Quadtrees

Outline:

I. Meshing

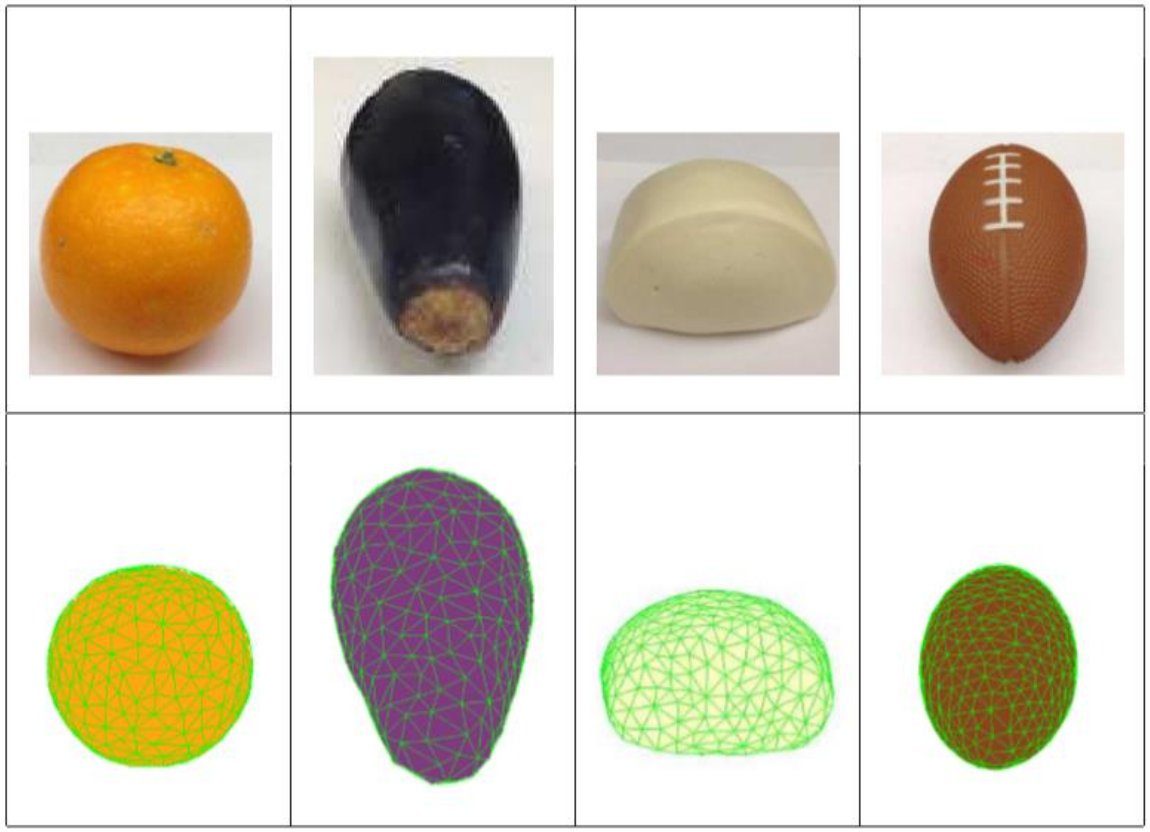
II. Definition of a quadtree

III. Tree height and size

IV. Search for a neighbor

V. Balancing a quadtree

I. Meshing



Modeling with the finite element methods (FEMs)

- mechanics-based
- a finer mesh results in
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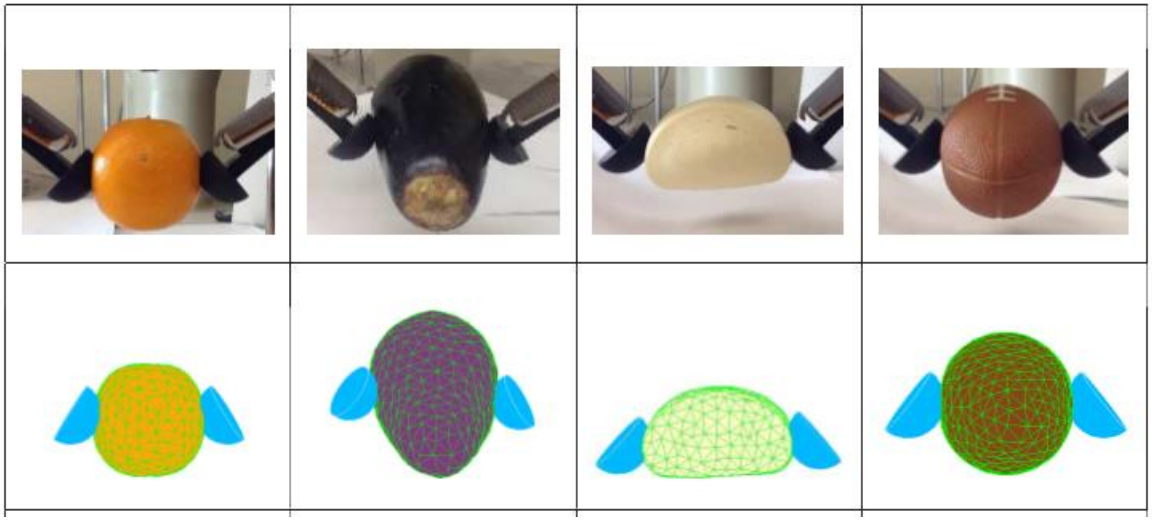

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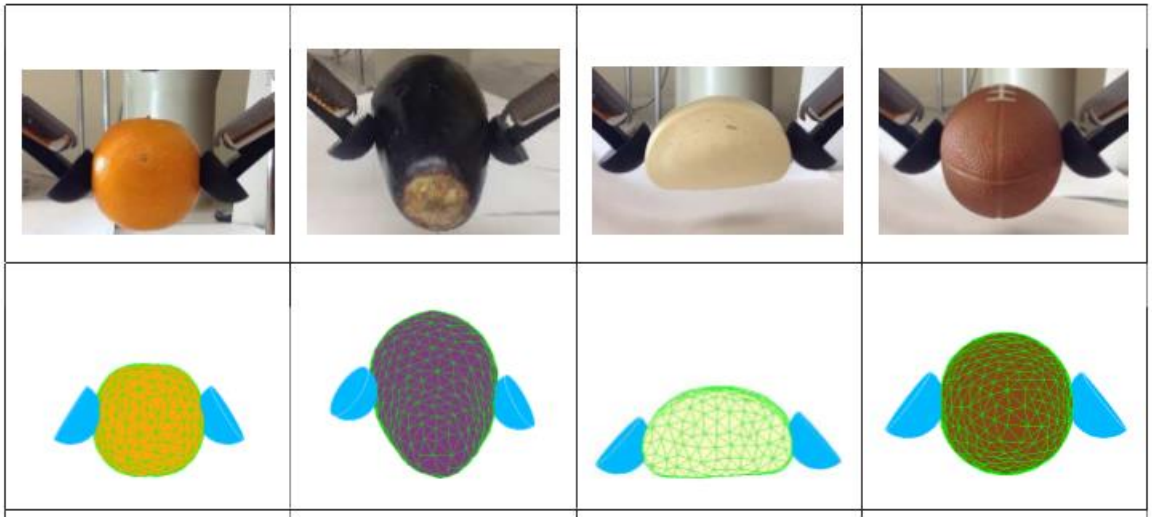

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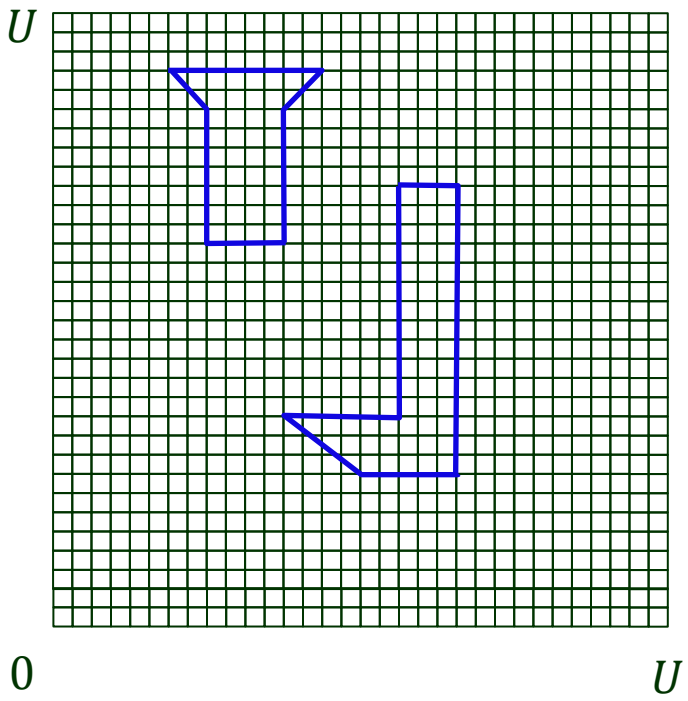
Applications:

- ♦ structural analysis & product design
- ♦ heat transfer
- ♦ fluid flow
- ♦ electromagnetism
- ♦ graphics & gaming, etc.
- ♦ robotics

Mesh Generation



Domain: $U \times U$ grid

where $U = 2^j$ for some integer j

Assumptions:

- Vertices of components have integer coordinates between 0 and U.
- Edges of components have four different orientations: $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$.

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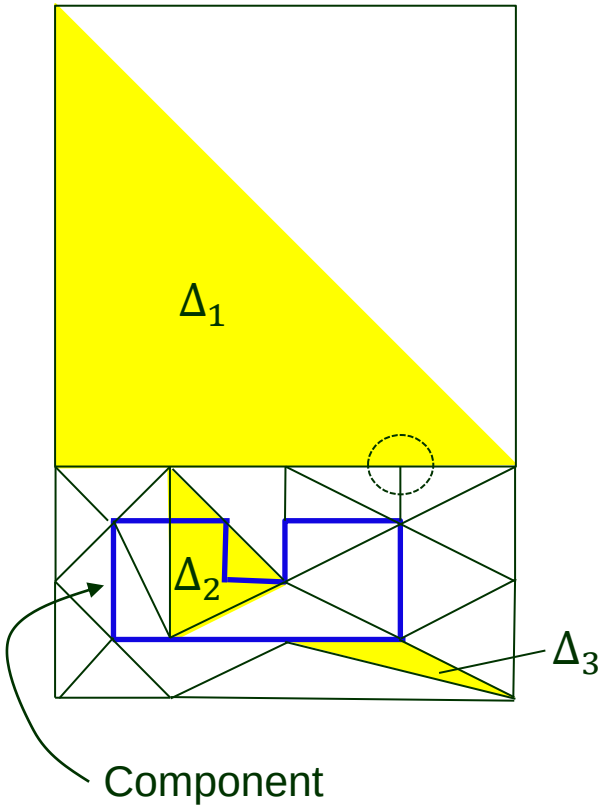

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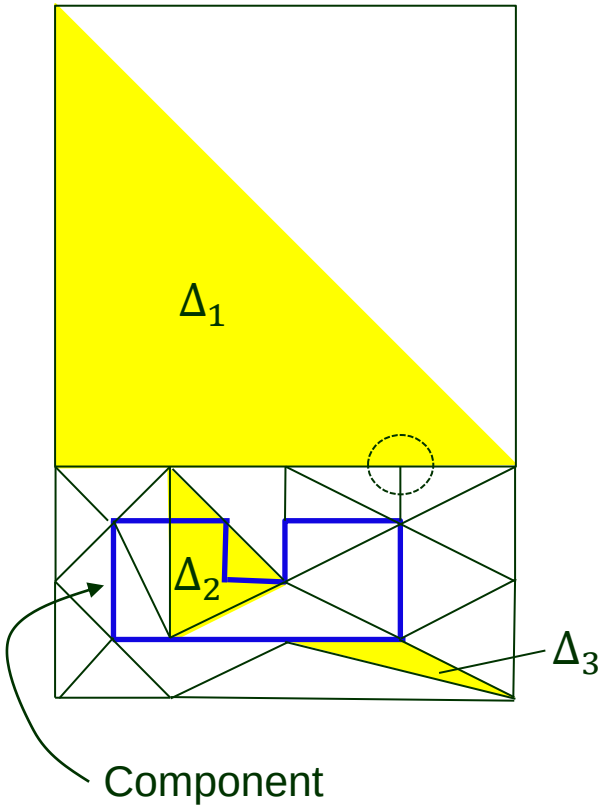

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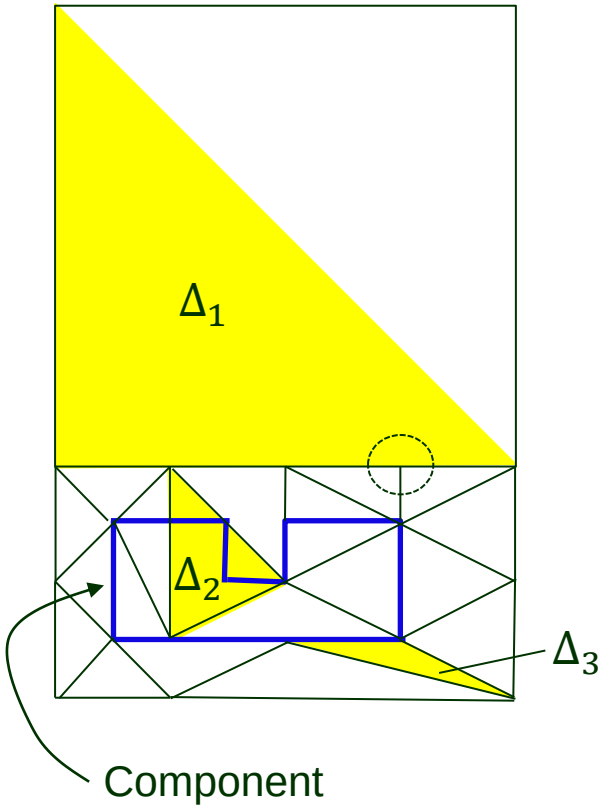

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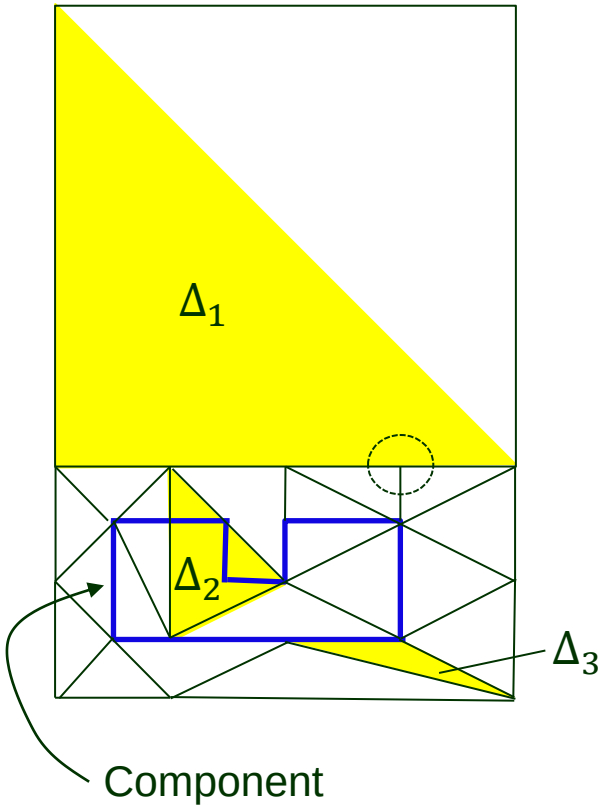

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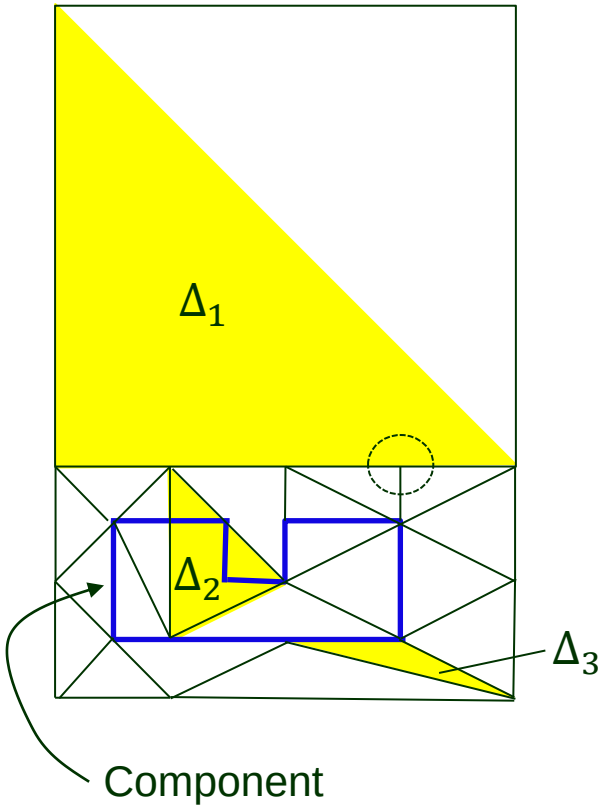

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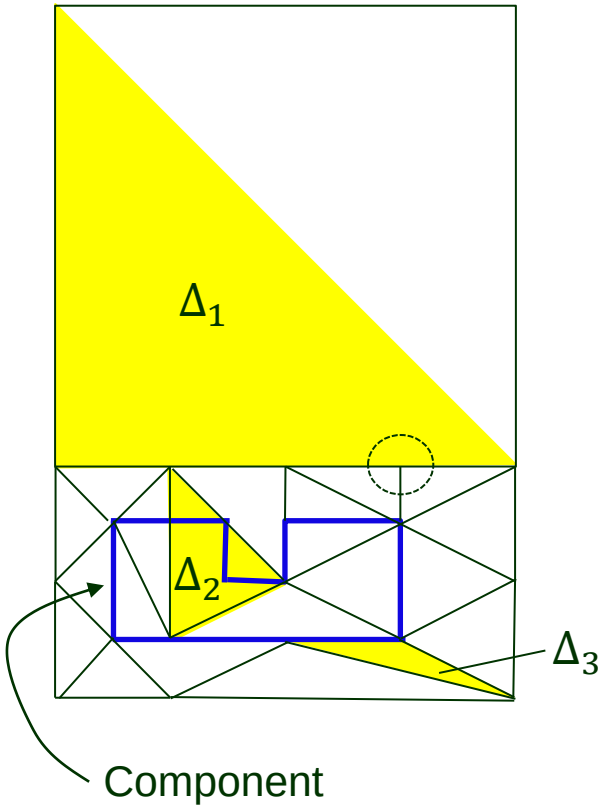

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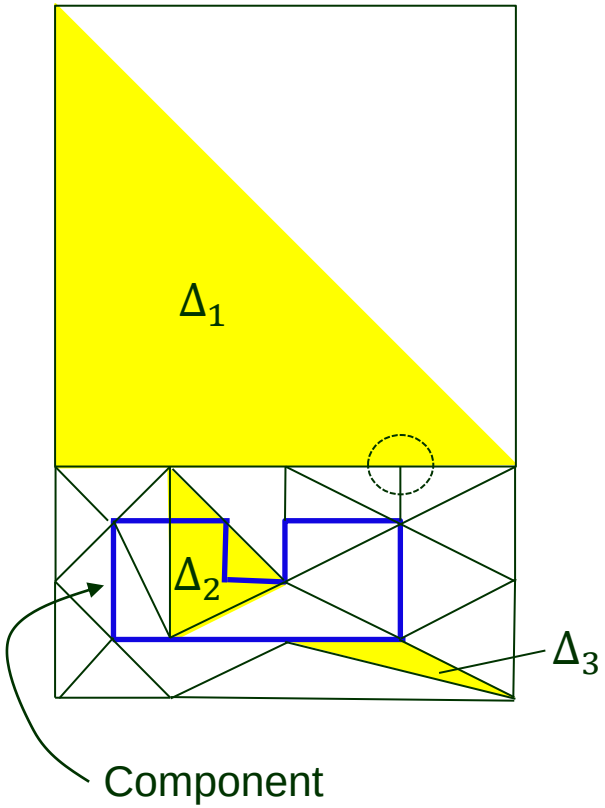

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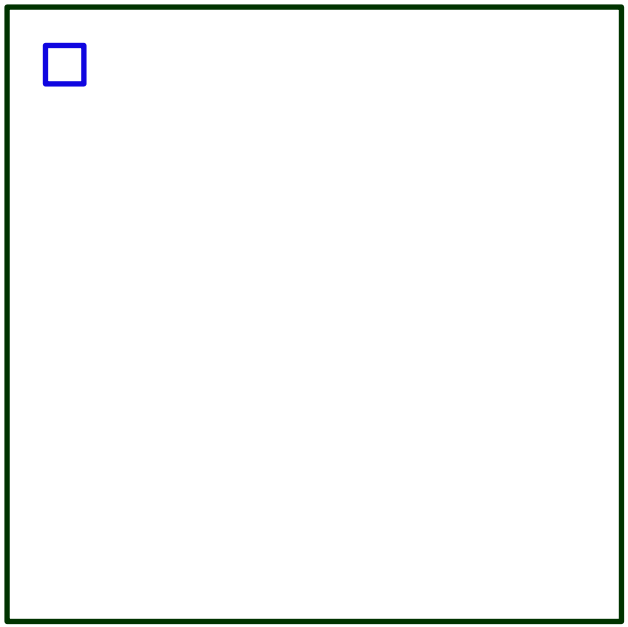
- *non-uniform* (mesh should be fine near the edges of components and coarse far away from them)

Steiner Triangulation

1×1 component located at $(1,1)$ in a 16×16 grid.



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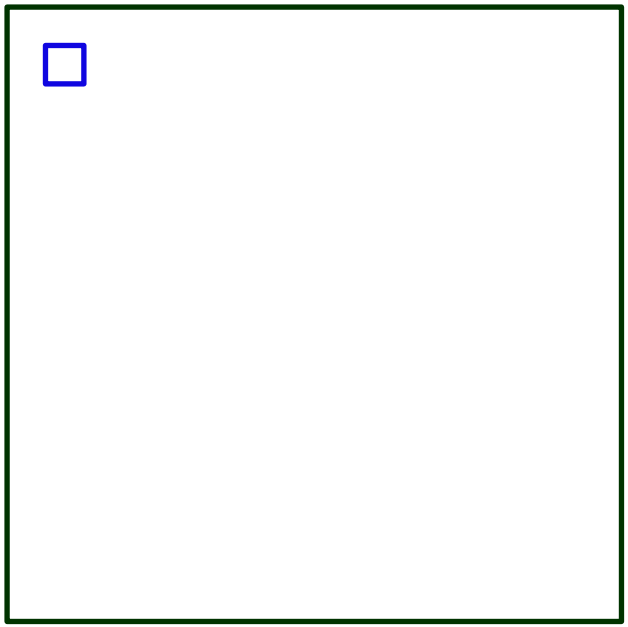

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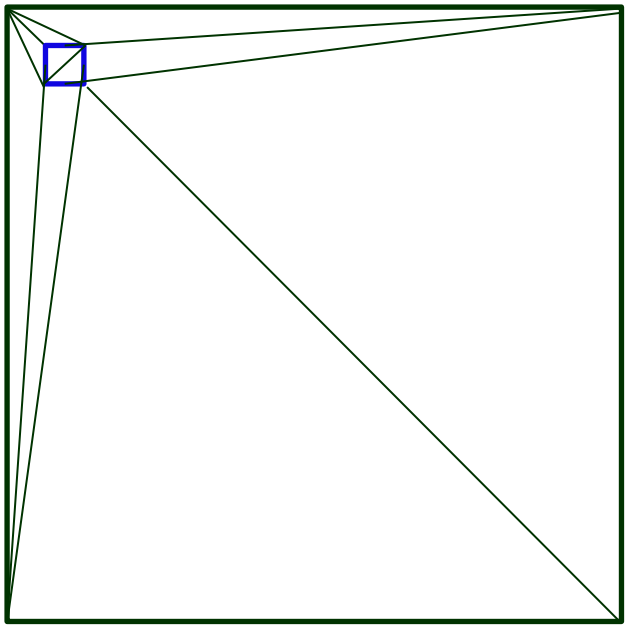

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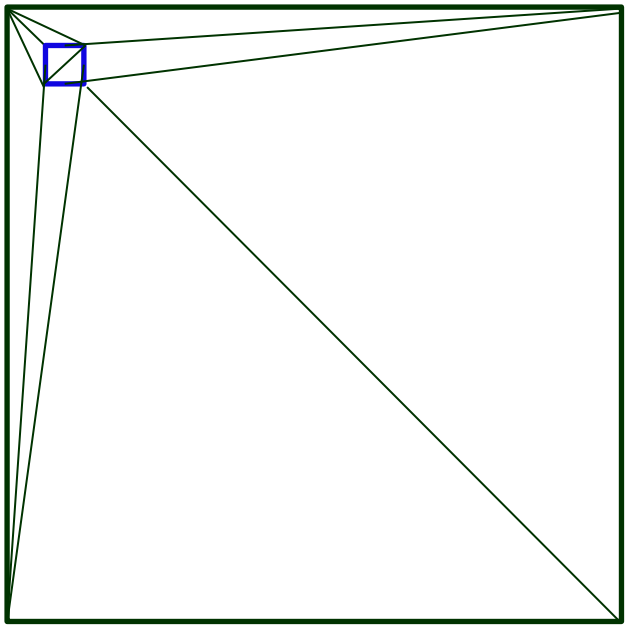

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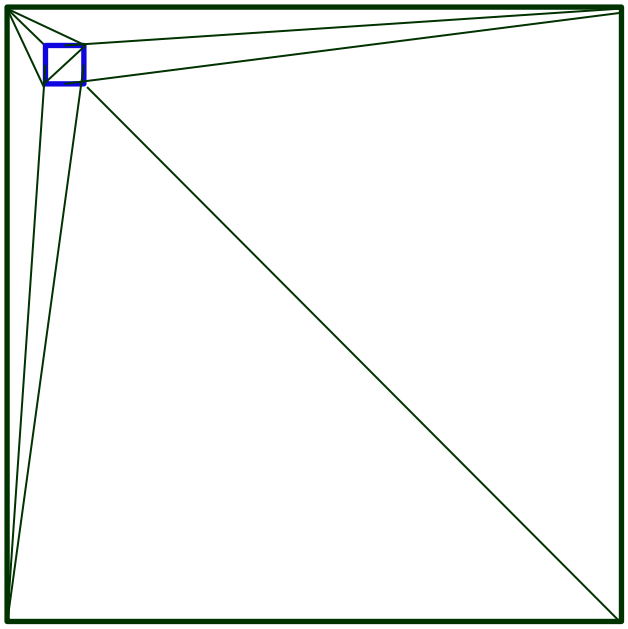

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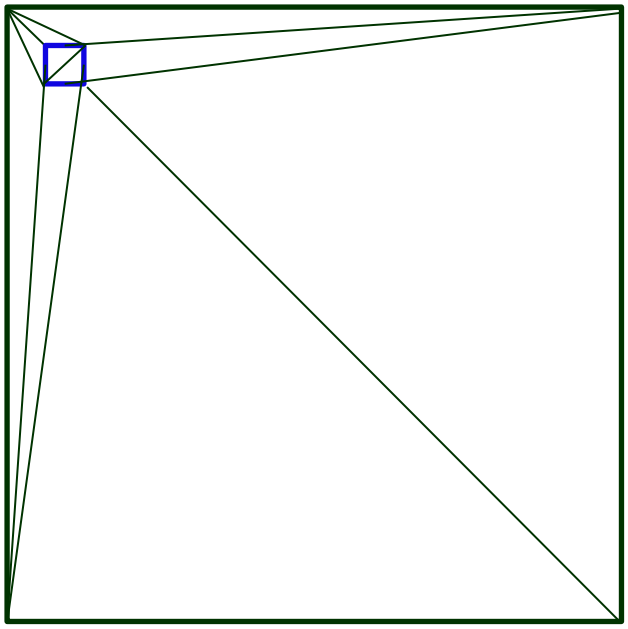

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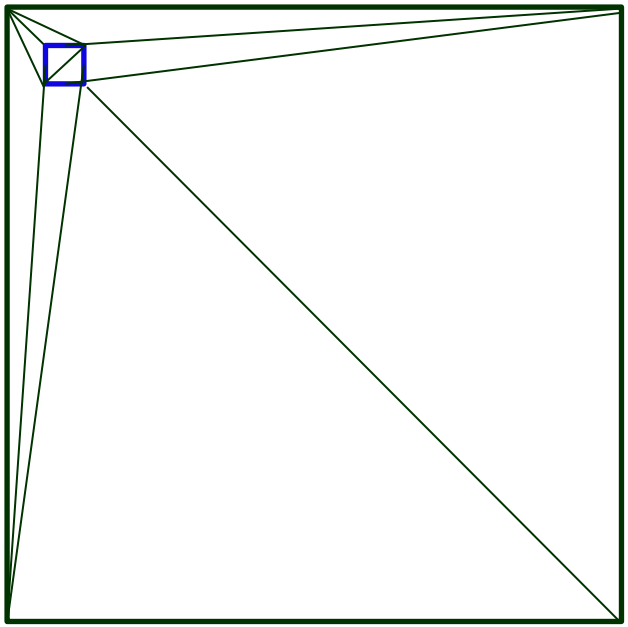

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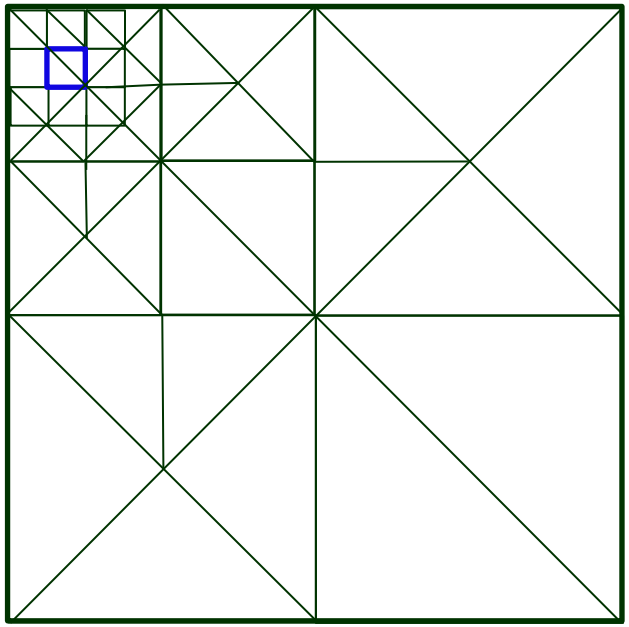
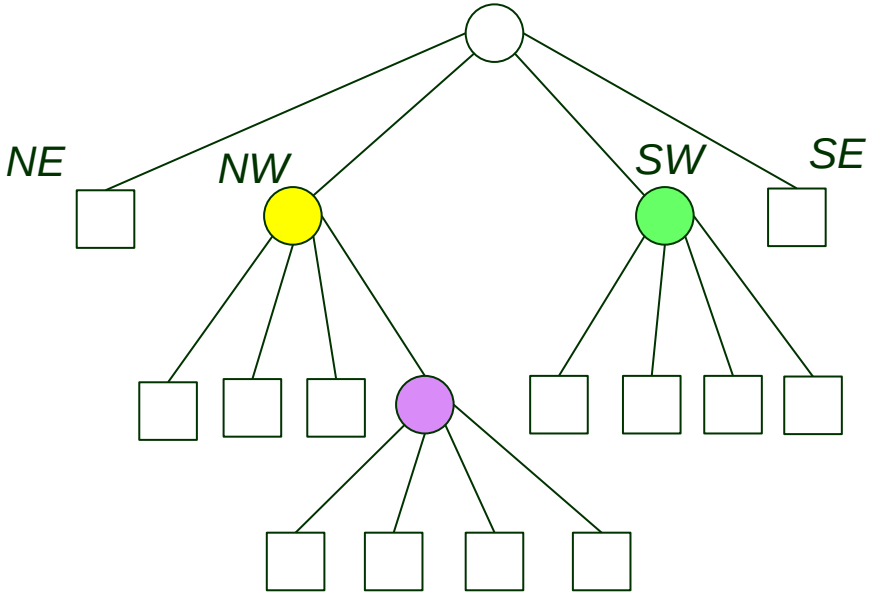


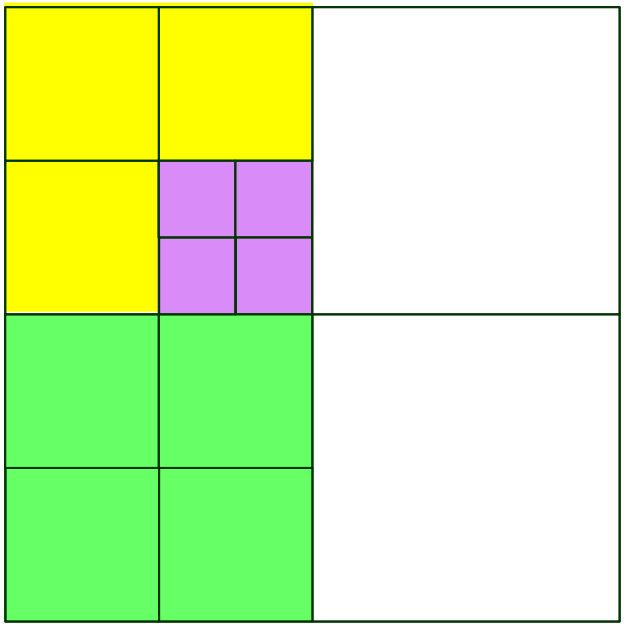
52 triangles (down from 512 uniform triangles)

II. Quadtree

A rooted tree in which

- every node corresponds to a square;
- every internal node v has four children which represent the four quadrants of the node's corresponding square.





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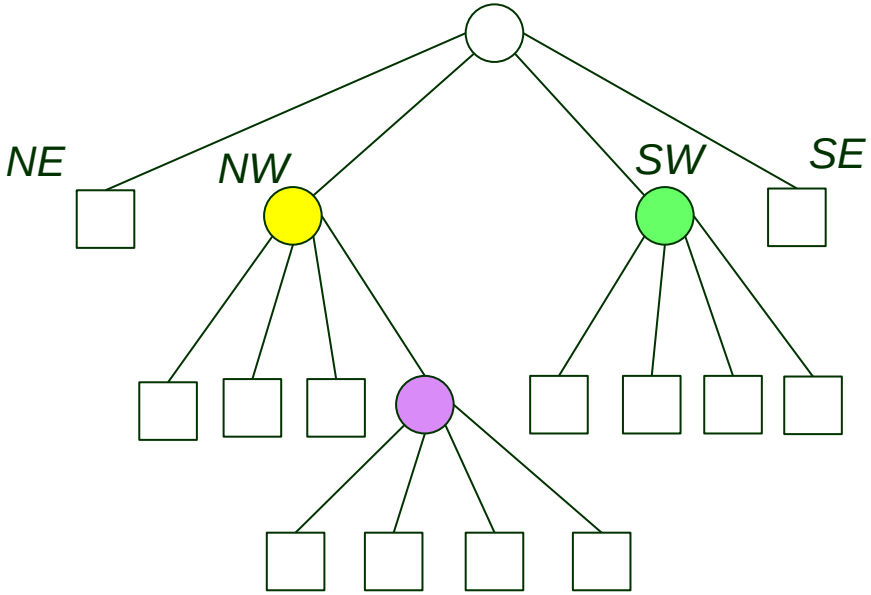
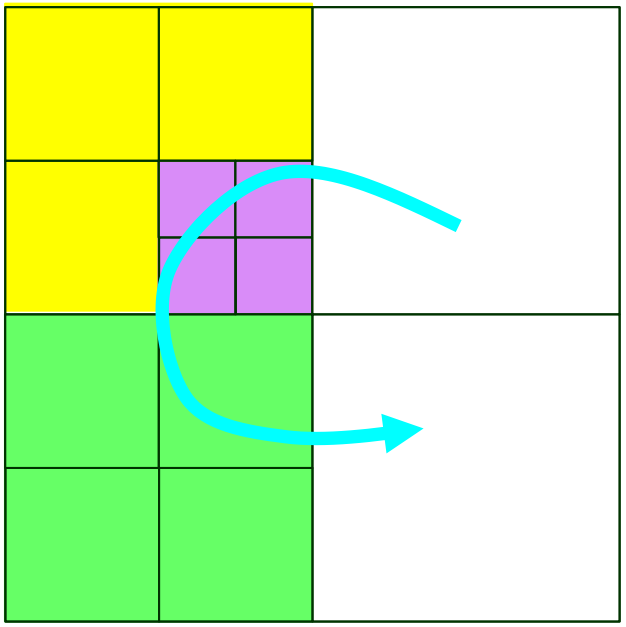
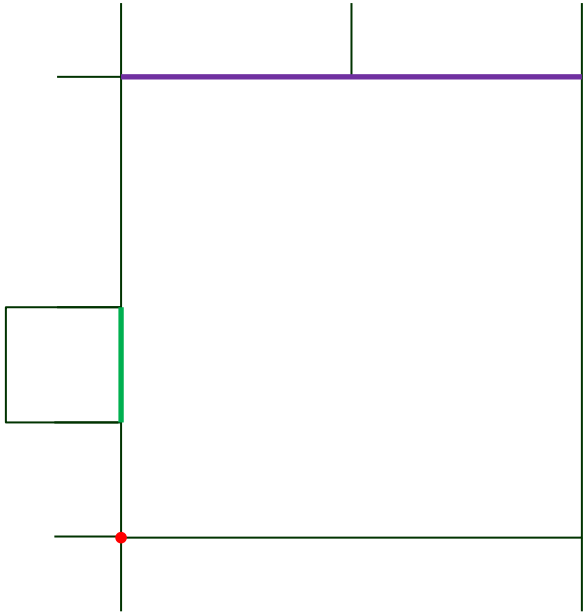




Quadtree subdivision

More Terminology



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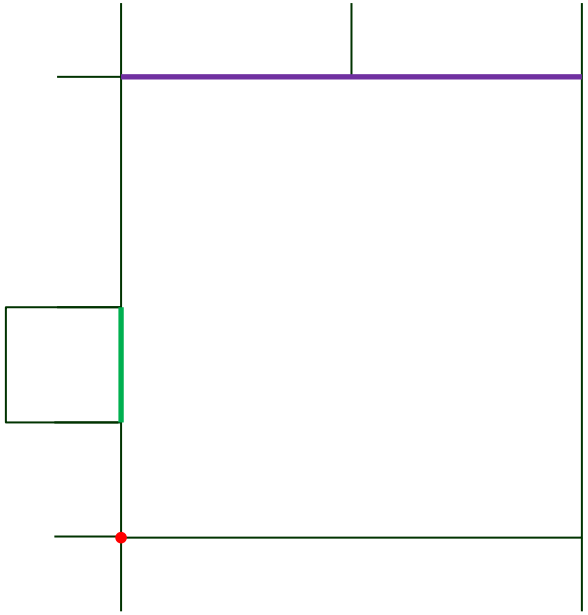

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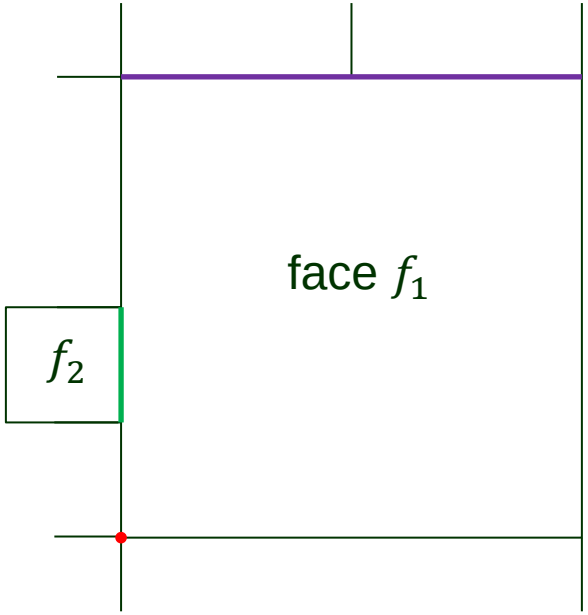

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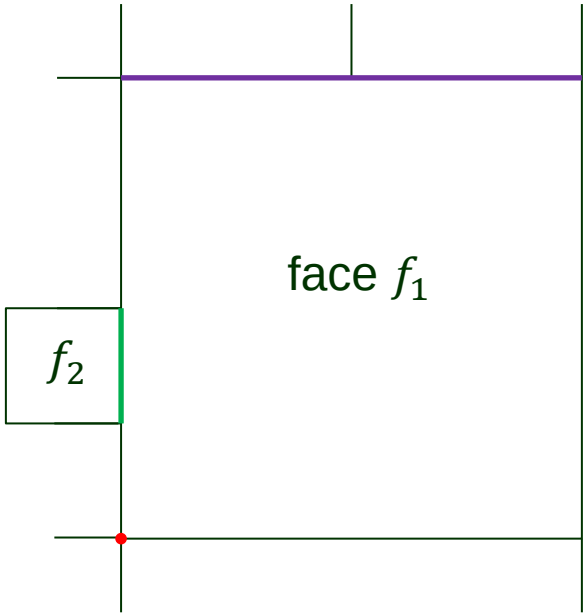

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Corner: a vertex at the corner of the square.

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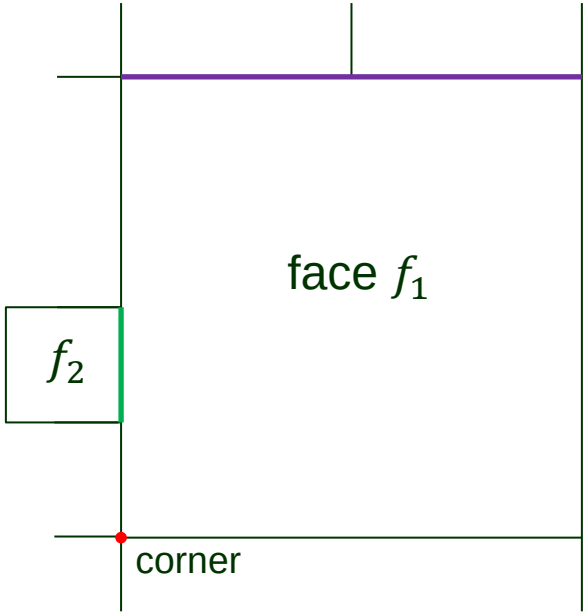

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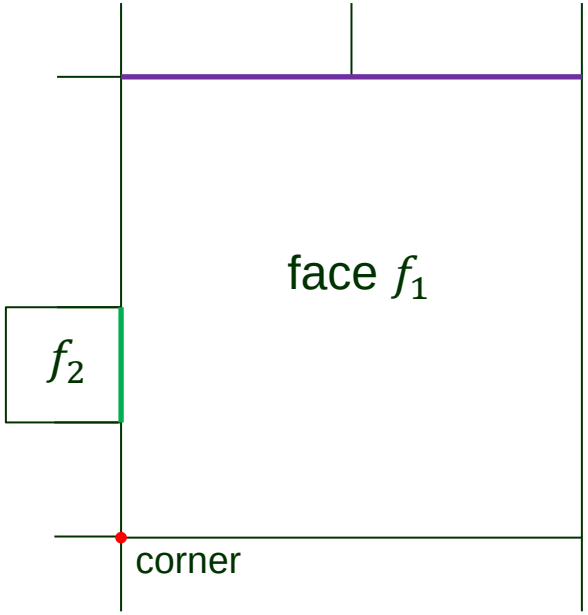

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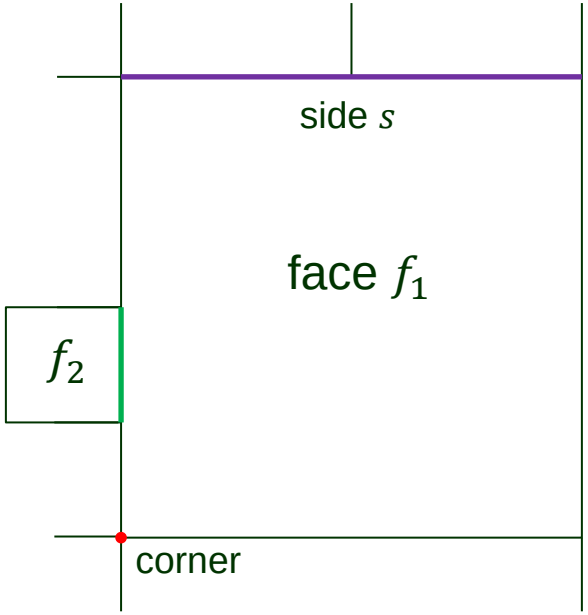

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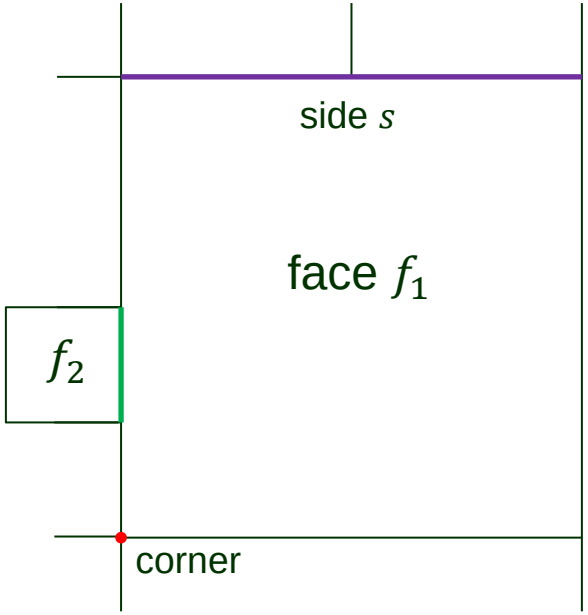

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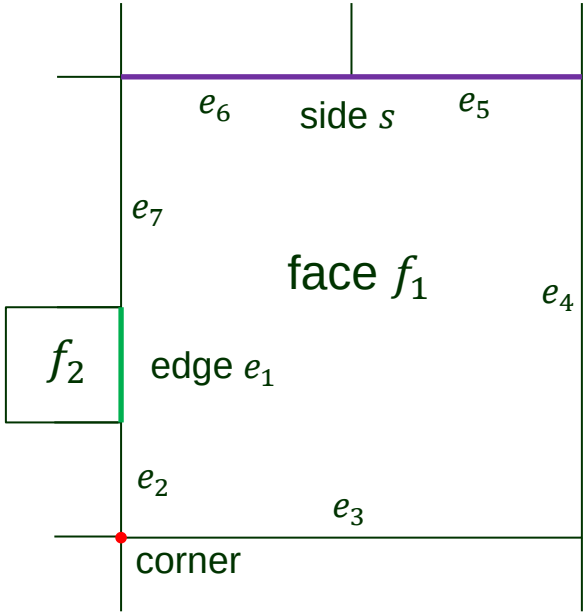

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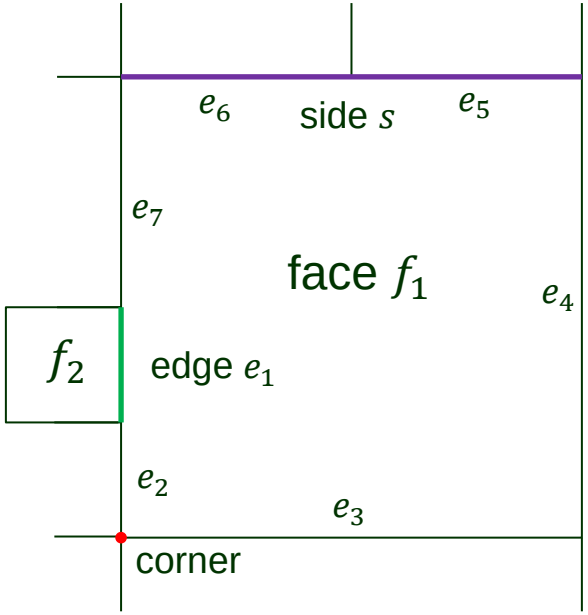

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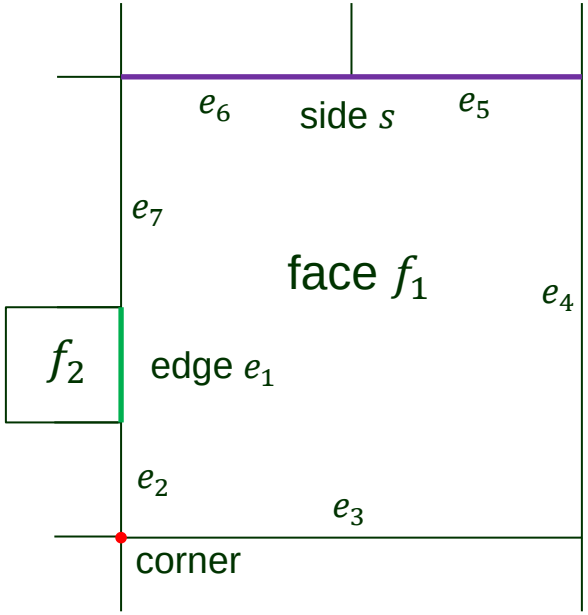

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Two squares are *neighbors* if they share an edge.

Point Storage

P: a set of points.

$\sigma = [x_\sigma, x'_\sigma] \times [y_\sigma, y'_\sigma]$: a square to store P.

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$$x_{mid} = \frac{x_\sigma + x'_\sigma}{2} \qquad y_{mid} = \frac{y_\sigma + y'_\sigma}{2}$$

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$$P_{NE} = \{p \in P \mid p_x > x_{mid} \text{ and } p_y > y_{mid}\}$$
$$P_{Nw} = \{p \in P \mid p_x \leq x_{mid} \text{ and } p_y > y_{mid}\}$$
$$P_{SW} = \{p \in P \mid p_x \leq x_{mid} \text{ and } p_y \leq y_{mid}\}$$
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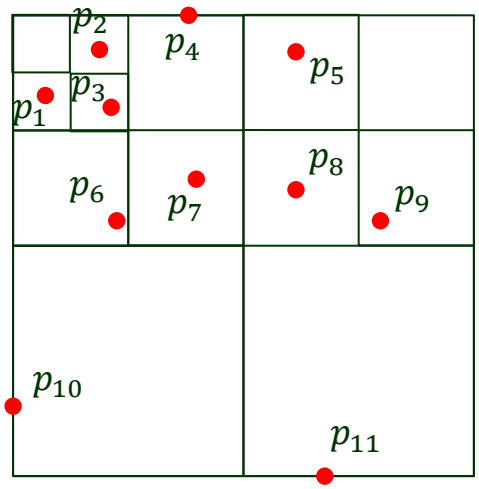
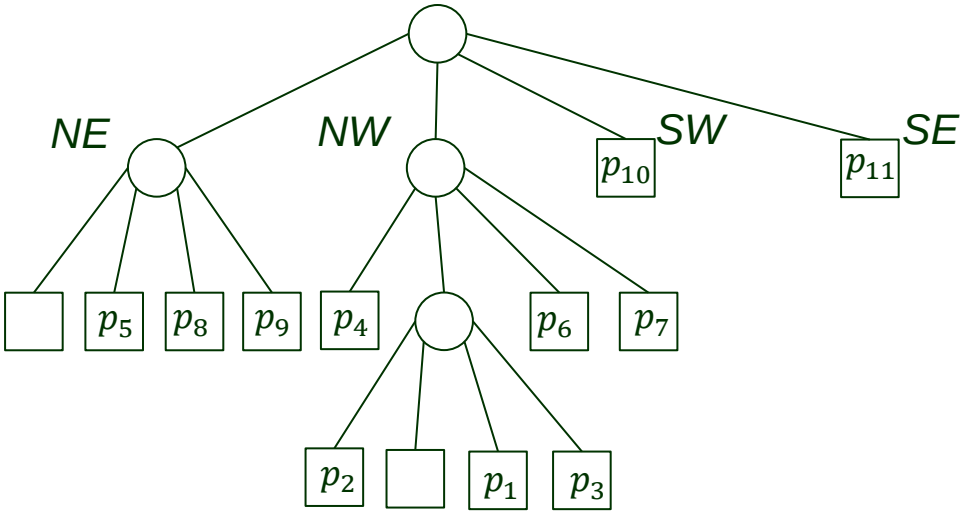
Store P_{NE}, P_{NW}, P_{SW}, P_{SE} in the four quadrants σ_{NE}, σ_{NW}, σ_{SW}, σ_{SE} of σ, respectively.

Recursive Construction



- Compute the smallest enclosing square for the point set P.



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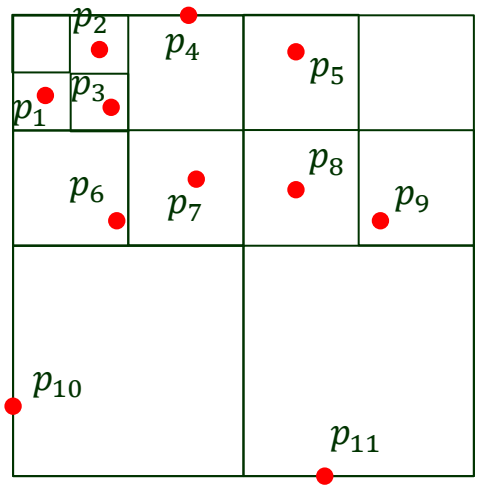

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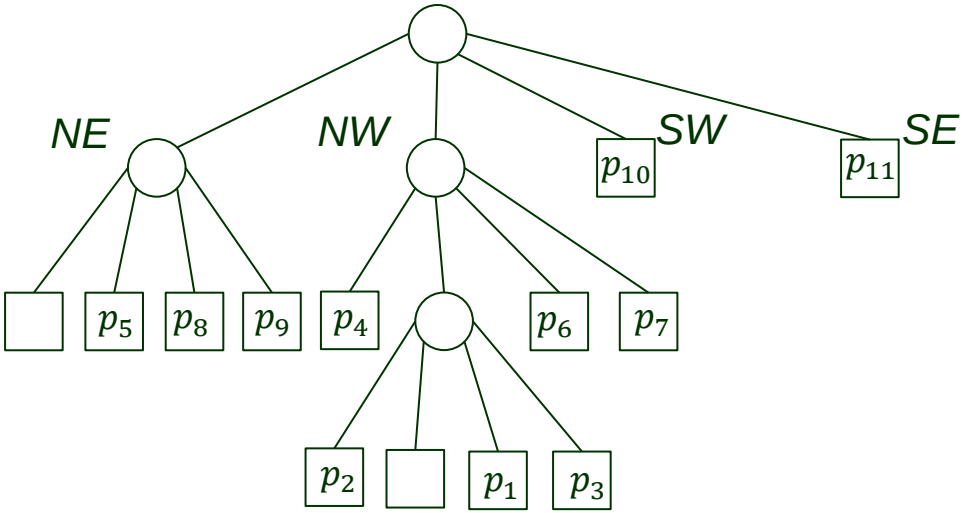
- Split the square into four quadrants: NE, NW, SW, and SE.



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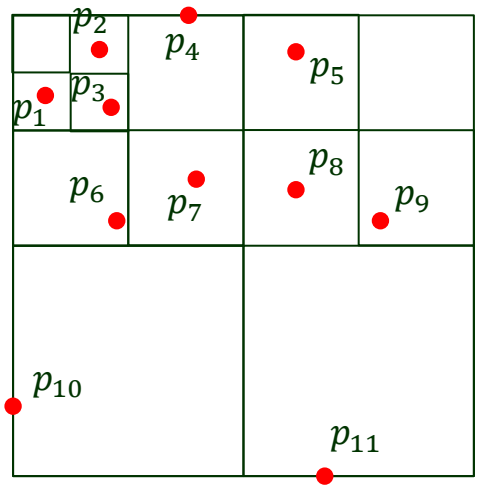

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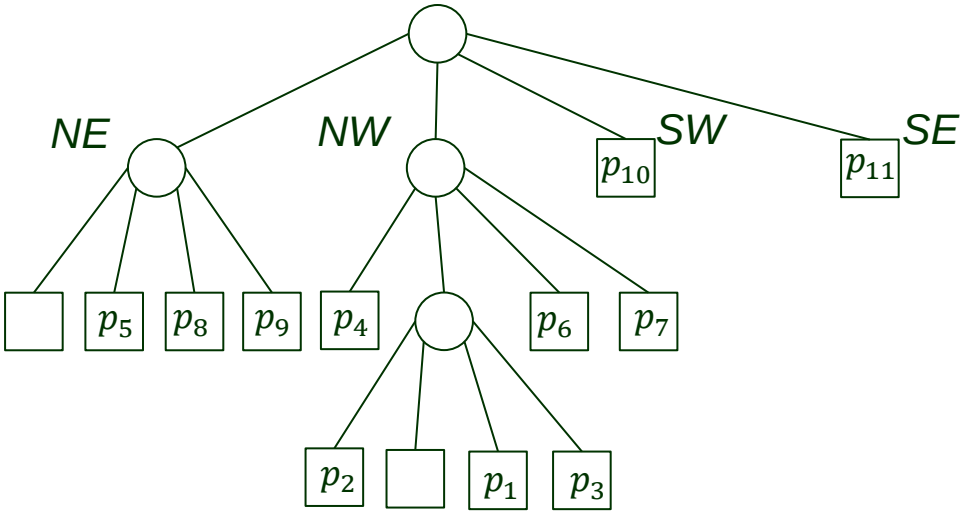

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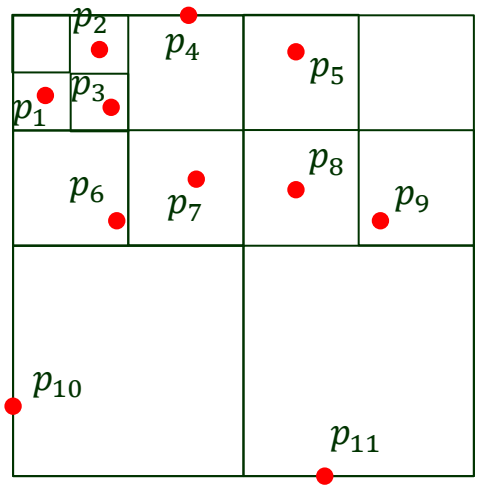

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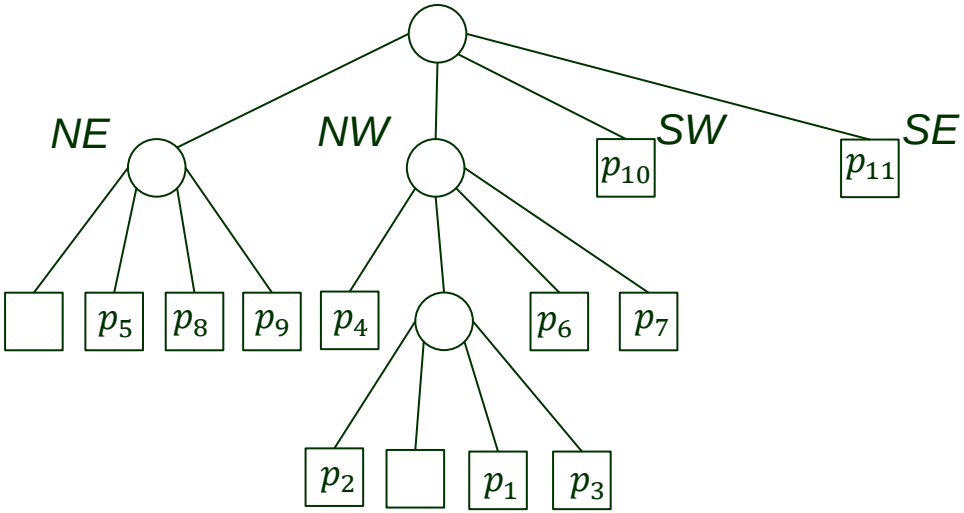

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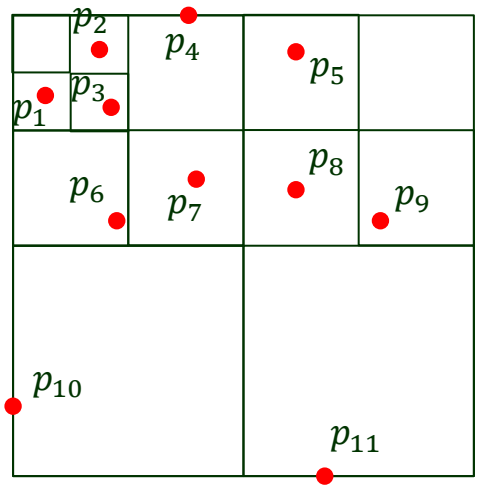

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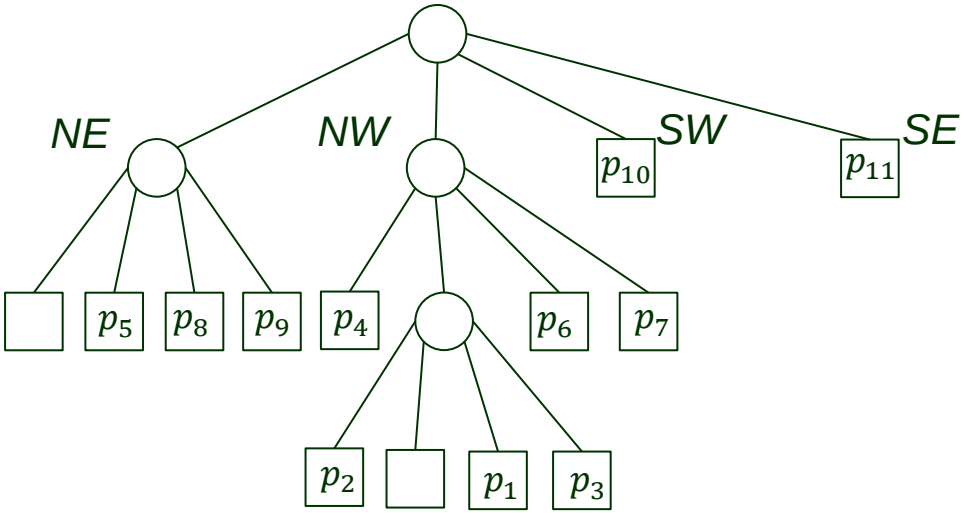

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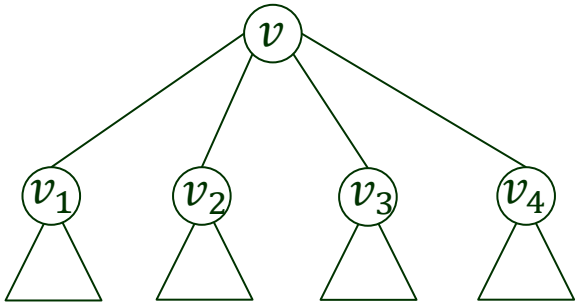

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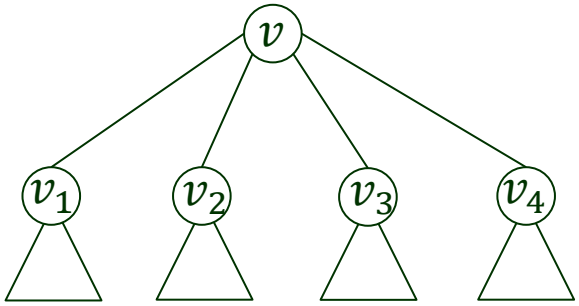

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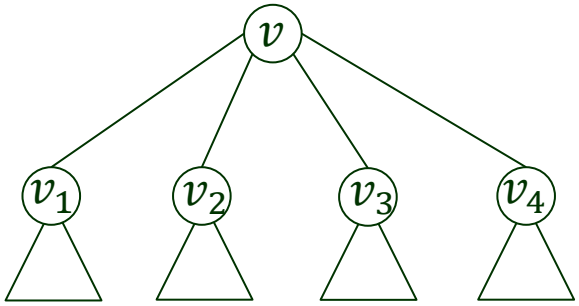

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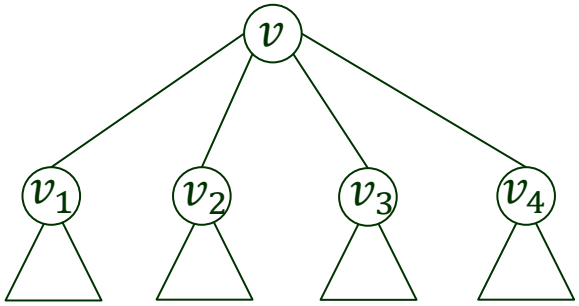

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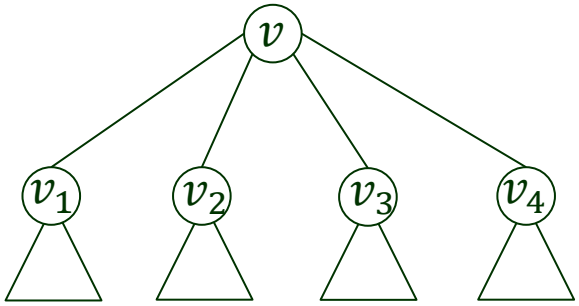

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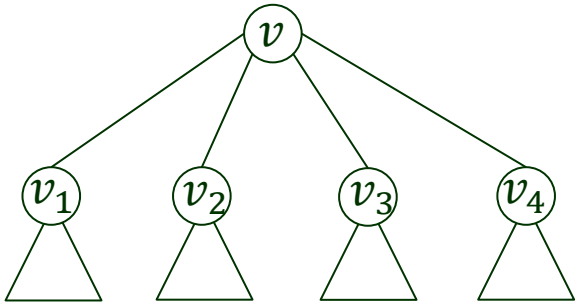

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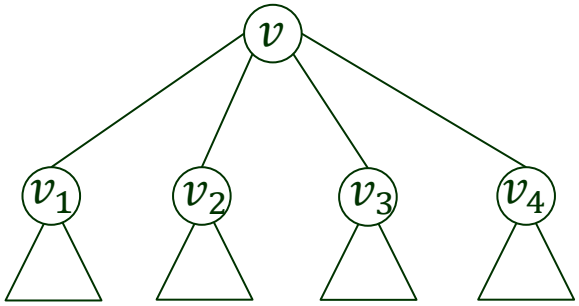

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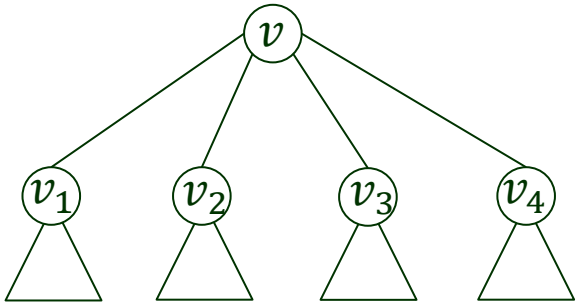

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Thus, it suffices to bound $\mu(v)$ for the size of the quadtree.

Proof (cont'd)

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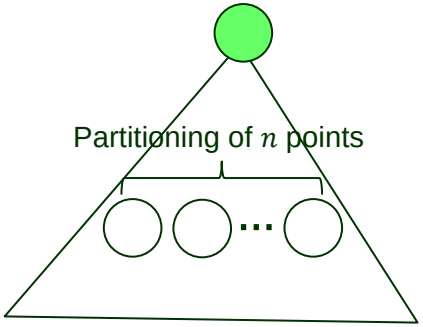

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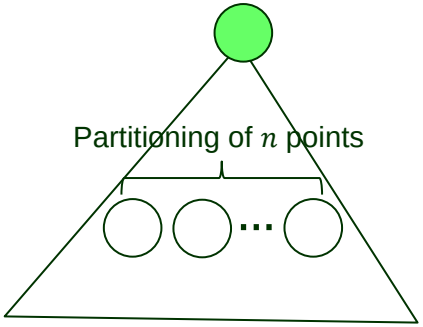

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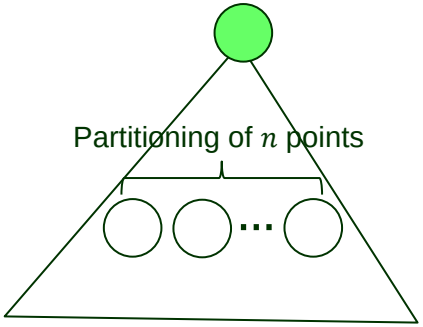

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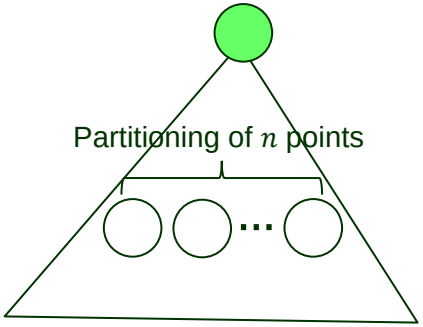

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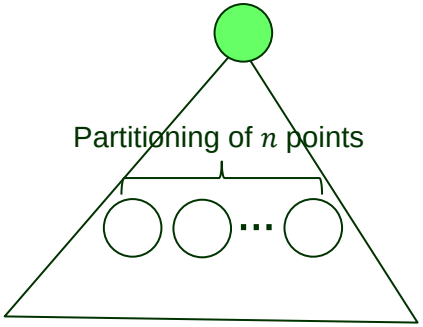

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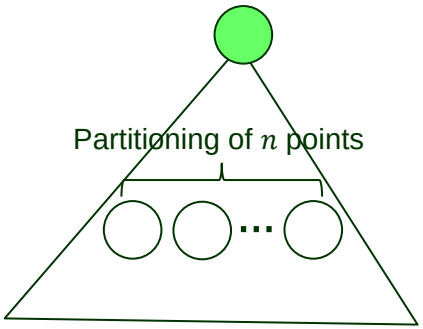

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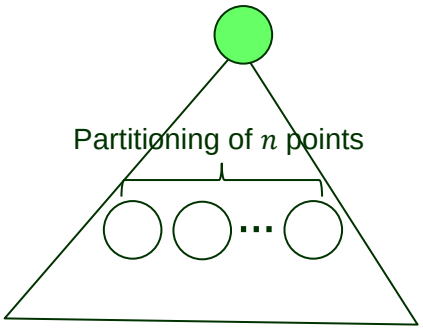

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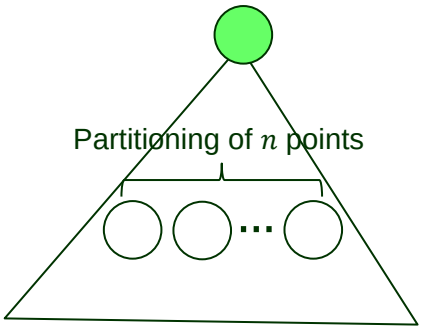

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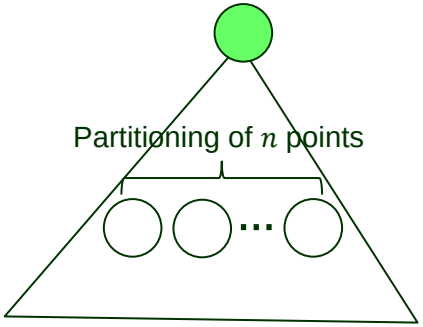

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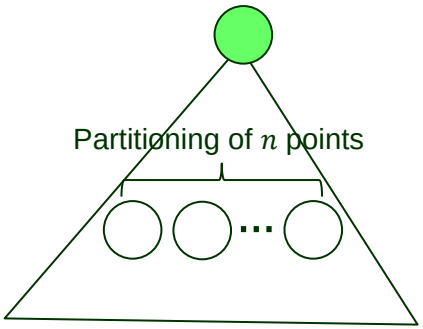

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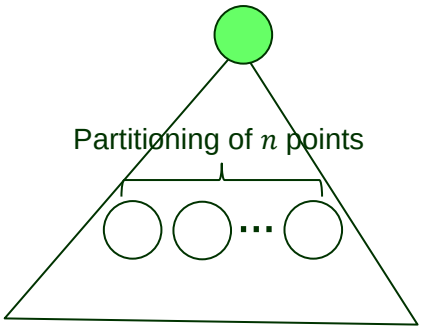
Total construction work $O((h+1)n)$.

Proof (cont'd)

- The associated square of any internal node contains ≥ 2 points (otherwise, it would be a leaf not an internal node).
- The squares of all internal nodes at the same depth are disjoint.
- There are n points in total distributed among these squares.



There are $\leq n/2$ internal nodes at the depth.

⇩

There are $O\big((h+1)n\big)$ internal nodes in the tree.
(We include 1 in the Big-O in case $h = 0$.)

⇩

The quadtree has $O\big((h+1)n\big)$ nodes.

The amount of time spent at an internal node is linear in the number of points in its associated square.

n points are distributed among the squares at the same depth.

$O(n)$ work at each depth.

⇩

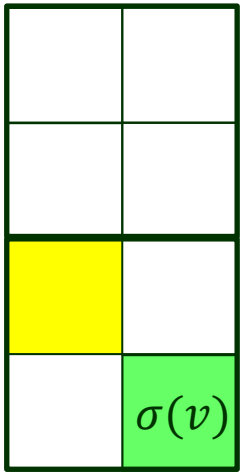
Total construction work $O((h+1)n)$.

□

IV. Neighbor Finding

Problem Given a node v and a direction (north, east, south, or west), find the deepest node v' with depth $\leq$ the depth of v such that its associate square $\sigma(v')$ is a neighbor of the associate square $\sigma(v)$ of v in the given direction.

Returns nil if no adjacent square in the direction (e.g., $\sigma(v)$'s edge is part of the edge of the bounding square in that direction).



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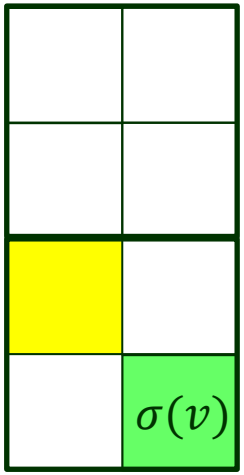
Suppose we want to find the north neighbor of v.



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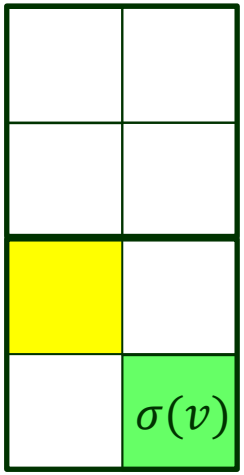
Suppose we want to find the north neighbor of v.

- v is the SE or SW-child of its parent.



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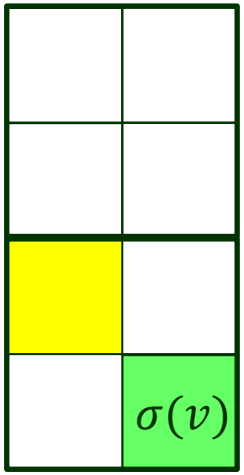

Suppose we want to find the north neighbor of v.

- v is the SE or SW-child of its parent.

⇩

The north neighbor is the NE or NW-child of the parent.

IV. Neighbor Finding

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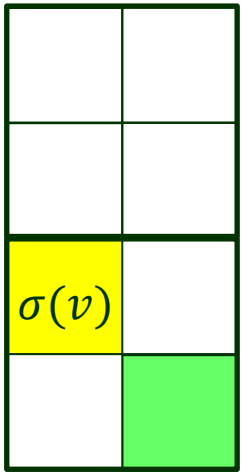

Suppose we want to find the north neighbor of v.

- v is the SE or SW-child of its parent.

⇩

The north neighbor is the NE or NW-child of the parent.

- v is the NE or NW-child of its parent.

Recursively finds the north neighbor, μ, of the parent node.

IV. Neighbor Finding

Problem Given a node v and a direction (north, east, south, or west), find the deepest node v' with depth $\leq$ the depth of v such that its associate square $\sigma(v')$ is a neighbor of the associate square $\sigma(v)$ of v in the given direction.

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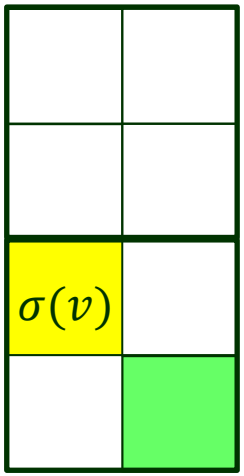

Suppose we want to find the north neighbor of v.

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⇩

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- ♣ If μ is an internal node, then the neighbor of v is its SE or SW-child.

IV. Neighbor Finding

Problem Given a node v and a direction (north, east, south, or west), find the deepest node v' with depth $\leq$ the depth of v such that its associate square $\sigma(v')$ is a neighbor of the associate square $\sigma(v)$ of v in the given direction.

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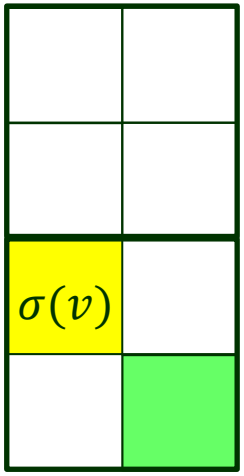

Suppose we want to find the north neighbor of v.

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⇩

The north neighbor is the NE or NW-child of the parent.

- v is the NE or NW-child of its parent.

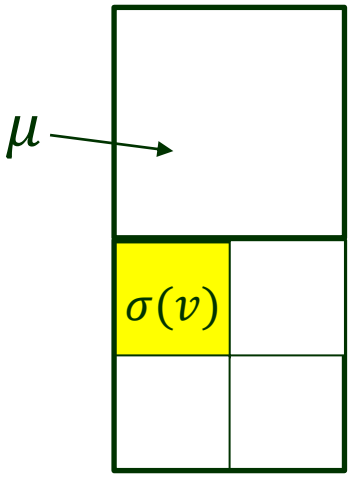
Recursively finds the north neighbor, μ, of the parent node.

- ♣ If μ is an internal node, then the neighbor of v is its SE or SW-child.
- ♣ If μ is a leaf, then it is the neighbor we are looking for.

Searching for the North Neighbor

NorthNeighbor(v,$\mathcal{T}$)

```
1.  if v = root(T)
2.       then return nil
3.   u ← parent(v)
4.   if v is the SW-child of u
5.       then return the NW-child of u
6.   if v is the SE-child of u
7.       then return the NE-child of u
8.   μ ← NorthNeighbor(u, T)
9.   if μ =nil or μ is a leaf
10.      then return μ
11.      else if v is the NW-child of u
12.           then return the SW-child of μ
13.           else return the SE-child of μ
```



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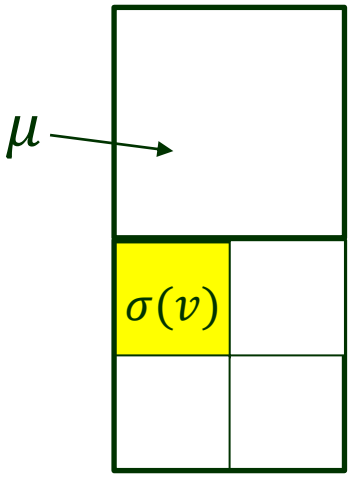
```



♣ The call does not necessarily return a leaf node.

Searching for the North Neighbor

NorthNeighbor(v,$\mathcal{T}$)

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1.  if v = root(T)
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12.           then return the SW-child of μ
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```
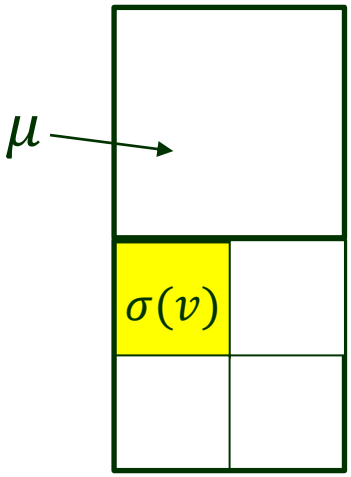
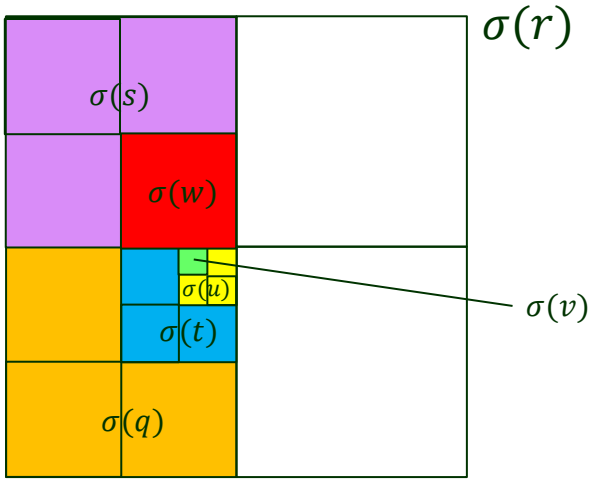
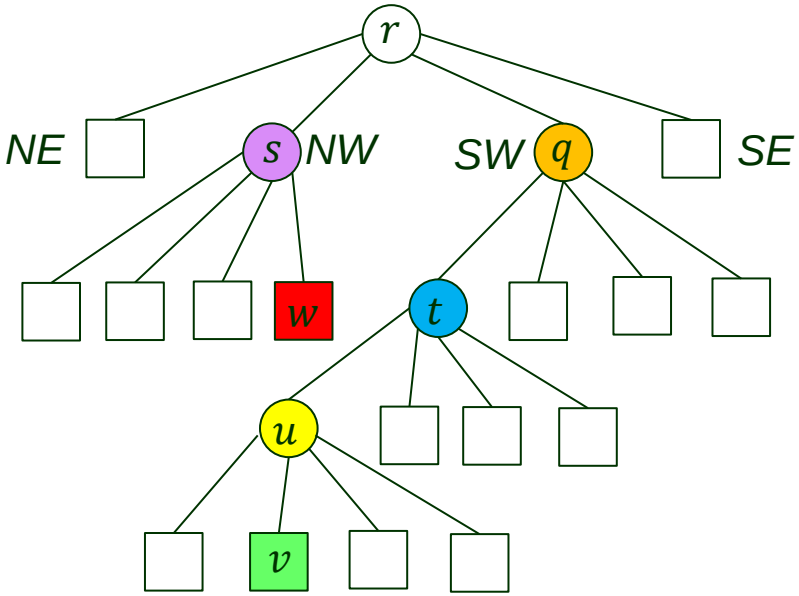



♣ The call does not necessarily return a leaf node.

♣ To find a leaf node, we need to walk down the quadtree from the returned node, always proceeding to a south-child.

A Bigger Example






```
NorthNeighbor(v,T)

1.    if v = root(T)
2.        then return nil
3.    u ← parent(v)
4.    if v is the SW-child of u
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10.       then return μ
11.       else if v is the NW-child of u
12.            then return the SW-child of μ
13.            else return the SE-child of μ
```
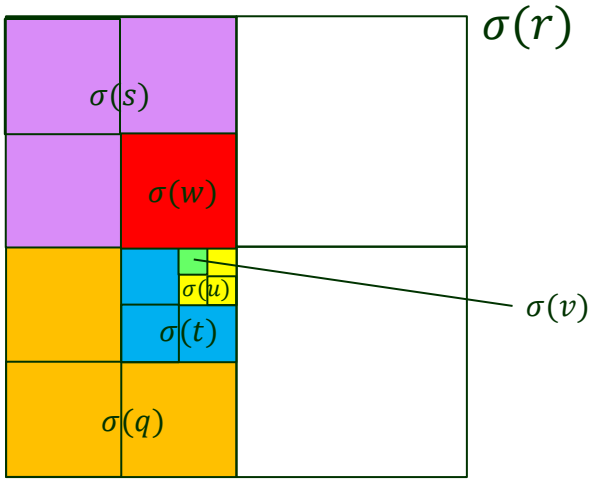
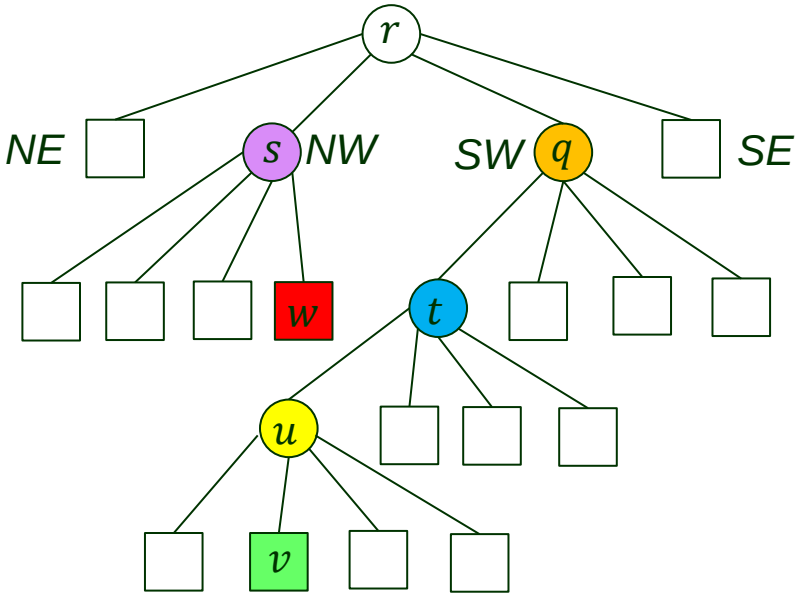

A Bigger Example





NorthNeighbor(v,$\mathcal{T}$)

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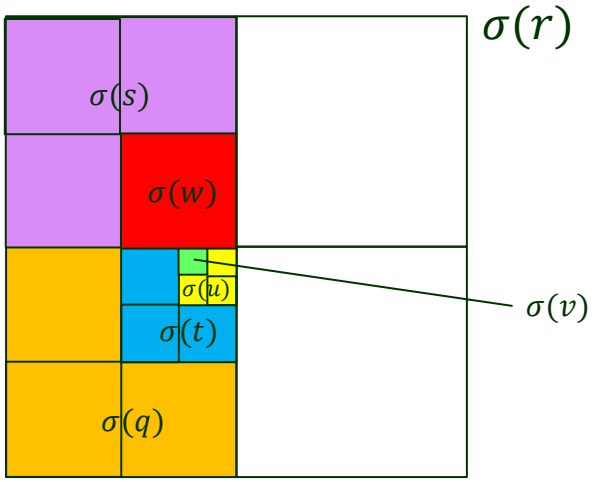
7.        then return the NE-child of u
8.    μ ← NorthNeighbor(u, T)
9.    if μ =nil or μ is a leaf
10.       then return μ
11.       else if v is the NW-child of u
12.            then return the SW-child of μ
13.            else return the SE-child of μ
```

NorthNeighbor(v,$\mathcal{T}$)

A Bigger Example

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10.       then return μ
11.       else if v is the NW-child of u
12.            then return the SW-child of μ
13.            else return the SE-child of μ
```
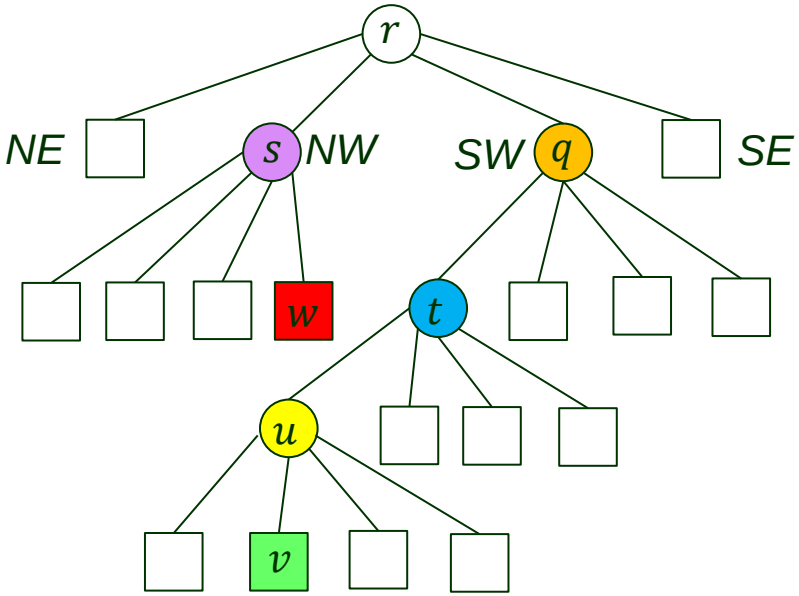
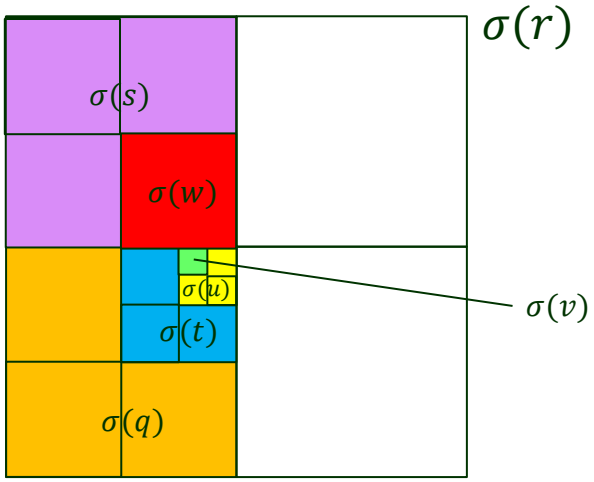



NorthNeighbor(v,$\mathcal{T}$)

|

NorthNeighbor(u,$\mathcal{T}$)

A Bigger Example




```
NorthNeighbor(v,T)

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11.       else if v is the NW-child of u
12.            then return the SW-child of μ
13.            else return the SE-child of μ
```
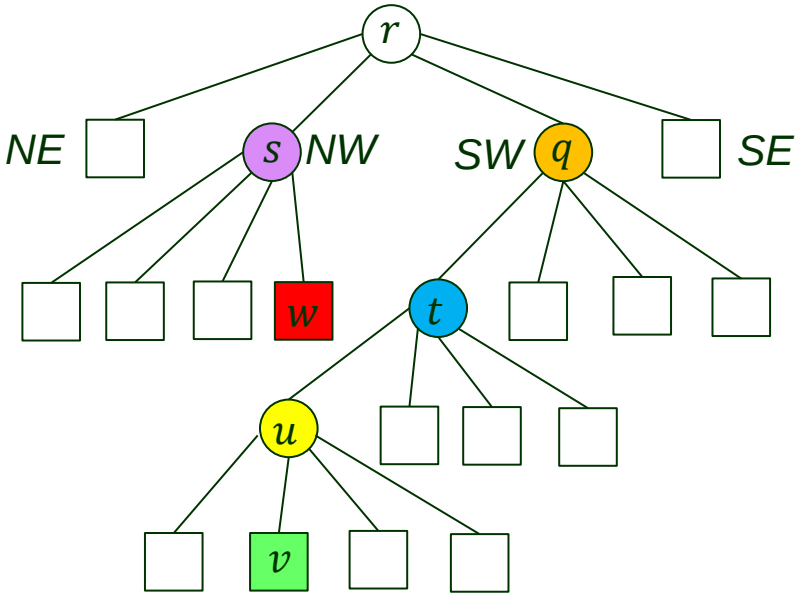
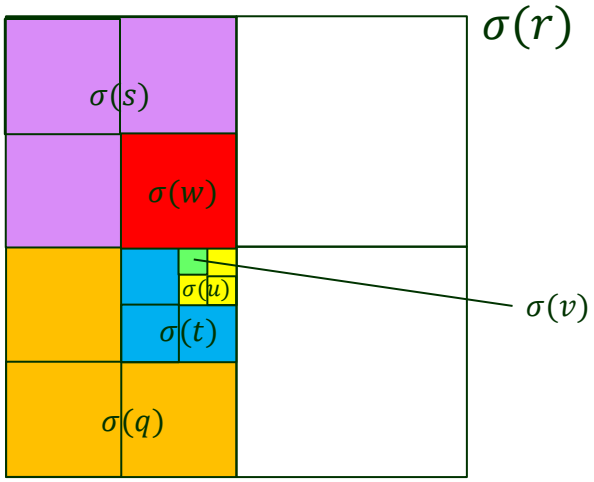
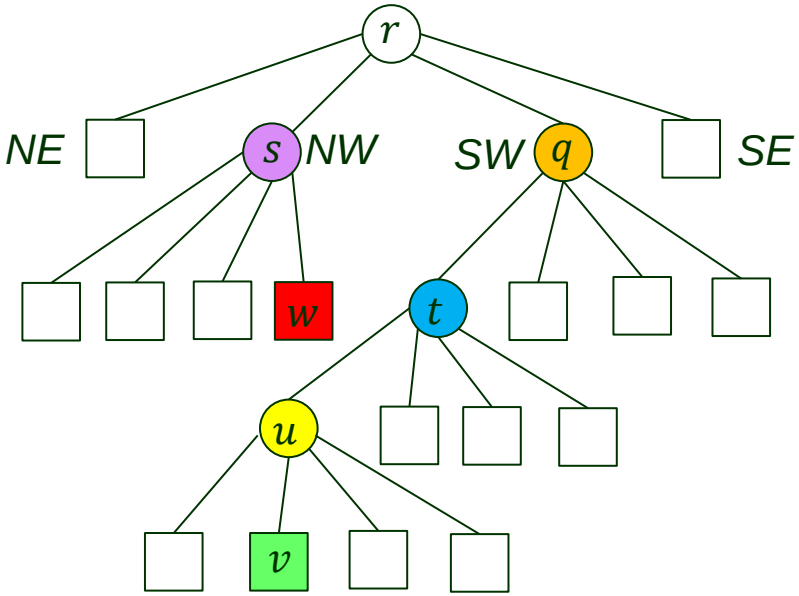



NorthNeighbor(v,$\mathcal{T}$)
|
NorthNeighbor(u,$\mathcal{T}$)
|
NorthNeighbor(t,$\mathcal{T}$)

A Bigger Example






```
NorthNeighbor(v,T)

1.    if v = root(T)
2.        then return nil
3.    u ← parent(v)
4.    if v is the SW-child of u
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6.    if v is the SE-child of u
7.        then return the NE-child of u
8.    μ ← NorthNeighbor(u, T)
9.    if μ =nil or μ is a leaf
10.       then return μ
11.       else if v is the NW-child of u
12.            then return the SW-child of μ
13.            else return the SE-child of μ
```
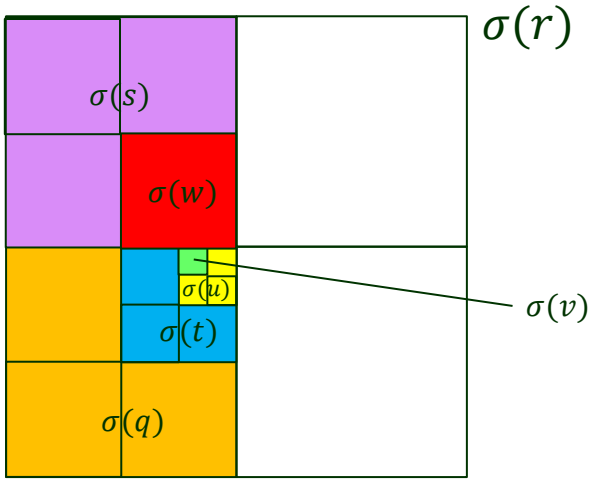
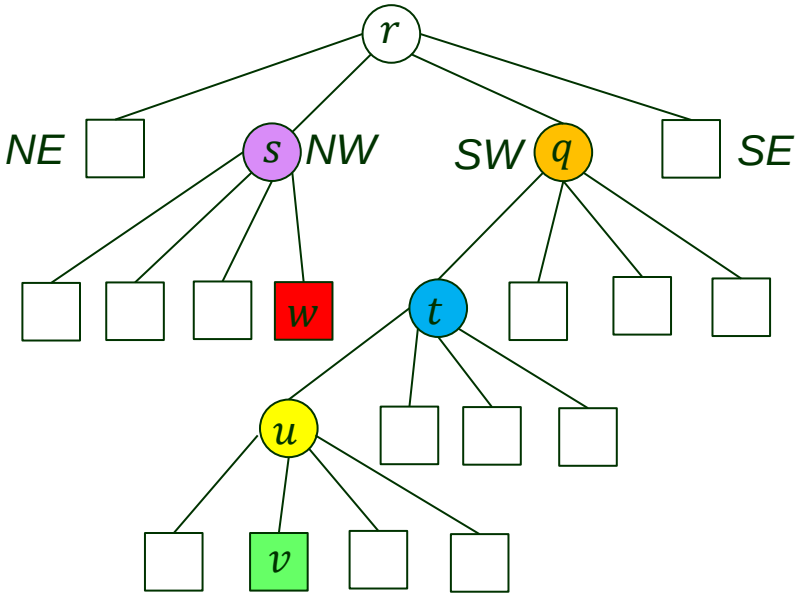

NorthNeighbor(v,$\mathcal{T}$)
|
NorthNeighbor(u,$\mathcal{T}$)
|
NorthNeighbor(t,$\mathcal{T}$)
|
NorthNeighbor(q,$\mathcal{T}$)

A Bigger Example

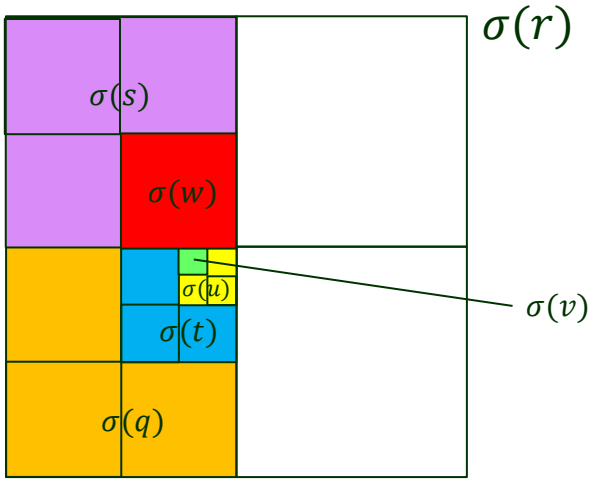



```
NorthNeighbor(v,𝒯)

1.    if v = root(𝒯)
2.        then return nil
3.    u ← parent(v)
4.    if v is the SW-child of u
5.        then return the NW-child of u
6.    if v is the SE-child of u
7.        then return the NE-child of u
8.    μ ← NorthNeighbor(u, 𝒯)
9.    if μ =nil or μ is a leaf
10.       then return μ
11.       else if v is the NW-child of u
12.            then return the SW-child of μ
13.            else return the SE-child of μ
```

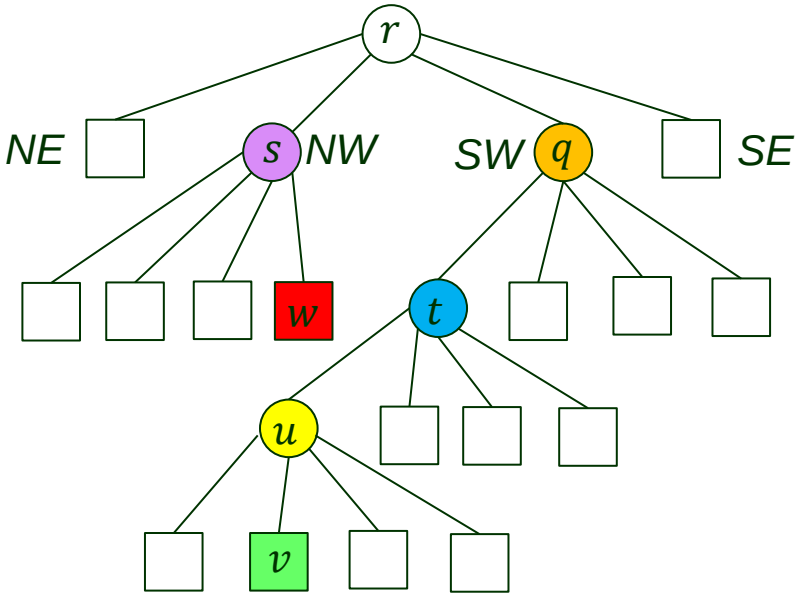


NorthNeighbor(v,$\mathcal{T}$)
|
NorthNeighbor(u,$\mathcal{T}$)
|
NorthNeighbor(t,$\mathcal{T}$)
|
NorthNeighbor(q,$\mathcal{T}$) returns s

A Bigger Example

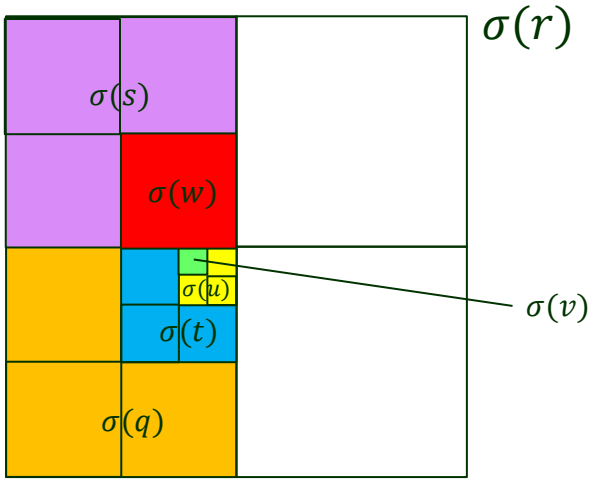



```
NorthNeighbor(v,𝒯)

1.    if v = root(𝒯)
2.        then return nil
3.    u ← parent(v)
4.    if v is the SW-child of u
5.        then return the NW-child of u
6.    if v is the SE-child of u
7.        then return the NE-child of u
8.    μ ← NorthNeighbor(u, 𝒯)
9.    if μ =nil or μ is a leaf
10.       then return μ
11.       else if v is the NW-child of u
12.            then return the SW-child of μ
13.            else return the SE-child of μ
```

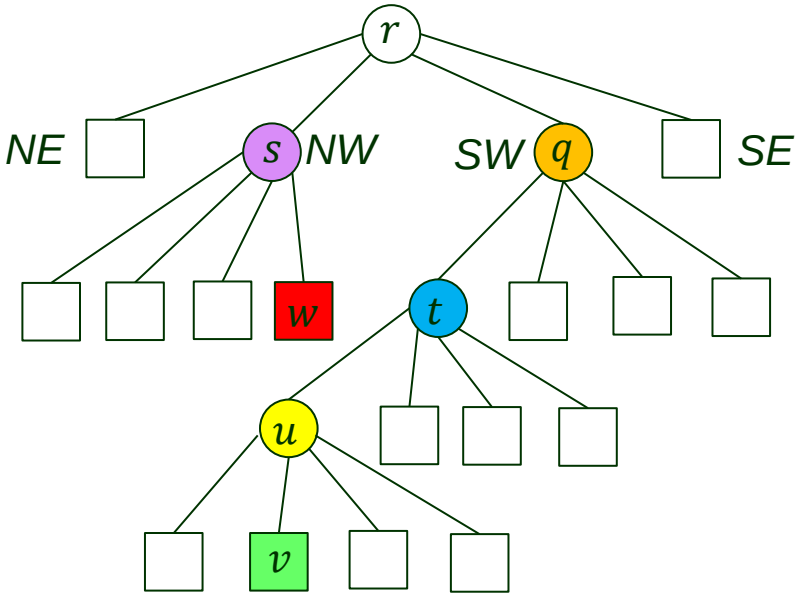


NorthNeighbor(v,$\mathcal{T}$)
|
NorthNeighbor(u,$\mathcal{T}$)
|
NorthNeighbor(t,$\mathcal{T}$) $\mu \leftarrow s$; returns w (SW-child of s)
|
NorthNeighbor(q,$\mathcal{T}$) returns s

A Bigger Example

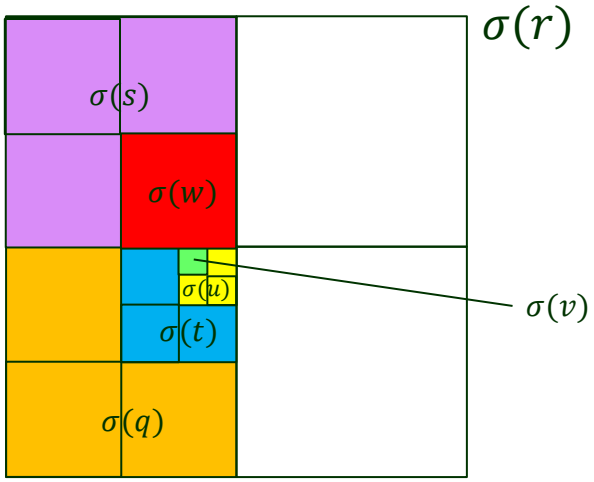



```
NorthNeighbor(v,T)

1.    if v = root(T)
2.        then return nil
3.    u ← parent(v)
4.    if v is the SW-child of u
5.        then return the NW-child of u
6.    if v is the SE-child of u
7.        then return the NE-child of u
8.    μ ← NorthNeighbor(u, T)
9.    if μ =nil or μ is a leaf
10.       then return μ
11.       else if v is the NW-child of u
12.            then return the SW-child of μ
13.            else return the SE-child of μ
```



NorthNeighbor(v,$\mathcal{T}$)
|
NorthNeighbor(u,$\mathcal{T}$) $\mu \leftarrow w$; returns w (leaf)
|
NorthNeighbor(t,$\mathcal{T}$) $\mu \leftarrow s$; returns w (SW-child of s)
|
NorthNeighbor(q,$\mathcal{T}$) returns s

A Bigger Example

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NorthNeighbor(v,T)

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9.    if μ =nil or μ is a leaf
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12.            then return the SW-child of μ
13.            else return the SE-child of μ
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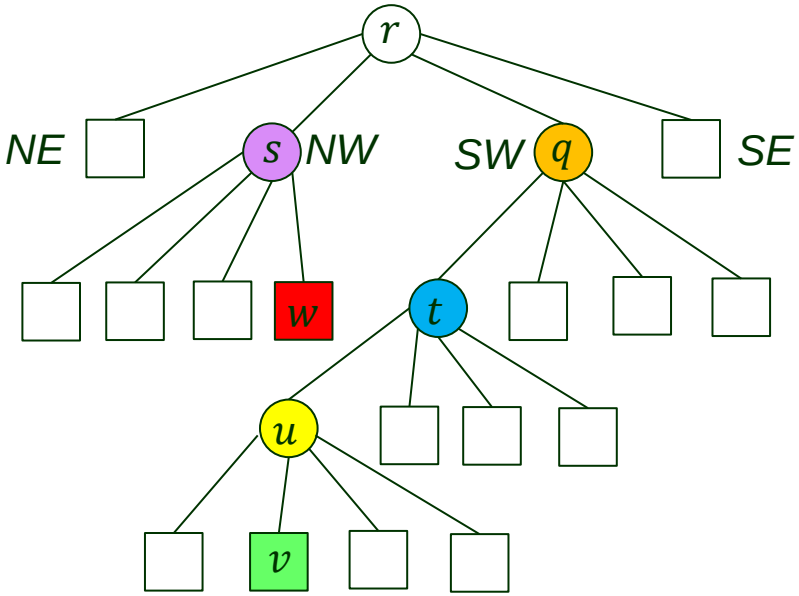


NorthNeighbor(v,$\mathcal{T}$) $\mu \leftarrow w$; returns w (leaf)

NorthNeighbor(u,$\mathcal{T}$) $\mu \leftarrow w$; returns w (leaf)

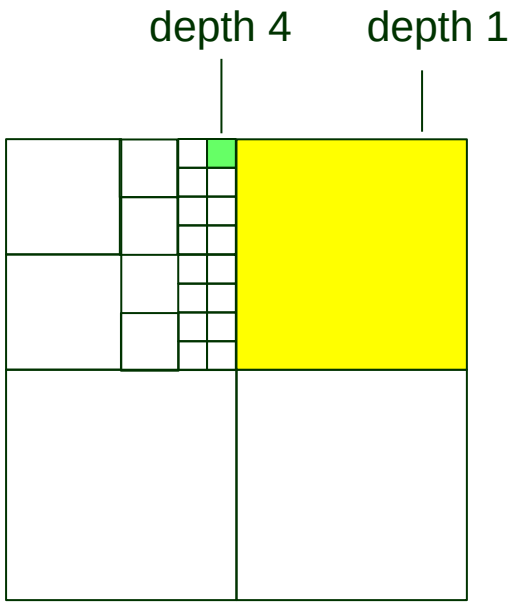
NorthNeighbor(t,$\mathcal{T}$) $\mu \leftarrow s$; returns w (SW-child of s)

NorthNeighbor(q,$\mathcal{T}$) returns s

V. Balanced Quadtree

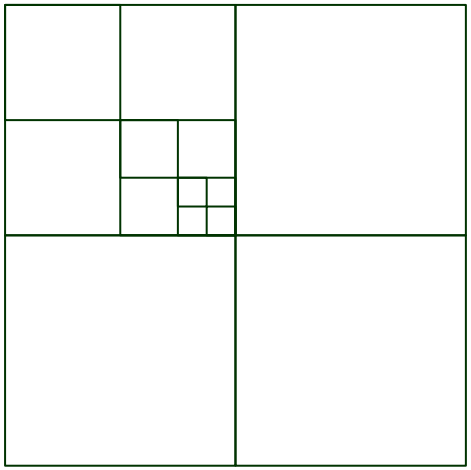
A quadtree subdivision is *balanced* if any two neighboring squares differ by *a factor of one or two* in size (as measured by the side length not by area).

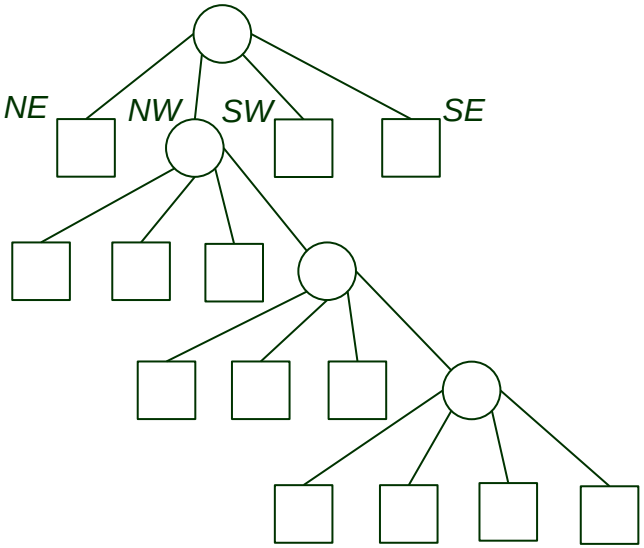
This implies that any two leaves whose squares are neighbors can differ *at most one* in depth.



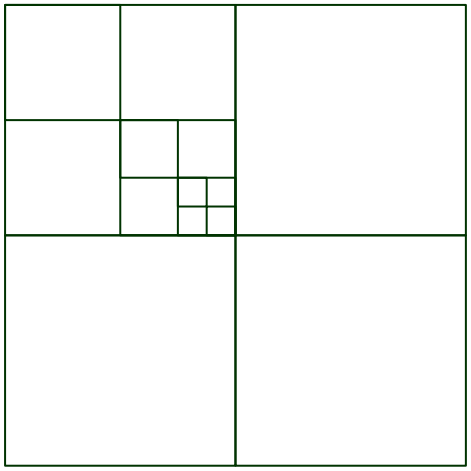
Unbalanced subdivision and
Corresponding quadtree.

Example

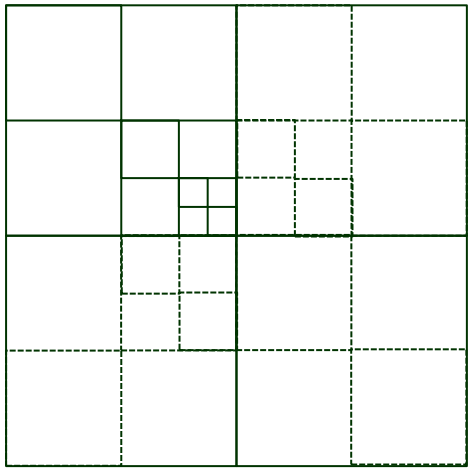


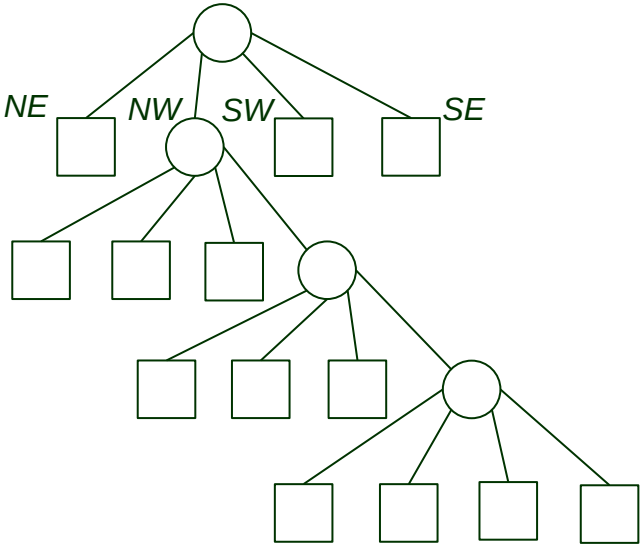


Example

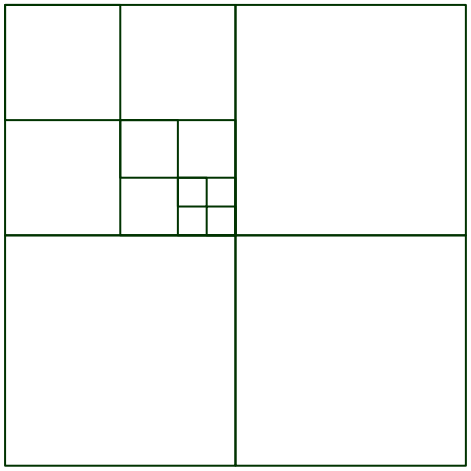


balancing

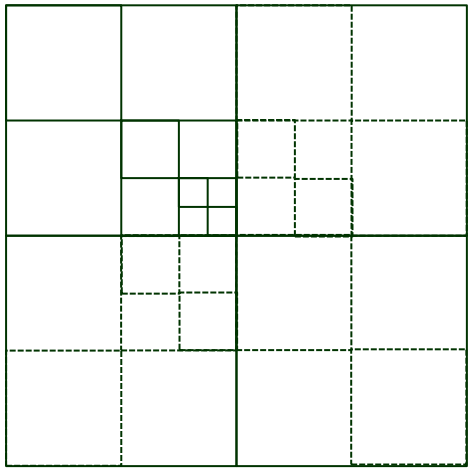


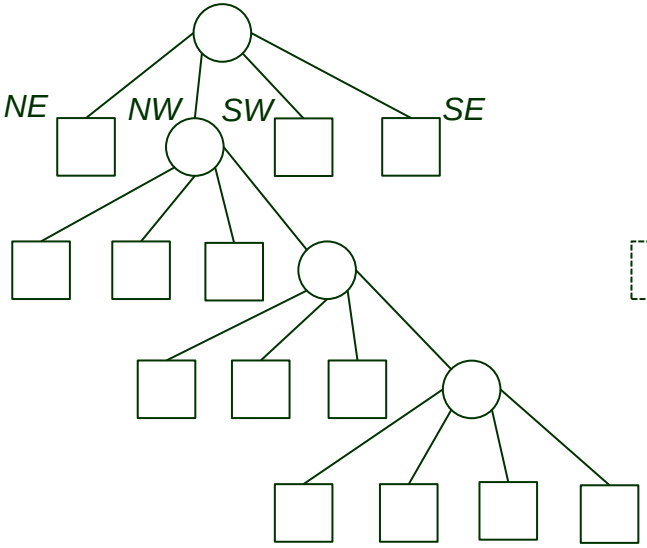


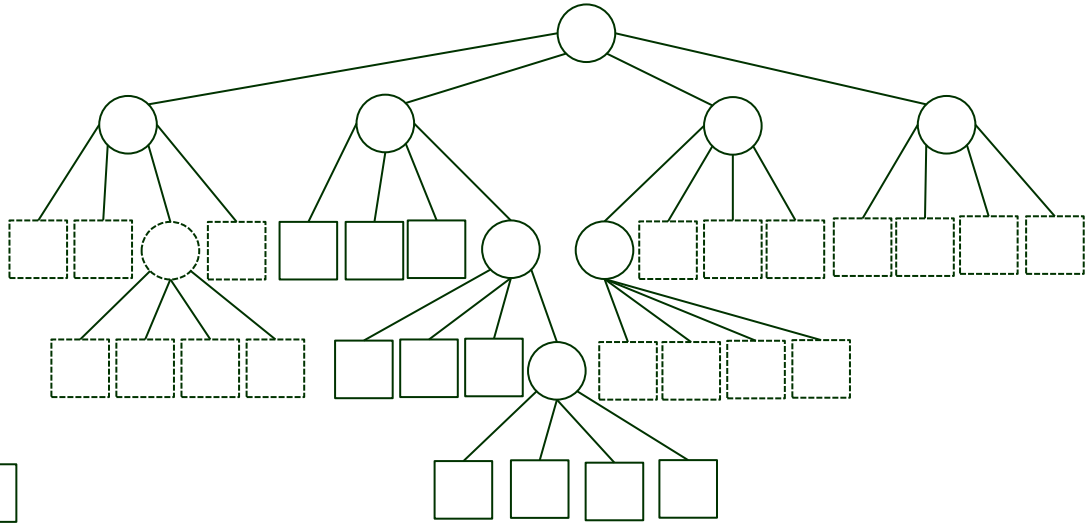
Example



balancing







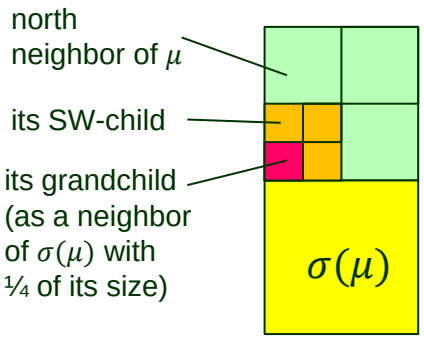
Balancing Algorithm

```
BalanceQuadTree(𝒯)

1.  insert all the leaves of 𝒯 into a linear list ℒ
2.  while ℒ is not empty
3.      do remove a leaf μ from ℒ
4.         if σ(μ) has to be split
5.         then make μ an internal node with four new leaves
6.              if μ stores a point
7.                  then stores it in the correct new leaf
8.              insert the four new leaves into ℒ
9.              if σ(μ) had neighbors that now need to be split
10.                 then insert them into ℒ
```

First Issue to Settle

On line 4 of the algorithm, how to check if a leaf $\sigma(\mu)$ needs to be split?

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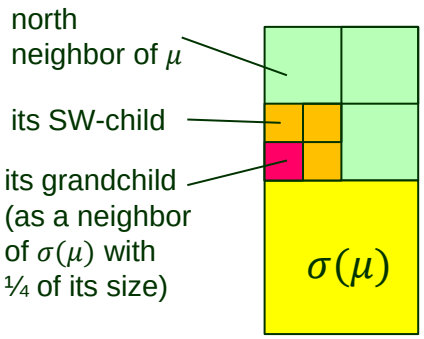
```

First Issue to Settle

On line 4 of the algorithm, how to check if a leaf $\sigma(\mu)$ needs to be split?

- Check if $\sigma(\mu)$ has a neighboring square less than half its size.
- Employ the earlier introduced neighbor finding algorithm.

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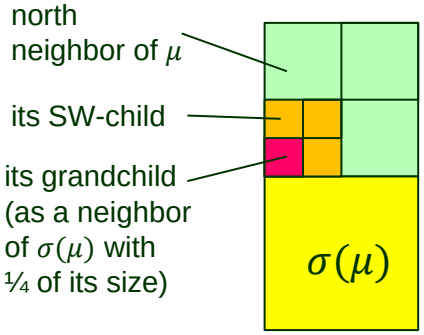

- Such a small neighbor in the north exists iff `NorthNeighbor(`$\mu, \mathcal{T}$`)` returns a node that has a SW- or SE-child that is not a leaf.

`BalanceQuadTree(`$\mathcal{T}$`)`

```
1.   insert all the leaves of T into a linear list L
2.   while L is not empty
3.       do remove a leaf μ from L
4.          if σ(μ) has to be split
5.          then make μ an internal node with four new leaves
6.              if μ stores a point
7.                  then stores it in the correct new leaf
8.              insert the four new leaves into L
9.              if σ(μ) had neighbors that now need to be split
10.                 then insert them into L
```

Second Issue to Settle

```
BalanceQuadTree(𝒯)

1.   insert all the leaves of 𝒯 into a linear list ℒ
2.   while ℒ is not empty
3.       do remove a leaf μ from ℒ
4.          if σ(μ) has to be split
5.          then make μ an internal node with four new leaves
6.               if μ stores a point
7.                   then stores it in the correct new leaf
8.               insert the four new leaves into ℒ
9.               if σ(μ) had neighbors that now need to be split
10.                  then insert them into ℒ
```

On line 9, check if $\sigma(\mu)$, already split, had a neighbor that needs to be split.

Second Issue to Settle

```
BalanceQuadTree(𝒯)

1.   insert all the leaves of 𝒯 into a linear list ℒ
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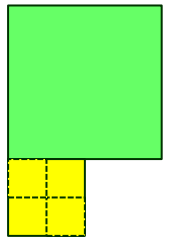
9.               if σ(μ) had neighbors that now need to be split
10.                  then insert them into ℒ
```

On line 9, check if $\sigma(\mu)$, already split, had a neighbor that needs to be split.

- Again use the neighbor finding algorithm.
  - Such a neighbor exists to the north iff `NorthNeighbor`$(\mu, \mathcal{T})$ returns a node corresponding to a square larger than $\sigma(\mu)$.



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```
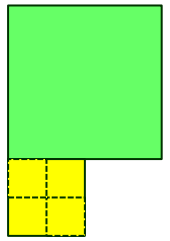

On line 9, check if $\sigma(\mu)$, already split, had a neighbor that needs to be split.

- Again use the neighbor finding algorithm.
  - Such a neighbor exists to the north iff `NorthNeighbor`$(\mu, \mathcal{T})$ returns a node corresponding to a square larger than $\sigma(\mu)$.
  - Such a neighbor would be more than twice the size of each of the four children from splitting of μ on line 5.

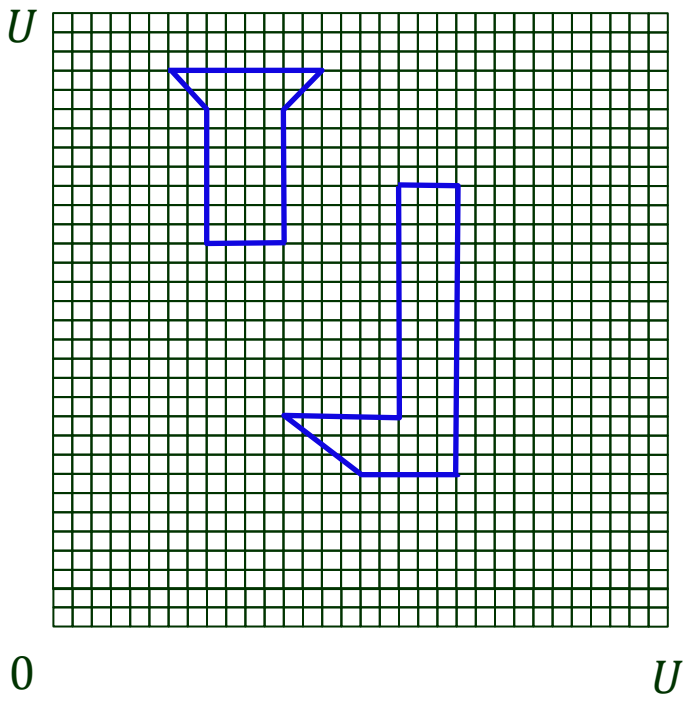


Cost of Balancing

Theorem 2 Let $\mathcal{T}$ be a quadtree with m nodes. Then the balanced version of $\mathcal{T}$ has $O(m)$ nodes and can be constructed in $O\big((h+1)m\big)$ time.

Proof Omitted.

From Quadtrees to Meshes

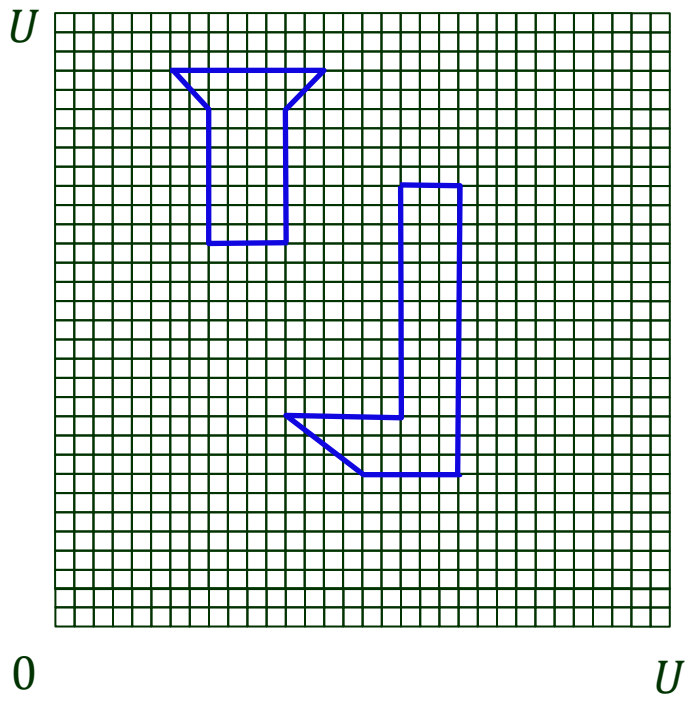


Mesh generation from a set of disjoint polygons

- whose vertices have integer coordinates and
- whose edges makes angles $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ with the x-axis.

Algorithm

From Quadtrees to Meshes



Mesh generation from a set of disjoint polygons

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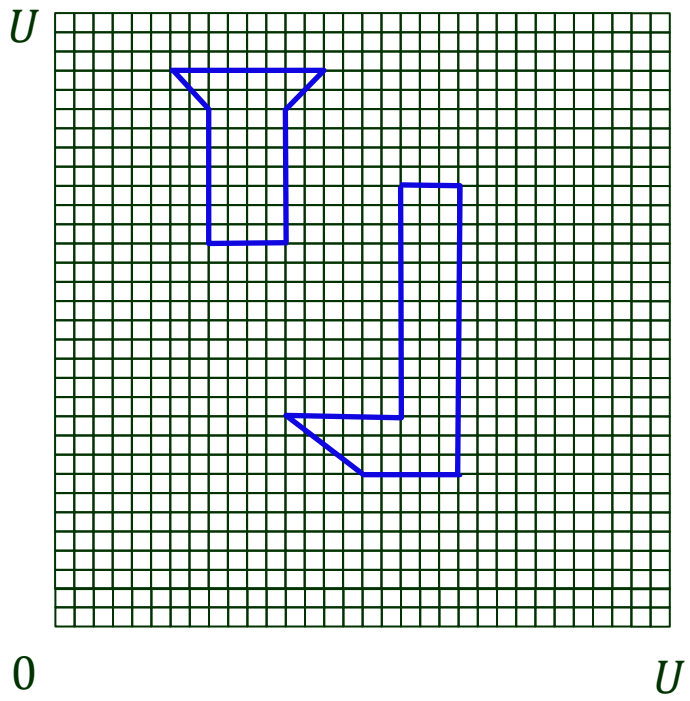
Algorithm

1. Construct a quadtree subdivision of the polygon vertices.

From Quadtrees to Meshes



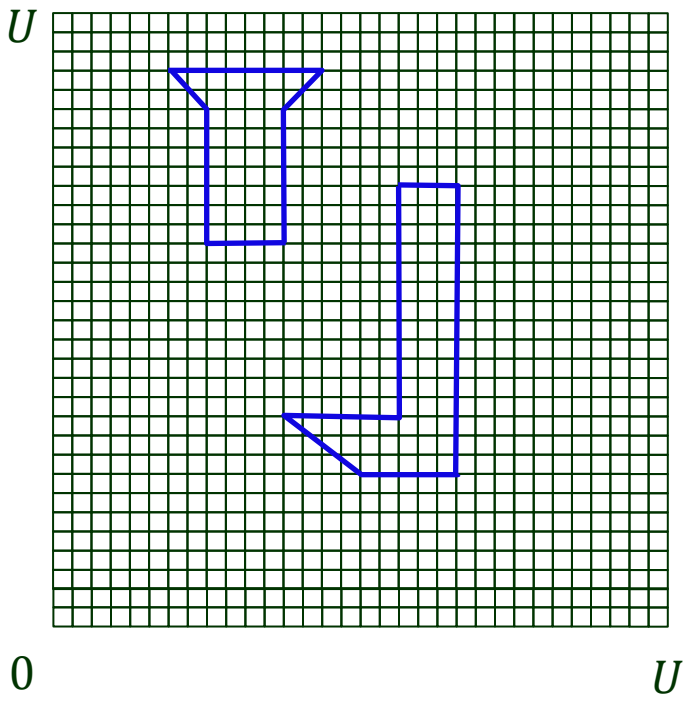
Mesh generation from a set of disjoint polygons

- ♦ whose vertices have integer coordinates and
- ♦ whose edges makes angles $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ with the x-axis.

Algorithm

1. Construct a quadtree subdivision of the polygon vertices.
   - ♣ Stop splitting a square when it is no longer intersected by a polygon edge, or when it has unit size.

From Quadtrees to Meshes



Mesh generation from a set of disjoint polygons

- ♦ whose vertices have integer coordinates and
- ♦ whose edges makes angles $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ with the x-axis.

Algorithm

1. Construct a quadtree subdivision of the polygon vertices.
   - ♣ Stop splitting a square when it is no longer intersected by a polygon edge, or when it has unit size.
2. Balance the quadtree subdivision.
3. Triangulate the balanced quadtree subdivision (adding Steiner points).

Other Applications of Quadtrees

- Image processing
- Point location
- Collision detection
- Data storage & visualization
- Computational fluid dynamics

Patch tree for a pebble
(based on the Gauss map)