Lecture notes for *Principles of Artificial Intelligence* (COMS 4720/5720)
Yan-Bin Jia, Iowa State University

Exact Inference in Bayesian Networks

Outline

I. Probabilistic query using a BN

II. Variable Elimination

III. Variable ordering and relevance

* Figures are from the textbook site.

II. Probabilistic Query

X: query variable

$\boldsymbol{E} = \{E_1, \ldots . E_m\}$: evidence variables

$\boldsymbol{e} = \{e_1, \ldots . e_m\}$: an observed event

$\boldsymbol{Y} = \{Y_1, \ldots . Y_l\}$: hidden variables

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We will discuss exact algorithms for posterior probability computation.

Inference by Enumeration

$$\boldsymbol{P}(X, \boldsymbol{e}) = \boldsymbol{P}(X \mid \boldsymbol{e}) P(\boldsymbol{e})$$

Vector

Scalar

Vector of evidences

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computable as products of conditional probabilities from the BN.

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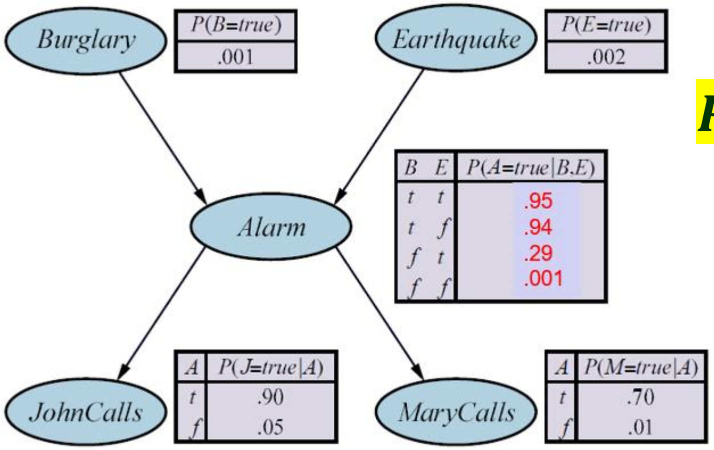
- Answer the query $\mathbf{P}(X \mid \mathbf{e})$ using a BN.

Burglary Example (revisited)



$\mathbf{P}(Burglary \mid JohnCalls = true, MaryCalls = true)$?

Hidden variables: $Y = \{Earthquake, Alarm\}$

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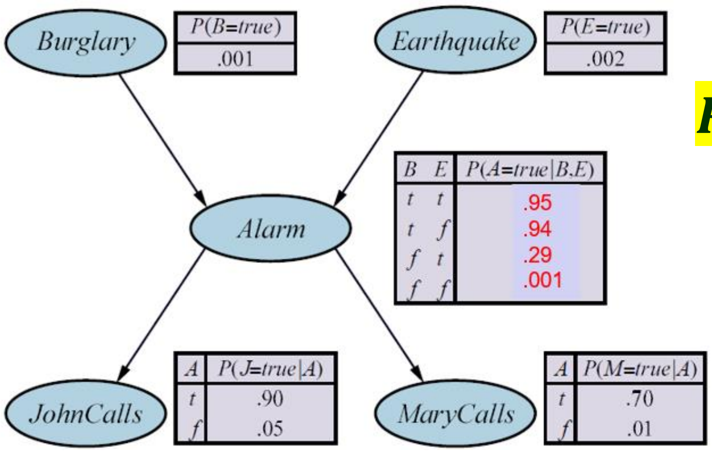

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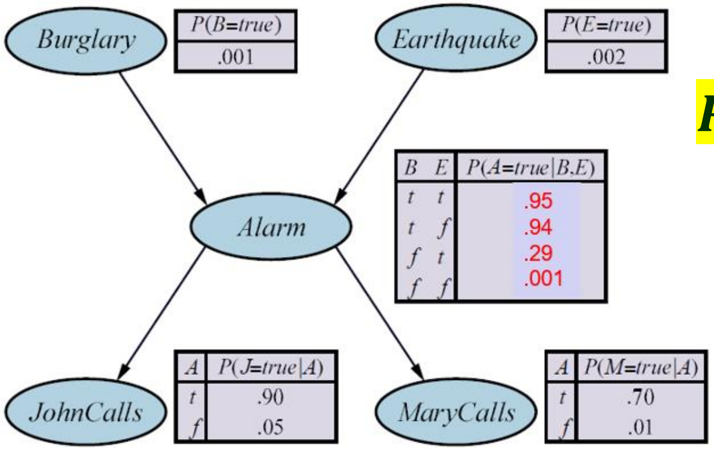

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// because e is a constant value standing for
// $Earthquake = true$. e cannot be overloaded. the index
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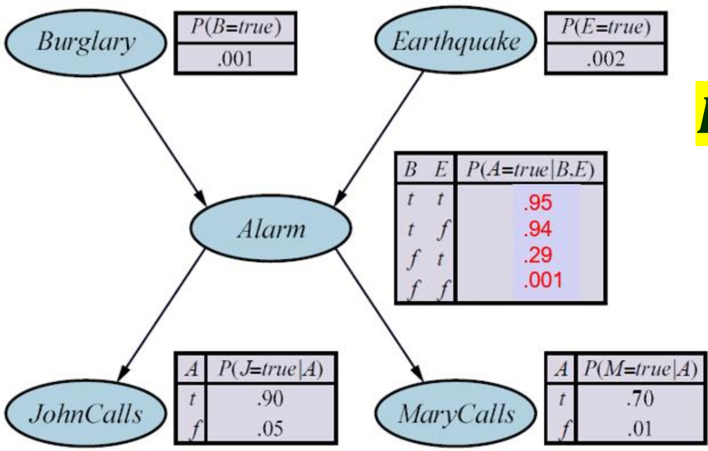

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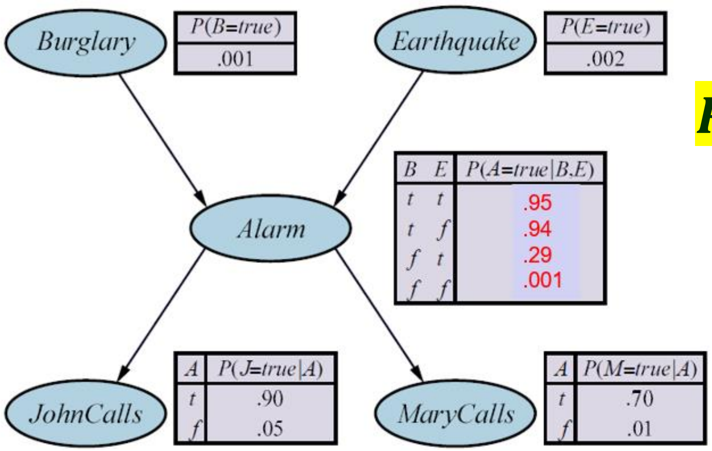

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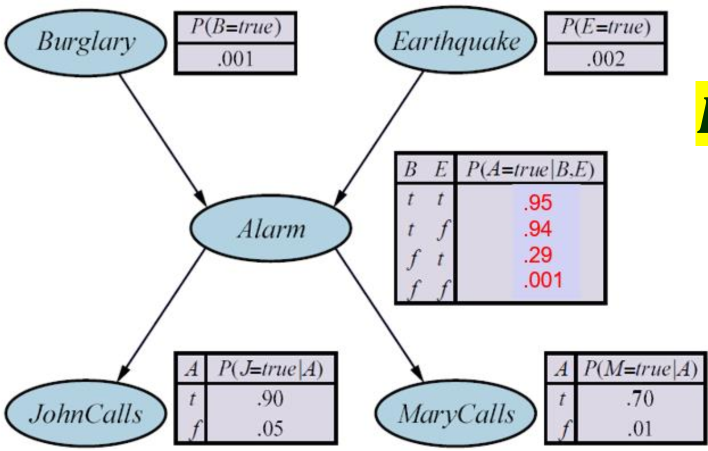

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In the general case with n hidden variables, there are 2^n summands, each as a product requires $O(n)$ computation time.

Expression Tree

Take advantage of the nested structure to move summations inwards as far as possible.

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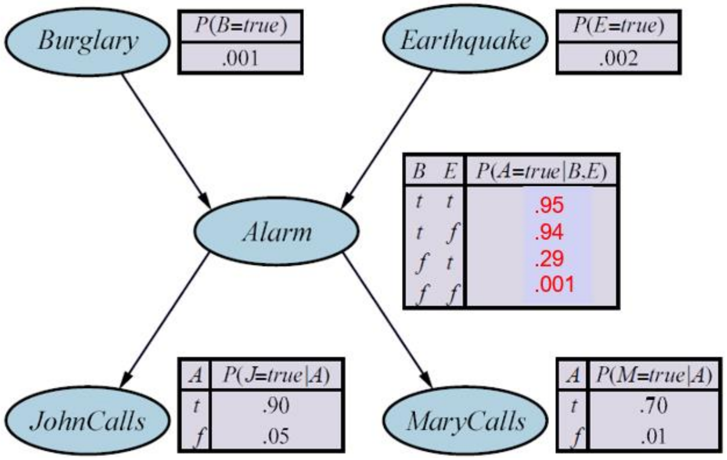

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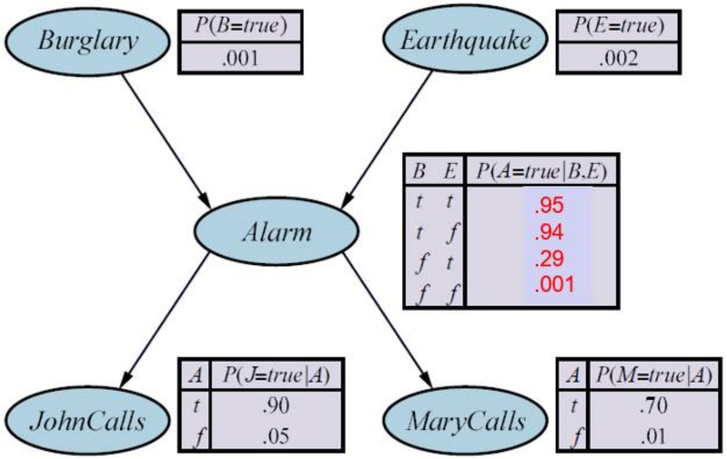

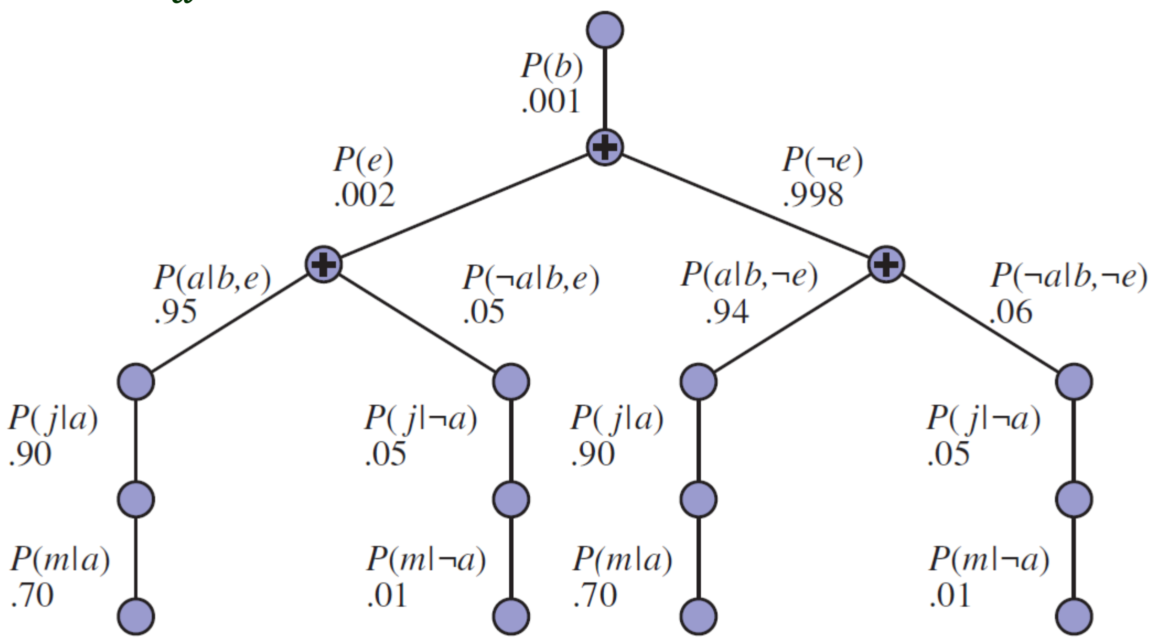

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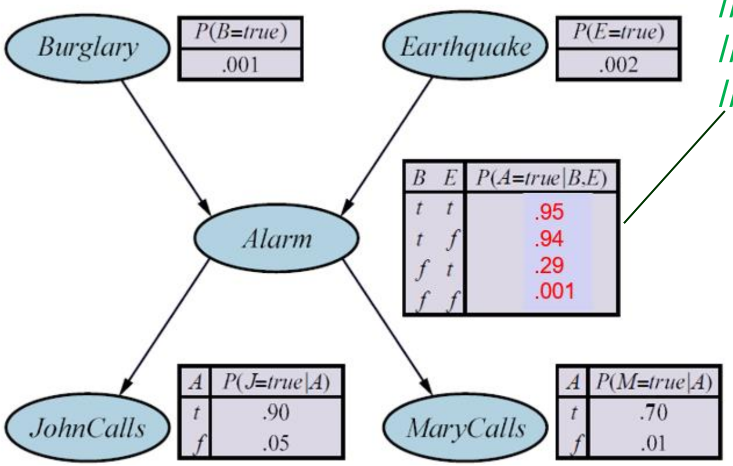

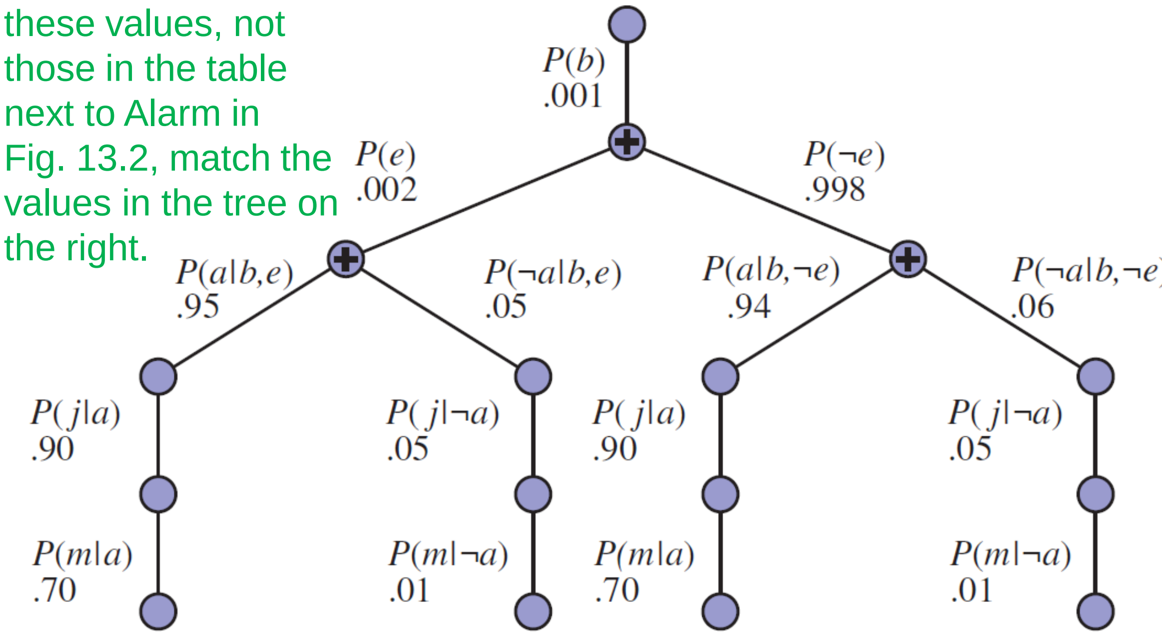

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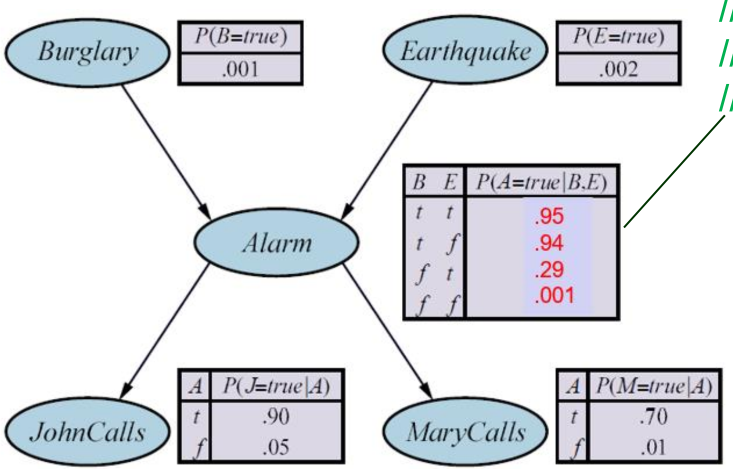

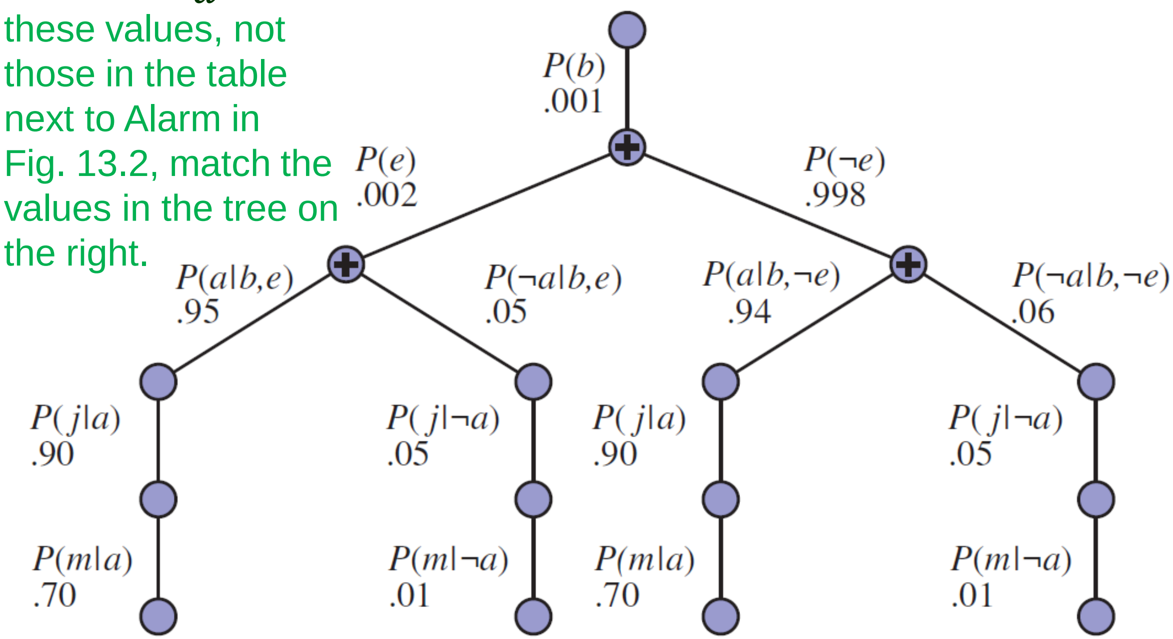

- Multiply values along each path.
- Sum at the "+" nodes.

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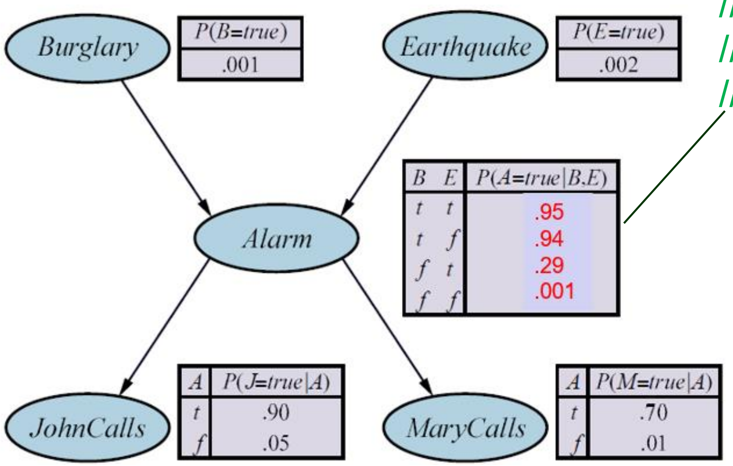

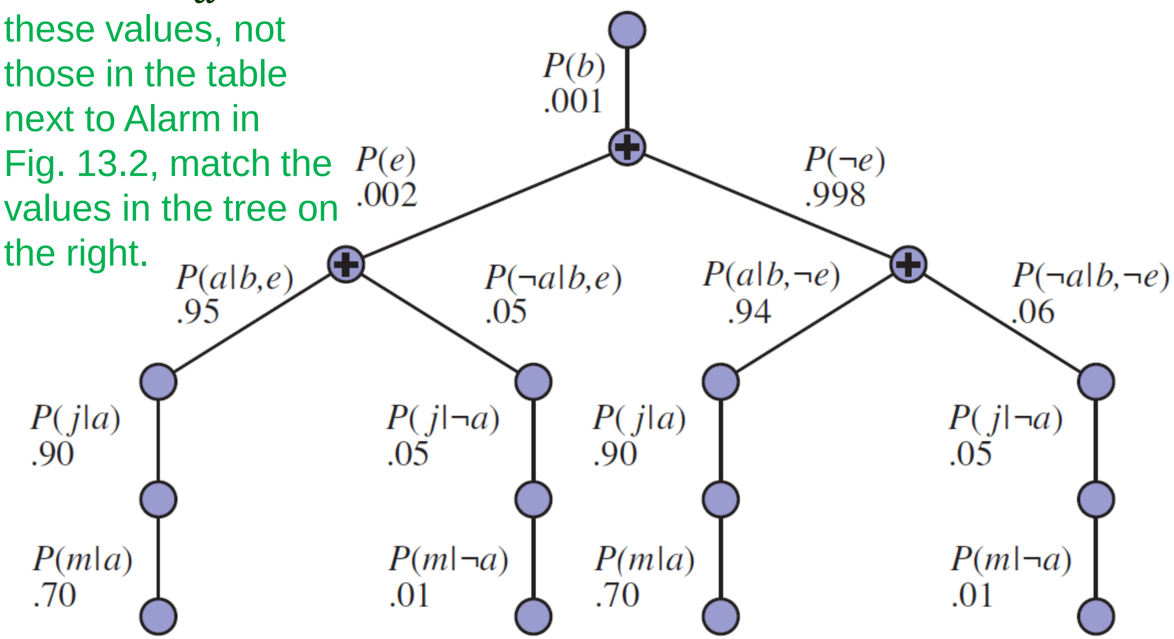

- Multiply values along each path.
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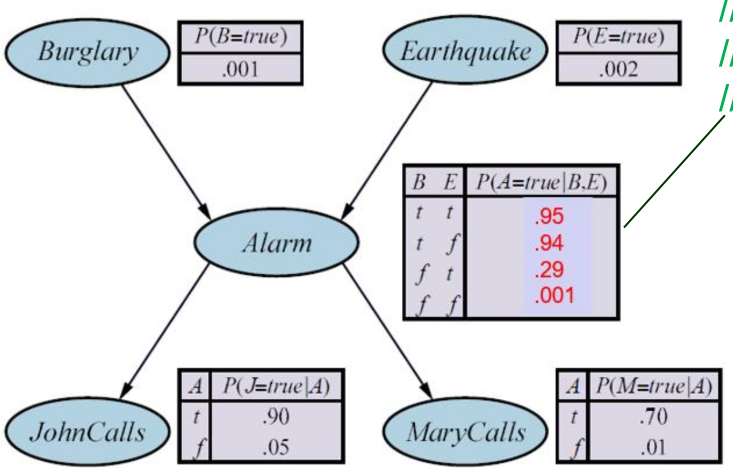

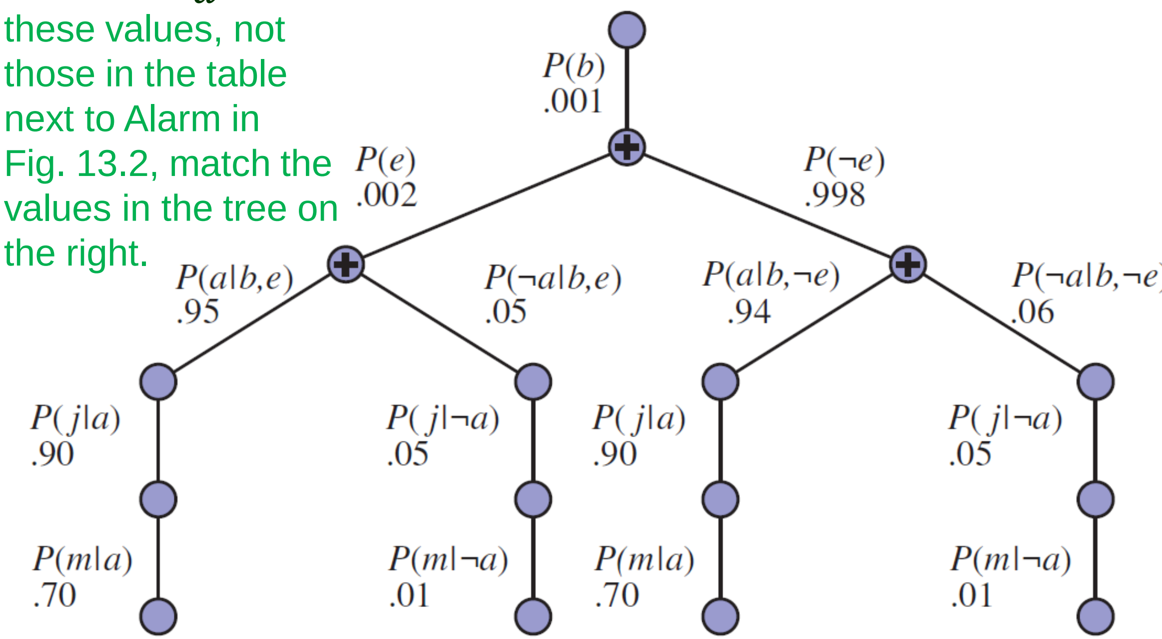

- Multiply values along each path.
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$$P(b \mid j,m) = \alpha \times 0.00059224$$

$$P(\neg b \mid j,m) = \alpha \times 0.0014919$$

(computed using a tree of the same structure but with b replaced by $\neg b$ and probabilities changed)

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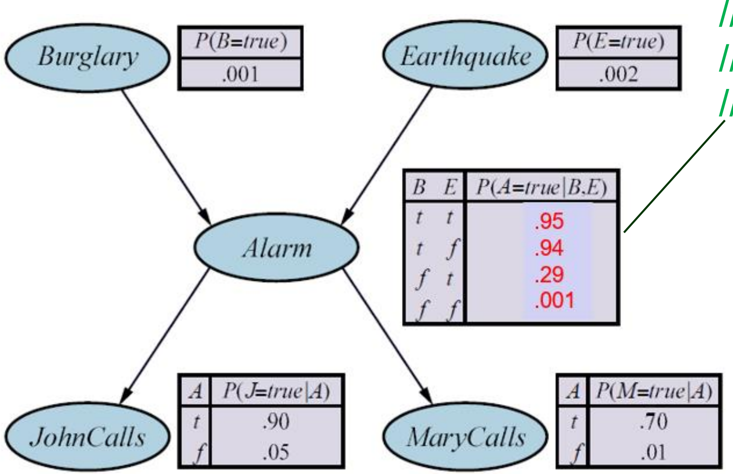

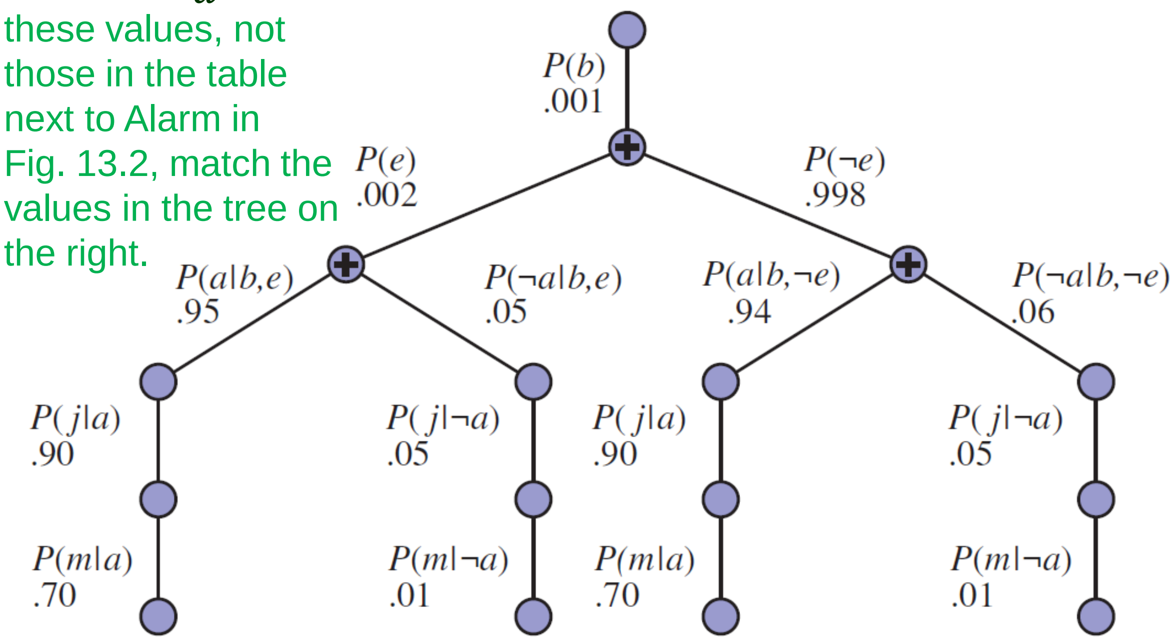

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$\Longrightarrow$

$$\mathbf{P}(B \mid j,m) = \alpha\langle 0.00059224, 0.0014919\rangle \approx \langle 0.284, 0.716\rangle$$

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Enumeration Algorithm

function ENUMERATION-ASK(X, **e**, bn) **returns** a distribution over X
  inputs: X, the query variable
    e, observed values for variables **E**
    bn, a Bayes net with variables $vars$

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL($vars$, $\mathbf{e}_{x_i}$)
      where $\mathbf{e}_{x_i}$ is **e** extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

function ENUMERATE-ALL($vars$, **e**) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $V \leftarrow$ FIRST($vars$)
  if V is an evidence variable with value v in **e**
    then return $P(v \mid parents(V)) \times$ ENUMERATE-ALL(REST($vars$), **e**)
    else return $\sum_v P(v \mid parents(V)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_v$)
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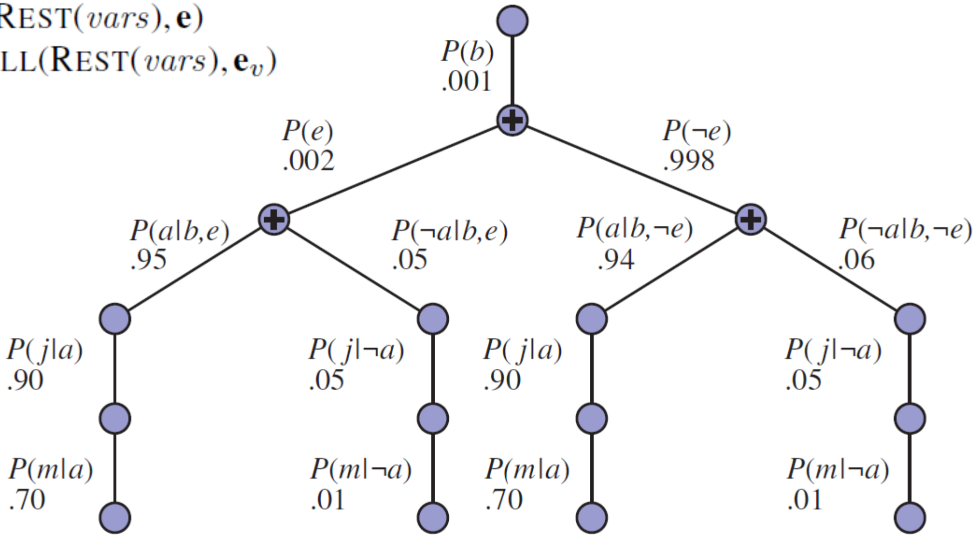

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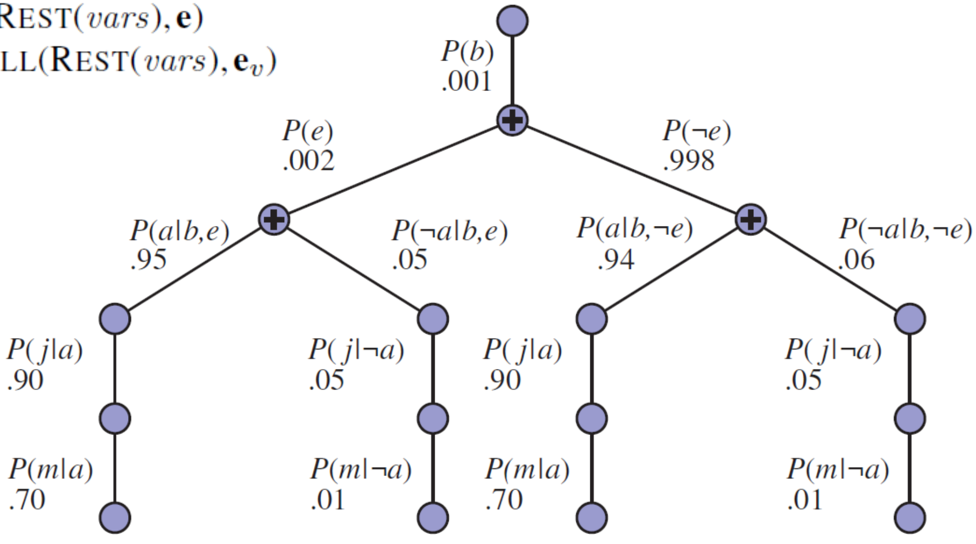
- Depth-first recursion



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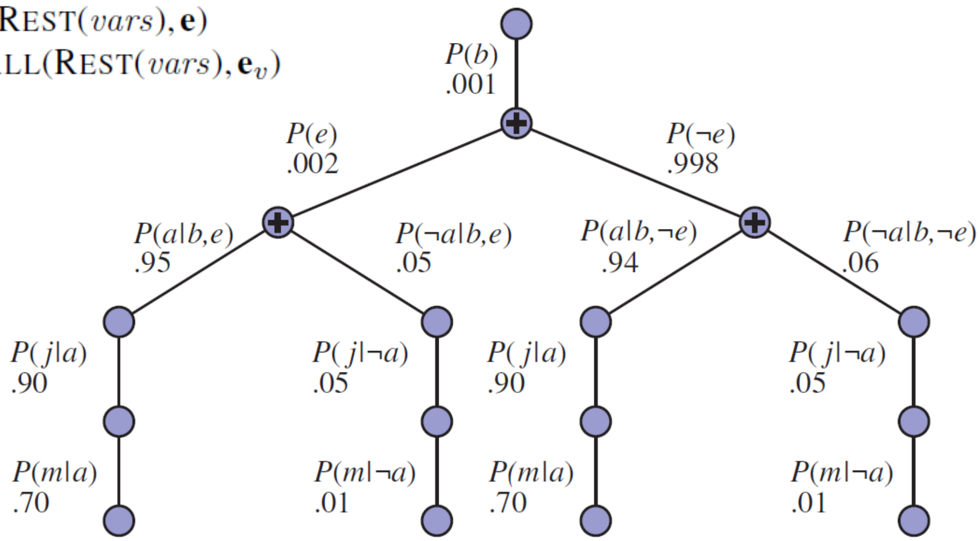
- Depth-first recursion
- $O(2^n)$ time for n variables



Enumeration Algorithm

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  for each value x_i of X do
      Q(x_i) ← ENUMERATE-ALL(vars, e_{x_i})
         where e_{x_i} is e extended with X = x_i
  return NORMALIZE(Q(X))

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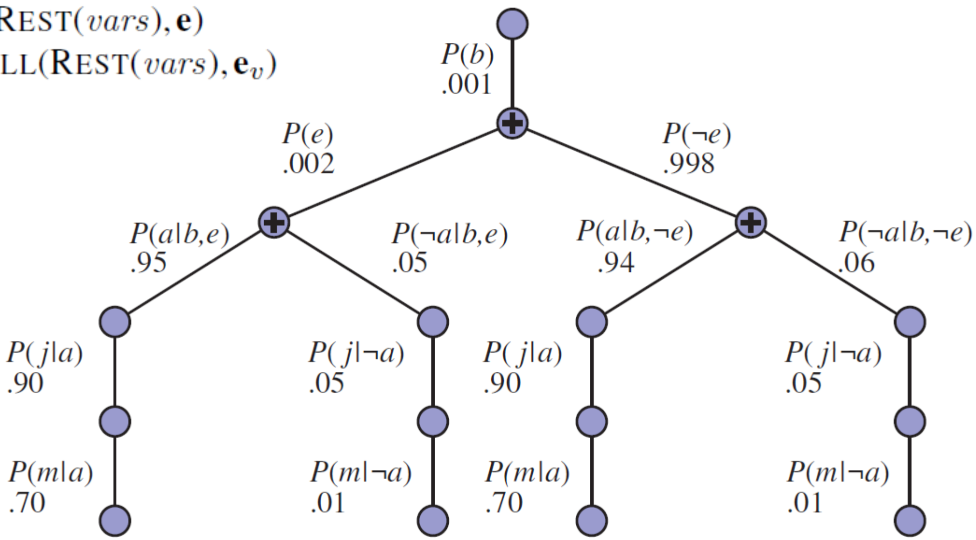

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- $O(2^n)$ time for n variables
- Repeated evaluations of the same subexpressions

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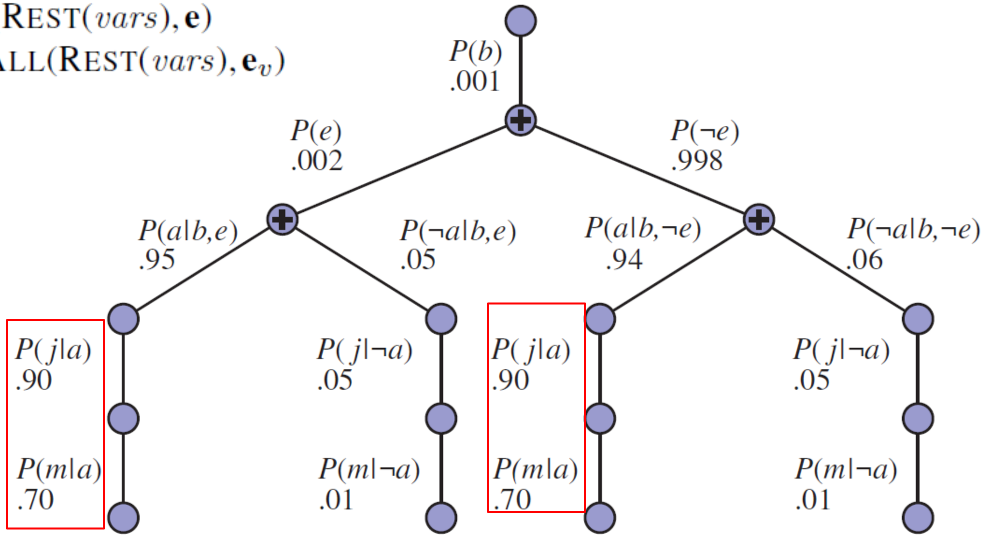
    then return $P(v \mid parents(V)) \times$ ENUMERATE-ALL(REST($vars$), **e**)
    else return $\sum_v P(v \mid parents(V)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_v$)
      where $\mathbf{e}_v$ is **e** extended with $V = v$

- Depth-first recursion
- $O(2^n)$ time for n variables
- Repeated evaluations of the same subexpressions



Enumeration Algorithm

function ENUMERATION-ASK(X, **e**, bn) **returns** a distribution over X
  inputs: X, the query variable
    e, observed values for variables **E**
    bn, a Bayes net with variables $vars$

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL($vars$, $\mathbf{e}_{x_i}$)
      where $\mathbf{e}_{x_i}$ is **e** extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

function ENUMERATE-ALL($vars$, **e**) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $V \leftarrow$ FIRST($vars$)
  if V is an evidence variable with value v in **e**
    then return $P(v \mid parents(V)) \times$ ENUMERATE-ALL(REST($vars$), **e**)
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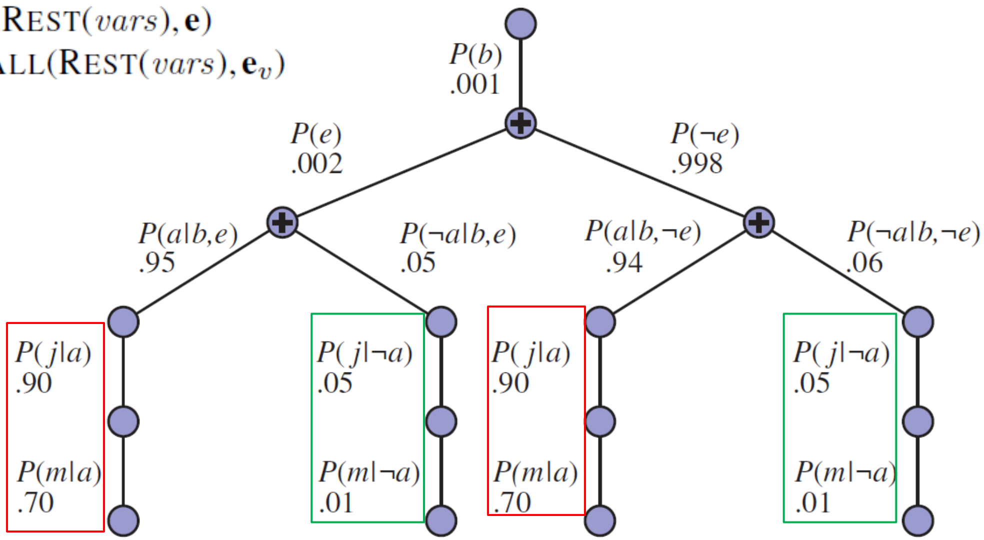

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$$P(b \mid j, m) = \alpha P(b) \sum_{e'} P(e') \sum_{a'} P(a' \mid b, e') P(j \mid a') P(m \mid a')$$

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$$\boldsymbol{f}_5(A) = \begin{pmatrix} P(m \mid a) \\ P(m \mid \neg a) \end{pmatrix} = \begin{pmatrix} 0.70 \\ 0.01 \end{pmatrix}$$

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factors: $\boldsymbol{f}_1(B)$ $\boldsymbol{f}_2(E)$ $\boldsymbol{f}_3(A, B, E)$ $\boldsymbol{f}_4(A)$ $\boldsymbol{f}_5(A)$

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// a multi-dimensional array).

$$\boldsymbol{f}_4(A) = \begin{pmatrix} P(j \mid a) \\ P(j \mid \neg a) \end{pmatrix} = \begin{pmatrix} 0.90 \\ 0.05 \end{pmatrix} \quad \boldsymbol{f}_5(A) = \begin{pmatrix} P(m \mid a) \\ P(m \mid \neg a) \end{pmatrix} = \begin{pmatrix} 0.70 \\ 0.01 \end{pmatrix}$$

Evaluation Example

Pointwise product operation

$$\boldsymbol{P}(B \mid j, m) = \alpha \boldsymbol{f_1}(B) \times \sum_{e'} \boldsymbol{f_2}(e') \times \underbrace{\sum_{a'} \boldsymbol{f_3}(a', B, e') \times \boldsymbol{f_4}(a') \times \boldsymbol{f_5}(a')}_{\boldsymbol{f}_6(B, e')}$$

Evaluation Example

Pointwise product operation

$$\boldsymbol{P}(B \mid j, m) = \alpha \boldsymbol{f_1}(B) \times \sum_{e'} \boldsymbol{f_2}(e') \times \underbrace{\sum_{a'} \boldsymbol{f_3}(a', B, e') \times \boldsymbol{f_4}(a') \times \boldsymbol{f_5}(a')}_{\boldsymbol{f}_6(B, e')}$$

- Sum out a' from the product of $\boldsymbol{f}_3, \boldsymbol{f}_4, \boldsymbol{f}_5$.

$$\boldsymbol{f}_6(B, e') = \sum_{a' \in \{a, \neg a\}} \boldsymbol{f_3}(a', B, e') \times \boldsymbol{f_4}(a') \times \boldsymbol{f_5}(a')$$

2×2

Evaluation Example

Pointwise product operation

$$\boldsymbol{P}(B \mid j,m) = \alpha \boldsymbol{f_1}(B) \times \sum_{e'} \boldsymbol{f_2}(e') \times \underbrace{\sum_{a'} \boldsymbol{f_3}(a',B,e') \times \boldsymbol{f_4}(a') \times \boldsymbol{f_5}(a')}_{\boldsymbol{f}_6(B,e')}$$

- Sum out a' from the product of $\boldsymbol{f}_3, \boldsymbol{f}_4, \boldsymbol{f}_5$.

$$\underset{2\times 2}{\boldsymbol{f}_6(B,e')} = \sum_{a' \in \{a,\neg a\}} \boldsymbol{f_3}(a',B,e') \times \boldsymbol{f_4}(a') \times \boldsymbol{f_5}(a')$$

$$= \left(\underset{2\times 2}{\boldsymbol{f}_3(a,B,e')} \times P(j \mid a) \times P(m \mid a)\right) + \left(\underset{2\times 2}{\boldsymbol{f}_3(\neg a,B,e')} \times P(j \mid \neg a) \times P(m \mid \neg a)\right)$$

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Pointwise product operation

$$\mathbf{P}(B \mid j,m) = \alpha \mathbf{f_1}(B) \times \sum_{e'} \mathbf{f_2}(e') \times \underbrace{\sum_{a'} \mathbf{f_3}(a',B,e') \times \mathbf{f_4}(a') \times \mathbf{f_5}(a')}_{\mathbf{f}_6(B,e')}$$

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Pointwise product operation

$$\mathbf{P}(B \mid j,m) = \alpha \boldsymbol{f}_{\mathbf{1}}(B) \times \sum_{e'} \boldsymbol{f}_{\mathbf{2}}(e') \times \underbrace{\sum_{a'} \boldsymbol{f}_{\mathbf{3}}(a',B,e') \times \boldsymbol{f}_{\mathbf{4}}(a') \times \boldsymbol{f}_{\mathbf{5}}(a')}_{\boldsymbol{f}_6(B,e')}$$

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$$\mathbf{P}(B \mid j,m) = \alpha \boldsymbol{f}_{\mathbf{1}}(B) \times \underbrace{\sum_{e'} \boldsymbol{f}_2(e') \times \boldsymbol{f}_6(B,e')}_{\boldsymbol{f}_7(B,e')}$$

Evaluation Example

Pointwise product operation

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- Sum out a' from the product of $\boldsymbol{f}_3, \boldsymbol{f}_4, \boldsymbol{f}_5$.

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- Sum out e' from the product of $\boldsymbol{f}_2$ and $\boldsymbol{f}_6$.

$$\underset{2 \times 1}{\boldsymbol{f}_7(B)} = \sum_{e' \in \{e, \neg e\}} \boldsymbol{f_2}(e') \times \boldsymbol{f_6}(B, e') = P(e) \times \underset{2 \times 1}{\boldsymbol{f}_6(B, e)} + P(\neg e) \times \underset{2 \times 1}{\boldsymbol{f}_6(B, \neg e)}$$

Evaluation Example

Pointwise product operation

$$\boldsymbol{P}(B \mid j, m) = \alpha \boldsymbol{f}_1(B) \times \sum_{e'} \boldsymbol{f}_2(e') \times \underbrace{\sum_{a'} \boldsymbol{f}_3(a', B, e') \times \boldsymbol{f}_4(a') \times \boldsymbol{f}_5(a')}_{\boldsymbol{f}_6(B, e')}$$

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- Finally, carry out the following pointwise product:

$$\underset{2\times 1}{\boldsymbol{P}(B \mid j, m)} = \alpha \underset{2\times 1}{\boldsymbol{f}_1(B)} \times \underset{2\times 1}{\boldsymbol{f}_7(B)}$$

Pointwise Product of Two Factors

$$\boldsymbol{f}(X_1, \ldots, X_j, Y_1, \ldots, Y_k) \times \boldsymbol{g}(Y_1, \ldots, Y_k, Z_1, \ldots, Z_l) = \boldsymbol{h}(X_1, \ldots, X_j, Y_1, \ldots, Y_k, Z_1, \ldots, Z_l)$$

Entries under $\boldsymbol{f}(X, Y)$, $\boldsymbol{g}(Y, Z)$, and $\boldsymbol{h}(X, Y, Z)$ do not represent probabilities.

X	Y	$\mathbf{f}(X, Y)$	Y	Z	$\mathbf{g}(Y, Z)$	X	Y	Z	$\mathbf{h}(X, Y, Z)$
t	t	.3	t	t	.2	t	t	t	$.3 \times .2 = .06$
t	f	.7	t	f	.8	t	t	f	$.3 \times .8 = .24$
f	t	.9	f	t	.6	t	f	t	$.7 \times .6 = .42$
f	f	.1	f	f	.4	t	f	f	$.7 \times .4 = .28$
						f	t	t	$.9 \times .2 = .18$
						f	t	f	$.9 \times .8 = .72$
						f	f	t	$.1 \times .6 = .06$
						f	f	f	$.1 \times .4 = .04$

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Summing Out a Variable

This operation over a product of factors is carried out by adding up the submatrices, each for one value of the same variable.

$$\boldsymbol{l}(Y,Z) = \sum_{x' \in \{x, \neg x\}} \boldsymbol{h}(x', Y, Z) = \boldsymbol{h}(x, Y, Z) + \boldsymbol{h}(\neg x, Y, Z)$$

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t	t	$.3$	t	t	$.2$	t	t	t	$.3 \times .2 = .06$
t	f	$.7$	t	f	$.8$	t	t	f	$.3 \times .8 = .24$
f	t	$.9$	f	t	$.6$	t	f	t	$.7 \times .6 = .42$
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Variable Elimination Algorithm

Move outside the summation any factor independent of the variable to be summed out.

$$\sum_{x'} \boldsymbol{f}(x', Y) \times \boldsymbol{g}(Y, Z) = \boldsymbol{g}(Y, Z) \times \sum_{x'} \boldsymbol{f}(x', Y)$$

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- Use a greedy heuristic: eliminate whichever variable minimizes the size of the next factor to be constructed.

Irrelevant Variable

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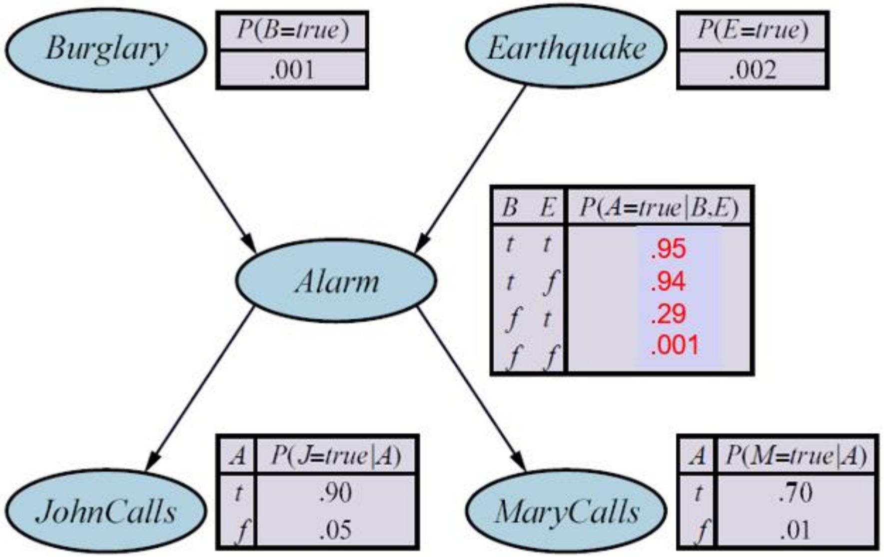

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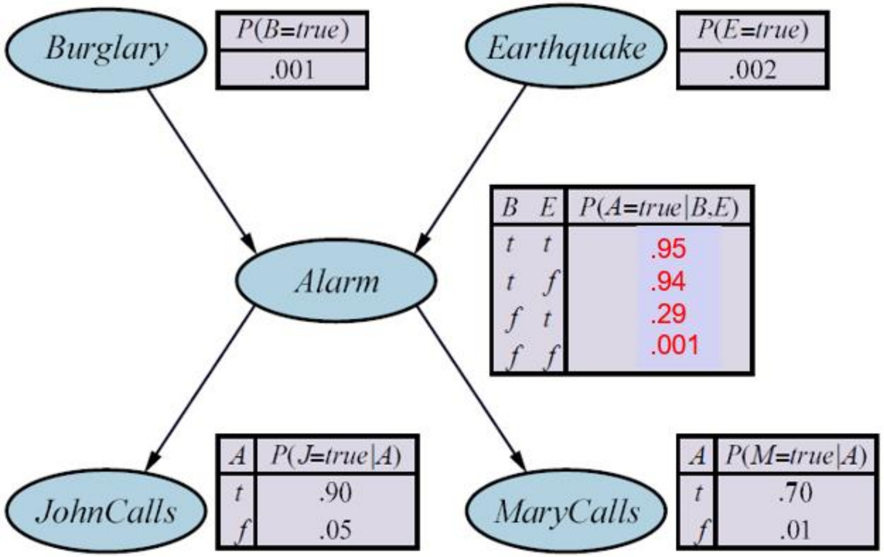

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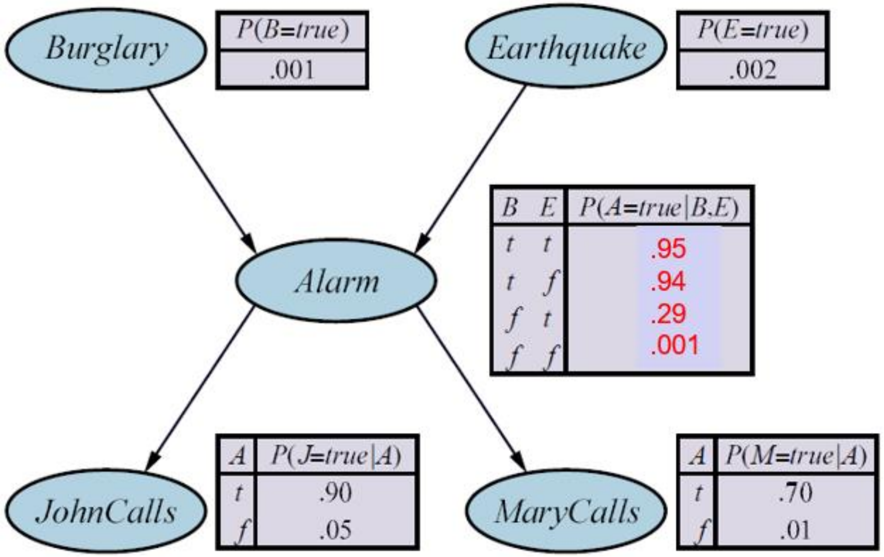

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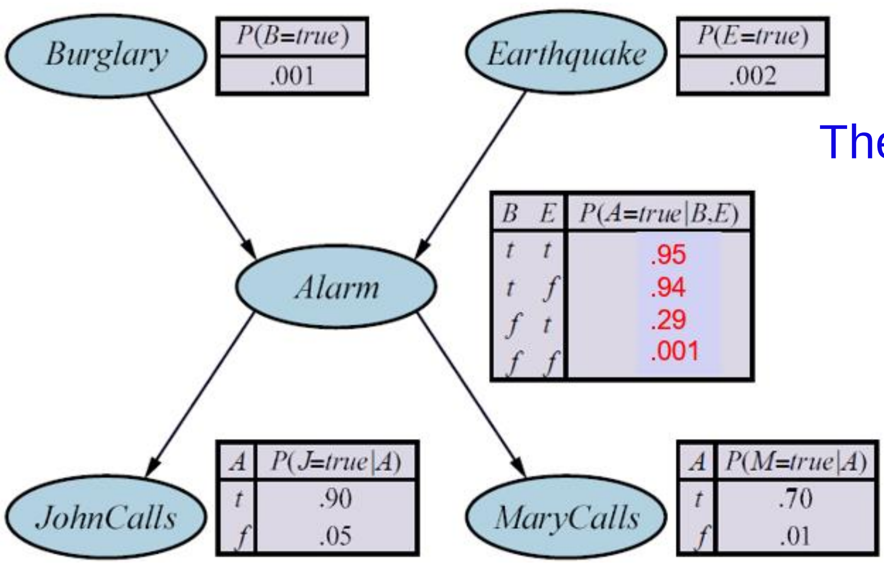

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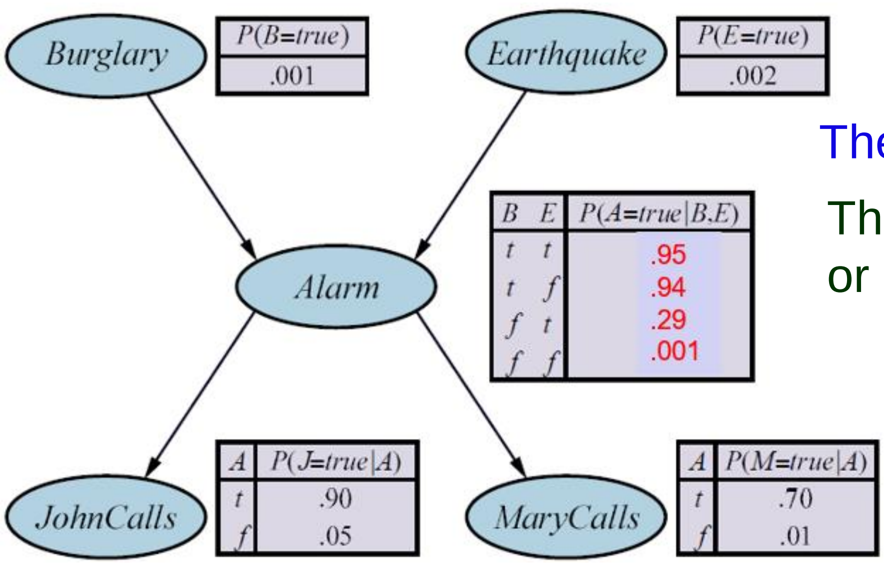

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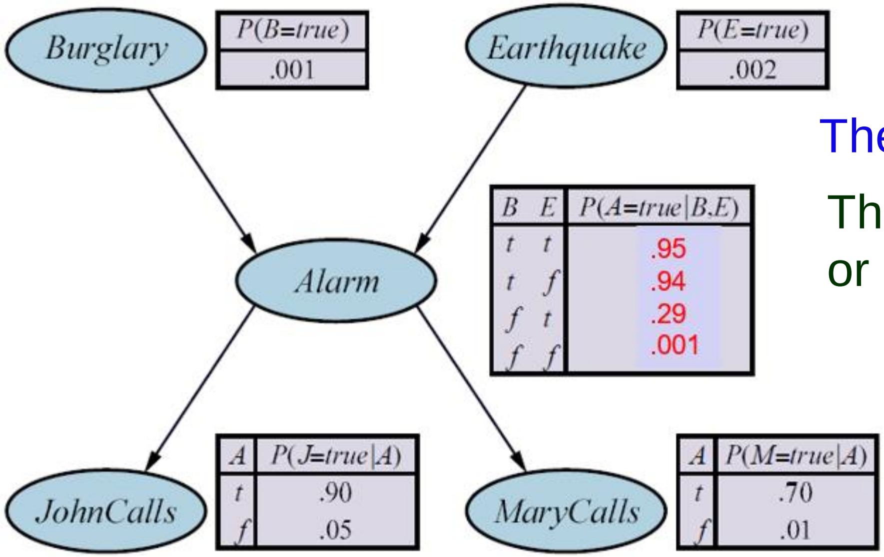

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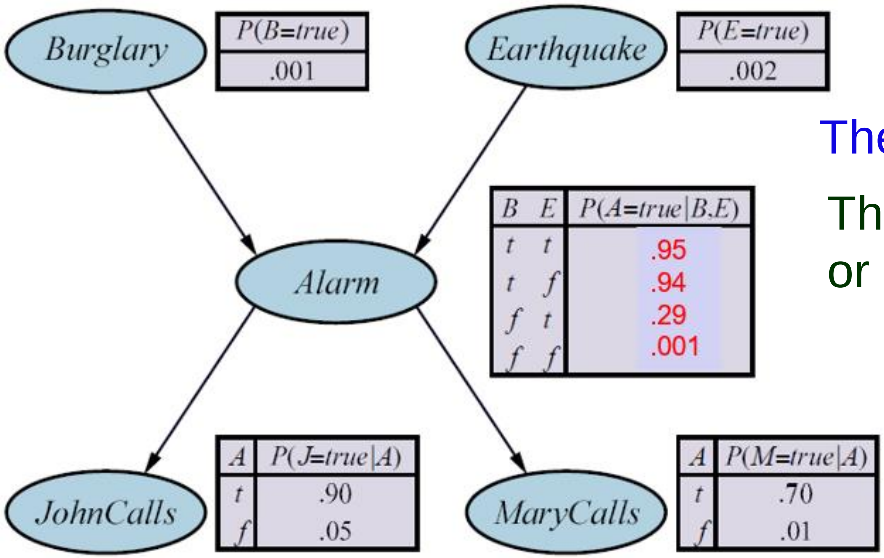
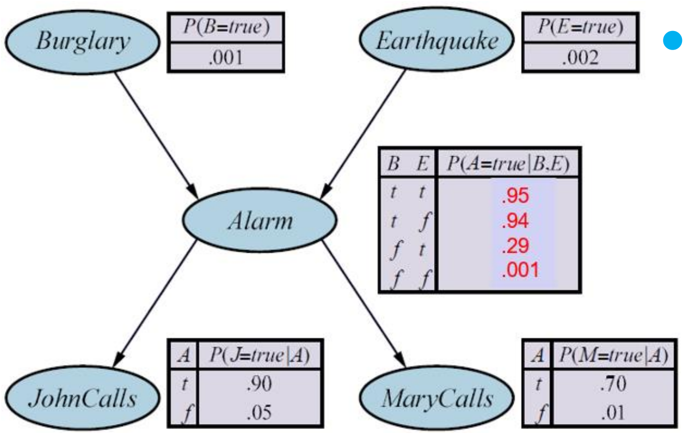
- *J* and *M* are not d-separated by *B*. (The moral graph would contain the network plus an edge connecting *B* and *E*.)



Inference with Elimination



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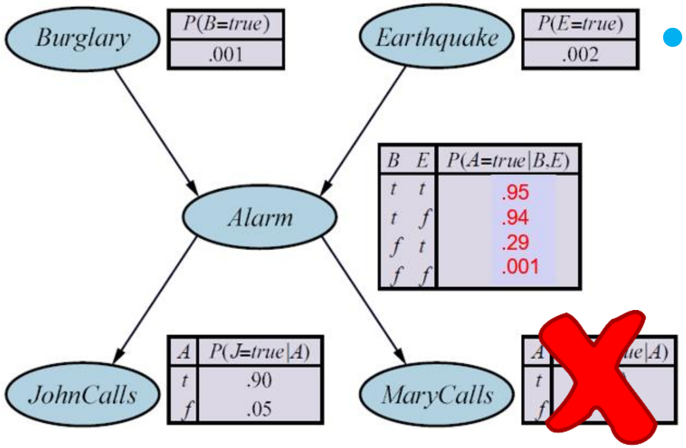

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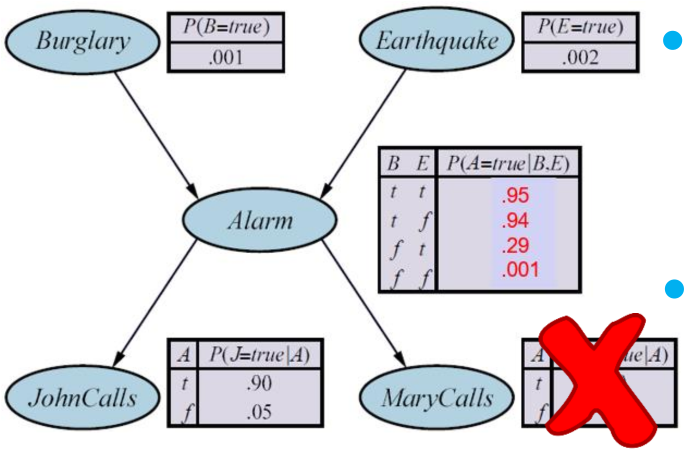

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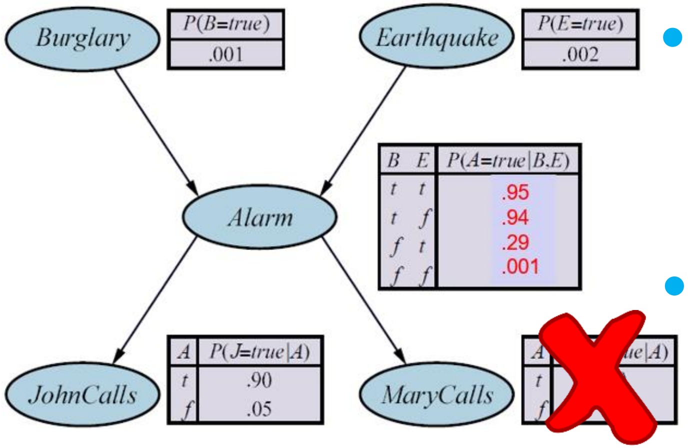

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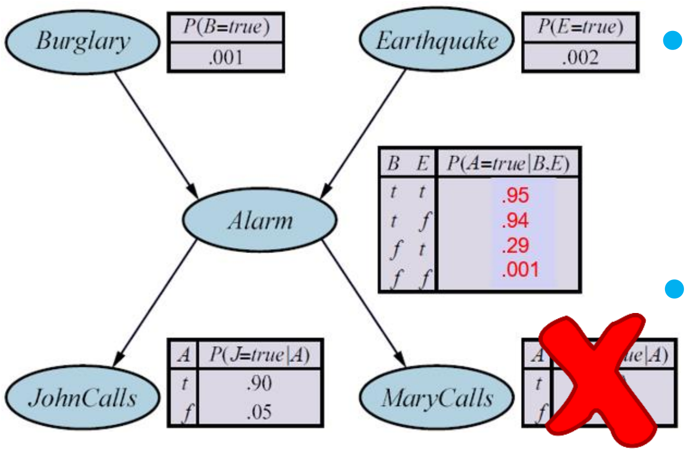

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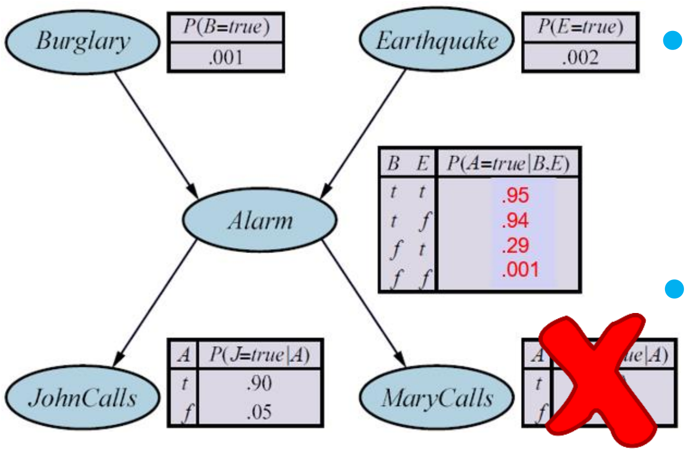

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♦ If we want to compute posterior probabilities for all the variables rather than answer individual queries, we can use clustering algorithms (i.e., join tree algorithms).