Lecture notes for *Principles of Artificial Intelligence* (COMS 4720/5720)
Yan-Bin Jia, Iowa State University

# Bayes' Rule

## Outline

I. Bayes' rule

II. Conditional independence

* Figures are either from the textbook site or by the instructor.

# I. Bayes' Rule

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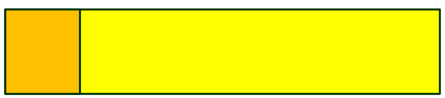

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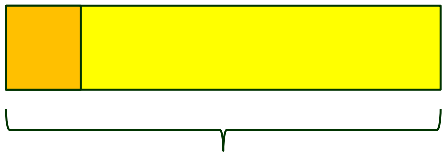

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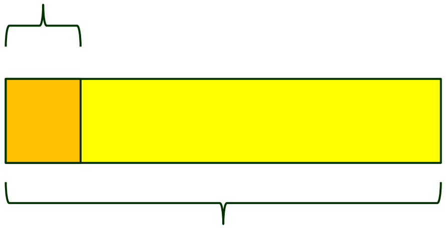

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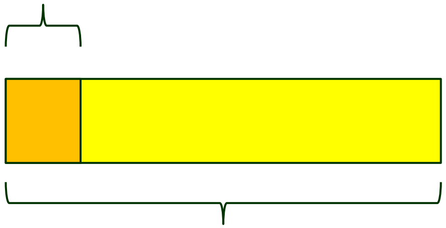

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It tells us how often $b$ happens given that $a$ happens, when we know:

- how often $a$ happens given that $b$ happens, and
- how likely $a$ is on its own, and
- how likely $b$ is on its own.

# Bayes' Rule for Multivalued Variables

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The above is a set of equations, each for a pair of possible values of $X$ and $Y$.

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A more generalized version conditionalized on some evidence $\boldsymbol{e}$:

$$\boldsymbol{P}(Y \mid X, \boldsymbol{e}) = \frac{\boldsymbol{P}(X \mid Y, \boldsymbol{e})\boldsymbol{P}(Y \mid \boldsymbol{e})}{\boldsymbol{P}(X \mid \boldsymbol{e})} = \frac{\boldsymbol{P}(X, Y \mid \boldsymbol{e})}{\boldsymbol{P}(X \mid \boldsymbol{e})}$$

# Applying Bayes' Rule

- Perceive as the evidence the *effect* of some unknown *cause*.
- Determine the cause.

$$P(\mathit{cause} \mid \mathit{effect}) = \frac{P(\mathit{effect} \mid \mathit{cause})P(\mathit{cause})}{P(\mathit{effect})}$$

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The doctor knows $P(symptoms \mid disease)$ and wants to derive a diagnosis $P(disease \mid symptoms)$.

# Example 1

The doctor knows:

- The disease meningitis causes a patient to have a stiff neck 70% of time.
- The prior probability that any patient has meningitis is 1/50,000.
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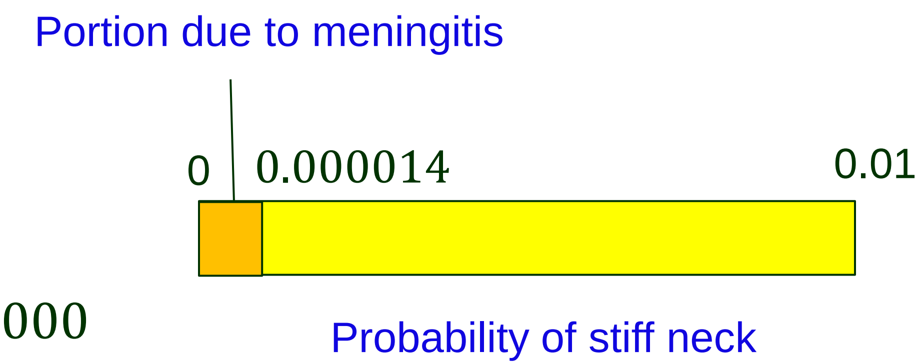


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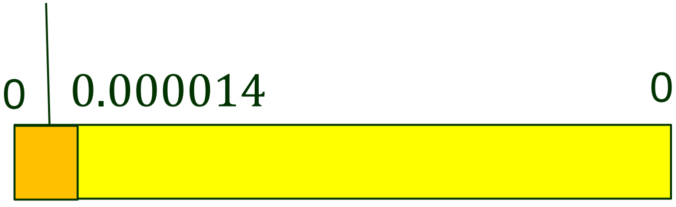


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Identify the probability of meningitis given a stiff neck with the disease' portion of contribution to the symptom.

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$$P(m \mid s) + P(\neg m \mid s) = 1$$

$$P(s) = P(s \mid m)\, P(m) + P(s \mid \neg m)\, P(\neg m)$$

# Example 2

**Beliefs**

- If a patient has lung cancer, there is a 60% chance that an X-ray test will come back positive; and a 40% chance negative.
- If a patient does not have lung cancer, there is 2% percent chance that an X-ray test will come back positive ; and a 98% percent chance negative.
- Population cancer rate is 1/1000.

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$$P(c \mid x) = \frac{P(x \mid c)P(c)}{P(x)} = \frac{P(x \mid c)P(c)}{P(x \mid c)P(c) + P(x \mid \neg c)P(\neg c)}$$

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$$= \frac{0.6 \cdot 0.001}{0.6 \cdot 0.001 + 0.02 \cdot 0.999} \approx 0.02915$$

# Bayes' Rule (General Form)

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normalization constant to make the entries in $\mathbf{P}(Y \mid X)$ sum to 1.

# II. Combining Evidence

What happens when we have two or more pieces of evidences?

- Suppose we know the full joint distribution.

| | $toothache$ | | $\neg toothache$ | |
|---|---|---|---|---|
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♠ Does not scale up to a large number of evidence variables.

# Exponential Growth

- Apply Bayes' rule:

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namely,

$$\langle P(cavity \mid toothache \wedge catch), P(\neg cavity \mid toothache \wedge catch) \rangle$$

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- $n$ (Boolean) evidence variables: *Toothache*, *Catch*, *X-rays*, *Diet*, …

⇒ $2^n$ possible combinations of observed values $\boldsymbol{P}(toothache \wedge catch \mid Cavity \wedge Toothache \wedge \cdots)$ are needed!

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$$\boldsymbol{P}(\mathit{Toothache}, \mathit{Catch} \mid \mathit{Cavity}) = \boldsymbol{P}(\mathit{Toothache} \mid \mathit{Cavity})\, \boldsymbol{P}(\mathit{Catch} \mid \mathit{Cavity})$$

// asserts independence for all possible combinations of values of
// *Toothache* and *Catch*.

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Conditional independence of two variables $X$ and $Y$ given a third variable $Z$:

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Decomposition into smaller conditional assertions.

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- Conditional independence assertions allow probabilistic systems to scale up.
- They are more commonly available than absolute independence assertions.
- Decomposition of large probabilistic domains through conditional independence is one of the most important recent developments in AI.