Lecture notes for *Principles of Artificial Intelligence* (COMS 4720/5720)
Yan-Bin Jia, Iowa State University

Probabilistic Inference

Outline

I. Probability for continuous variables

II. Inference by enumeration

* Figures are either from the textbook site or by the instructor.

I. Probability Density Function

A *continuous random variable* has its outcome take on a continuous set of values.

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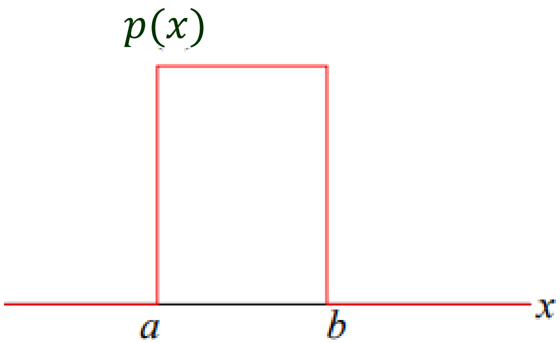
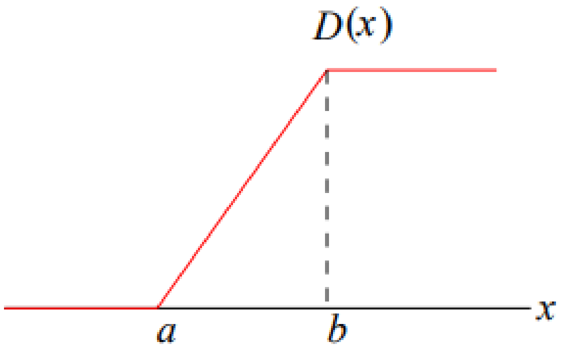
Uniform Distribution

A *uniform distribution* has constant pdf.

Example Continuous uniform distribution over the interval $[a, b]$.

$$p(x) = \begin{cases} 0, & \text{for } x < a, \\ \frac{1}{b-a}, & \text{for } a \leq x < b, \\ 0, & \text{for } x \geq b. \end{cases} \qquad D(x) = \begin{cases} 0, & \text{for } x < a, \\ \frac{x-a}{b-a}, & \text{for } a \leq x < b, \\ 1, & \text{for } x \geq b. \end{cases}$$





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Example Roll a die an infinite number of times. Each number appears 1/6 of the time.

$$E(X) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{6} i \cdot \frac{n}{6} = \frac{7}{2}$$

Variance and Standard Deviation

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The *standard deviation* of X is $\sigma = \sqrt{\text{var}(X)}$.

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Gaussian Distribution

A continuous random variable X with mean μ and variance σ^2 has *Gaussian distribution* (or *normal distribution*) if its pdf is

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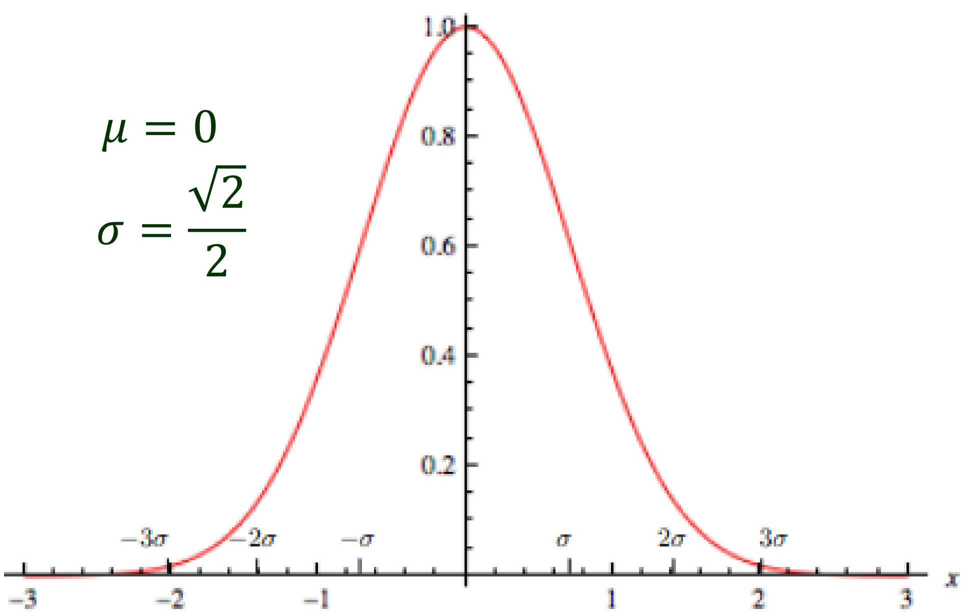

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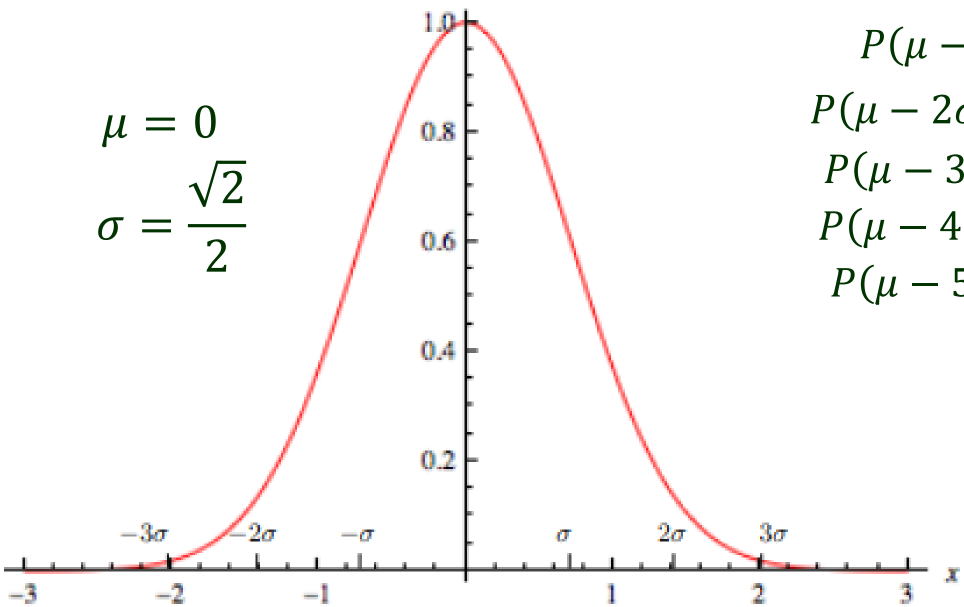

$$P(\mu - \sigma \le X \le \mu + \sigma) \approx 0.6826895$$

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$$P(\mu - 3\sigma \le X \le \mu + 3\sigma) \approx 0.9973002$$

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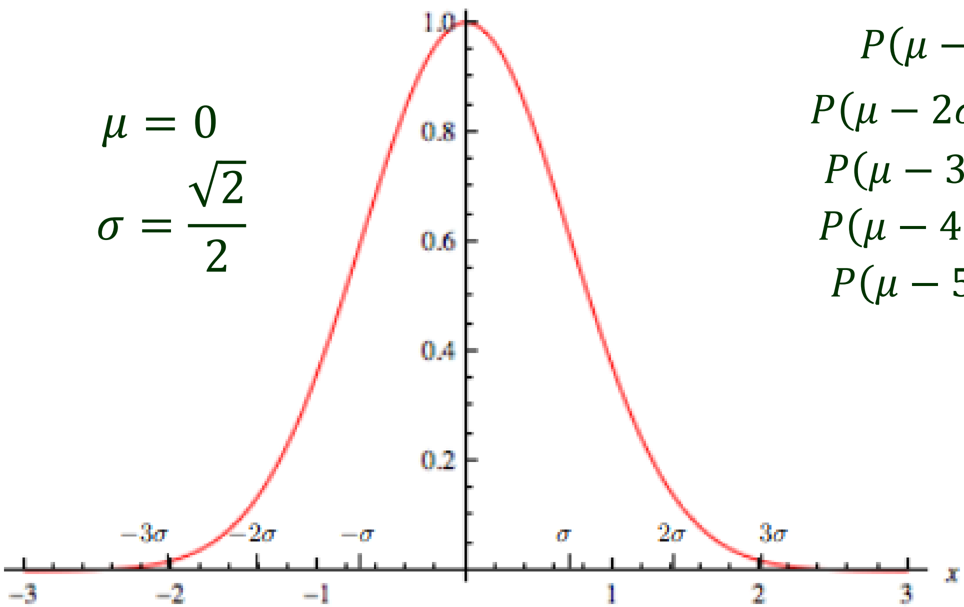

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Three-sigma rule (in practice): Consider population within $(\mu - 3\sigma, \mu + 3\sigma)$.

Why Gaussian Distribution?

- It is the most important distribution because it fits many natural phenomena (e.g., human characteristics such as weight, height, body temperature, etc.).

- It is the limiting distribution of $X_1 + \cdots + X_n$ of n independent random variables $X_1, \dots, X_n$, as $n \to \infty$ (the central limit theorem), explaining a characteristic impacted by numerous independent factors.

- It is the foundation for important methods such as least-squares, Kalman filters, etc., which are used in statistics and engineering.

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Joint distribution table:

	$toothache$		$\neg toothache$	
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$$P(\textit{cavity} \vee \textit{toothache}) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$$

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Sums over all possible combinations of the values of the set of variables $\boldsymbol{Z}$.

Conditioning

Abbreviate $\boldsymbol{P}(\boldsymbol{Y}, \boldsymbol{Z} = \boldsymbol{z})$ as $\boldsymbol{P}(\boldsymbol{Y}, \boldsymbol{z})$.

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random variable beginning with an **uppercase letter**

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not bold-faced

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$$\mathbf{P}(Y) = \sum_z \mathbf{P}(Y \mid z) P(z)$$

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

Conditioning

Abbreviate $\mathbf{P}(\mathbf{Y}, \mathbf{Z} = \mathbf{z})$ as $\mathbf{P}(\mathbf{Y}, \mathbf{z})$.

values

$$\mathbf{P}(Cavity) = \mathbf{P}(Cavity, toothache, catch) + \mathbf{P}(Cavity, toothache, \neg catch)$$
$$+ \mathbf{P}(Cavity, \neg toothache, catch) + \mathbf{P}(Cavity, \neg toothache, \neg catch)$$

random variable beginning with an **uppercase letter**

$$= \langle 0.108, 0.016\rangle + \langle 0.012, 0.064\rangle + \langle 0.072, 0.144\rangle + \langle 0.008, 0.576\rangle$$
$$= \langle 0.2, 0.8\rangle$$

cavity *¬cavity*

not bold-faced

Rule of *conditioning*:

$$\mathbf{P}(\mathbf{Y}) = \sum_{\mathbf{z}} \mathbf{P}(\mathbf{Y} \mid \mathbf{z}) P(\mathbf{z})$$

$$= \mathbf{P}(\mathbf{Y}, \mathbf{z})$$

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
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Computing Conditional Probabilities

1. $P(a \mid b) = \dfrac{P(a \wedge b)}{P(b)}$ $\Longrightarrow$ an expression in terms of unconditional probabilities.
2. Evaluate the expression from full joint distribution.

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$$P(\mathit{cavity} \mid \mathit{toothache}) = \frac{P(\mathit{cavity} \wedge \mathit{toothache})}{P(\mathit{toothache})}$$

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$$= \frac{0.108 + 0.012}{0.108 + 0.012 + 0.016 + 0.064} = 0.6$$

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$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
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	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
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$$= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$$

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
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Computing Conditional Probabilities

1. $P(a \mid b) = \dfrac{P(a \land b)}{P(b)}$ $\Longrightarrow$ an expression in terms of unconditional probabilities.
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$P(cavity \mid toothache) + P(\neg cavity \mid toothache) = 1$

	toothache		¬toothache	
	catch	¬catch	catch	¬catch
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

Normalization

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
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$$P(cavity \mid toothache) = \frac{P(cavity \wedge toothache)}{P(toothache)} = 0.6$$

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Normalization

	toothache		¬toothache	
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- $P(toothache)$ is a normalization constant for the distribution $\mathbf{P}(Cavity \mid toothache)$ so it adds up to 1.

Normalization

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Normalization

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$$\mathbf{P}(Cavity \mid toothache) = \alpha\, \mathbf{P}(Cavity, toothache)$$

Normalization

	toothache		¬toothache	
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Normalization

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Normalization

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$$= \alpha\, [\langle 0.108, 0.016\rangle + \langle 0.012, 0.064\rangle]$$

$$= \alpha\, \langle 0.108 + 0.012, 0.016 + 0.064\rangle = \alpha\, \langle 0.12, 0.08\rangle$$

Normalization

	toothache		¬toothache	
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$$= \langle 0.6, 0.4\rangle$$

Normalization

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cavity	0.108	0.012	0.072	0.008
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$$= \langle 0.6, 0.4 \rangle$$

normalization constant $\alpha = 5 = 1/P(toothache)$.

Inference Procedure

- X: single query variable (e.g., *Cavity*).
- $\boldsymbol{E}$: evidence variables (e.g., *Toothache*).
- $\boldsymbol{e}$: their observed values.
- $\boldsymbol{Y}$: unobserved variables (e.g., *Catch*).

Inference Procedure

- X: single query variable (e.g., *Cavity*).
- $\boldsymbol{E}$: evidence variables (e.g., *Toothache*).
- $\boldsymbol{e}$: their observed values.
- $\boldsymbol{Y}$: unobserved variables (e.g., *Catch*).

$$\boldsymbol{P}(X \mid \boldsymbol{e}) \leftarrow \alpha \boldsymbol{P}(X, \boldsymbol{e}) = \alpha \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{e}, \boldsymbol{y})$$

- ♣ Summation is over all possible combinations $\boldsymbol{y}$ of values of variables in $\boldsymbol{Y}$.
- ♣ $\boldsymbol{P}(X, \boldsymbol{e}, \boldsymbol{y})$ is a subset of probabilities from the full joint distribution.
- ♣ The full joint distribution has size exponential in # variables and is rarely computed.

Independence

Add a fourth variable *Weather* with domain { *sun*, *rain*, *cloud*, *snow* }.

$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$

	$toothache$		$\neg toothache$	
	$catch$	$\neg catch$	$catch$	$\neg catch$
$cavity$	0.108	0.012	0.072	0.008
$\neg cavity$	0.016	0.064	0.144	0.576

Independence

Add a fourth variable *Weather* with domain { *sun*, *rain*, *cloud*, *snow* }.

$\boldsymbol{P}(Toothache, Catch, Cavity, Weather)$

	toothache		*¬toothache*	
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⇨ A 32-element table.

Independence

Add a fourth variable *Weather* with domain { *sun*, *rain*, *cloud*, *snow* }.

$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$

$$P(toothache, catch, cavity, cloud)$$

	toothache		$\neg$*toothache*	
	catch	$\neg$*catch*	*catch*	$\neg$*catch*
cavity	0.108	0.012	0.072	0.008
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Independence

Add a fourth variable *Weather* with domain { *sun*, *rain*, *cloud*, *snow* }.

$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$

$$P(toothache, catch, cavity, cloud)$$

$$= P(cloud \mid toothache, catch, cavity)\, P(toothache, catch, cavity)$$

	toothache		*¬toothache*	
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⇒ A 32-element table.

Independence

Add a fourth variable *Weather* with domain { *sun*, *rain*, *cloud*, *snow* }.

$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$

$$P(toothache, catch, cavity, cloud)$$

$$= P(cloud \mid toothache, catch, cavity)\, P(toothache, catch, cavity)$$

$$\Downarrow \quad P(cloud \mid toothache, catch, cavity) = P(cloud)$$

$$= P(cloud)\, P(toothache, catch, cavity)$$

	toothache		*¬toothache*	
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cavity	0.108	0.012	0.072	0.008
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$\Rightarrow$ A 32-element table.

Independence

Add a fourth variable *Weather* with domain { *sun*, *rain*, *cloud*, *snow* }.

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$$P(toothache, catch, cavity, cloud)$$

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$$\Downarrow \quad P(cloud \mid toothache, catch, cavity) = P(cloud)$$

$$= P(cloud)\ P(toothache, catch, cavity)$$

$$\mathbf{P}(Toothache, Catch, Cavity, Weather) = \mathbf{P}(Toothache, Catch, Cavity)\ \mathbf{P}(Weather)$$

	toothache		*¬toothache*	
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$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$

$$P(toothache, catch, cavity, cloud)$$

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$$= P(cloud\,)\, P(toothache, catch, cavity)$$

$$\mathbf{P}(Toothache, Catch, Cavity, Weather) = \mathbf{P}(Toothache, Catch, Cavity)\, \mathbf{P}(Weather)$$

8-element table + 4-element table

	toothache		*¬toothache*	
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cavity	0.108	0.012	0.072	0.008
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⇒ ~~A 32-element table.~~

Independent Variables

- Two propositions a and b are *independent* if

$$\underbrace{P(a \mid b) = P(a) \quad \text{or} \quad P(b \mid a) = P(b) \quad \text{or} \quad P(a \wedge b) = P(a)P(b)}_{\text{Equivalent}}$$

Independent Variables

- Two propositions a and b are *independent* if

$P(a \mid b) = P(a)$ or $P(b \mid a) = P(b)$ or $P(a \wedge b) = P(a)P(b)$

Equivalent

- Two variables X and Y are *independent* if

$\boldsymbol{P}(X \mid Y) = \boldsymbol{P}(X)$ or $\boldsymbol{P}(Y \mid X) = \boldsymbol{P}(Y)$ or $\boldsymbol{P}(X \wedge Y) = \boldsymbol{P}(X)P(Y)$

Independent Variables

- Two propositions a and b are *independent* if

$$\underbrace{P(a \mid b) = P(a) \quad \text{or} \quad P(b \mid a) = P(b) \quad \text{or} \quad P(a \wedge b) = P(a)P(b)}_{\text{Equivalent}}$$

- Two variables X and Y are *independent* if

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- Two continuous variables X and Y are *independent* if, for all $x, y \in \mathbb{R}$,

$$P(X \leq x \wedge Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$$

Independent Variables

- Two propositions a and b are *independent* if

$$P(a \mid b) = P(a) \quad \text{or} \quad P(b \mid a) = P(b) \quad \text{or} \quad P(a \wedge b) = P(a)P(b)$$

Equivalent

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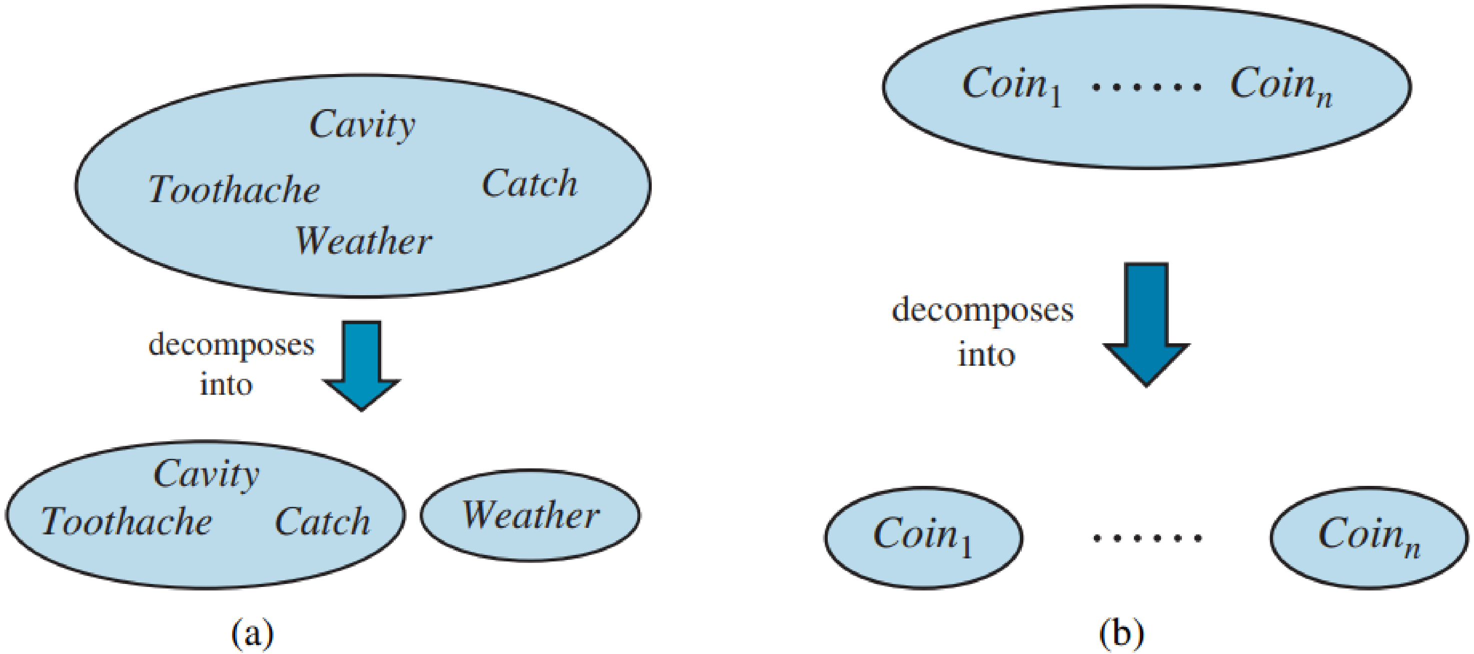
$$P(X \leq x \wedge Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$$

The joint probability density function $p(x, y)$ satisfies

$$p(x, y) = \int_{-\infty}^{\infty} p(x, y)\, dy \cdot \int_{-\infty}^{\infty} p(x, y)\, dx$$

Factoring a Joint Distribution

The full joint distribution can be factored into *separate* joint distributions on subsets of variables that are *independent*.



Weather and dental problems are independent.

Coin flips are independent.