayes Net for the Umbrella World

time // t is overloaded

R_t = *true*



true

R_{t-1}	$P(R_t \| R_{t-1})$
t	0.7
f	0.3

R_t	$P(U_t \| R_t)$
t	0.9
f	0.2

Transition model:

$P(Rain_t \mid Rain_{t-1})$

Sensor model:

$P(Umbrella_t \mid Rain_t)$

1^{st} order Markov Process

The state (*Rain*) causes the sensor to take on a particular value (*Umbrella*).

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- ♦ Discrete time models can handle an infinite set of variables due to
  - ♣ use of integer indices
  - ♣ use of implicit universal quantification to define sensor and transition models

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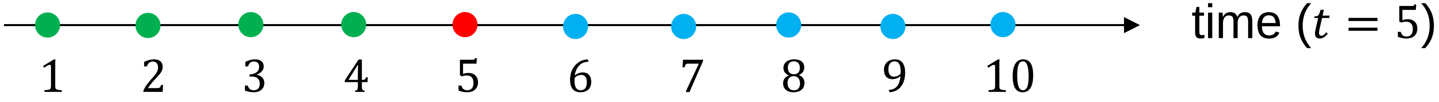

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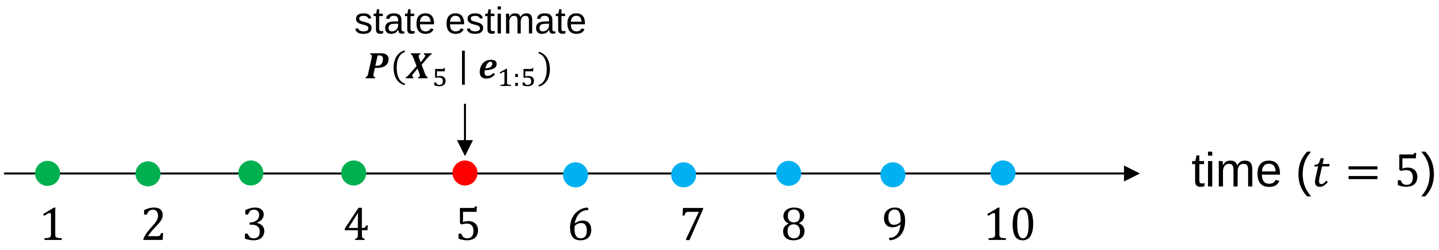

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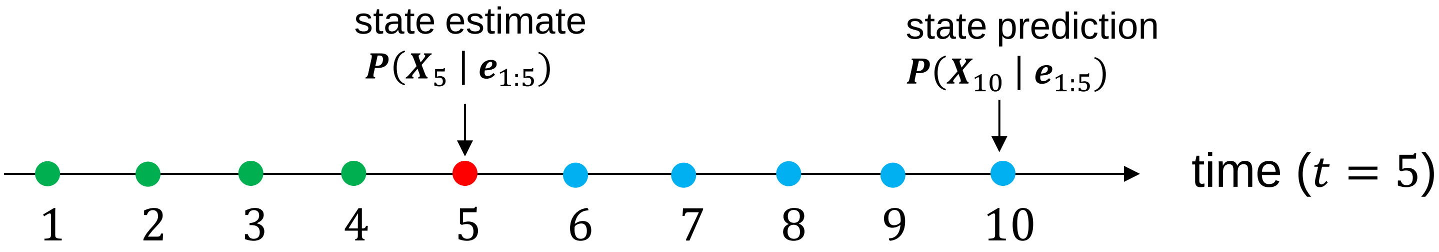

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$\boldsymbol{P}(\boldsymbol{X}_{10} \mid \boldsymbol{e}_{1:5})$

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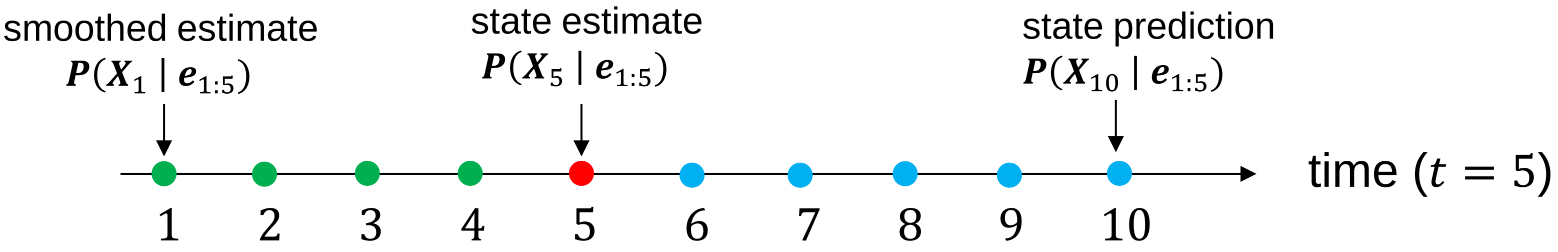

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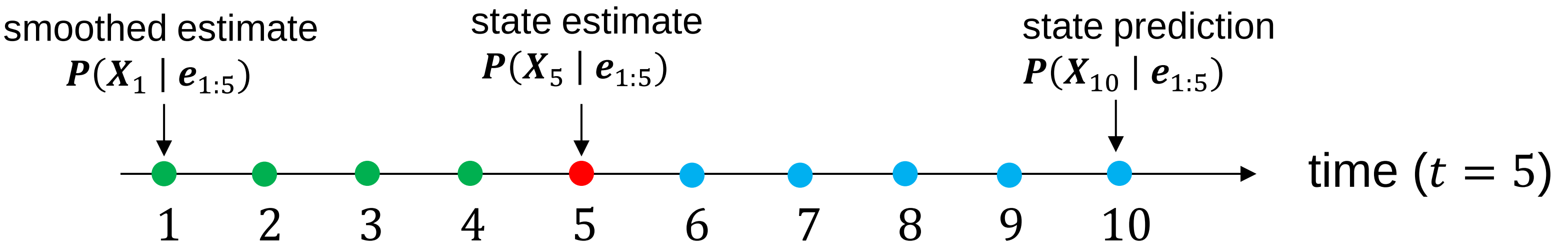


- *Most likely explanation* is the task of computing the sequence of states $\boldsymbol{x}_{1:t}$ to maximize $\boldsymbol{P}(\boldsymbol{X}_{1:t} \mid \boldsymbol{e}_{1:t})$.

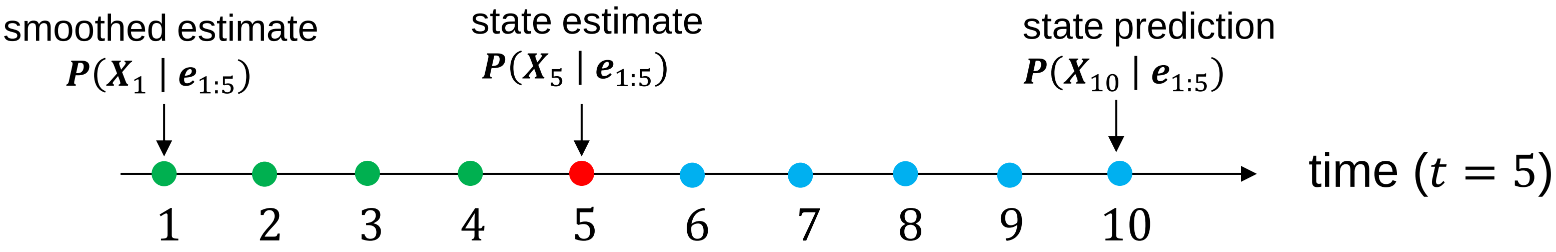
  Given a sequence of observations, find the sequence of states that is mostly likely to have generated those observations.

Smoothing & Most Likely Explanation

- *Smoothing* is the task of computing $\mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:t})$ for some k, $0 \leq k < t$.

  Compute the posterior distribution over a past state, given all the evidence to date.

  E.g., the probability that it rained last Wednesday.



smoothed estimate $\mathbf{P}(\mathbf{X}_1 \mid \mathbf{e}_{1:5})$

state estimate $\mathbf{P}(\mathbf{X}_5 \mid \mathbf{e}_{1:5})$

state prediction $\mathbf{P}(\mathbf{X}_{10} \mid \mathbf{e}_{1:5})$

1 2 3 4 5 6 7 8 9 10

time ($t = 5$)

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  Given a sequence of observations, find the sequence of states that is mostly likely to have generated those observations.

  E.g., If the umbrella appears on each of the first three days and is absent on the fourth, then the most likely explanation is that it rained on the first three days and did not rain on the fourth.

Filtering

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$$= \underbrace{\alpha}_{\text{normalizing constant}} \boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1}, \boldsymbol{e}_{1:t})\ \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t}) \qquad \text{(Bayes' rule, given } \boldsymbol{e}_{1:t}\text{)}$$

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normalizing constant

$$= \alpha \boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1})\, \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t}) \quad \text{(by the sensor Markov assumption)}$$

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One-Step Prediction

$$\boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t+1}) = \alpha \boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1})\ \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t})$$

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update

$$= \alpha \underbrace{\mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})}_{\text{sensor model}} \sum_{\mathbf{x}_t} \underbrace{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t)}_{\text{transition model}} \underbrace{P(\mathbf{x}_t \mid \mathbf{e}_{1:t})}_{\text{recursion}}$$

prediction

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$$= \alpha \overset{\text{update}}{\underbrace{\boxed{\mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})}}_{\text{sensor model}}} \boxed{\sum_{\mathbf{x}_t} \underbrace{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t)}_{\text{transition model}} \underbrace{P(\mathbf{x}_t \mid \mathbf{e}_{1:t})}_{\text{recursion}}} \ \text{prediction}$$

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// Initial estimate (prior)

One-Step Prediction

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$$\mathbf{P}(\mathbf{X}_0) \rightarrow \mathbf{P}(\mathbf{X}_1 \mid \mathbf{e}_{1:1}) \rightarrow \cdots \rightarrow \mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:k}) \rightarrow \mathbf{P}(\mathbf{X}_{k+1} \mid \mathbf{e}_{1:k+1}) \rightarrow \cdots$$

One-Step Prediction

$$\boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t+1}) = \alpha \boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1})\, \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t})$$

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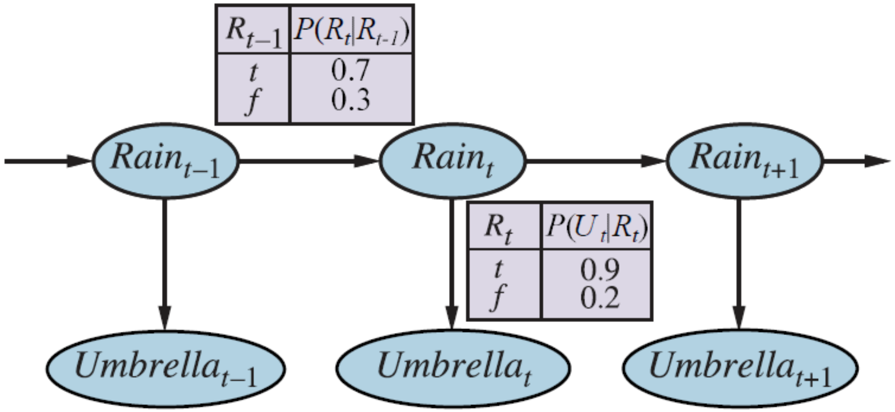
Time and space for the update at t must be *constant* in order to keep track of the current state distribution indefinitely.

Filtering in the Umbrella World



Compute $\boldsymbol{P}(R_2 \mid u_{1:2})$ as follows:

$U_1 = true \wedge U_2 = true$

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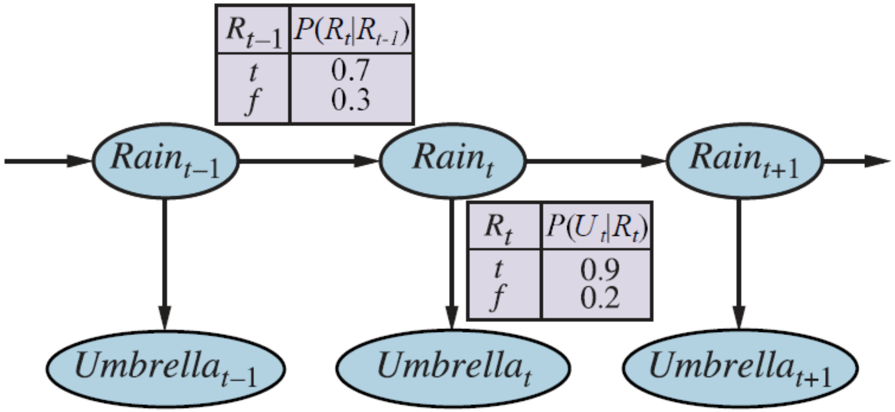

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- On day 0, no observation.

$$\boldsymbol{P}(R_0) = \langle 0.5, 0.5 \rangle$$

$$r_0, \neg r_0$$

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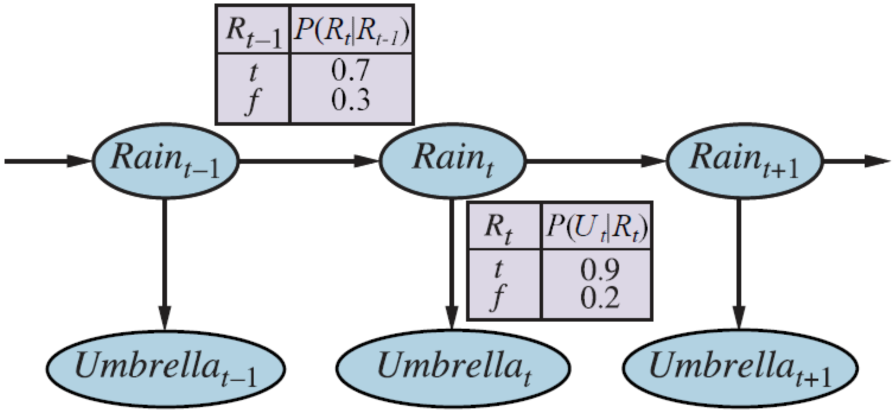

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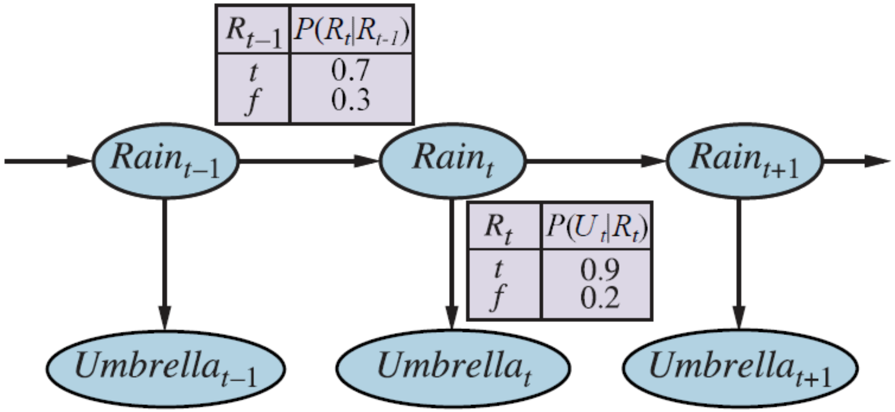

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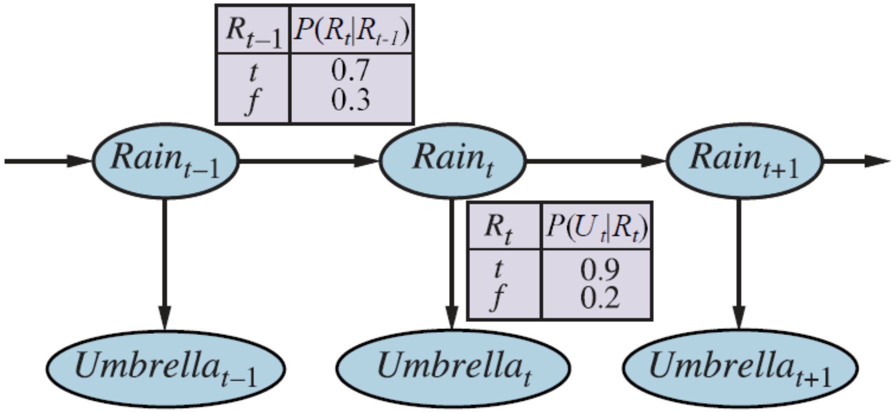

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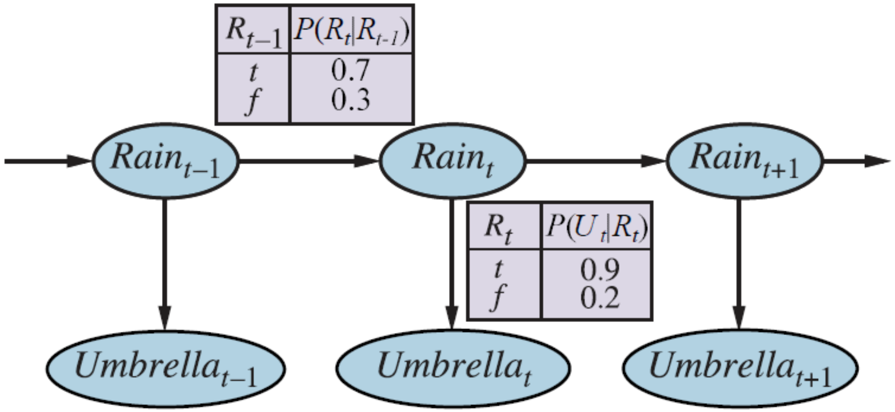

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b) Update with evidence for $t = 1$ followed by normalization:

$$\mathbf{P}(R_1 | \, u_1) = \alpha \mathbf{P}(u_1 \mid R_1) \mathbf{P}(R_1)$$

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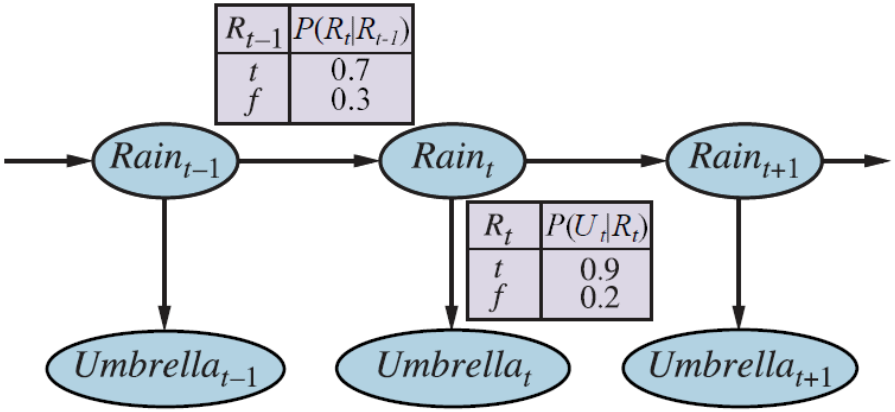

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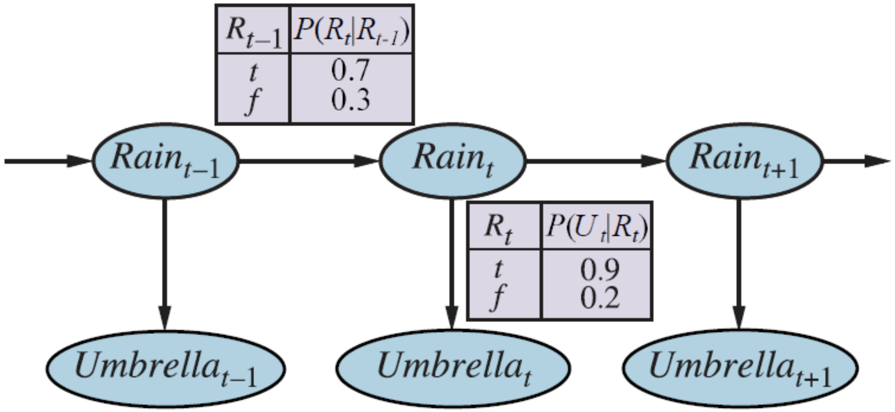
$$= \langle 0.7, 1-0.7 \rangle \times 0.5 + \langle 0.3, 1-0.3 \rangle \times 0.5 = \langle 0.5, 0.5 \rangle$$

b) Update with evidence for $t = 1$ followed by normalization:

$$\boldsymbol{P}(R_1 | \, u_1) = \alpha \boldsymbol{P}(u_1 \mid R_1) \boldsymbol{P}(R_1)$$

$$= \alpha \langle 0.9, 0.2 \rangle \, \langle 0.5, 0.5 \rangle$$

Filtering in the Umbrella World



R_{t-1}	$P(R_t \mid R_{t-1})$
t	0.7
f	0.3

R_t	$P(U_t \mid R_t)$
t	0.9
f	0.2

Compute $\boldsymbol{P}(R_2 \mid \underset{U_1 = true \wedge U_2 = true}{u_{1:2}})$ as follows:

- On day 0, no observation.

$$\boldsymbol{P}(R_0) = \underset{r_0,\, \neg r_0}{\langle 0.5, 0.5\rangle}$$

- On day 1, the umbrella appears.

$$u_1 \equiv (U_1 = true)$$

a) Prediction:

$$\boldsymbol{P}(R_1) = \sum_{r_0' \in \{r_0, \neg r_0\}} \boldsymbol{P}(R_1 \mid r_0') P(r_o') \qquad // \ r_0 \equiv (R_0 = true)$$

$$= \langle 0.7, 1 - 0.7\rangle \times 0.5 + \langle 0.3, 1 - 0.3\rangle \times 0.5 = \langle 0.5, 0.5\rangle$$

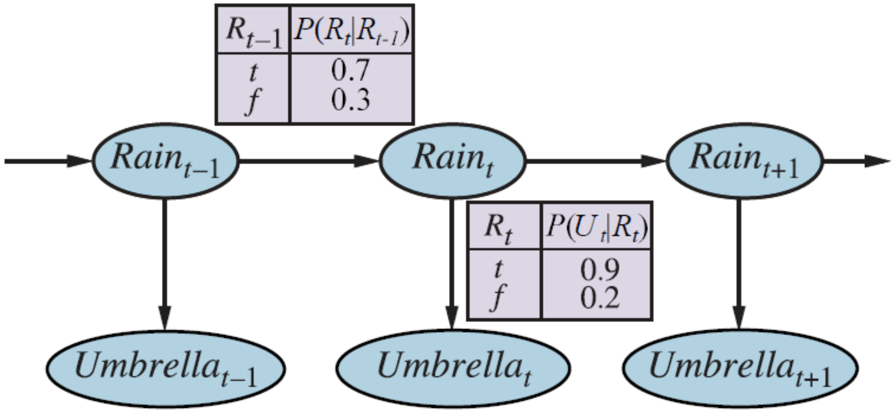
b) Update with evidence for $t = 1$ followed by normalization:

$$\boldsymbol{P}(R_1 \mid u_1) = \alpha \boldsymbol{P}(u_1 \mid R_1) \boldsymbol{P}(R_1)$$

$$= \alpha \langle 0.9, 0.2\rangle \langle 0.5, 0.5\rangle$$

$$= \alpha \langle 0.45, 0.1\rangle$$

Filtering in the Umbrella World



R_{t-1}	$P(R_t \mid R_{t-1})$
t	0.7
f	0.3

R_t	$P(U_t \mid R_t)$
t	0.9
f	0.2

Compute $\boldsymbol{P}(R_2 \mid u_{1:2})$ as follows:

$U_1 = \mathit{true} \wedge U_2 = \mathit{true}$

- On day 0, no observation.

$$\boldsymbol{P}(R_0) = \underset{r_0,\ \neg r_0}{\langle 0.5, 0.5 \rangle}$$

- On day 1, the umbrella appears.

$$u_1 \equiv (U_1 = \mathit{true})$$

a) Prediction:

$$\boldsymbol{P}(R_1) = \sum_{r_0' \in \{r_0, \neg r_0\}} \boldsymbol{P}(R_1 \mid r_0') P(r_o') \qquad // \ r_0 \equiv (R_0 = \mathit{true})$$

$$= \langle 0.7, 1 - 0.7 \rangle \times 0.5 + \langle 0.3, 1 - 0.3 \rangle \times 0.5 = \langle 0.5, 0.5 \rangle$$

b) Update with evidence for $t = 1$ followed by normalization:

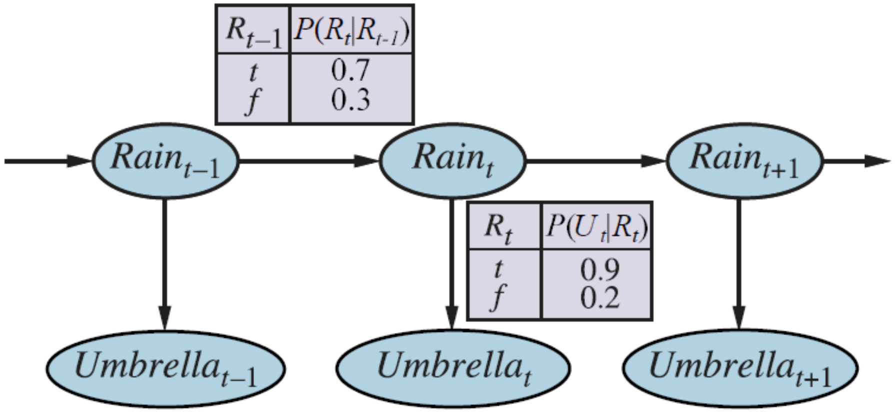
$$\boldsymbol{P}(R_1 \mid u_1) = \alpha \boldsymbol{P}(u_1 \mid R_1) \boldsymbol{P}(R_1)$$

$$= \alpha \langle 0.9, 0.2 \rangle \langle 0.5, 0.5 \rangle$$

$$= \alpha \langle 0.45, 0.1 \rangle$$

$$\approx \langle 0.818, 0.182 \rangle$$

Filtering in UW (cont'd)

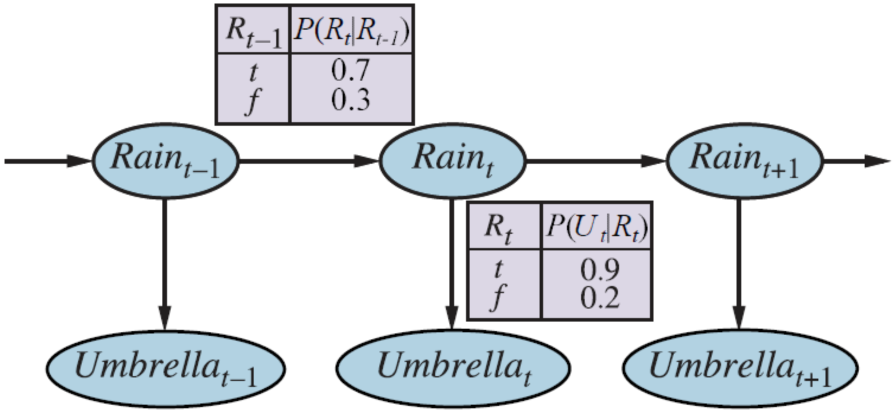


$$\boldsymbol{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182 \rangle$$

- On day 2, the umbrella appears.

$$u_2 \equiv (U_2 = \mathit{true})$$

Filtering in UW (cont’d)



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t	0.7
f	0.3

R_t	$P(U_t \mid R_t)$
t	0.9
f	0.2

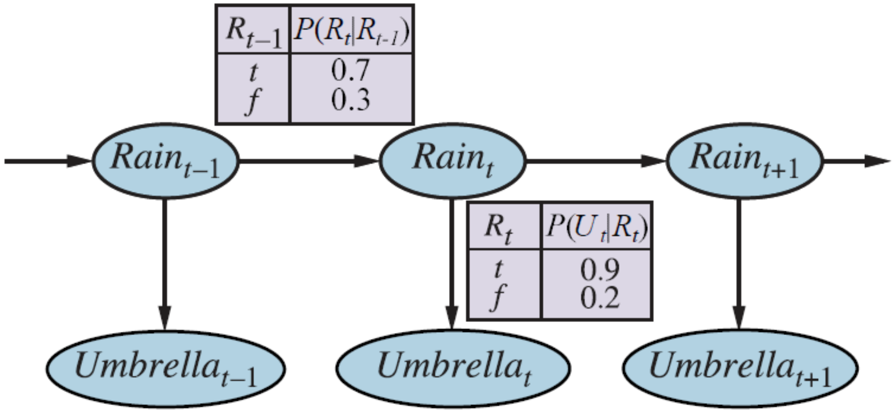
$$\boldsymbol{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182 \rangle$$

- On day 2, the umbrella appears.

$$u_2 \equiv (U_2 = true)$$

a) Prediction: $\boldsymbol{P}(R_2 \mid u_1) = \sum_{r_1' \in \{r_1, \neg r_1\}} \boldsymbol{P}(R_2 \mid r_1') P(r_1' \mid u_1)$ // $r_1 \equiv (R_1 = true)$

Filtering in UW (cont'd)



$$\boldsymbol{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182\rangle$$

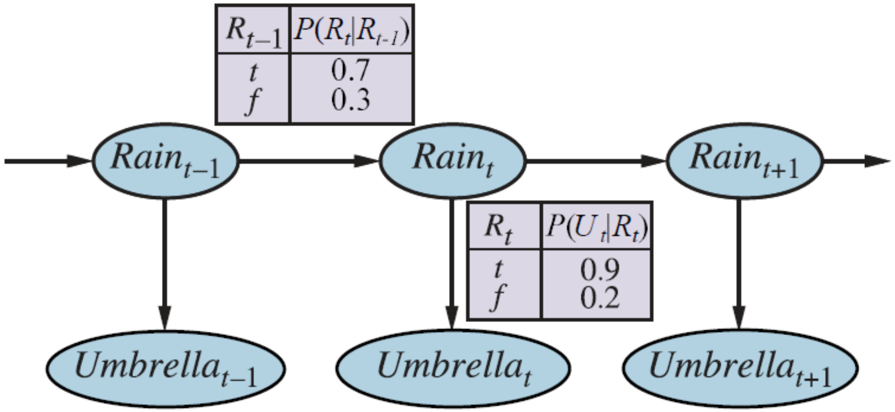
- On day 2, the umbrella appears.

$$u_2 \equiv (U_2 = \textit{true})$$

a) Prediction: $\boldsymbol{P}(R_2 \mid u_1) = \sum_{r_1' \in \{r_1, \neg r_1\}} \boldsymbol{P}(R_2 \mid r_1') P(r_1' \mid u_1)$ // $r_1 \equiv (R_1 = \textit{true})$

$$= \langle 0.7, 0.3\rangle \times 0.818 + \langle 0.3, 0.7\rangle \times 0.182 \approx \langle 0.627, 0.373\rangle$$

Filtering in UW (cont'd)



$$\boldsymbol{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182 \rangle$$

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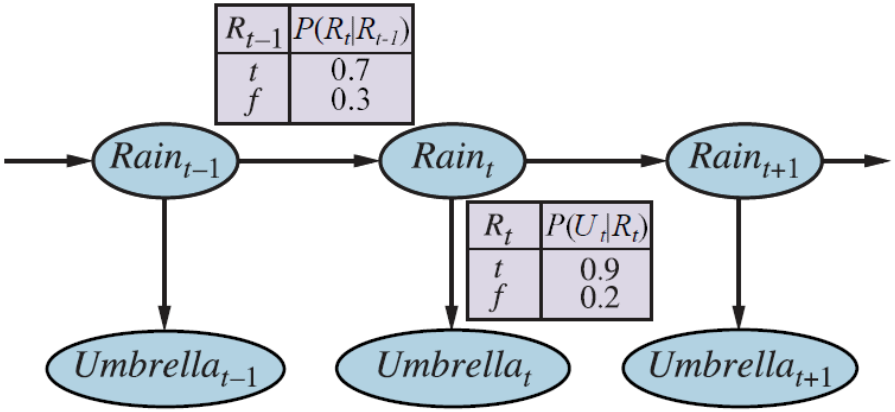
$$= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \approx \langle 0.627, 0.373 \rangle$$

b) Update with evidence for $t = 2$: $\boldsymbol{P}(R_2 \mid u_1, u_2) = \alpha \boldsymbol{P}(u_2 \mid R_2) \boldsymbol{P}(R_2 \mid u_1)$

Filtering in UW (cont'd)



$$\boldsymbol{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182 \rangle$$

- On day 2, the umbrella appears.

$$u_2 \equiv (U_2 = \mathit{true})$$

a) Prediction: $\boldsymbol{P}(R_2 \mid u_1) = \sum_{r_1' \in \{r_1, \neg r_1\}} \boldsymbol{P}(R_2 \mid r_1') P(r_1' \mid u_1)$ // $r_1 \equiv (R_1 = \mathit{true})$

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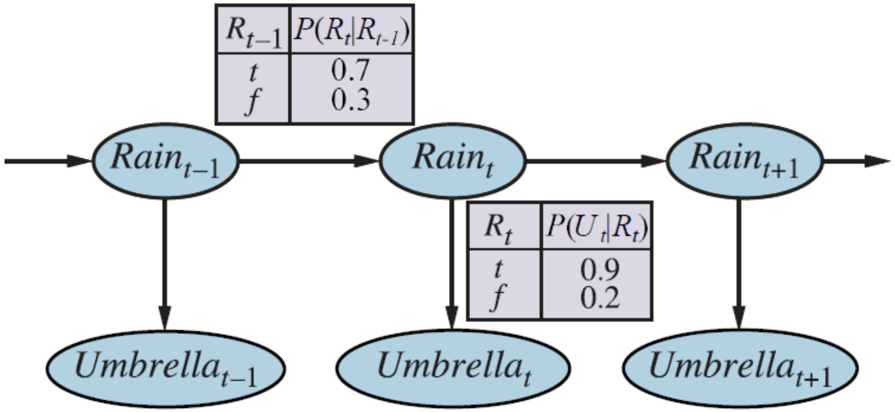
$$\approx \alpha \langle 0.9, 0.2 \rangle \langle 0.627, 0.373 \rangle$$

Filtering in UW (cont'd)



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- On day 2, the umbrella appears.

$$u_2 \equiv (U_2 = \mathit{true})$$

a) Prediction: $$\boldsymbol{P}(R_2 \mid u_1) = \sum_{r_1' \in \{r_1, \neg r_1\}} \boldsymbol{P}(R_2 \mid r_1') P(r_1' \mid u_1)$$ // $r_1 \equiv (R_1 = \mathit{true})$

$$= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \approx \langle 0.627, 0.373 \rangle$$

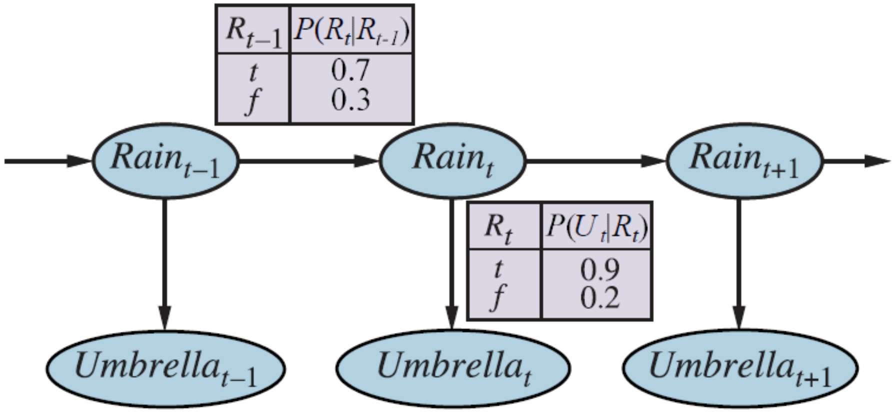
b) Update with evidence for $t = 2$: $$\boldsymbol{P}(R_2 \mid u_1, u_2) = \alpha \boldsymbol{P}(u_2 \mid R_2) \boldsymbol{P}(R_2 \mid u_1)$$

$$\approx \alpha \langle 0.9, 0.2 \rangle \langle 0.627, 0.373 \rangle$$

$$\approx \alpha \langle 0.565, 0.075 \rangle$$

$$\approx \langle 0.883, 0.117 \rangle$$

Filtering in UW (cont'd)



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t	0.9
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a) Prediction: $$\boldsymbol{P}(R_2 \mid u_1) = \sum_{r_1' \in \{r_1, \neg r_1\}} \boldsymbol{P}(R_2 \mid r_1') P(r_1' \mid u_1) \qquad // \; r_1 \equiv (R_1 = \mathit{true})$$

$$= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \approx \langle 0.627, 0.373 \rangle$$

b) Update with evidence for $t = 2$: $$\boldsymbol{P}(R_2 \mid u_1, u_2) = \alpha \boldsymbol{P}(u_2 \mid R_2) \boldsymbol{P}(R_2 \mid u_1)$$

$$\approx \alpha \langle 0.9, 0.2 \rangle \langle 0.627, 0.373 \rangle$$

$$\approx \alpha \langle 0.565, 0.075 \rangle$$

$$\approx \langle 0.883, 0.117 \rangle$$

For Kalman filtering and its subroutine recursive least-squares (updating states out of a **continuum**), we refer to
https://faculty.sites.iastate.edu/jia/files/inline-files/kalman-filter.pdf
https://faculty.sites.iastate.edu/jia/files/inline-files/recursive-least-squares.pdf