Lecture notes for *Principles of Artificial Intelligence* (COMS 4720/5720)
Yan-Bin Jia, Iowa State University

Bayes Models

Outline

I. Naïve Bayes model

II. Revisiting the wumpus world

* Figures are from the textbook site.

I. Naive Bayes Model

Full joint distribution:

$$\boldsymbol{P}(Cause, \underbrace{Effect_1, \ldots, Effect_n}) = \boldsymbol{P}(Cause) \prod_i \boldsymbol{P}(Effect_i \mid Cause)$$

Conditionally independent
given the cause

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Conditionally independent given the cause

- Often used in cases where the "effect" variables $Effect_1, \ldots, Effect_n$ are not strictly independent.
- Hence the model is called "naïve".

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- Unobserved effects: $\boldsymbol{Y} = \boldsymbol{y}$

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When the effects are not conditionally independent:

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(cont'd)

$$\mathbf{P}(Cause \mid \mathbf{e}) = \alpha \mathbf{P}(Cause) \left(\prod_j \mathbf{P}(e_j \mid Cause) \right)$$

Calculate the probability distribution of the causes from observed effects:

- Take each possible cause.
- Multiply its prior probability by the product of the conditional probabilities of the observed effects given that cause.
- Normalize the result.

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- The number of unobserved effects is irrelevant no matter how large it is (as in medicine).

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1. *Stocks rallied on Monday, with major indices gaining 1% as optimism persisted over the first quarter earnings season.*
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Classify each sentence into a Category.

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- Prior probabilities: $\boldsymbol{P}(Category)$
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    $P(\mathit{HasWord}_6 = \mathit{true} \mid \mathit{Category} = \mathit{business}) \approx 0.37$
    // 37% of articles about business contain word 6, "stocks".

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$$\mathbf{P}(\mathit{Category} \mid \mathit{HasWords}) = \alpha \mathbf{P}(\mathit{Category}) \left(\prod_j \mathbf{P}(\mathit{HasWord}_j \mid \mathit{Category}) \right)$$

$$\text{e.g., } \mathit{HasWords} = \mathit{HasWord}_1 = \mathit{true} \wedge \mathit{HasWord}_2 = \mathit{false} \wedge \cdots$$

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Other Applications of Naïve Bayes Models

- Language determination (to detect the language a text is written in)
- Spam filtering (to identify spam e-mails)
- Sentiment analysis (to identify positive and negative customer sentiments in social media)
- Real-time prediction (because they are very fast)
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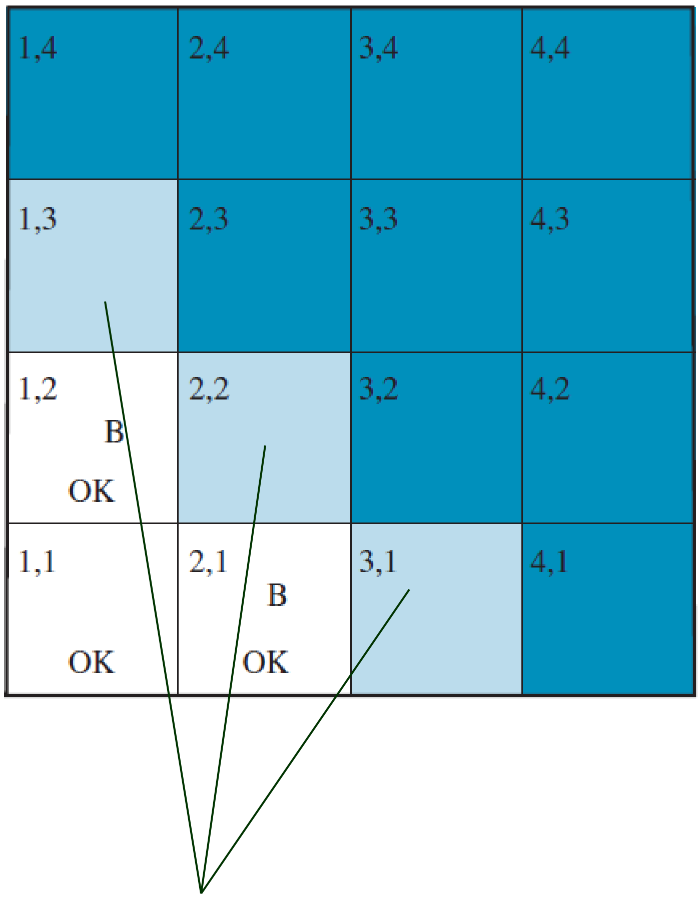
Naïve Bayes models are not used in

- ♠ Medical diagnosis (which requires more sophisticated models)

II. The Wumpus World Revisited



Each of the three squares might contain a pit.

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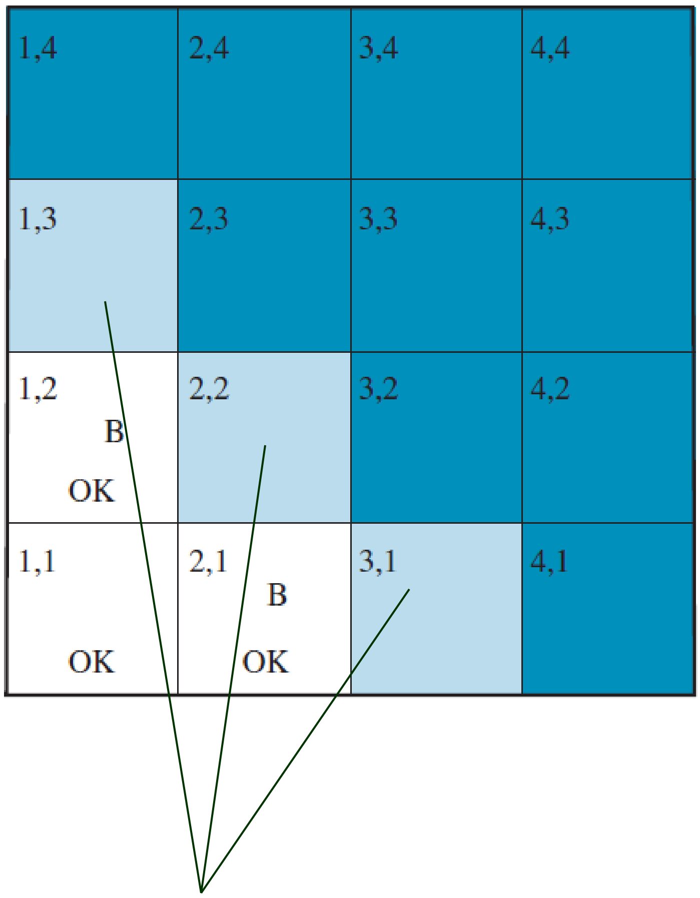

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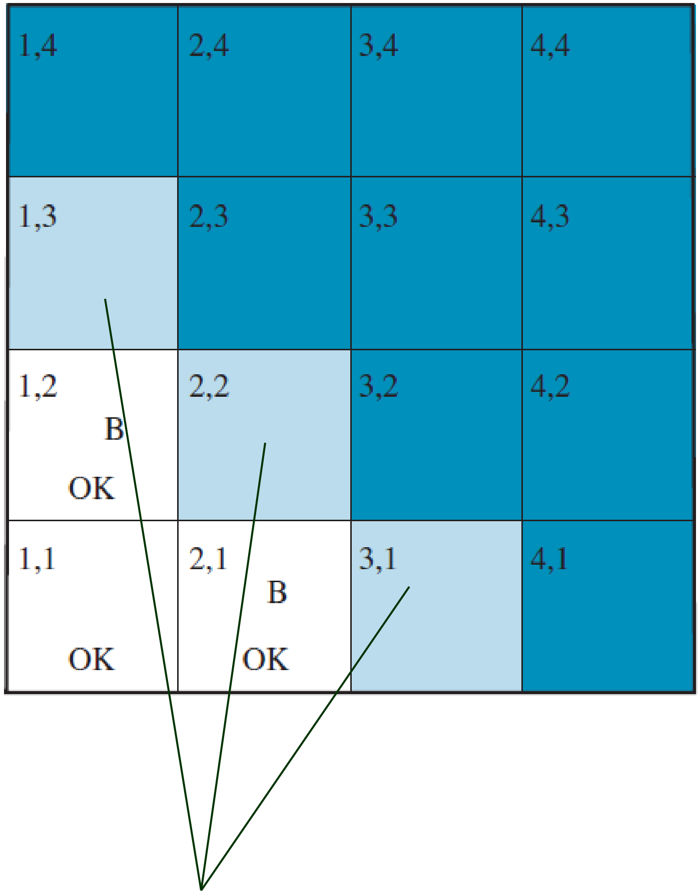

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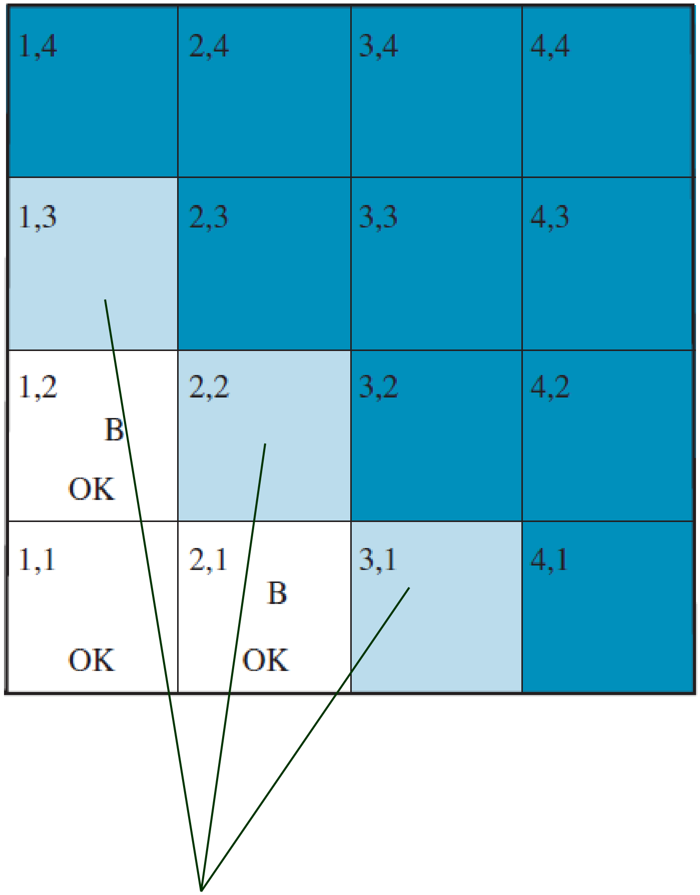

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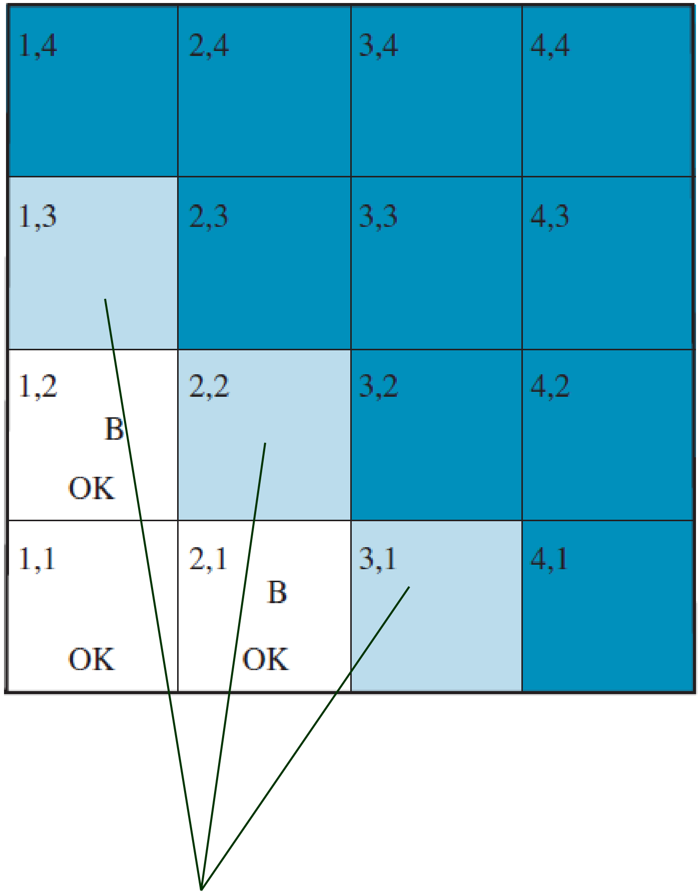

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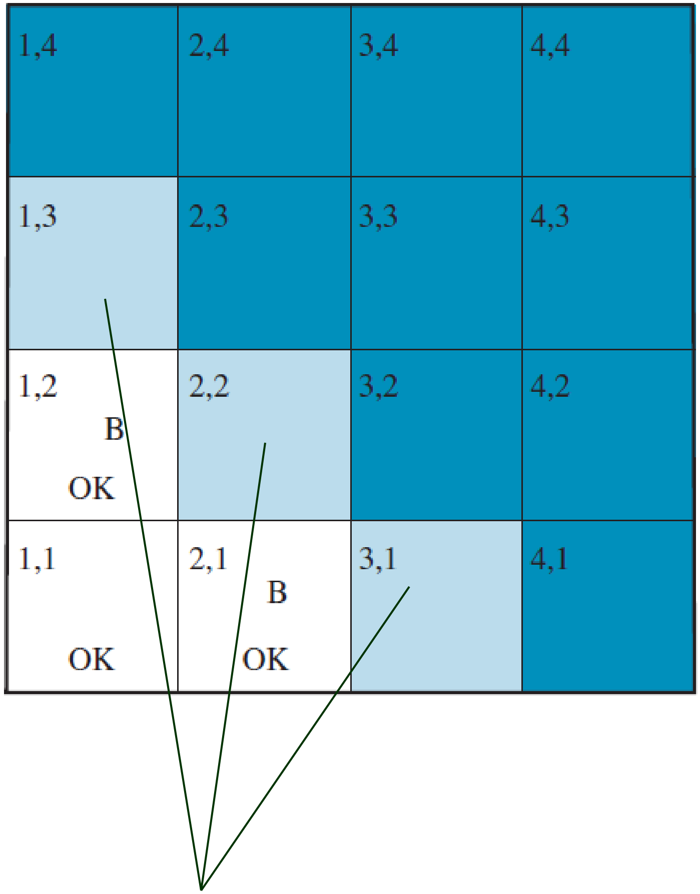

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- ♦ Identify the set of random variables.
  - $P_{i,j}$: true if square $[i,j]$ contains a pit.
  - $B_{i,j}$: true if square $[i,j]$ is breezy – included only for the observed squares, $[1,1]$, $[1,2]$, $[2,1]$.

Full Joint Distribution

$\boldsymbol{P}\left(P_{1,1}, \ldots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1}\right) =$ // $P_{1,1} \equiv$ false

$$\boldsymbol{P}\left(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \ldots, P_{4,4}\right) \boldsymbol{P}\left(P_{1,1}, \ldots, P_{4,4}\right)$$

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values in the distribution, for a given pit configuration, are 1 if all the breezy squares among $[1,1], [1,2], [2,1]$ are adjacent to pits, and 0 otherwise.

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prior probability of a pit configuration

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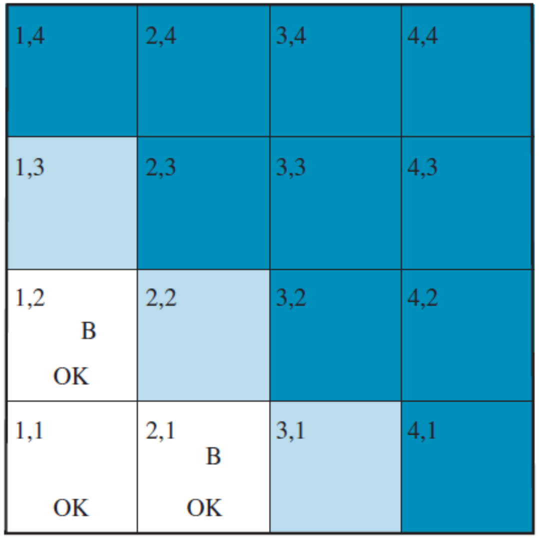
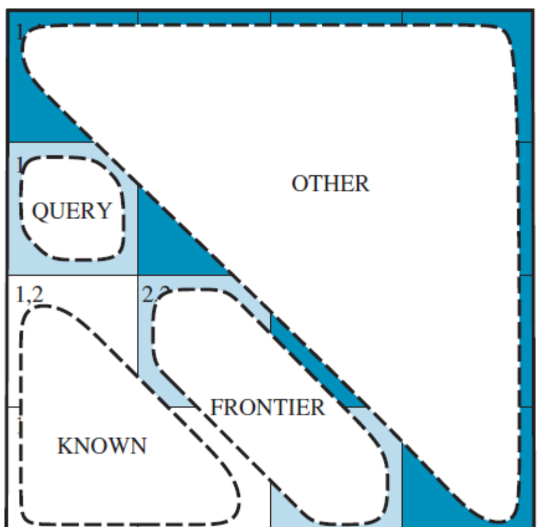
$0.2^n \times 0.8^{15-n}$ for a configuration with $n \leq 15$ pits.

Evidence





$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

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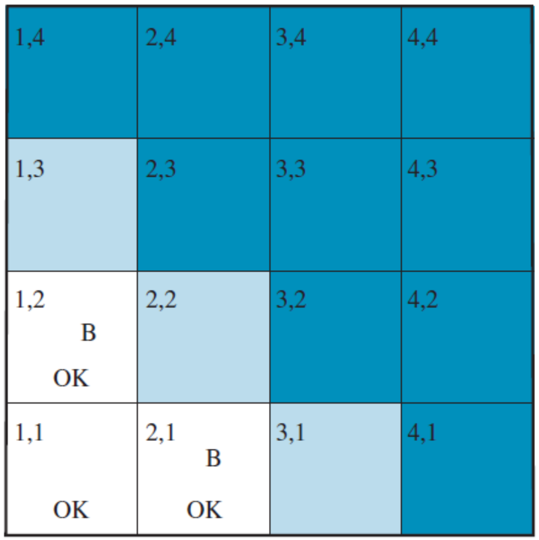

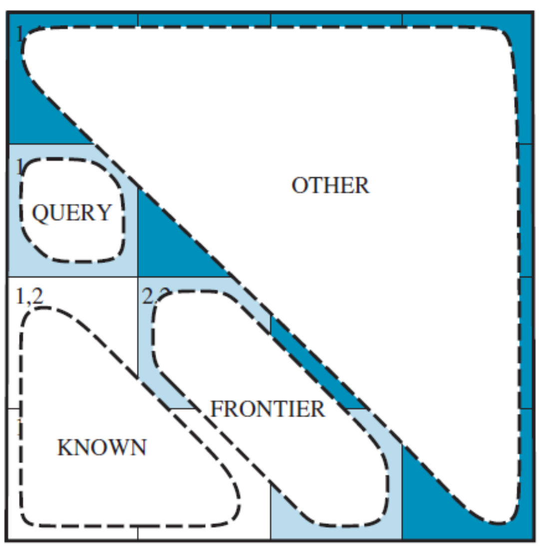

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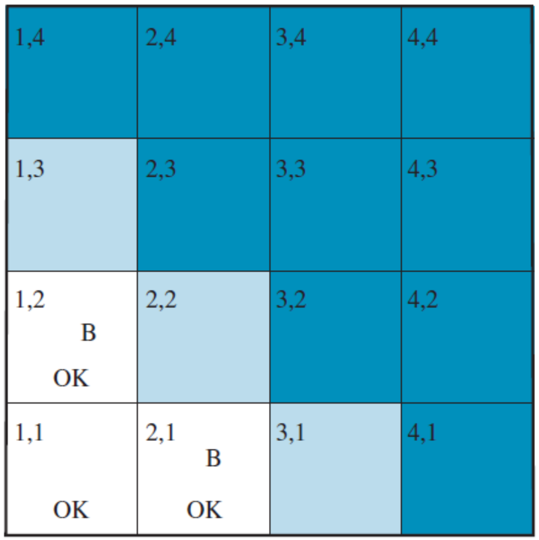

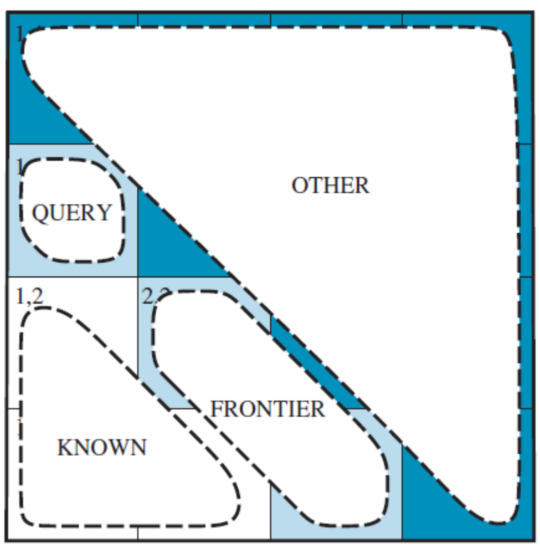

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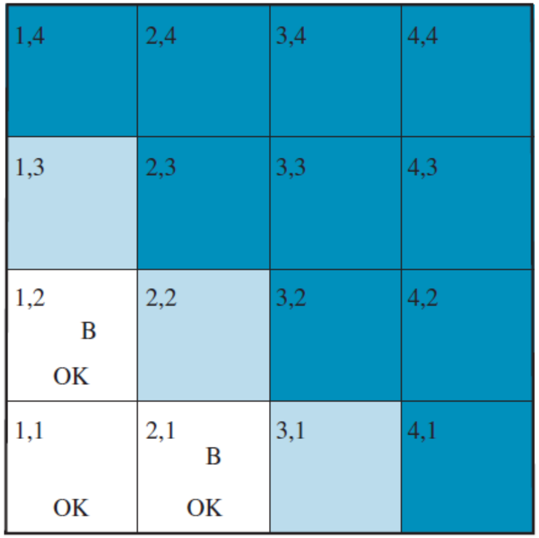

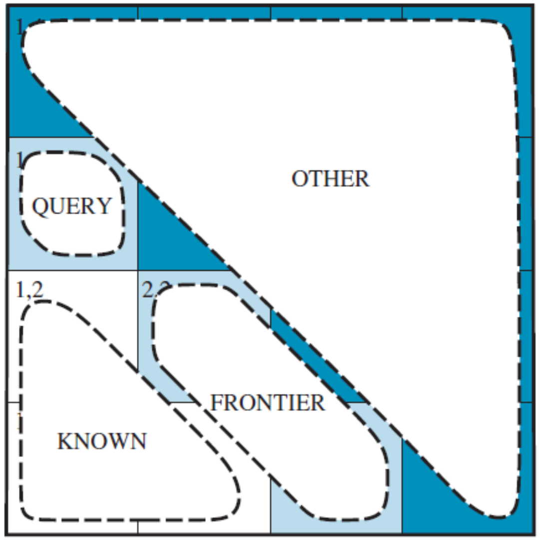

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$\in \{(p_{1,4}, p_{2,2}, \ldots, p_{4,4}), \ldots, (\neg p_{1,4}, \neg p_{2,2}, \ldots \neg p_{4,4})\}$

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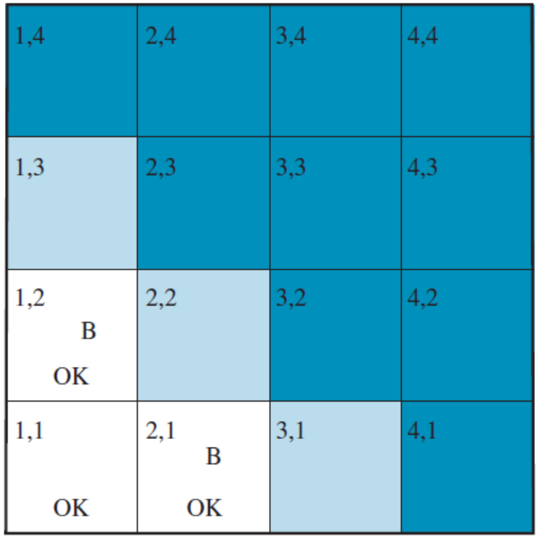

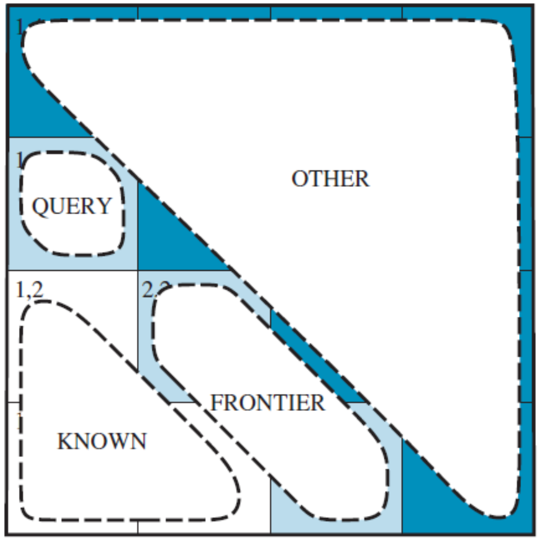

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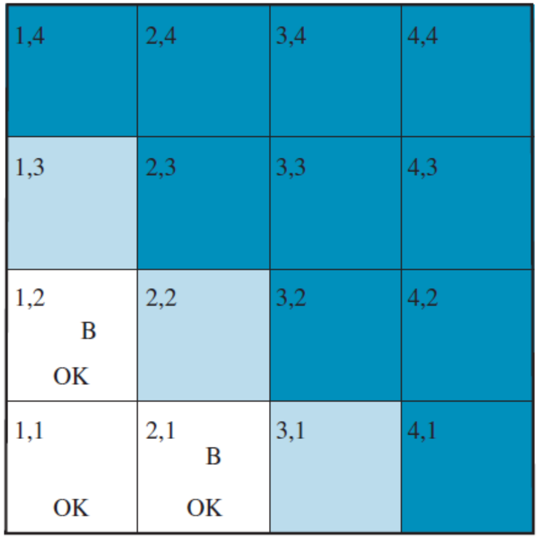
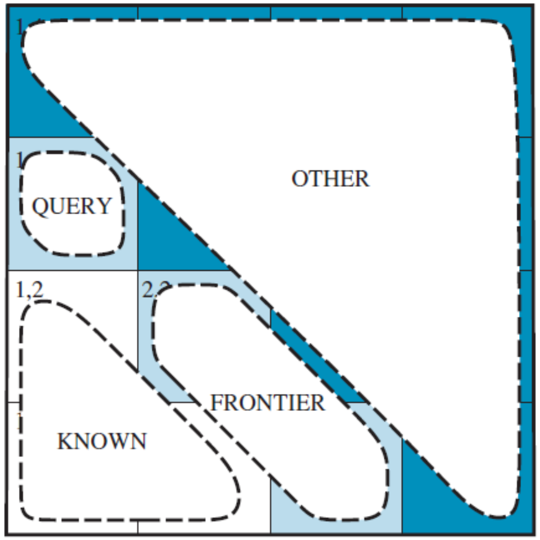
$\in \{(p_{1,4}, p_{2,2}, \ldots, p_{4,4}), \ldots, (\neg p_{1,4}, \neg p_{2,2}, \ldots \neg p_{4,4})\}$

♠ summation over $2^{12} = 4096$ terms (if the full joint probabilities are available).

Evidence





$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

$known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

Query $\boldsymbol{P}(P_{1,3} \mid known, b)$?

// how likely does [1,3] contain a pit,
// given the observations so far?

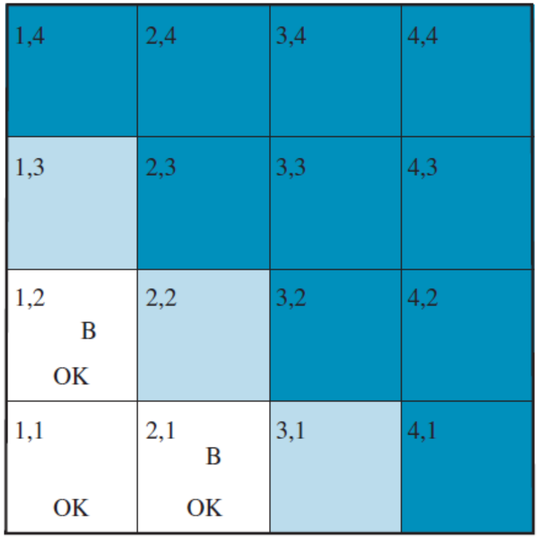
- *Unknown*: a set of 12 $P_{i,j}$s for squares other than the three known ones and the query one $[1, 3]$.

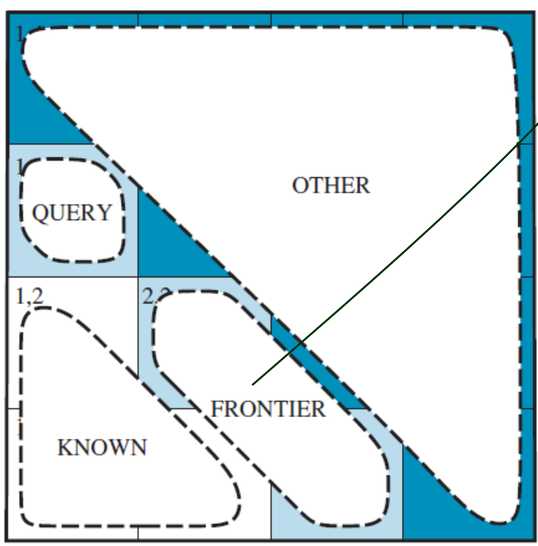
$$\boldsymbol{P}(P_{1,3} \mid known, b) = \alpha \sum_{unknown \in \{(p_{1,4}, p_{2,2}, \ldots, p_{4,4}), \ldots, (\neg p_{1,4}, \neg p_{2,2}, \ldots \neg p_{4,4})\}} \boldsymbol{P}(P_{1,3}, known, b, unknown)$$

- summation over $2^{12} = 4096$ terms (if the full joint probabilities are available).

Exponential in the number of squares!

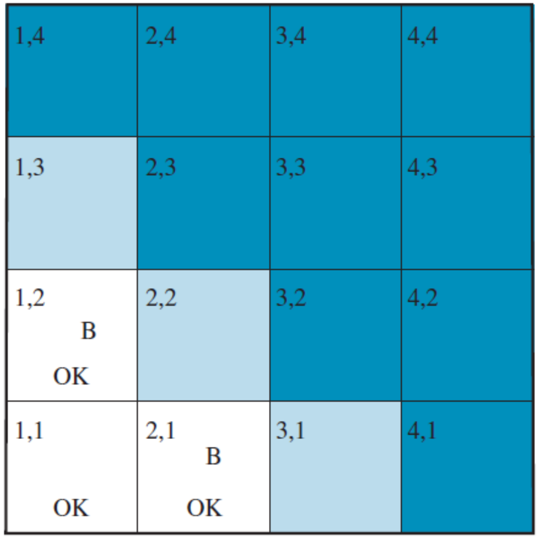
Irrelevant Squares?

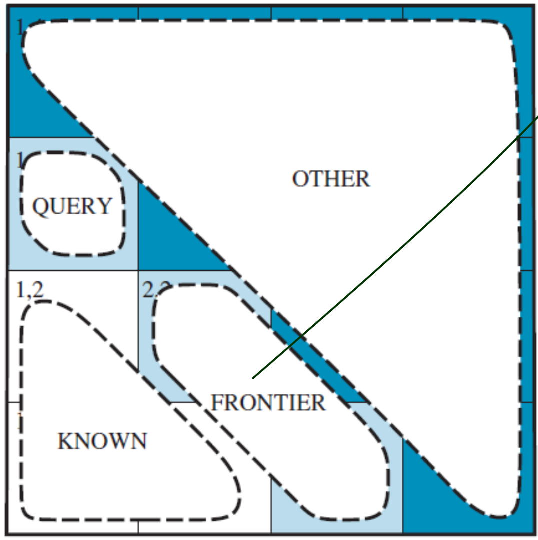




Frontier: pit variables for the squares adjacent to visited ones.

Irrelevant Squares?

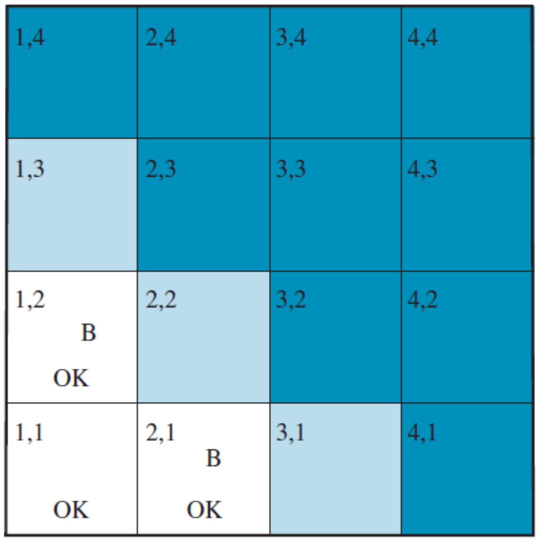


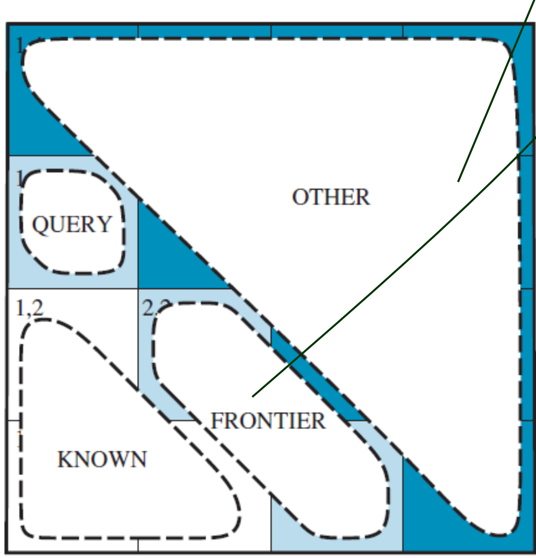


Frontier: pit variables for the squares adjacent to visited ones.

[2,2] and [3,1]

Irrelevant Squares?



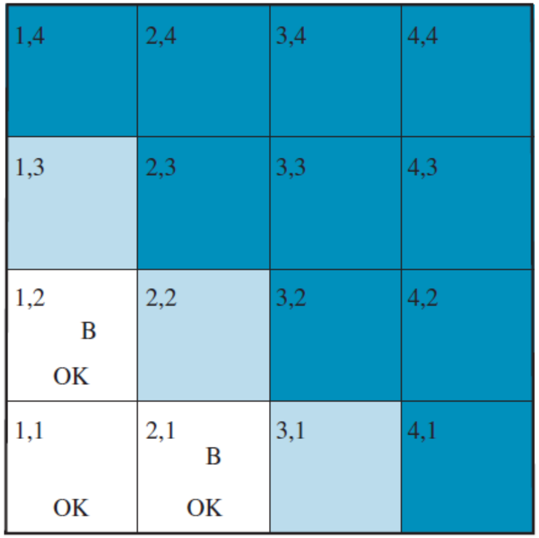


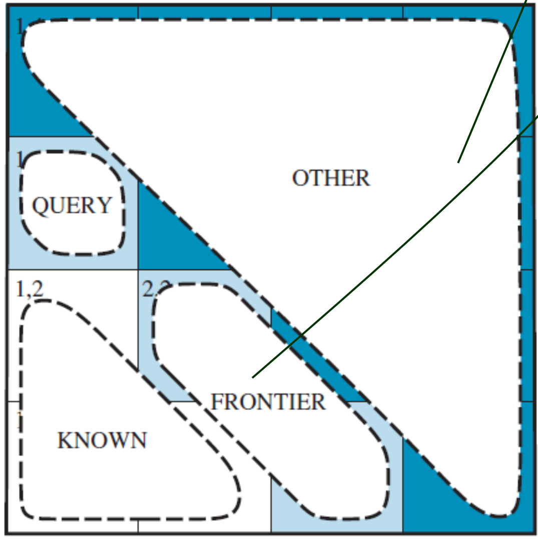
Frontier: pit variables for the squares adjacent to visited ones.

[2,2] and [3,1]

Other: pit variables for the other unknown squares.

Irrelevant Squares?





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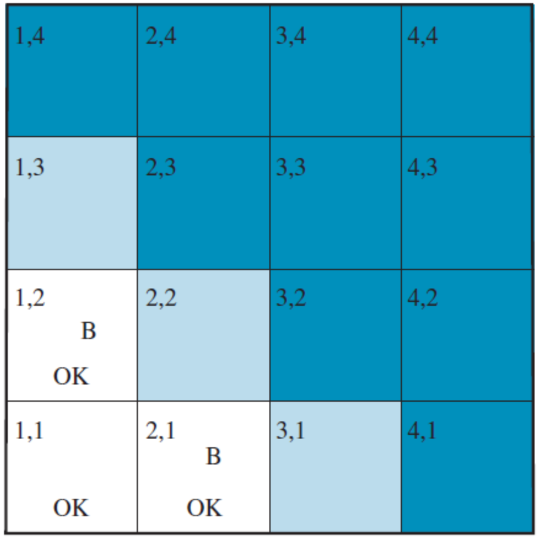
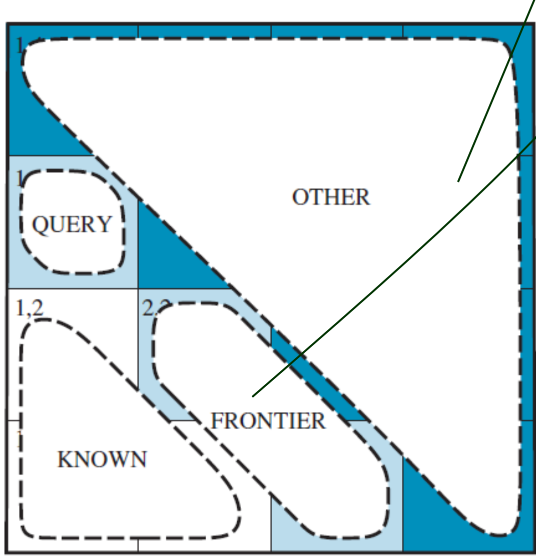
[2,2] and [3,1]

Other: pit variables for the other unknown squares.

10 other squares

Irrelevant Squares?





Frontier: pit variables for the squares adjacent to visited ones.

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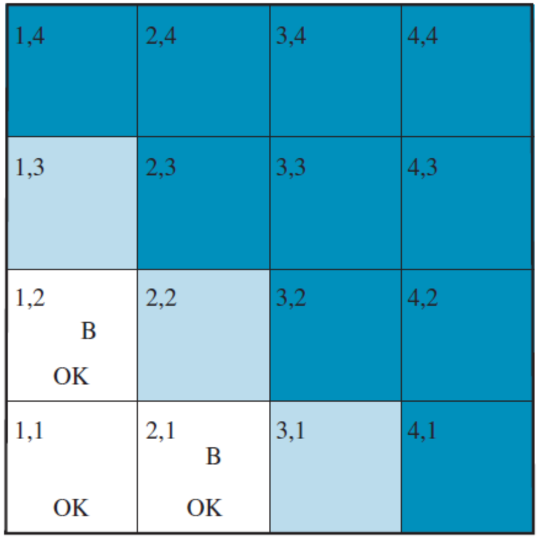
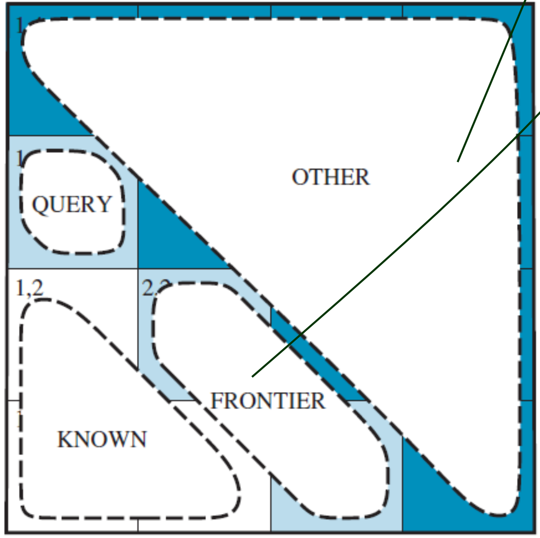
10 other squares

$Unknown = Frontier \cup Other$

Irrelevant Squares?





Frontier: pit variables for the squares adjacent to visited ones.

[2,2] and [3,1]

Other: pit variables for the other unknown squares.

10 other squares

Unknown = *Frontier* ∪ *Other*

Insight: The observed breezes are conditionally *independent of Other*, given *Known*, *Frontier*, and the query variable.

Applying Conditional Independence

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha \sum_{unknown \in \{p_{1,4} \wedge p_{2,2} \wedge \cdots \wedge p_{4,4}, \ldots, \neg p_{1,4} \wedge \neg p_{2,2} \wedge \cdots \wedge \neg p_{4,4}\}} \mathbf{P}(P_{1,3}, known, b, unknown)$$

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⇓ product rule

$$= \alpha \sum_{unknown} \mathbf{P}(b \mid P_{1,3}, known, unknown)\mathbf{P}(P_{1,3}, known, unknown)$$

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⇩ product rule

$$= \alpha \sum_{unknown} \mathbf{P}(b \mid P_{1,3}, known, unknown)\mathbf{P}(P_{1,3}, known, unknown)$$

$$= \alpha \sum_{frontier \in \left\{ \begin{matrix} p_{2,2} \wedge p_{3,1}, p_{2,2} \wedge \neg p_{3,1}, \\ \neg p_{2,2} \wedge p_{3,1}, \neg p_{2,2} \wedge \neg p_{3,1} \end{matrix} \right\}} \sum_{other} \mathbf{P}(b \mid P_{1,3}, known, frontier, other)\mathbf{P}(P_{1,3}, known, frontier, other)$$

Applying Conditional Independence

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha \sum_{unknown \in \{p_{1,4} \wedge p_{2,2} \wedge \cdots \wedge p_{4,4}, \ldots, \neg p_{1,4} \wedge \neg p_{2,2} \wedge \cdots \wedge \neg p_{4,4}\}} \mathbf{P}(P_{1,3}, known, b, unknown)$$

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$$= \alpha \sum_{frontier} \sum_{other} \mathbf{P}(b \mid P_{1,3}, known, frontier, other)\mathbf{P}(P_{1,3}, known, frontier, other)$$

$$\in \left\{ \begin{array}{c} p_{2,2} \wedge p_{3,1}, p_{2,2} \wedge \neg p_{3,1}, \\ \neg p_{2,2} \wedge p_{3,1}, \neg p_{2,2} \wedge \neg p_{3,1} \end{array} \right\}$$

⇓ b is independent of *other,* given *known*, $P_{1,3}$, and *frontier*.

$$= \alpha \sum_{frontier} \sum_{other} \mathbf{P}(b \mid P_{1,3}, known, frontier)\mathbf{P}(P_{1,3}, known, frontier, other)$$

Applying Conditional Independence

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha \sum_{unknown \in \{p_{1,4} \wedge p_{2,2} \wedge \cdots \wedge p_{4,4}, \ldots, \neg p_{1,4} \wedge \neg p_{2,2} \wedge \cdots \wedge \neg p_{4,4}\}} \mathbf{P}(P_{1,3}, known, b, unknown)$$

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⇓ b is independent of *other,* given *known*, $P_{1,3}$, and *frontier*.

$$= \alpha \sum_{frontier} \sum_{other} \underbrace{\mathbf{P}(b \mid P_{1,3}, known, frontier)}_{\text{independent of } other}\mathbf{P}(P_{1,3}, known, frontier, other)$$

Applying Conditional Independence

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha \sum_{unknown \in \{p_{1,4} \land p_{2,2} \land \cdots \land p_{4,4}, \ldots, \neg p_{1,4} \land \neg p_{2,2} \land \cdots \land \neg p_{4,4}\}} \mathbf{P}(P_{1,3}, known, b, unknown)$$

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⇩ b is independent of *other,* given *known*, $P_{1,3}$, and *frontier*.

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$$= \alpha \sum_{frontier} \mathbf{P}(b \mid P_{1,3}, known, frontier) \sum_{other} \mathbf{P}(P_{1,3}, known, frontier, other)$$

Elimination of Other Squares (*other*)

$\mathbf{P}\left(P_{1,3} \mid known, b\right)$

$$= \alpha \sum_{frontier} \mathbf{P}\left(b \mid P_{1,3}, known, frontier\right) \sum_{other} \mathbf{P}\left(P_{1,3}, known, frontier, other\right)$$

Elimination of Other Squares (*other*)

$\mathbf{P}(P_{1,3} \mid known, b)$

$$= \alpha \sum_{frontier} \mathbf{P}(b \mid P_{1,3}, known, frontier) \sum_{other} \underbrace{\mathbf{P}(P_{1,3}, known, frontier, other)}_{\text{factoring}}$$

Elimination of Other Squares (*other*)

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$$= \alpha \sum_{frontier} \mathbf{P}(b \mid P_{1,3}, known, frontier) \sum_{other} \mathbf{P}(P_{1,3}) P(known) P(frontier) P(other)$$

Elimination of Other Squares (*other*)

$$\mathbf{P}\left(P_{1,3} \mid known, b\right)$$

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$$= \alpha \sum_{frontier} \mathbf{P}\left(b \mid P_{1,3}, known, frontier\right) \sum_{other} \mathbf{P}\left(P_{1,3}\right) P(known) P(frontier) P(other)$$

$$= \alpha P(known) \mathbf{P}\left(P_{1,3}\right) \sum_{frontier} \mathbf{P}\left(b \mid P_{1,3}, known, frontier\right) P(frontier) \sum_{other} P(other)$$

Elimination of Other Squares (*other*)

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$$= \alpha P(known) \mathbf{P}(P_{1,3}) \sum_{frontier} \mathbf{P}(b \mid P_{1,3}, known, frontier) P(frontier) \sum_{other} P(other)$$

⇓ $\alpha' = \alpha P(known)$ and $\sum_{other} P(other) = 1$

$$= \alpha' \mathbf{P}(P_{1,3}) \sum_{frontier} \mathbf{P}(b \mid P_{1,3}, known, frontier) P(frontier)$$

Probability of Containing a Pit

$$\mathbf{P}\left(P_{1,3} \mid known,\ b\right) = \alpha' \mathbf{P}\left(P_{1,3}\right) \sum_{frontier} \mathbf{P}\left(b \mid P_{1,3}, known, frontier\right) P(frontier)$$

Probability of Containing a Pit

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha' \mathbf{P}(P_{1,3}) \sum_{frontier} \underbrace{\mathbf{P}(b \mid P_{1,3}, known, frontier)}_{= 1 \text{ or } 0} P(frontier)$$

- In the distribution $\mathbf{P}(b \mid P_{1,3}, known, frontier)$: a probability is 1 if b is consistent with the values of $P_{1,3}$ and the variables in *frontier*, and 0 otherwise.

Probability of Containing a Pit

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha' \mathbf{P}(P_{1,3}) \sum_{frontier} \underbrace{\mathbf{P}(b \mid P_{1,3}, known, frontier)}_{=\,1 \text{ or } 0} P(frontier)$$

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- For each value of $P_{1,3}$, we sum over the logical models for the frontier variables that are consistent with *known*.

Probability of Containing a Pit

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha' \mathbf{P}(P_{1,3}) \sum_{frontier} \underbrace{\mathbf{P}(b \mid P_{1,3}, known, frontier)}_{=\ 1 \text{ or } 0} P(frontier)$$

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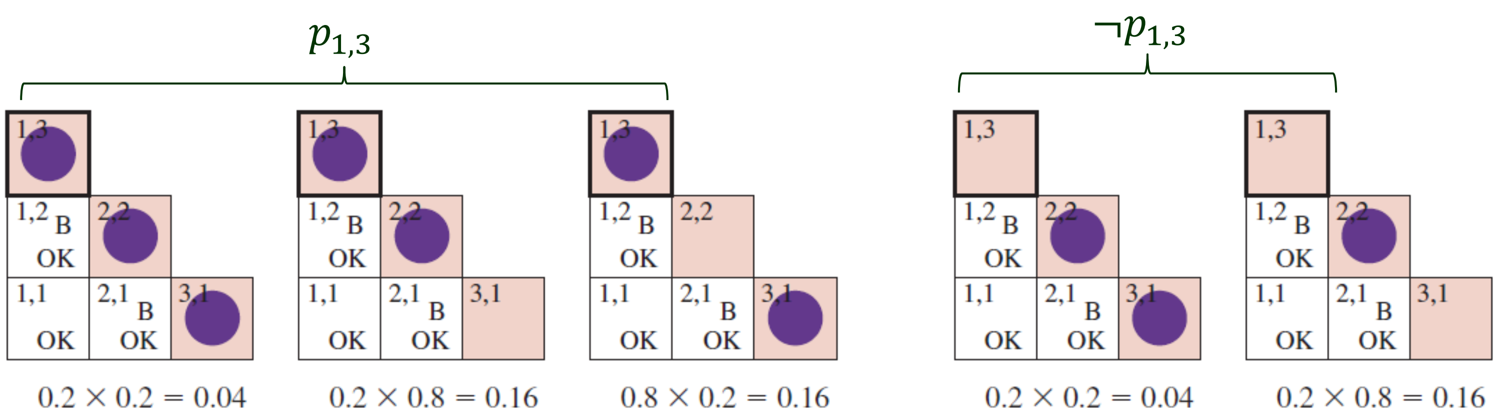
Five consistent models for *Frontier* $= \{P_{2,2}, P_{3,1}\}$:



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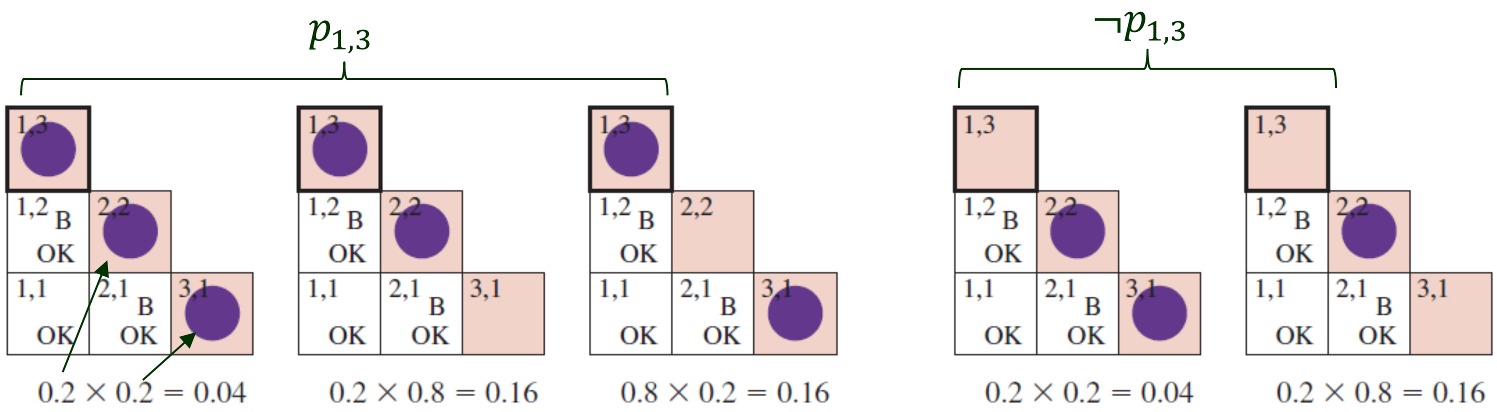

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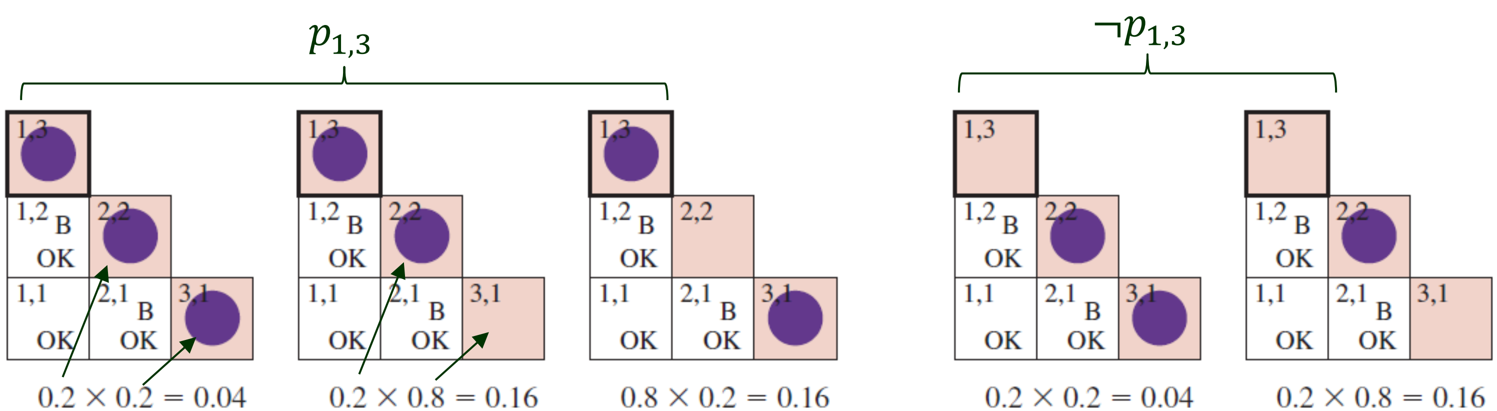

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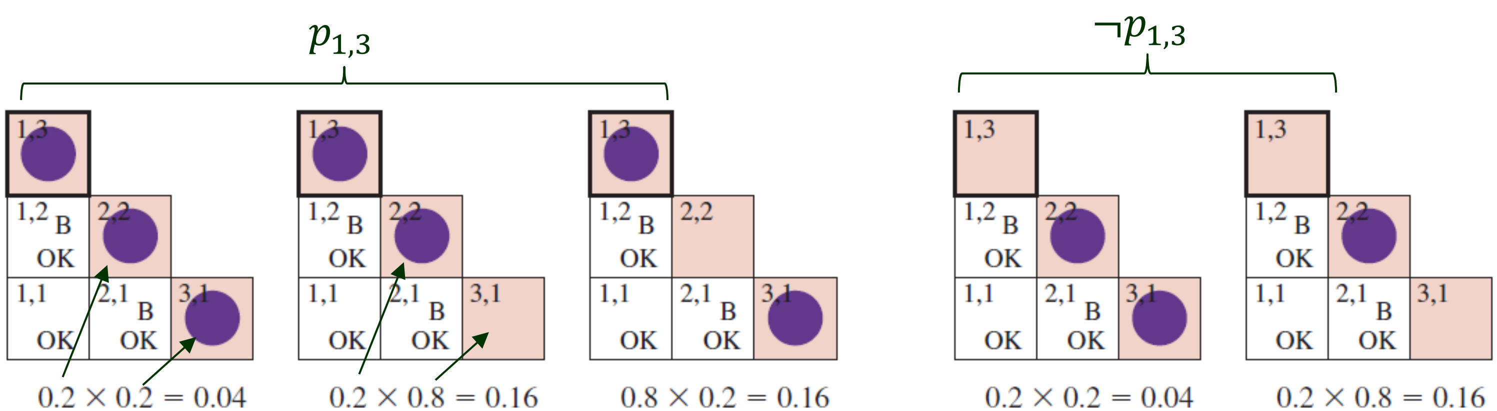

$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

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Probability of Containing a Pit

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha' \mathbf{P}(P_{1,3}) \sum_{frontier} \underbrace{\mathbf{P}(b \mid P_{1,3}, known, frontier)}_{= 1 \text{ or } 0} P(frontier)$$

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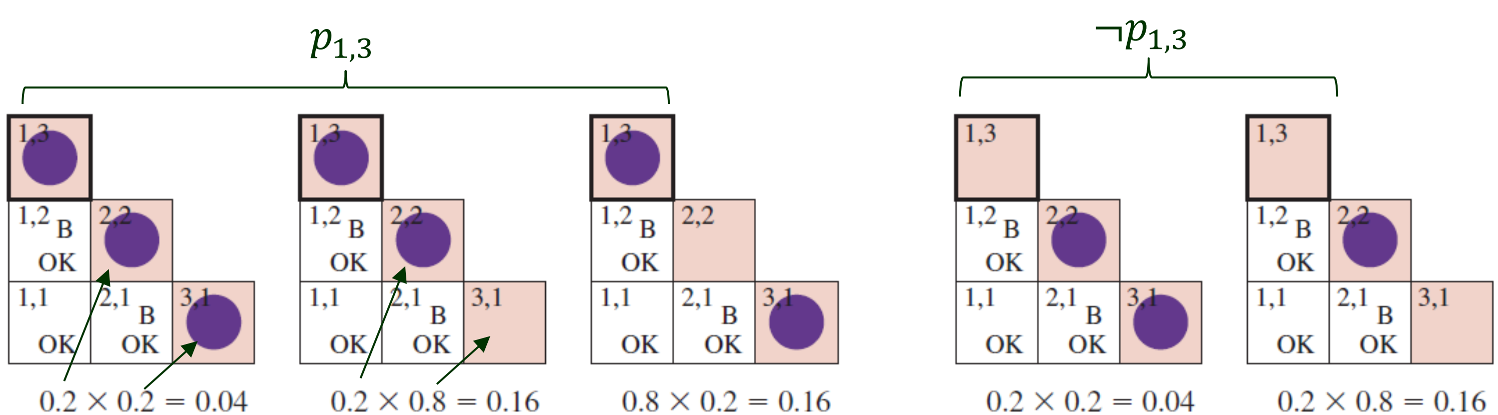

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$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha' \langle 0.2(0.04 + 0.16 + 0.16), 0.8(0.04 + 0.16) \rangle$$

Probability of Containing a Pit

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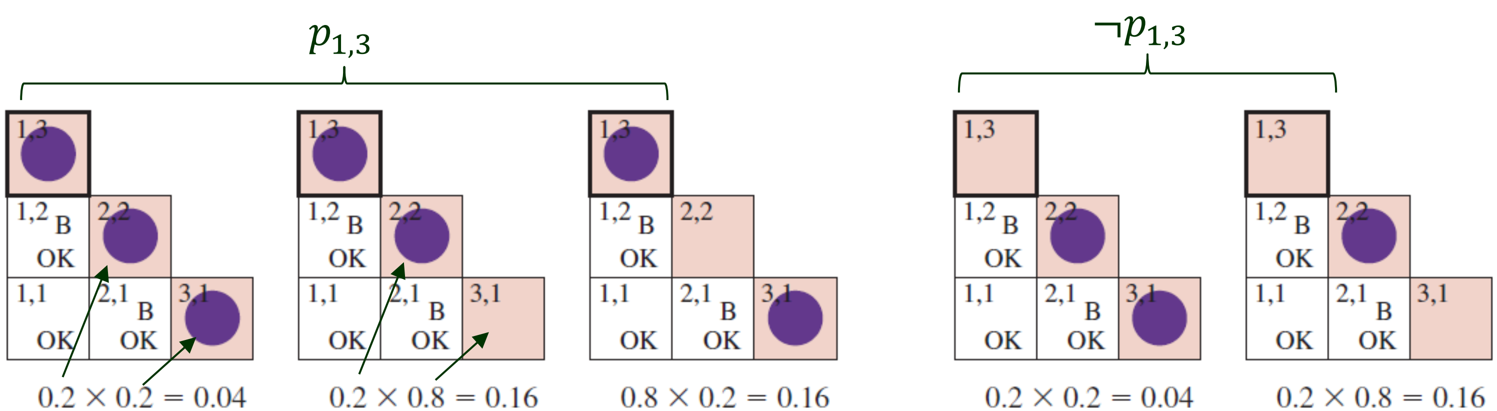

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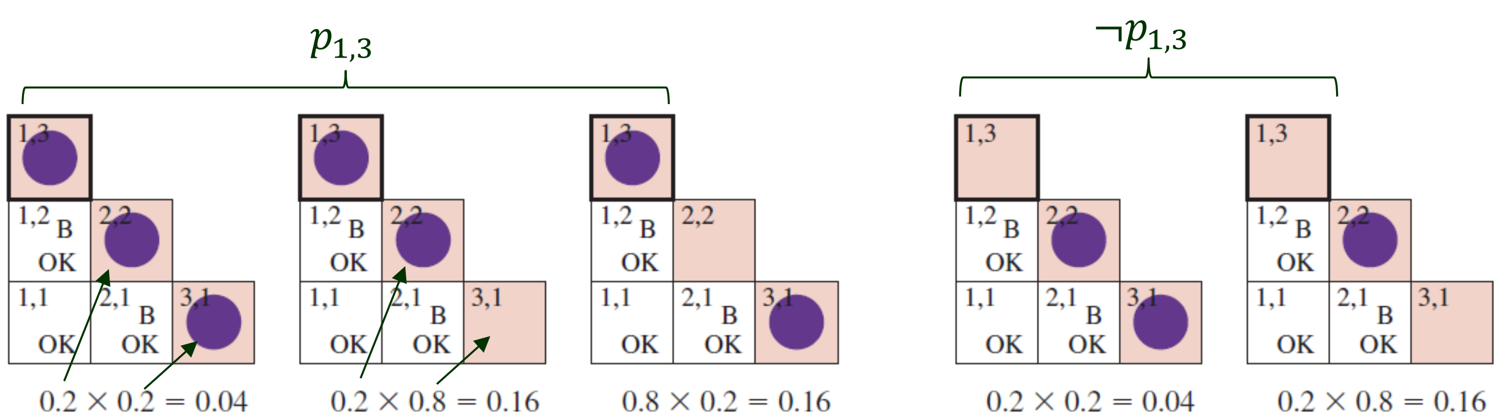

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$P(p_{1,3})$

Probability of Containing a Pit

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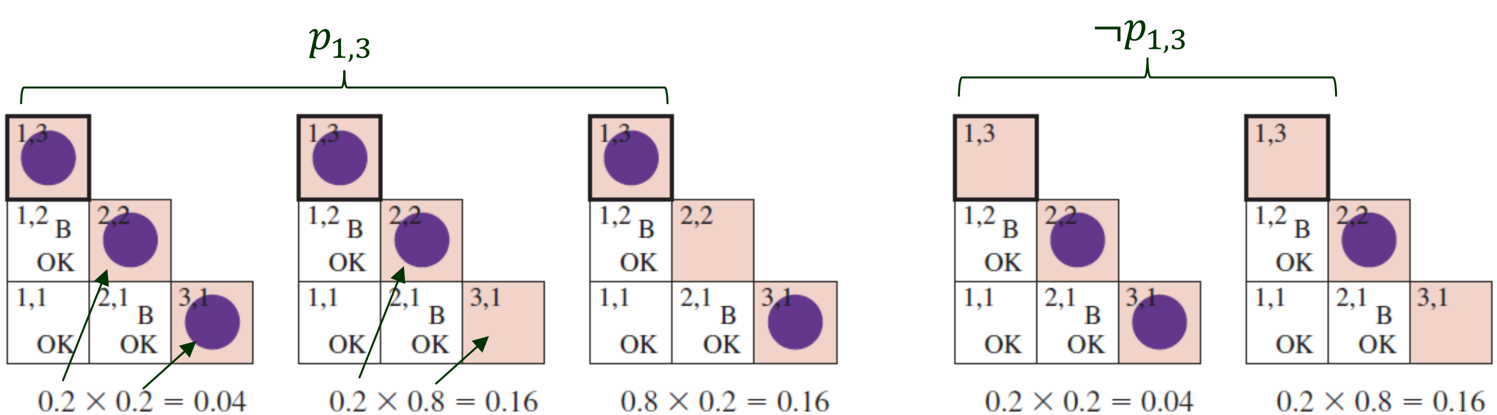

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$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha' \langle \underset{P(p_{1,3})}{0.2} \underbrace{(0.04 + 0.16 + 0.16)}_{P(b \mid known, p_{1,3})}, 0.8(0.04 + 0.16) \rangle$$

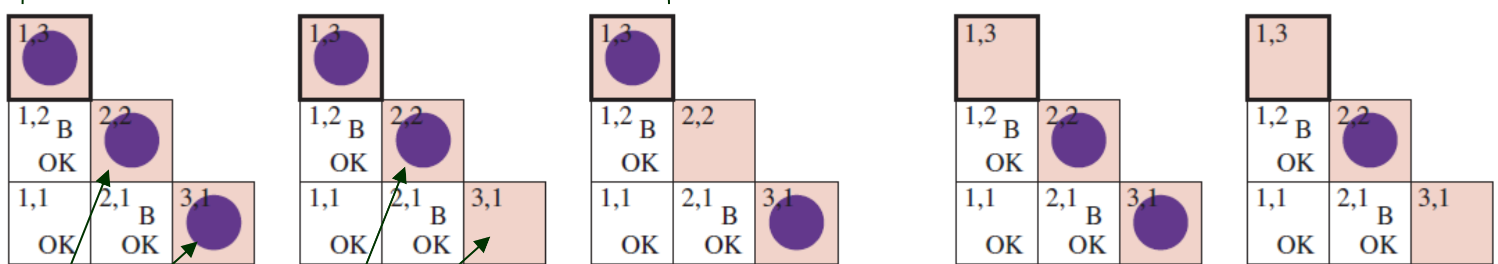
Probability of Containing a Pit

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha' \mathbf{P}(P_{1,3}) \sum_{frontier} \underbrace{\mathbf{P}(b \mid P_{1,3}, known, frontier)}_{=\ 1\text{ or }0} P(frontier)$$

- In the distribution $\mathbf{P}(b \mid P_{1,3}, known, frontier)$: a probability is 1 if b is consistent with the values of $P_{1,3}$ and the variables in *frontier*, and 0 otherwise.
- For each value of $P_{1,3}$, we sum over the logical models for the frontier variables that are consistent with *known*.

Five consistent models for *Frontier* = $\{P_{2,2}, P_{3,1}\}$:

$p_{1,3}$ $\qquad\qquad$ $\neg p_{1,3}$



$0.2 \times 0.2 = 0.04$ $\quad$ $0.2 \times 0.8 = 0.16$ $\quad$ $0.8 \times 0.2 = 0.16$ $\quad$ $0.2 \times 0.2 = 0.04$ $\quad$ $0.2 \times 0.8 = 0.16$

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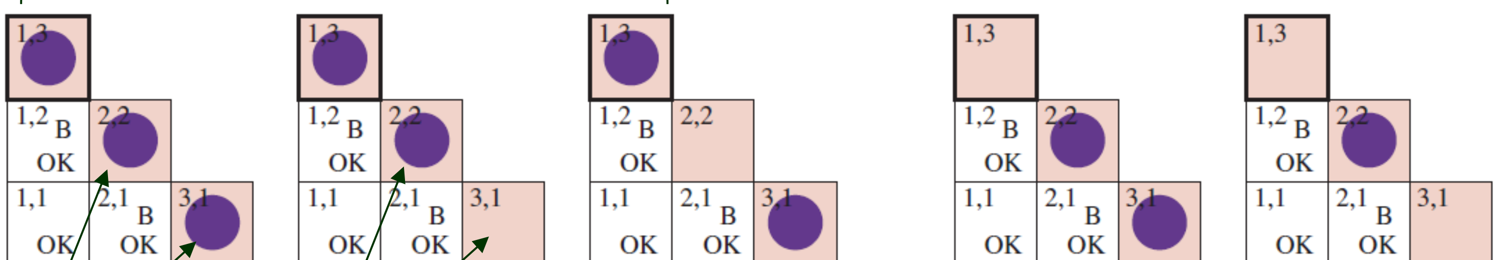
Five consistent models for *Frontier* $= \{P_{2,2}, P_{3,1}\}$:

$p_{1,3}$ (first three models) $\qquad \neg p_{1,3}$ (last two models)



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$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha' \langle \underbrace{0.2}_{P(p_{1,3})} \underbrace{(0.04 + 0.16 + 0.16)}_{P(b \mid known,\, p_{1,3})}, \underbrace{0.8}_{P(\neg p_{1,3})} \underbrace{(0.04 + 0.16)}_{P(b \mid known, \neg p_{1,3})} \rangle$$

Probability of Containing a Pit

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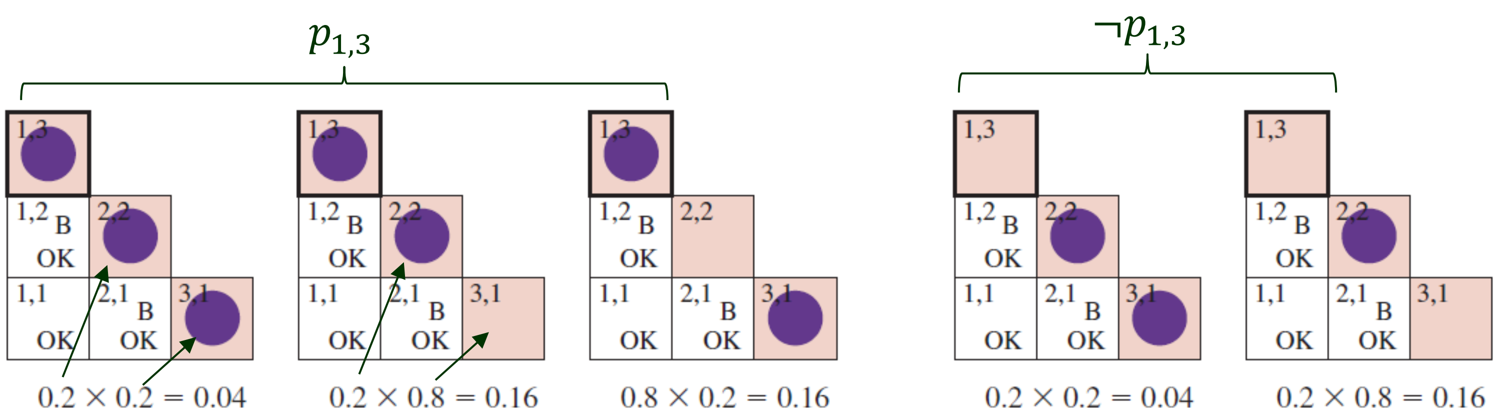

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$$\approx \langle 0.31, 0.69 \rangle$$

Probability of Containing a Pit

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- In the distribution $\mathbf{P}(b \mid P_{1,3}, known, frontier)$: a probability is 1 if b is consistent with the values of $P_{1,3}$ and the variables in *frontier*, and 0 otherwise.
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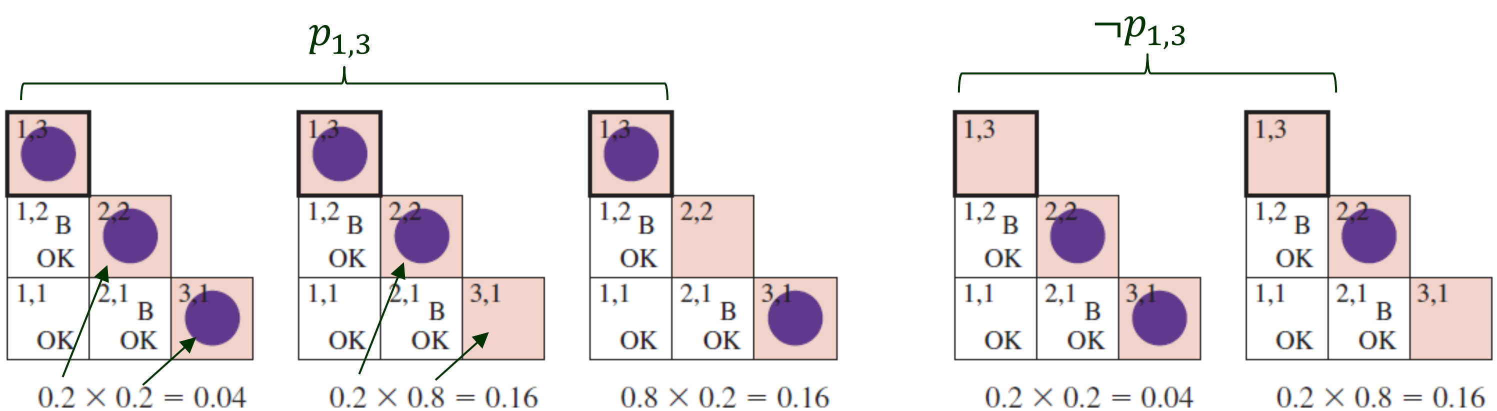
Five consistent models for *Frontier* $= \{P_{2,2}, P_{3,1}\}$:



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$$\approx \langle 0.31, 0.69 \rangle$$

[1,3] contains a pit with 31% probability.