Yan-Bin Jia, Iowa State University

Priority Search Trees

Outline:

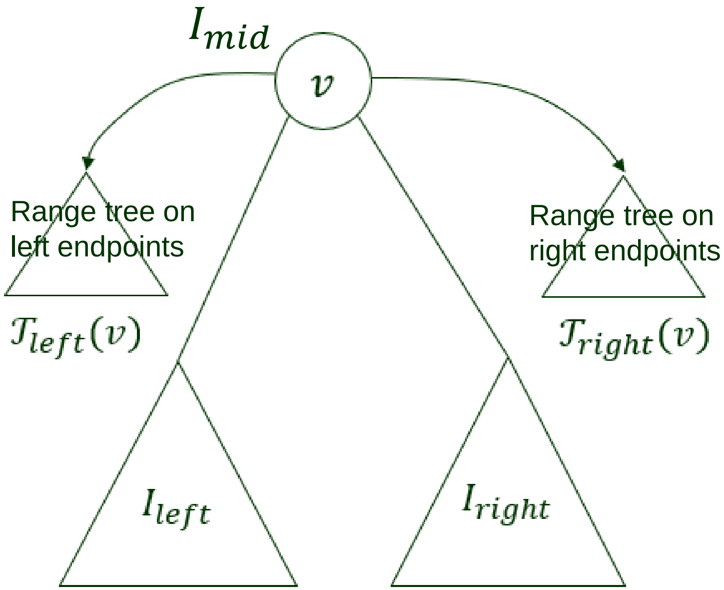
I. Heap-based point queries

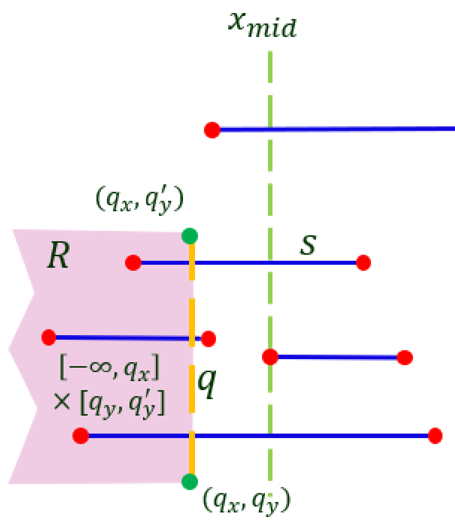
II. Structure of a PST

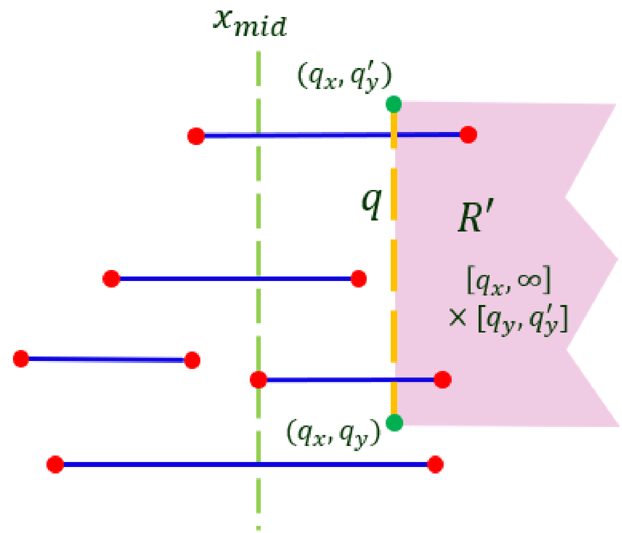
III. Query

IV. Correctness and running time

Windowing Queries with an Interval Tree







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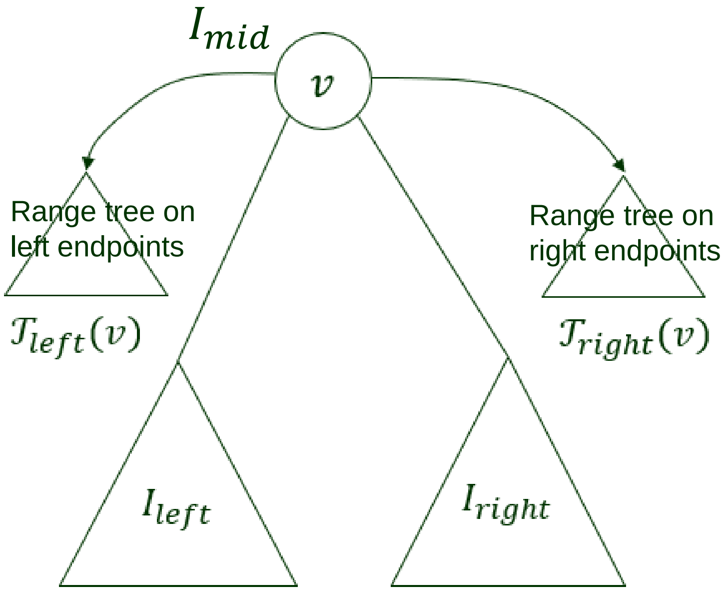

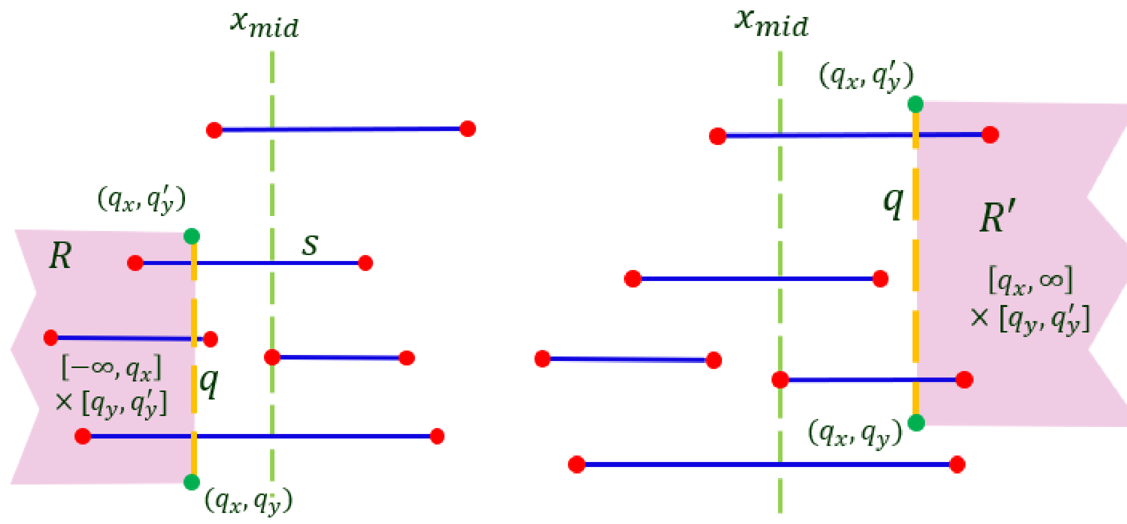

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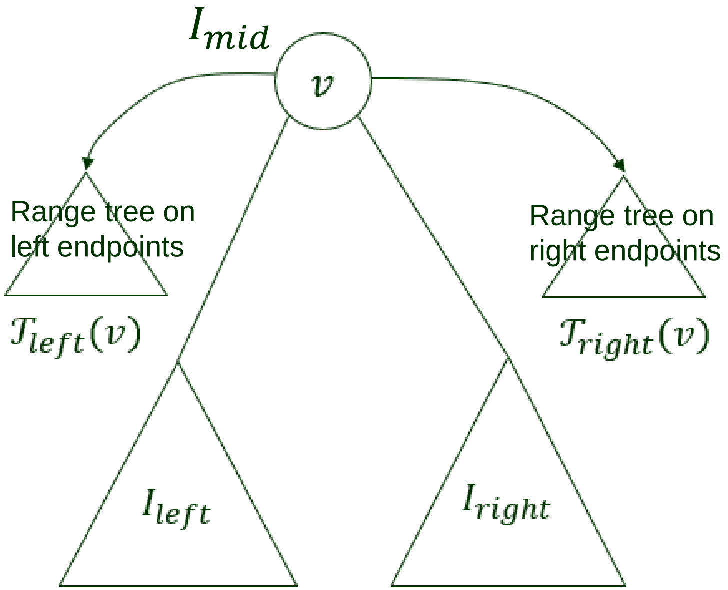

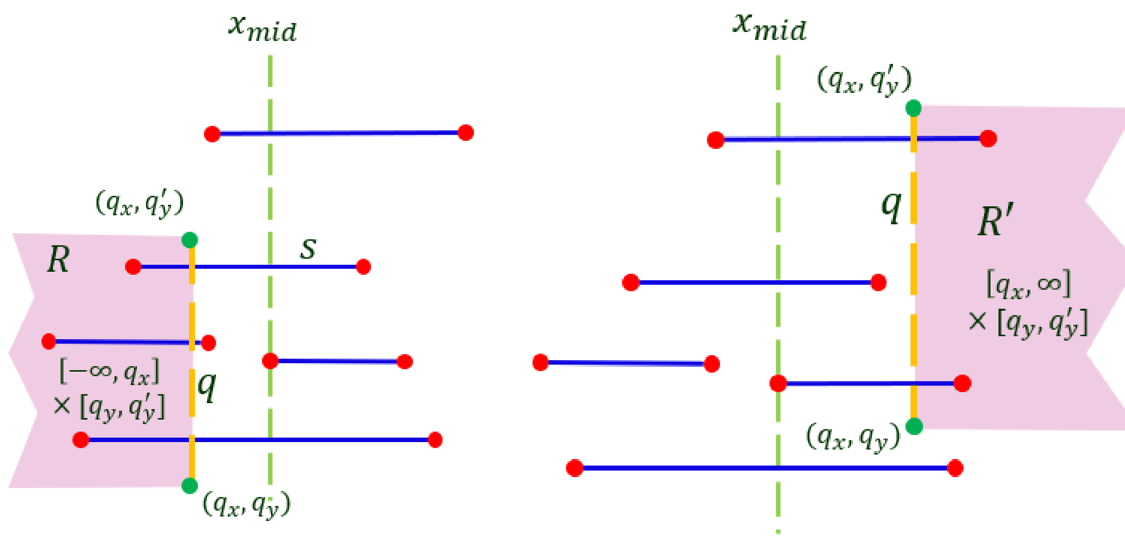

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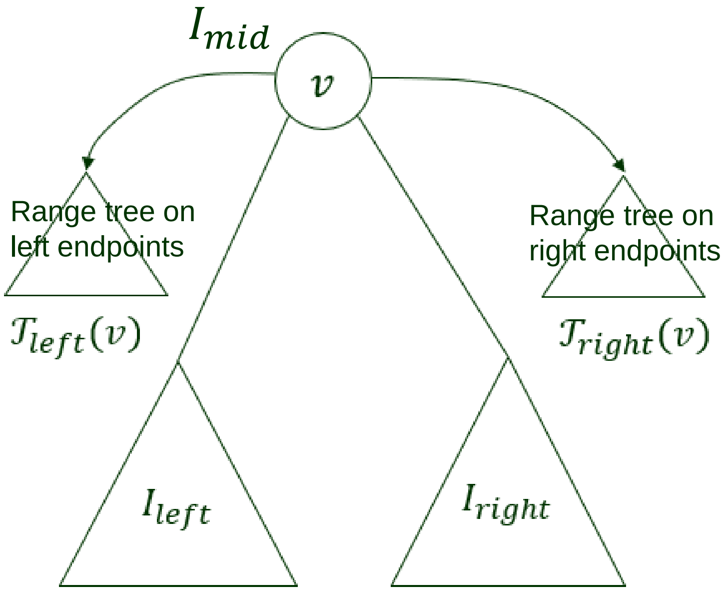

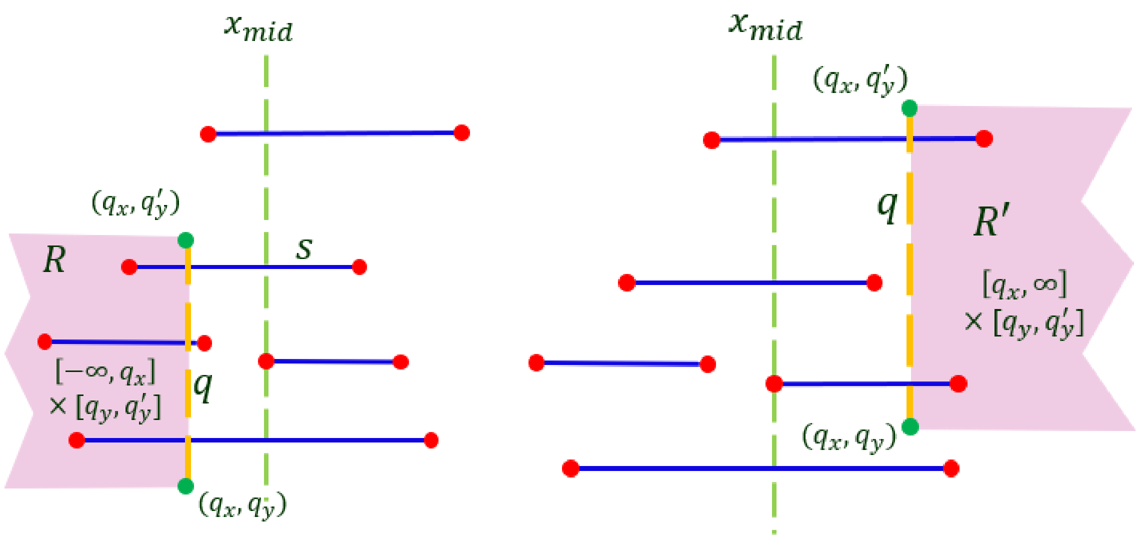

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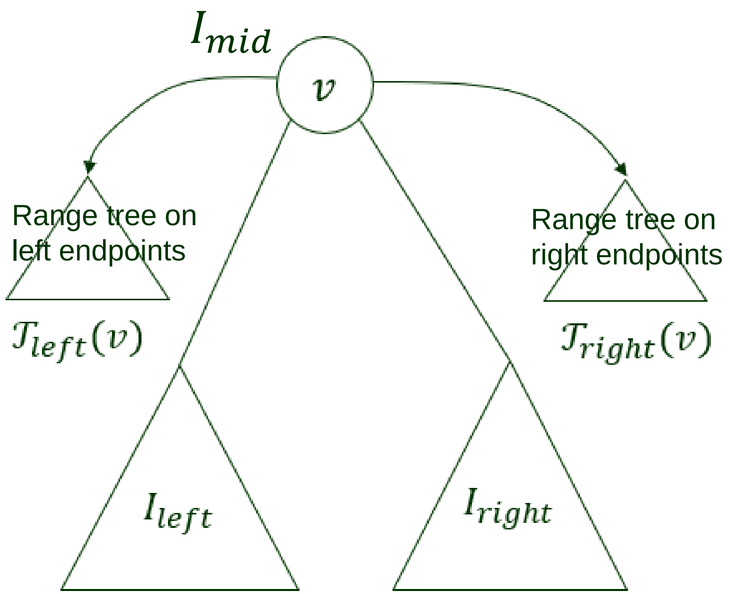

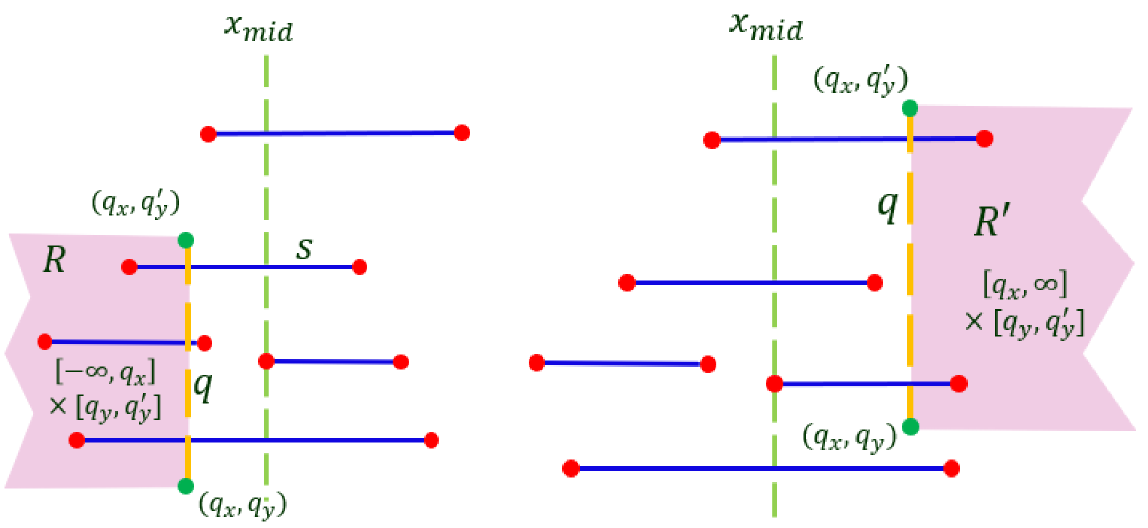

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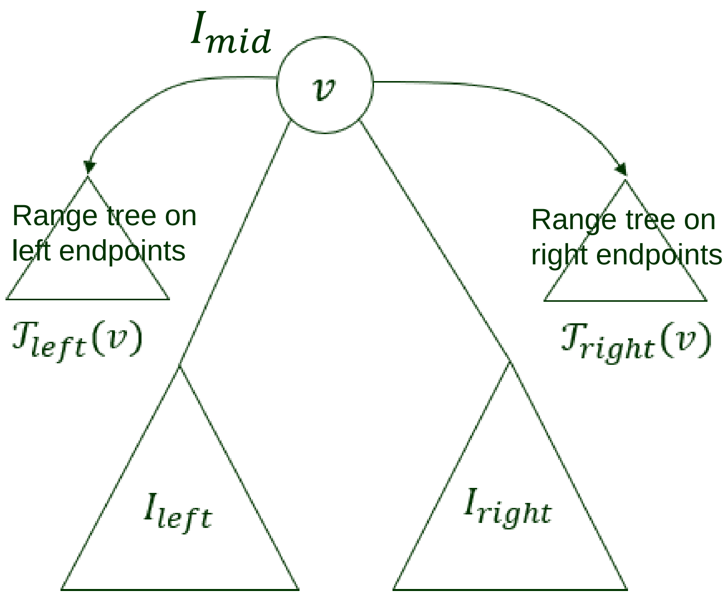

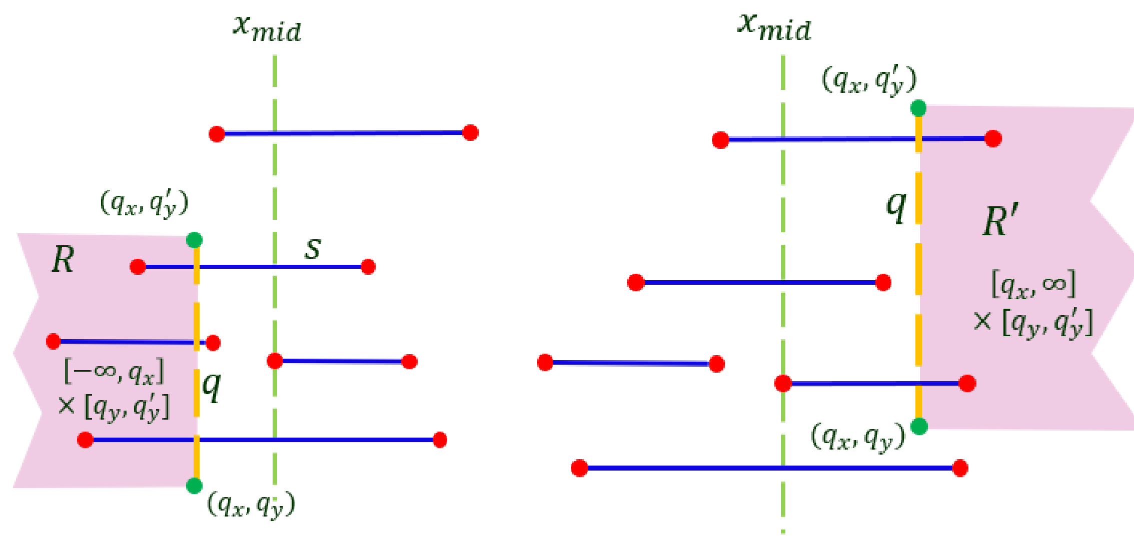

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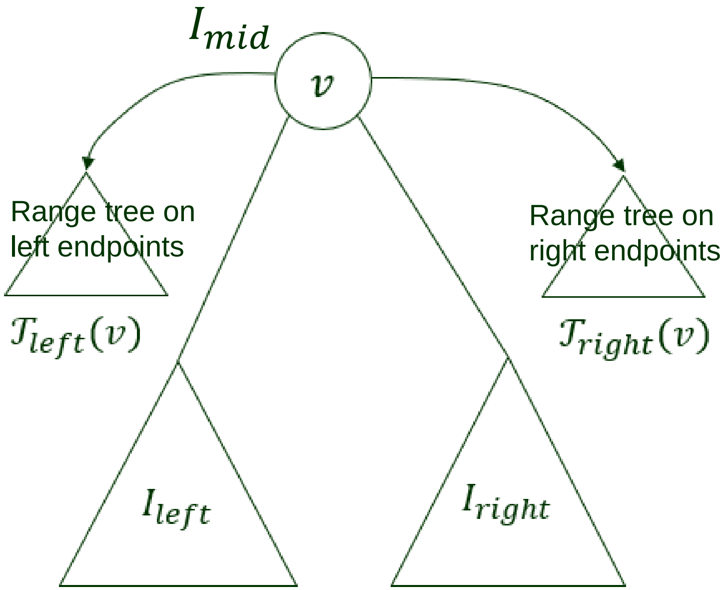

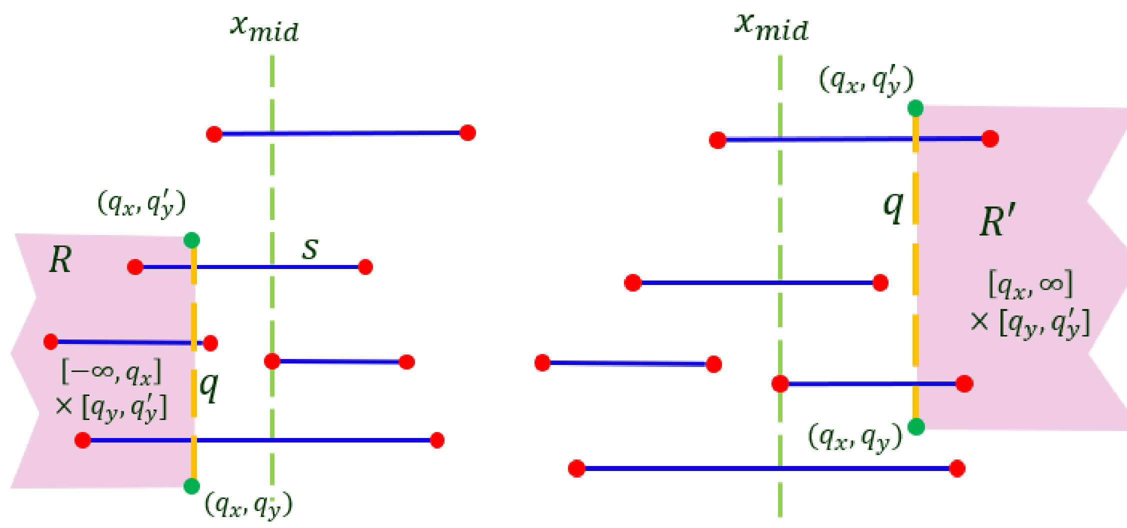

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Priority search tree

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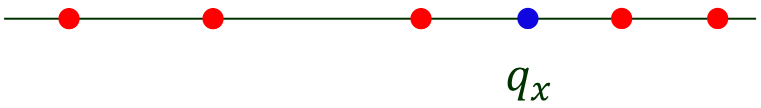

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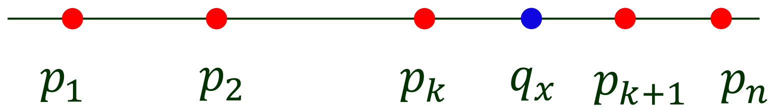

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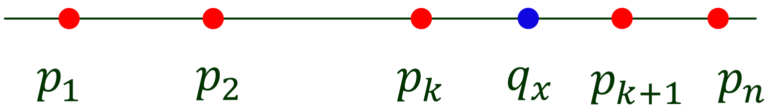

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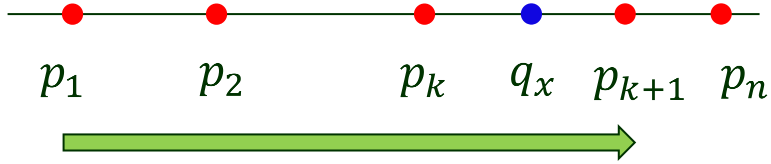

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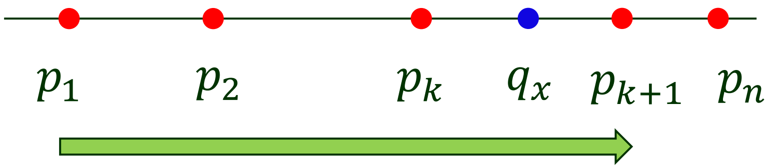
- Report $p_1, p_2, \dots, p_k$ during the walk



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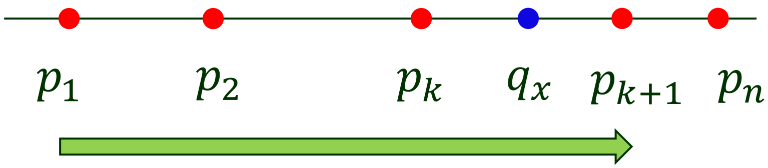

$O(1 + k)$

Moving on to 2D Query

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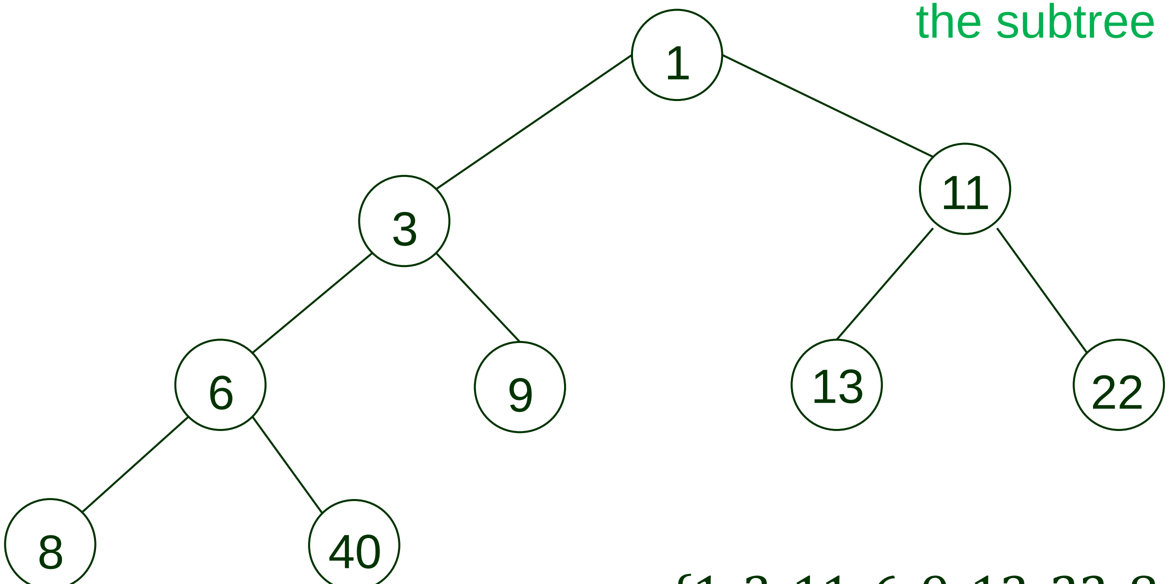

$\{1, 3, 11, 6, 9, 13, 22, 8, 40\}$

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Point set: $P = \{p_1, p_2, \ldots, p_n\}$

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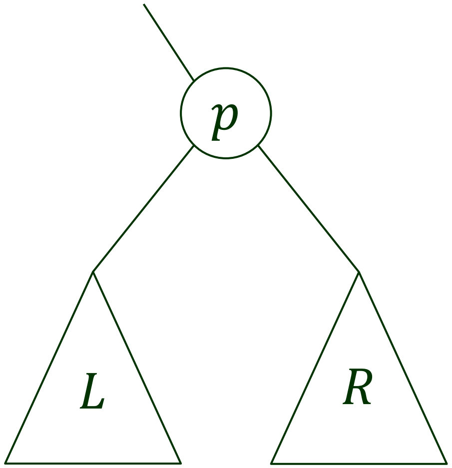

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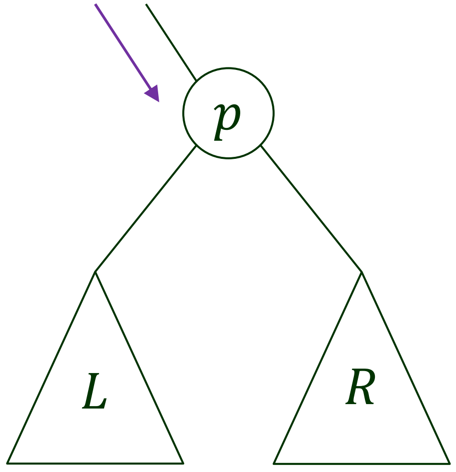

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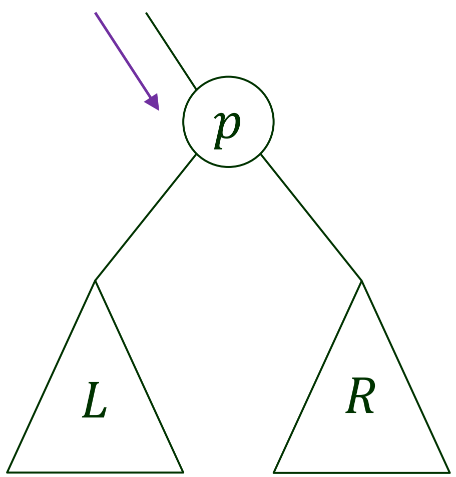

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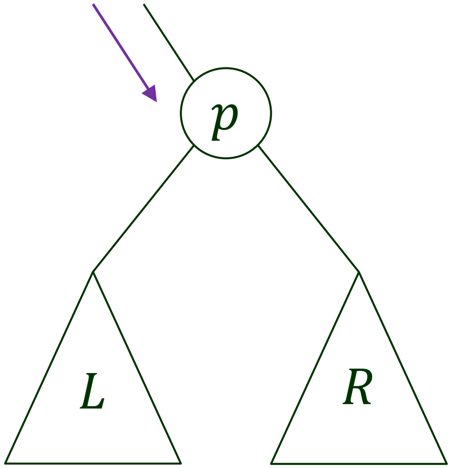

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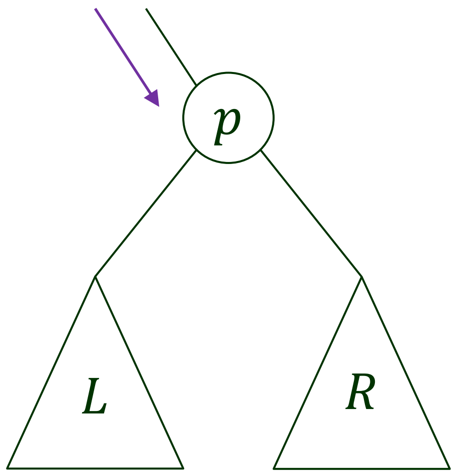

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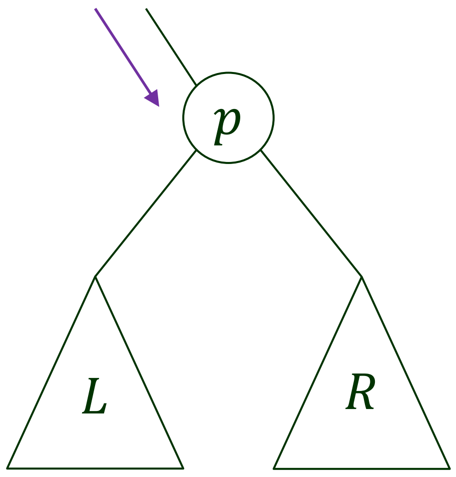

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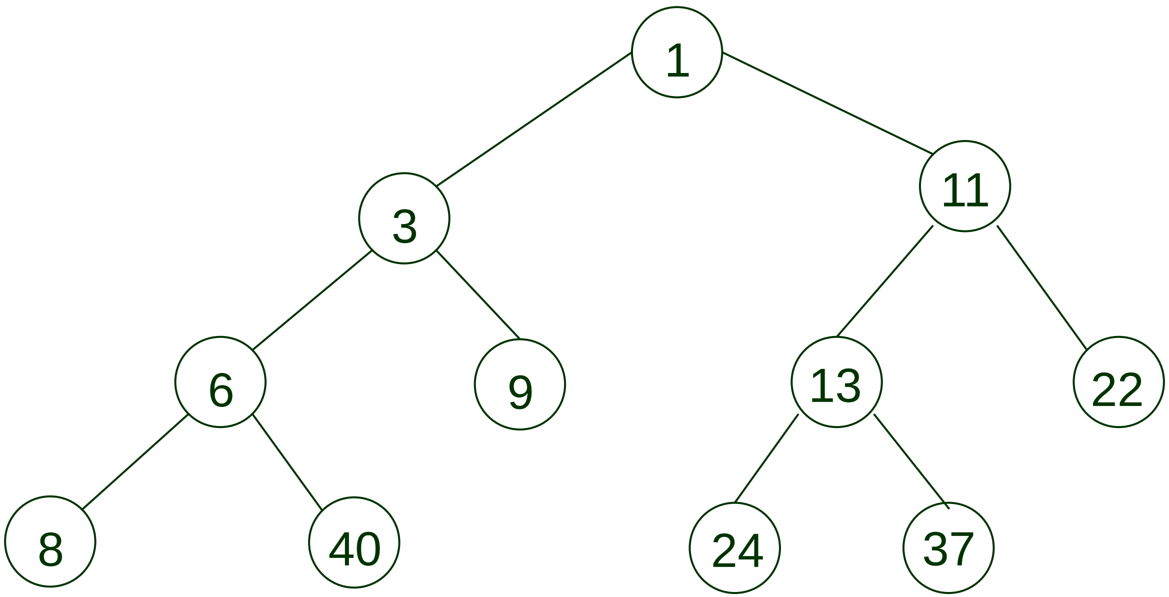
Every node $r = (r_x, r_y) \neq p$ in $\mathcal{T}(p)$ satisfies

$$r_x \ge p_x > q_x$$

Example

$(-\infty, 12]$ on the heap below.



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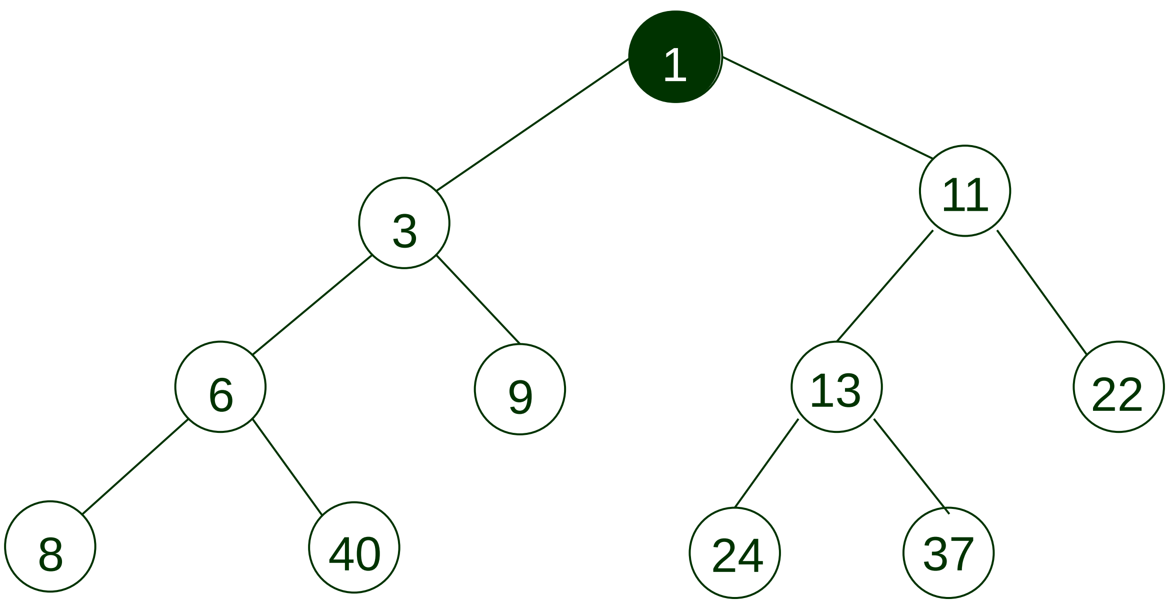

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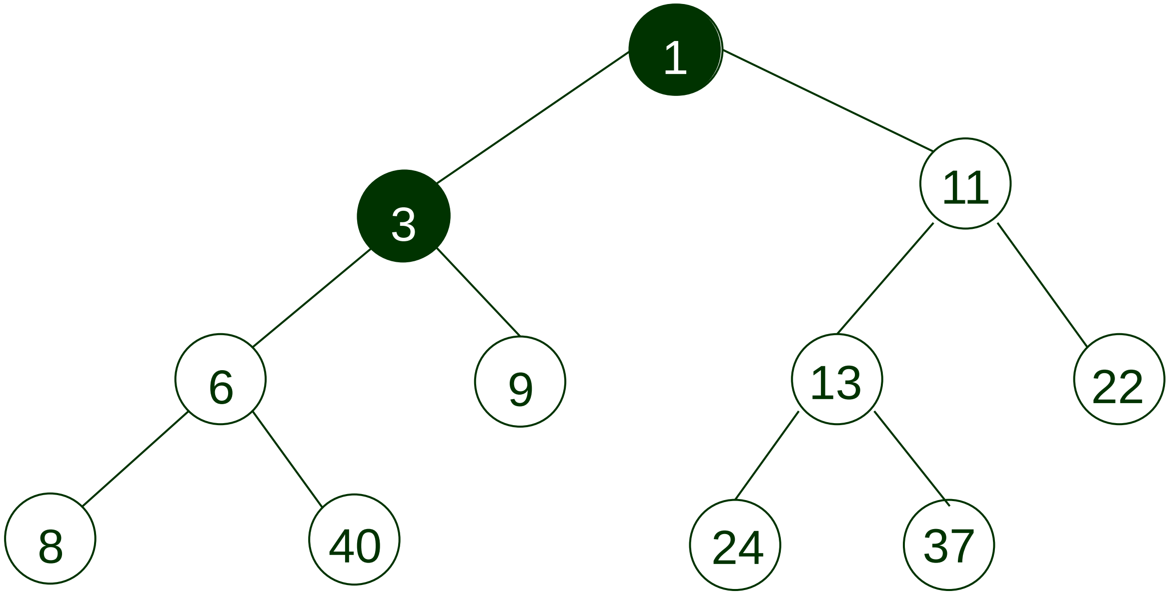

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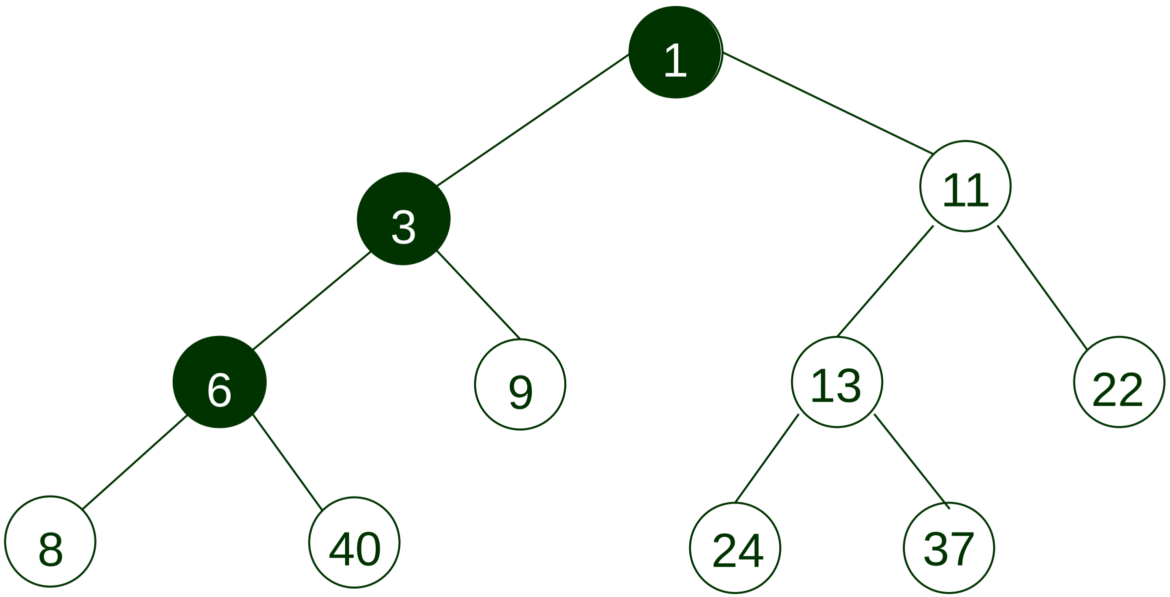

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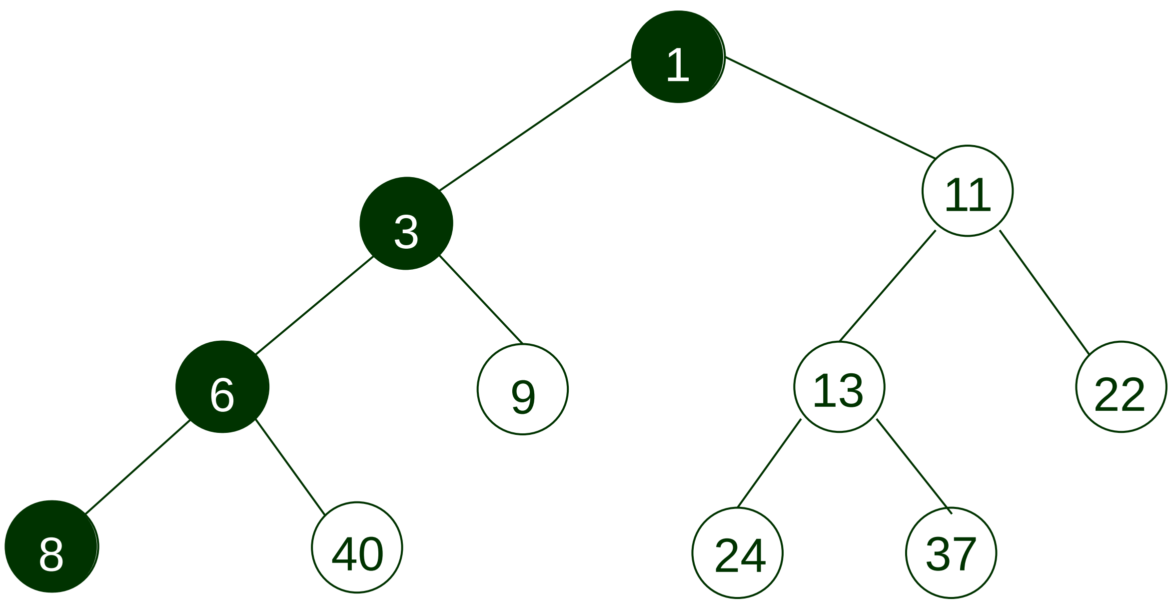

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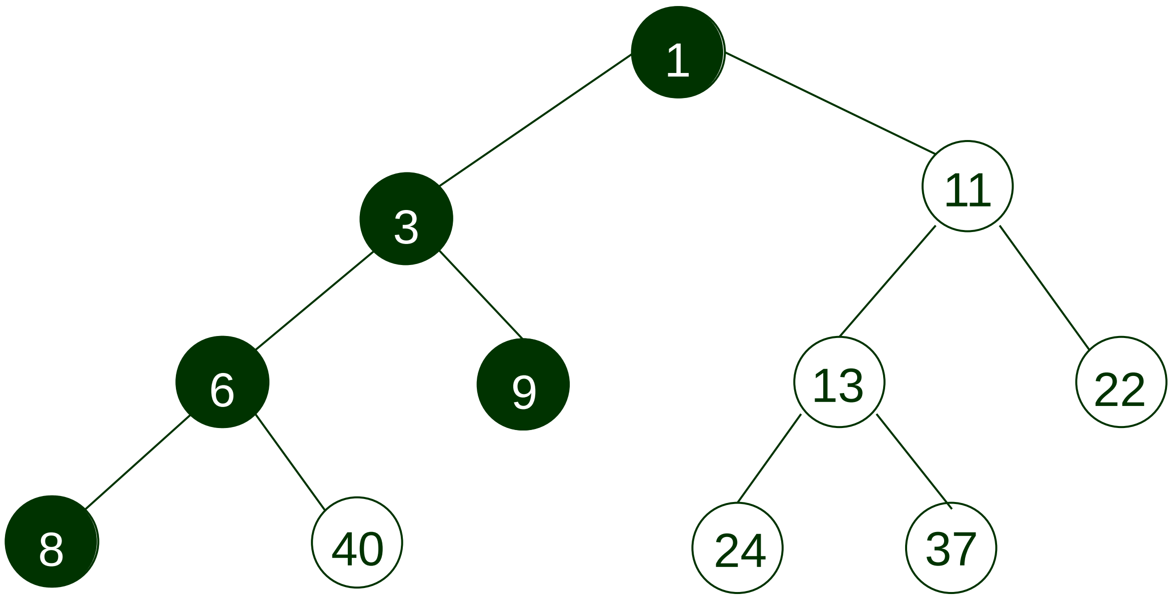

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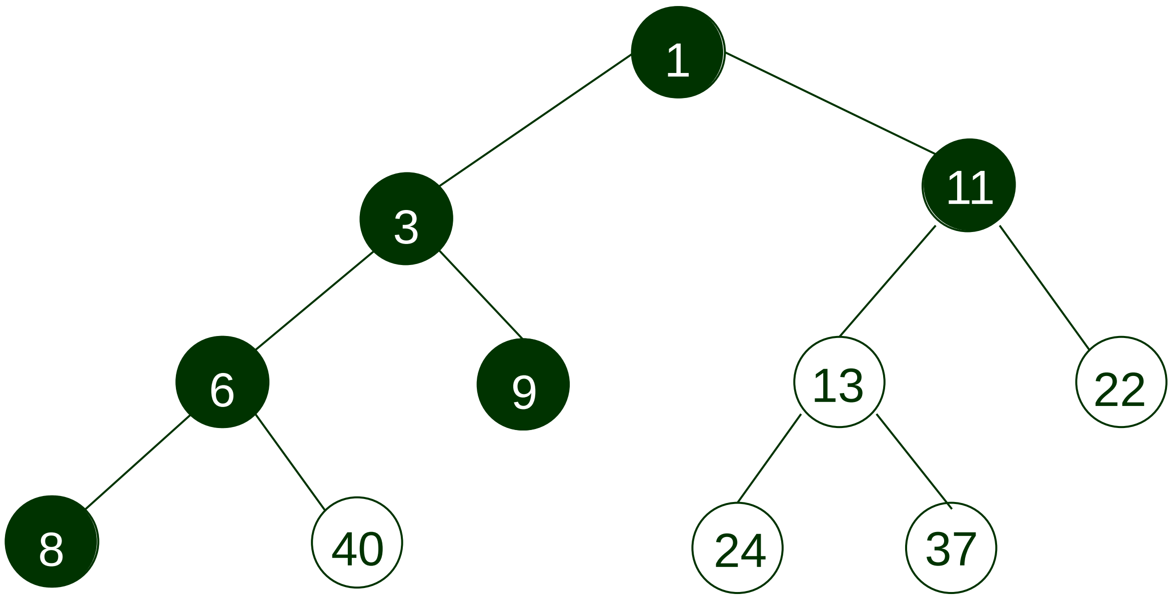

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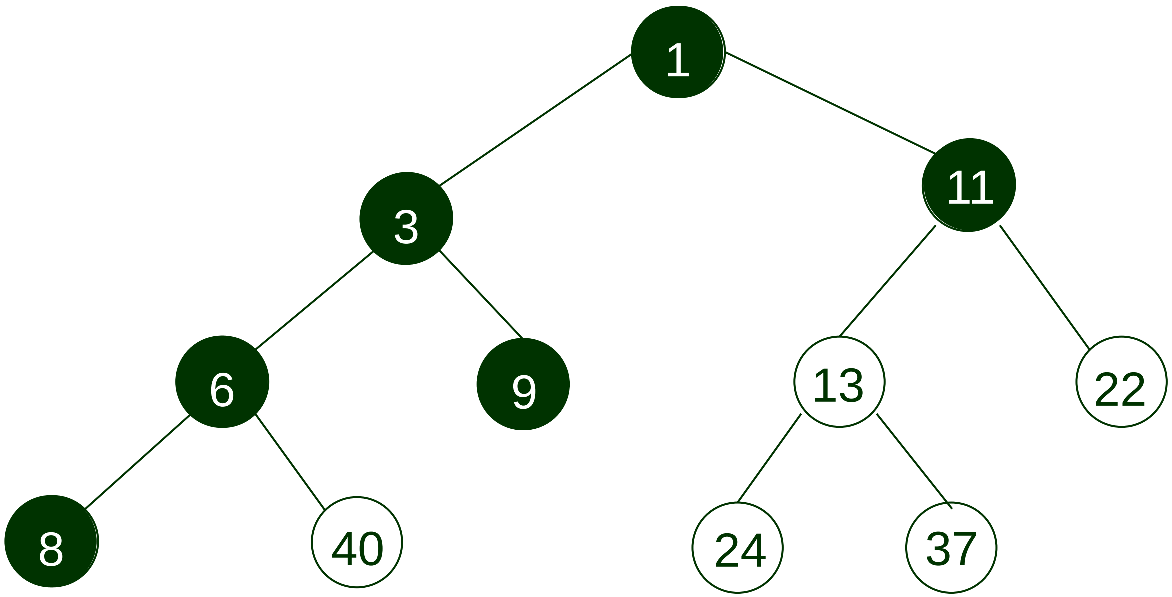

Report 1, 3, 6, 8, 9, 11.

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Split the remainder of the set into two subsets such that the points in one subset have their y-coordinates less than those of the points in the other subset.

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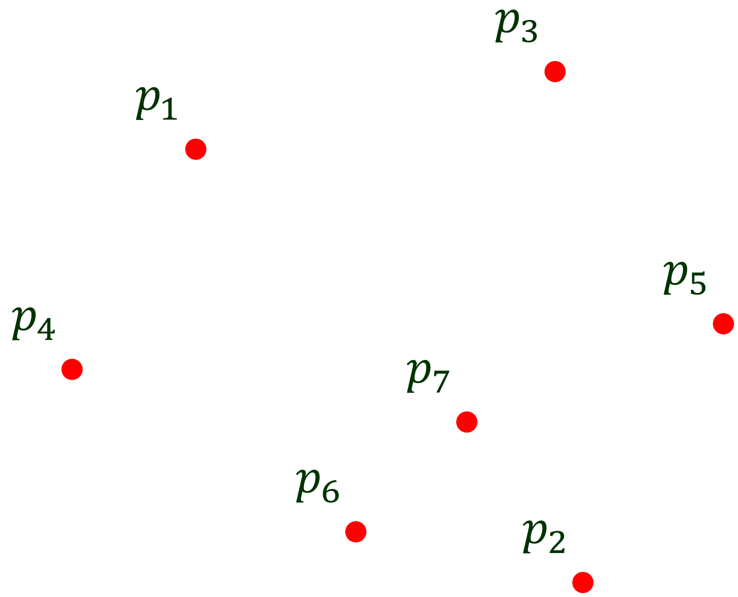

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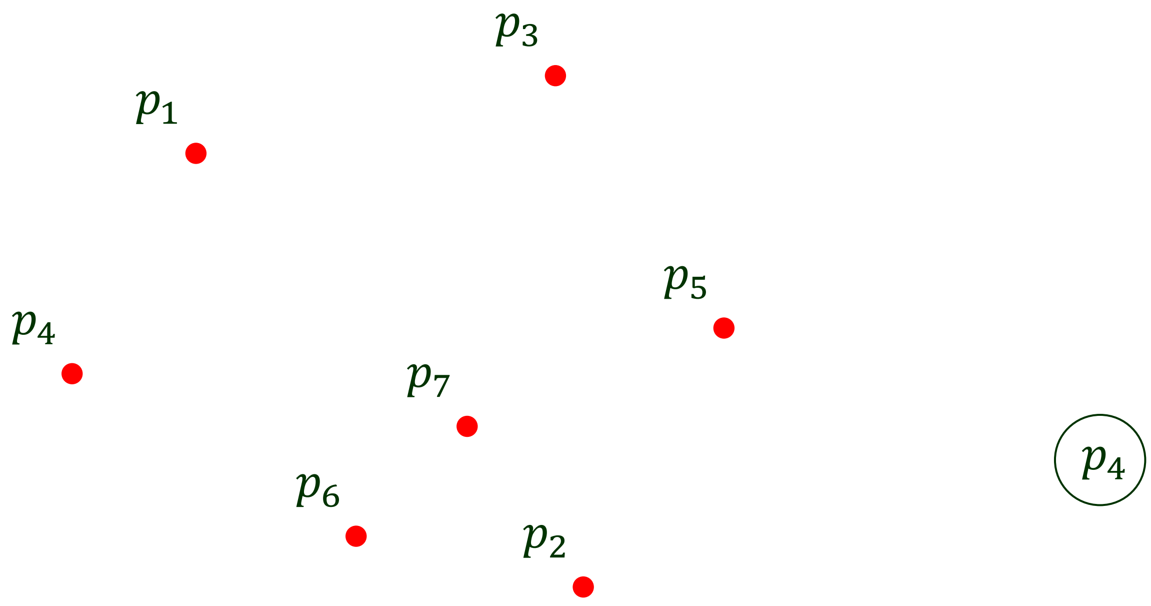

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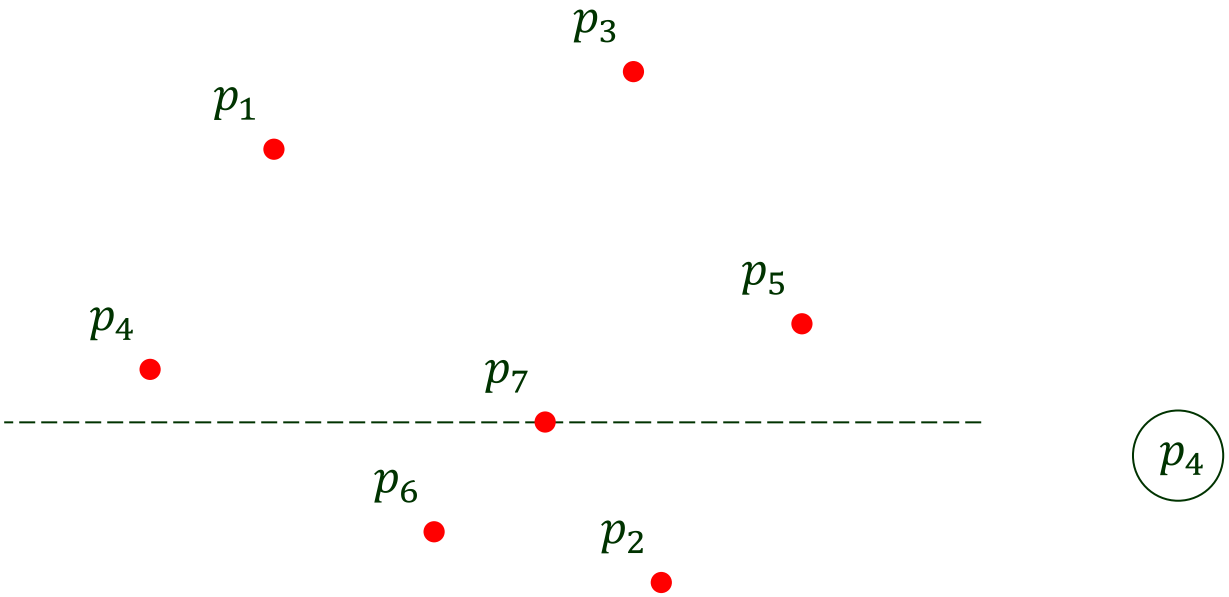

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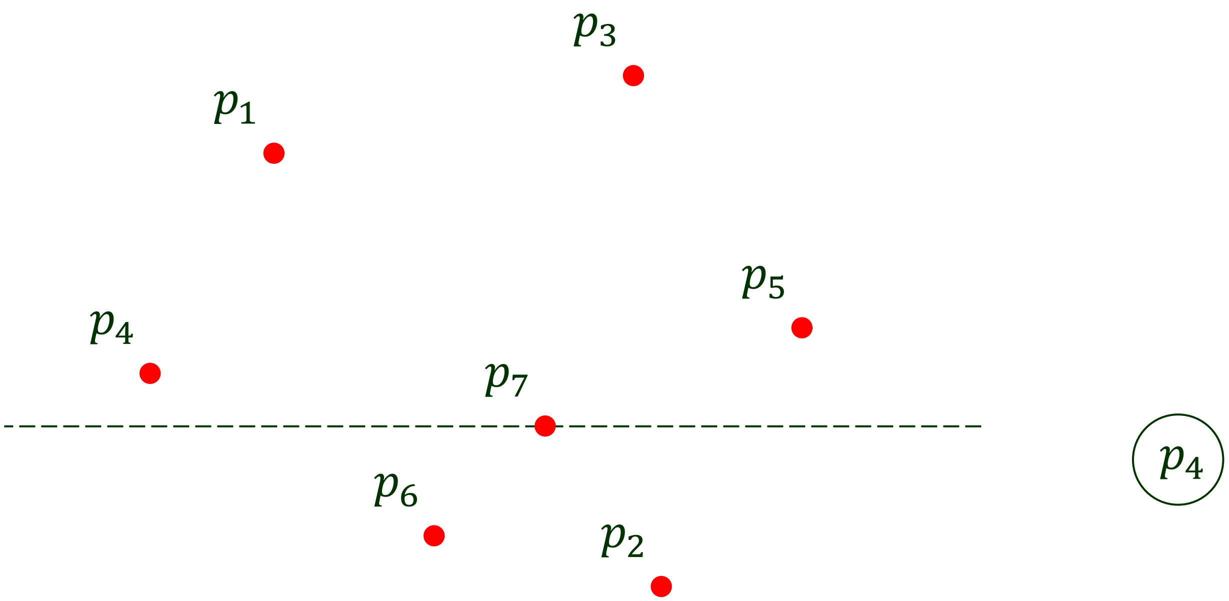

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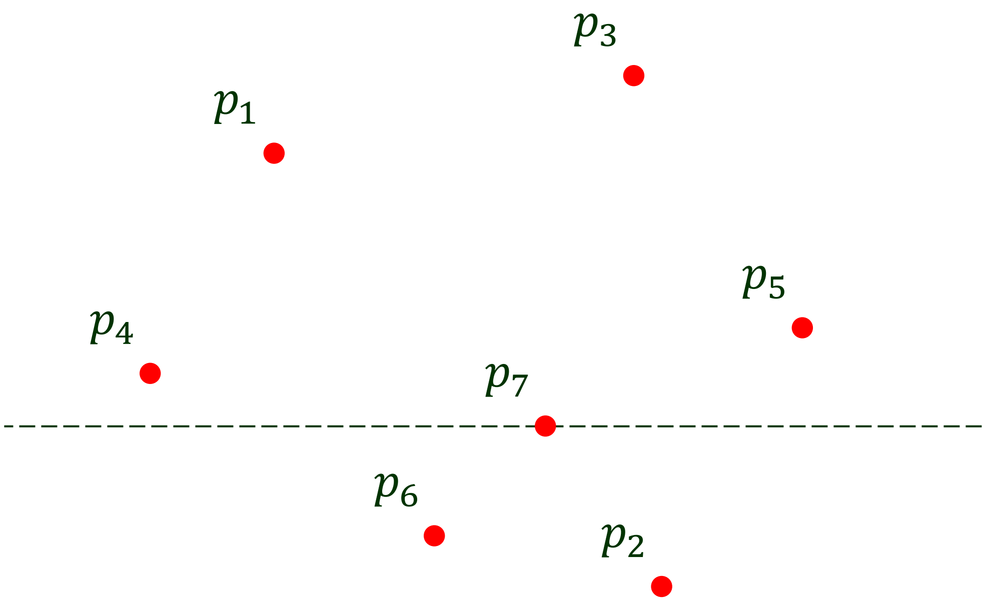

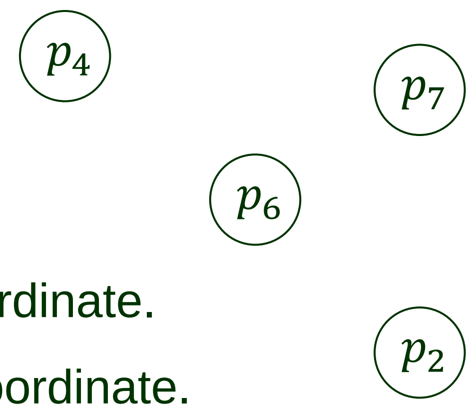

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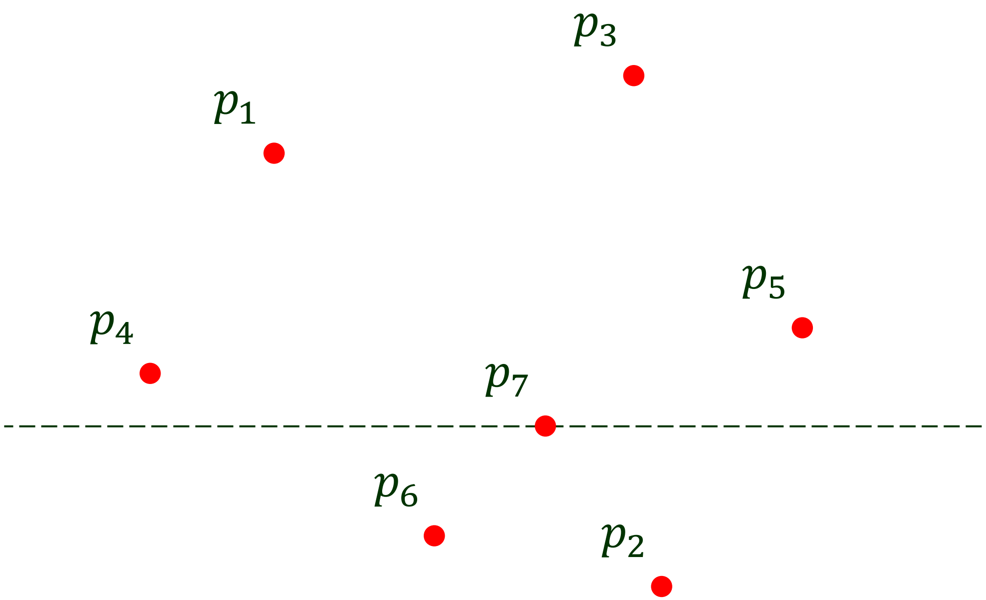

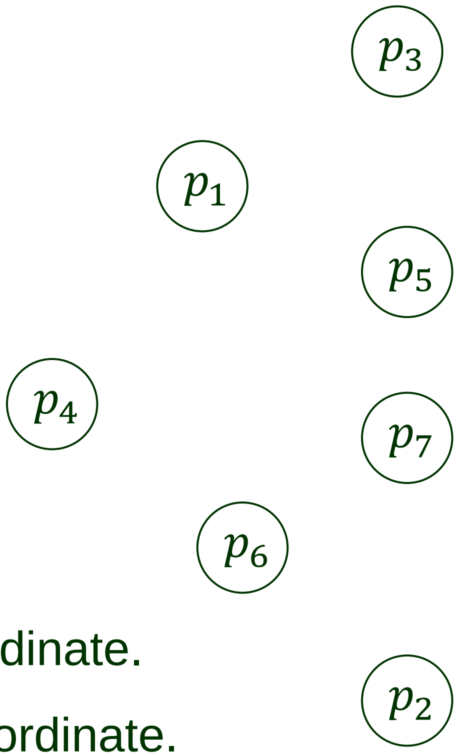

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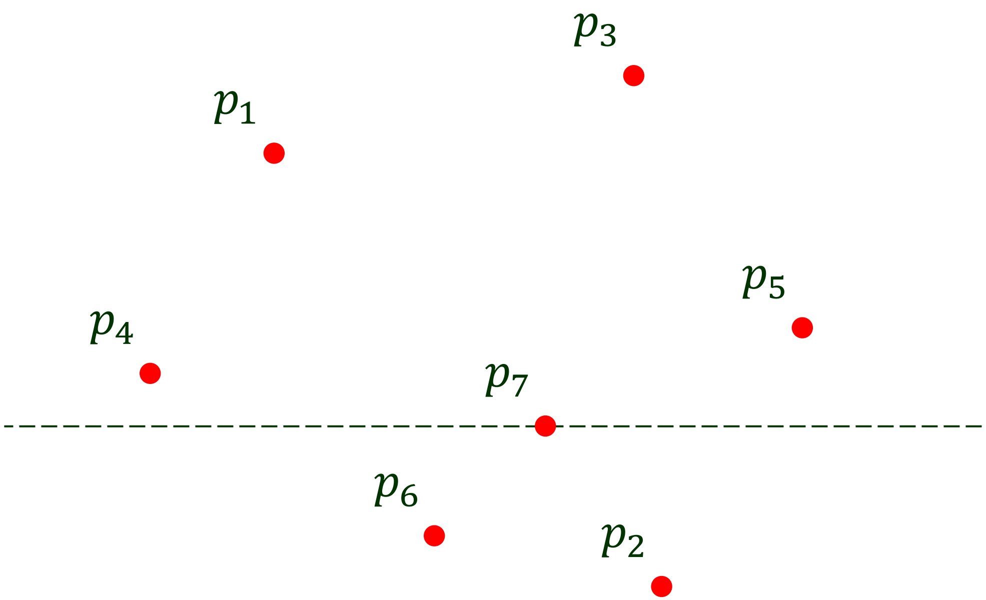
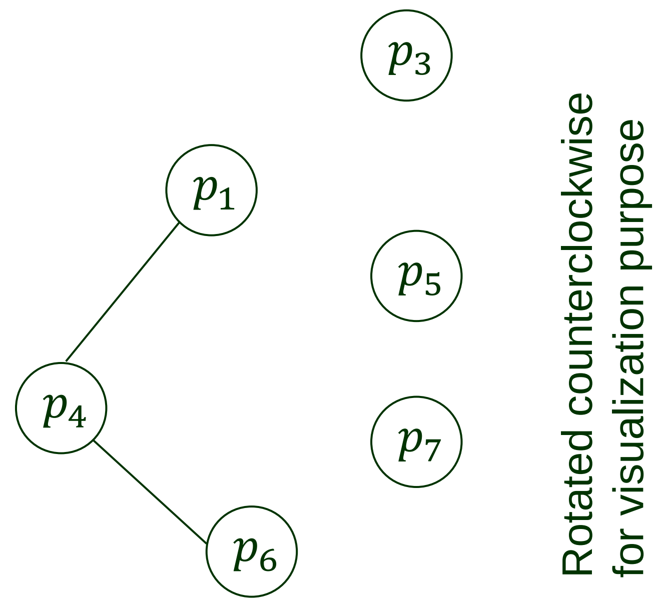
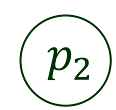
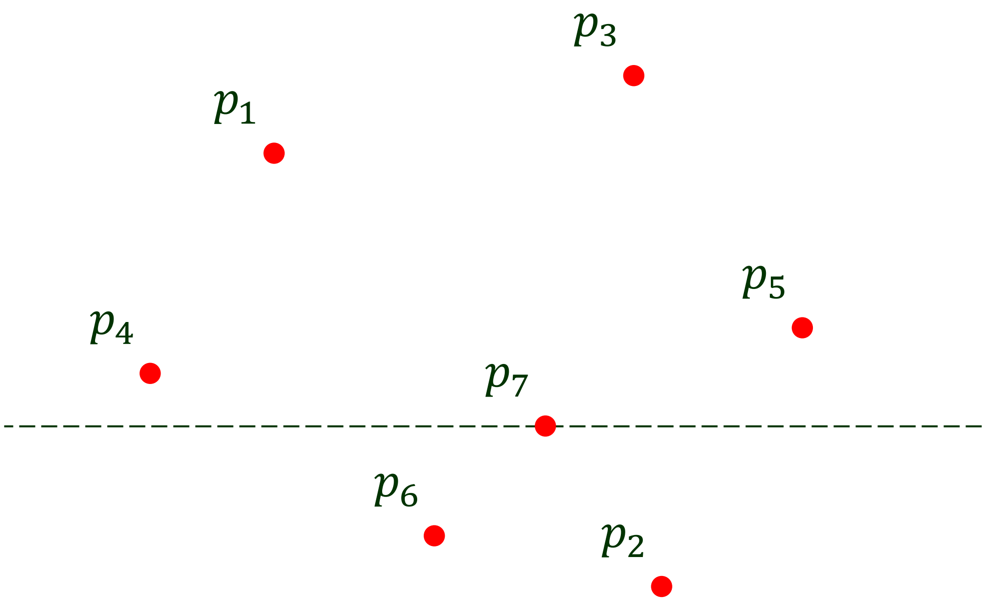


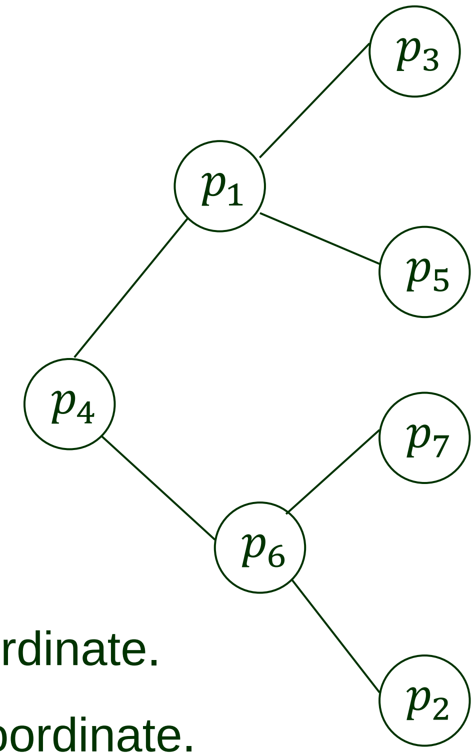




- p_4 is at the root because it has the smallest x-coordinate.
- Of the remaining 6 points, p_7 has the median y-coordinate.
- This median splits them into two groups: p_2, p_6, p_7 stored in the left (lower) subtree and p_1, p_3, p_5 stored in the right (upper) subtree.
- p_6 and p_1 are the roots of the two subtrees because they have the smallest x-coordinates in their groups, and so on.

II. Priority Search Tree (PST)





Rotated counterclockwise
for visualization purpose

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Assumption No two points have the same x- or y-coordinate.

(Easily removable with lexicographic ordering)

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  - Create v p_{min}, y_{mid}

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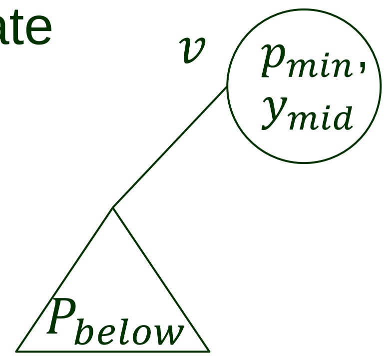

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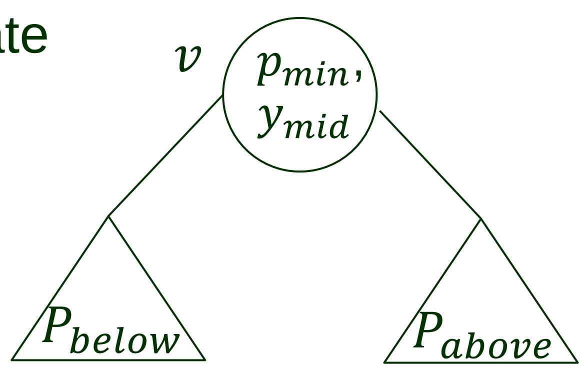

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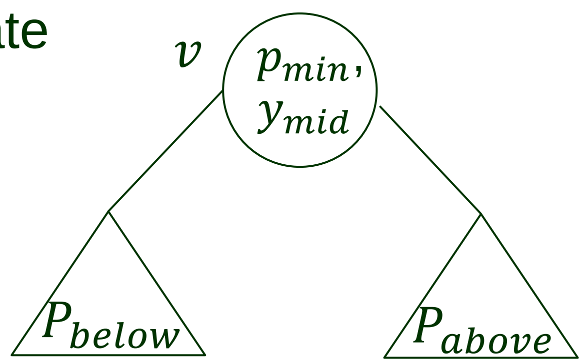
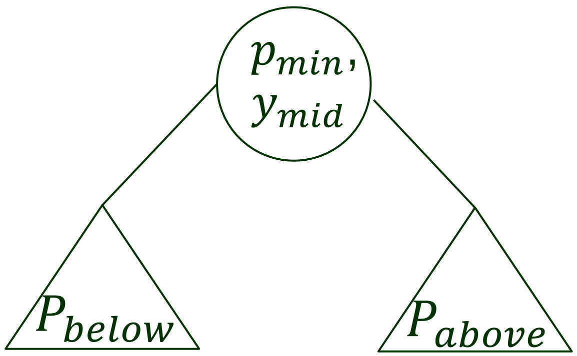


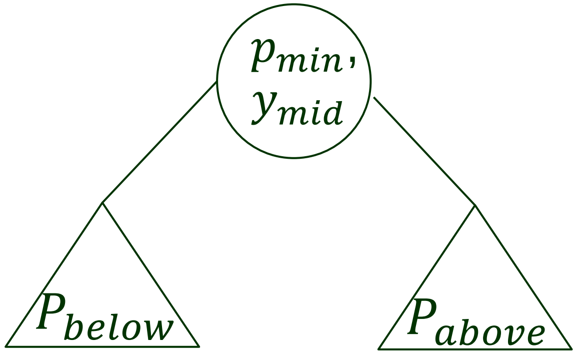
- Points are not stored at leaves only.
- Every node stores a different point.

Construction Time



$O(n \log n)$ if recursively (top-down)

Construction Time



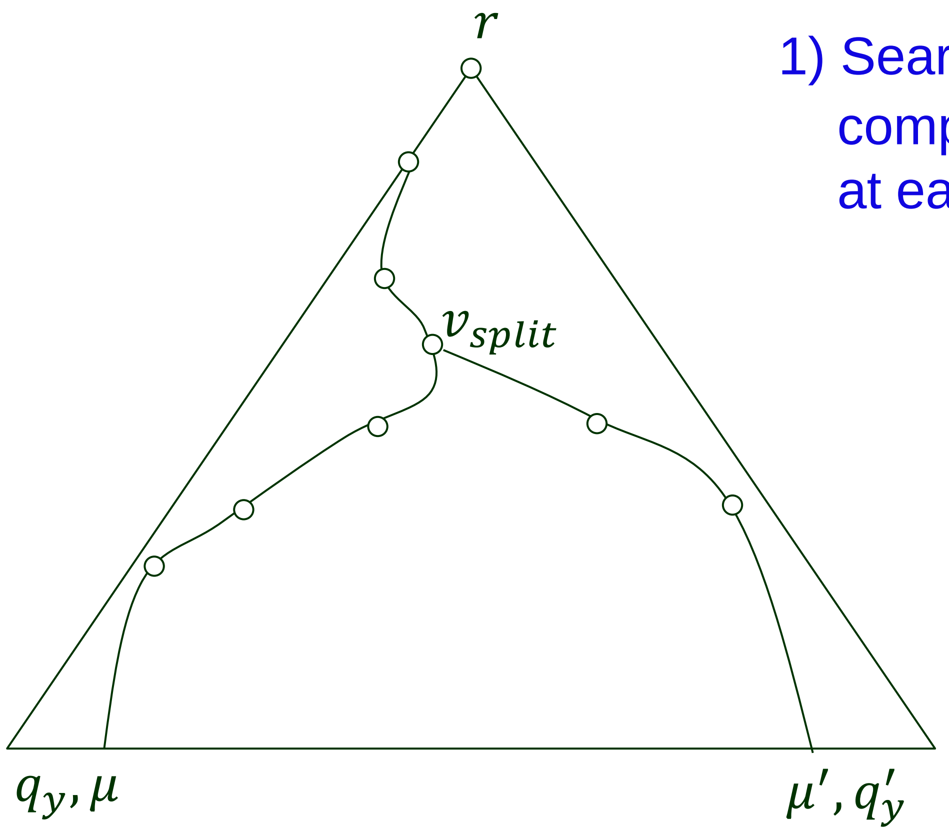
$O(n \log n)$ if recursively (top-down)

$O(n)$ if

- the points are pre-sorted on y-coordinate, and
- constructed bottom-up in the way of building a heap.

III. Query

Query range: $(-\infty, q_x] \times [q_y, q_y']$



1) Search the PST with q_y and q_y' by comparing them with the y_{mid} value at each node.

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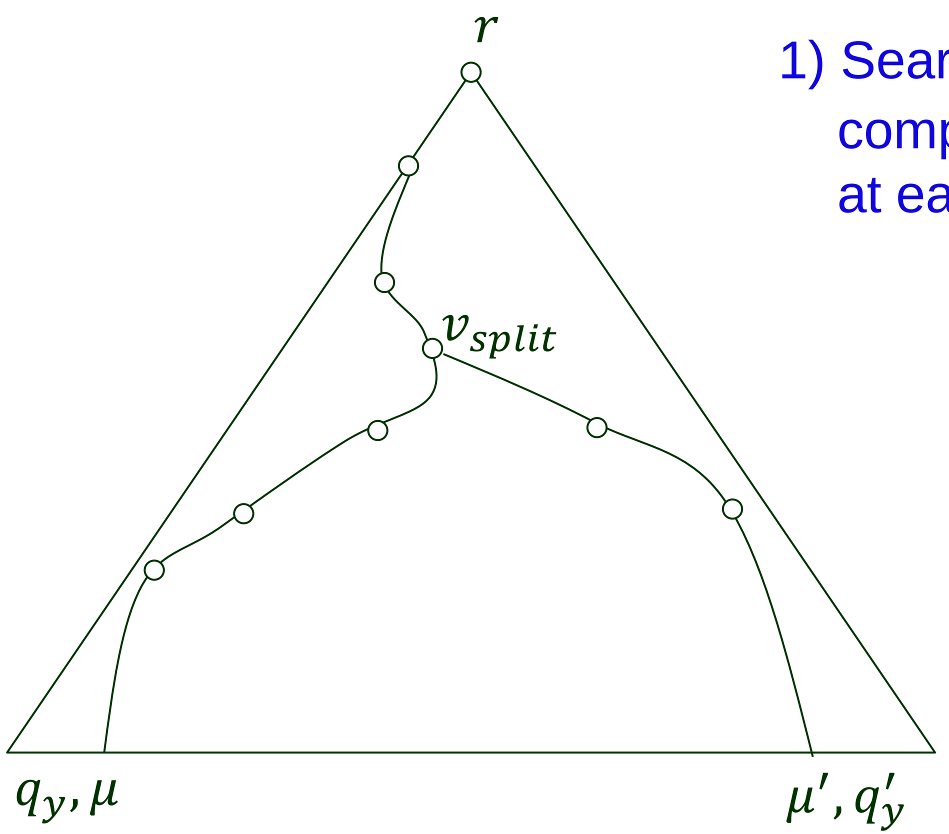

1) Search the PST with q_y and q_y' by comparing them with the y_{mid} value at each node.

1D range searching

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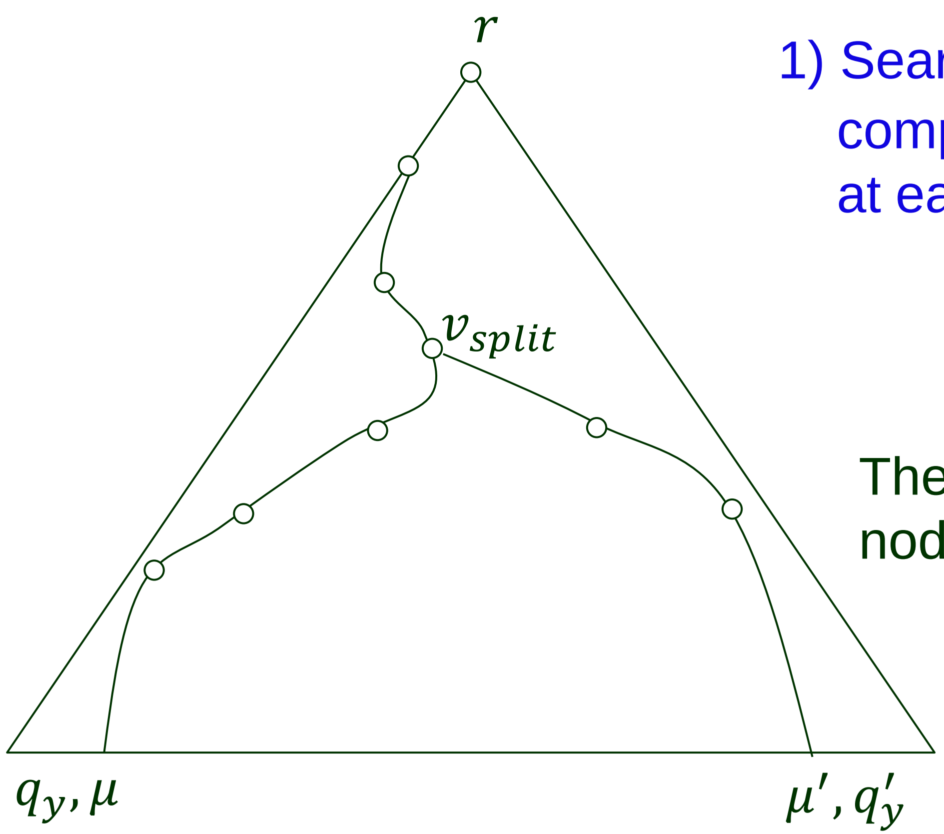
Query range: $(-\infty, q_x] \times [q_y, q'_y]$



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The two searches end at the nodes μ and μ', respectively.

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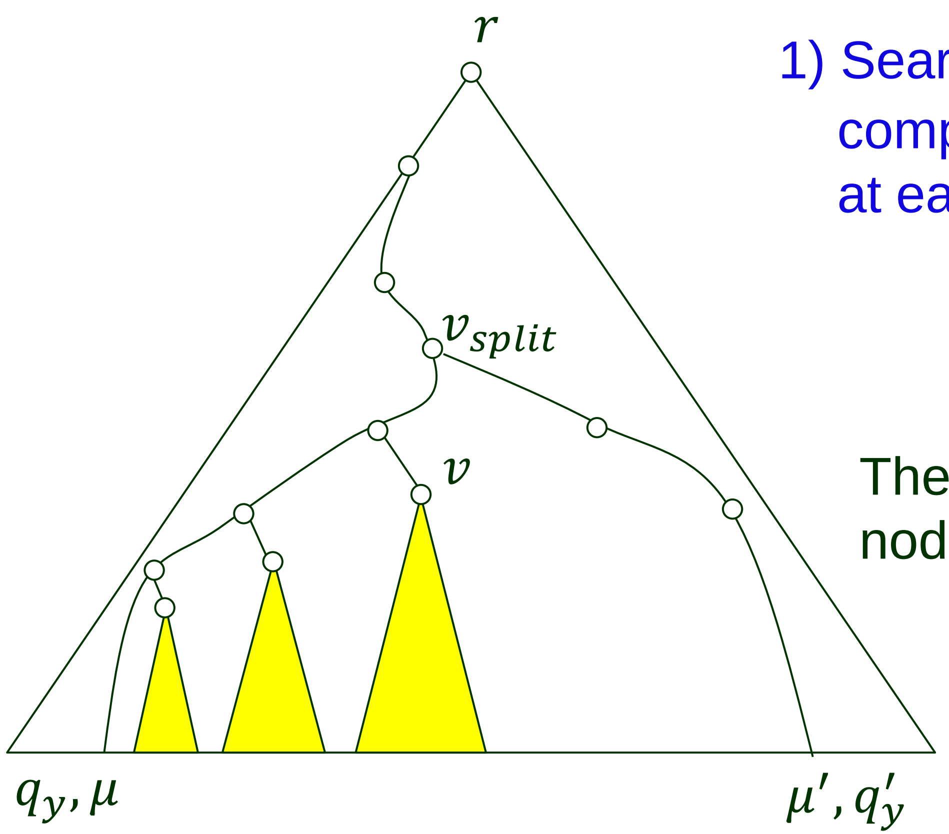

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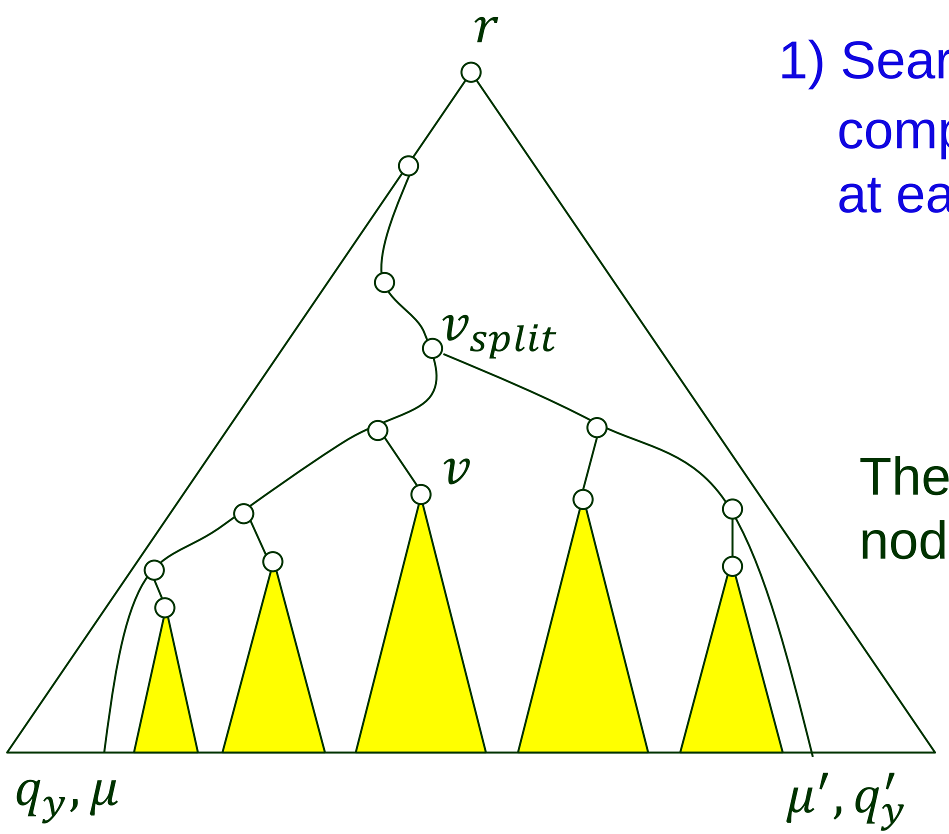

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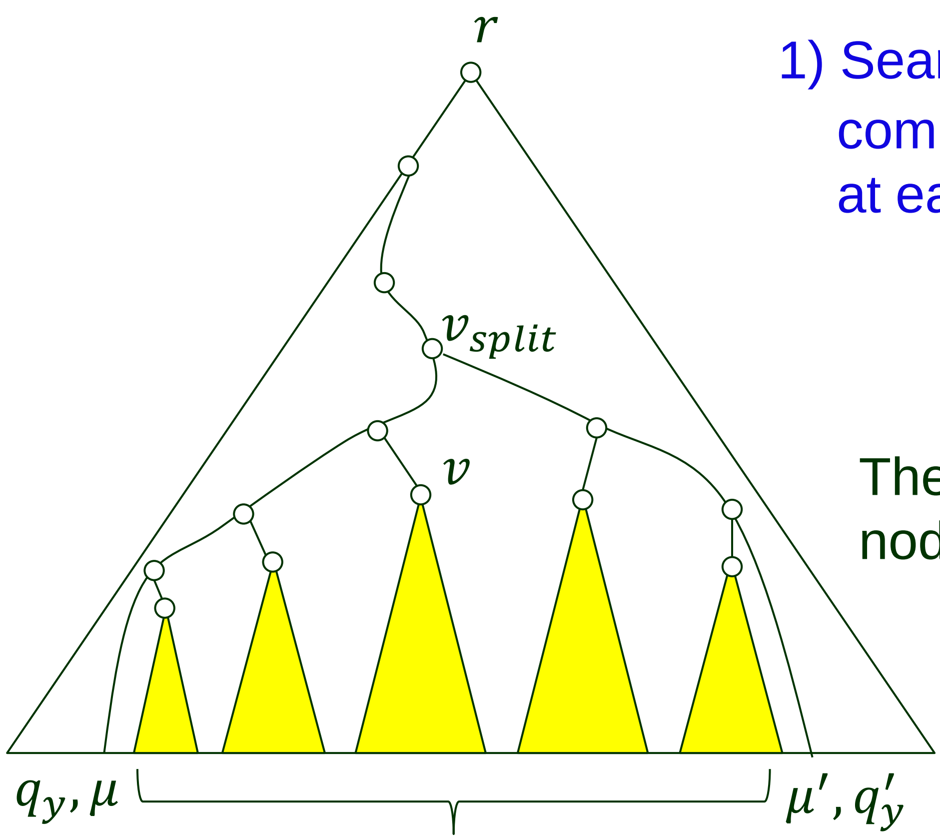


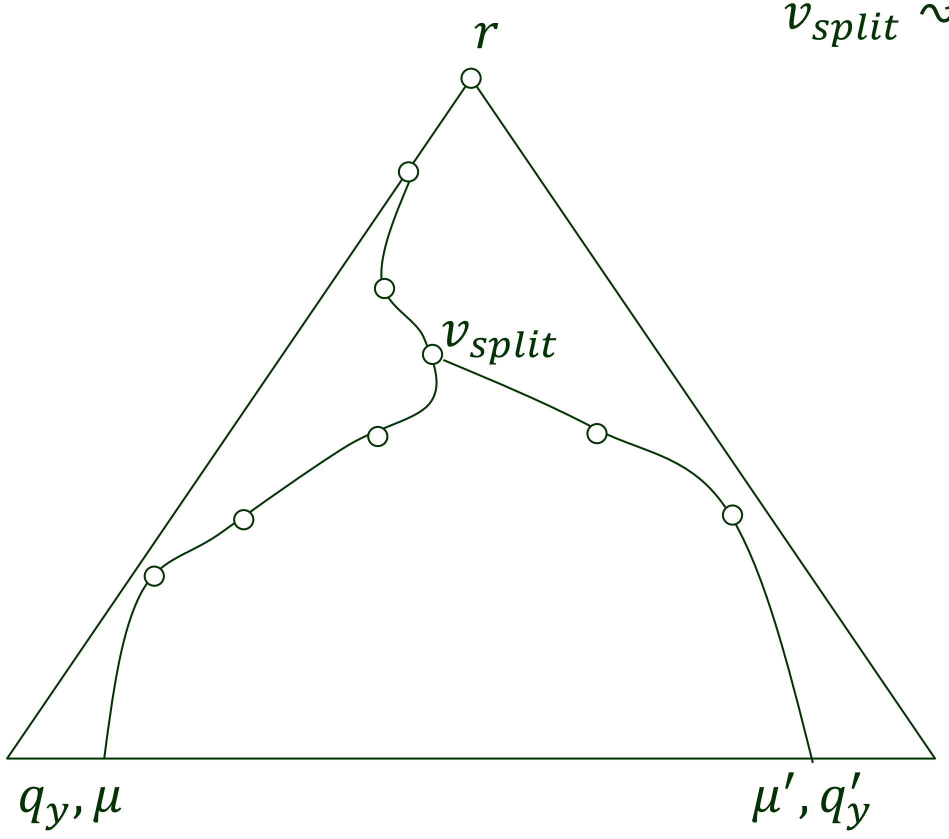
Selected subtrees for future searches based on x-coordinates

1) Search the PST with q_y and q_y' by comparing them with the y_{mid} value at each node.

1D range searching

The two searches end at the nodes μ and μ', respectively.

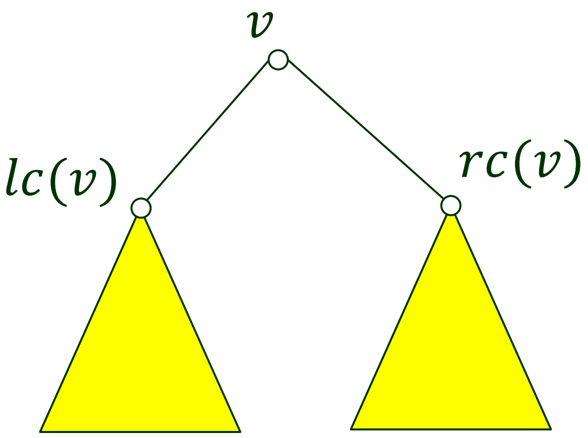
Nodes on the Search Paths



Check every node v on every one of the three paths, $r \rightsquigarrow v_{split}$, $v_{split} \rightsquigarrow \mu$ and $v_{split} \rightsquigarrow \mu'$ to see if

$$p(v) \in (-\infty, q_x] \times [q_y, q_y']$$

Search in the Selected Subtrees



min heap
on x-coordinate

2) Search every selected subtree based on x-coordinate as in a one-dimensional array.

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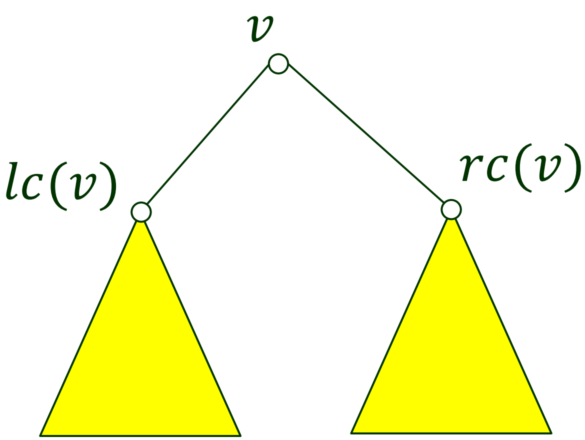

min heap
on x-coordinate

```
ReportInSubtree(v, q_x)
// incorrect in the text for omitting
// the case of v as a leaf
if (p(v))_x ≤ q_x
    then report p(v)
    if v is not a leaf
        then ReportInSubtree(lc(v), q_x)
             ReportInSubtree(rc(v), q_x)
```

IV. Correctness of `ReportInSubtree()`

Lemma `ReportInSubtree(`v, q_x`)` reports in $O(1 + k_v)$ time all the k_v points in the subtree $\mathcal{T}(v)$ whose x-coordinate is at most q_x.

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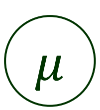
Proof Consider a node μ in $\mathcal{T}(v)$ such that its stored point $p(\mu)$ satisfies $\big(p(\mu)\big)_x \le q_x$.



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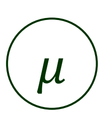

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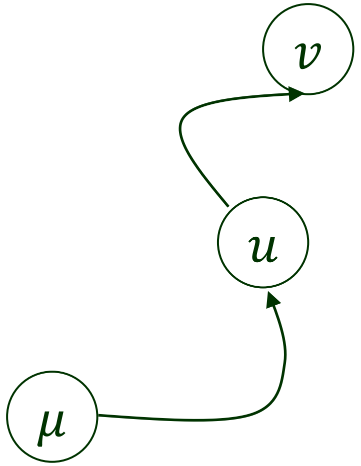

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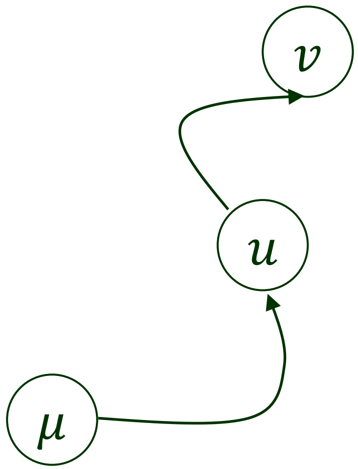

$$q_x \ge \big(p(\mu)\big)_x > \cdots > \big(p(u)\big)_x > \cdots > \big(p(v)\big)_x$$

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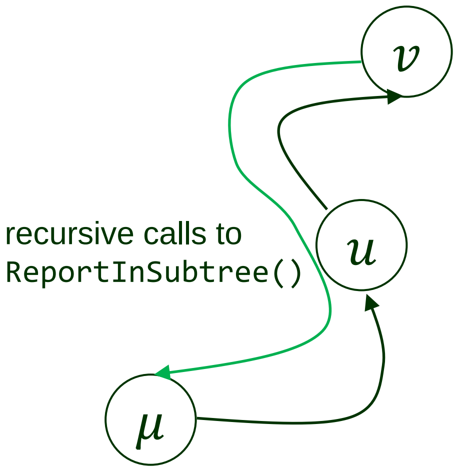
$$\Downarrow$$

Recursive calls to `ReportInSubtree` are invoked at all the nodes on the downward path $v \leadsto \mu$.



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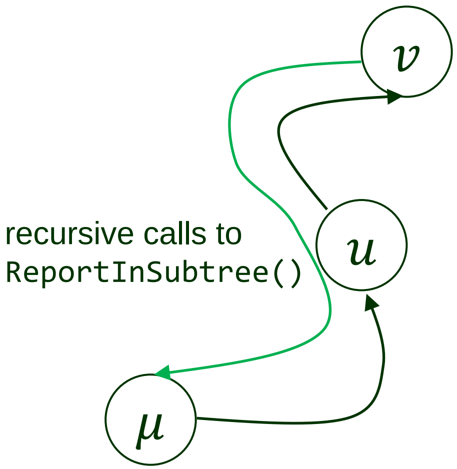
Recursive calls to `ReportInSubtree` are invoked at all the nodes on the downward path $v \rightsquigarrow \mu$.

$$\Downarrow$$

$p(\mu)$ is reported.



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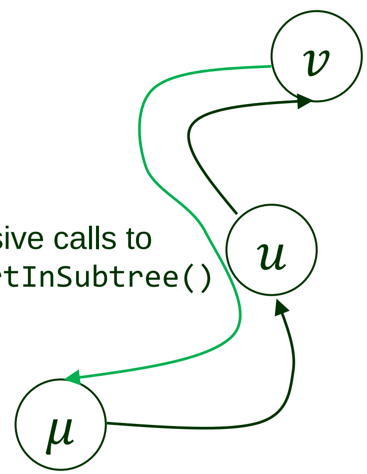

recursive calls to `ReportInSubtree()`

The time $O(1+k_v)$ follows from $O(1)$ effort spent on each node .

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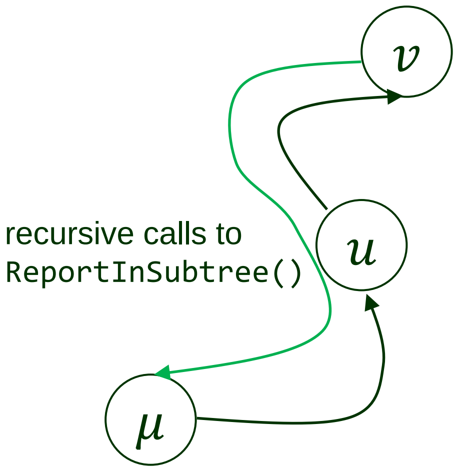

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The time $O(1 + k_v)$ follows from $O(1)$ effort spent on each node . □

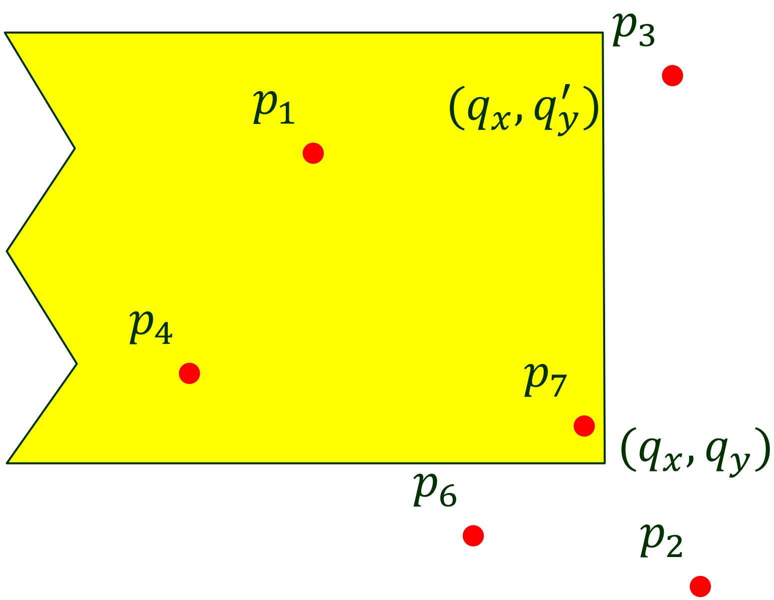
Query Algorithm

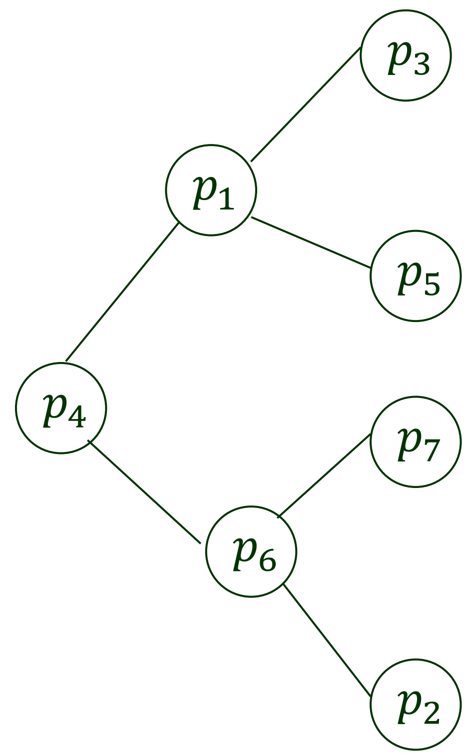
QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)

// r is the root of $\mathcal{T}$

1. search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'
2. let v_{split} be the node where the two paths split.
3. **for** each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
4.   **do if** $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$
5.   **then** report $p(v)$
6. **for** each node v on $v_{split} \rightsquigarrow \mu$
7.   **do if** the path goes left at v
8.   **then** ReportInSubtree($rc(v)$, q_x)
9. **for** each node v on $v_{split} \rightsquigarrow \mu'$
10.   **do if** the path goes right at v
11.   **then** ReportInSubtree($lc(v)$, q_x)

Example of Execution





QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q_y']$)
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ReportInSubtree(v, q_x)

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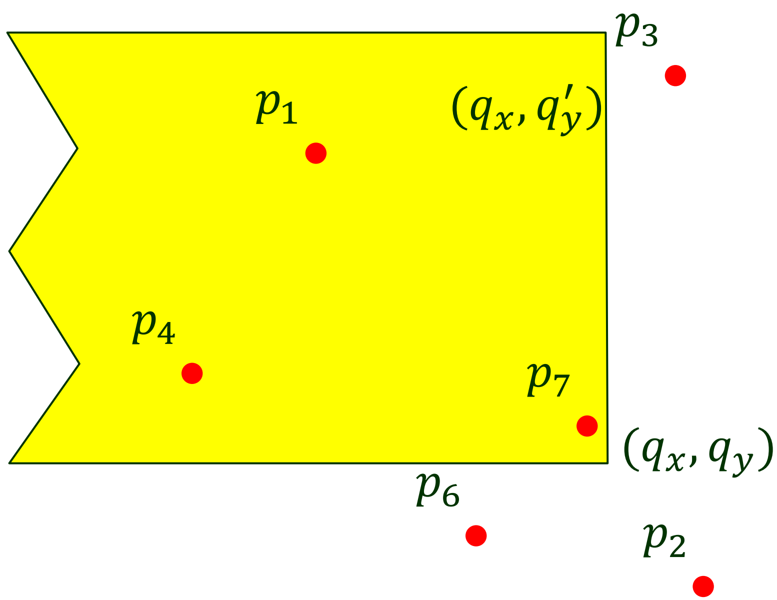
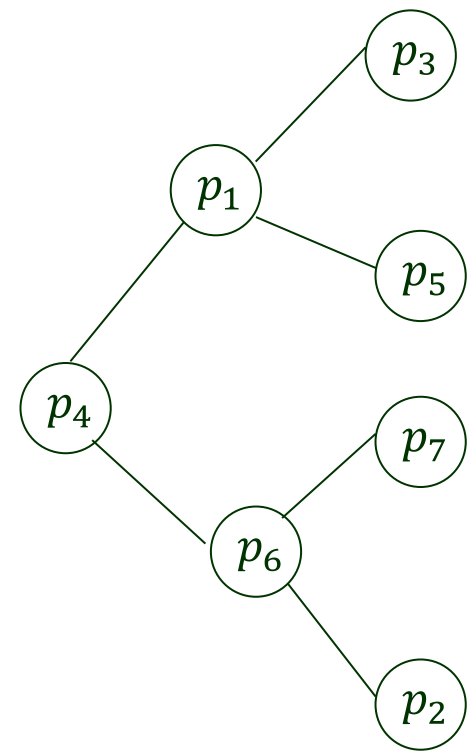


$v_{split} = p_4$



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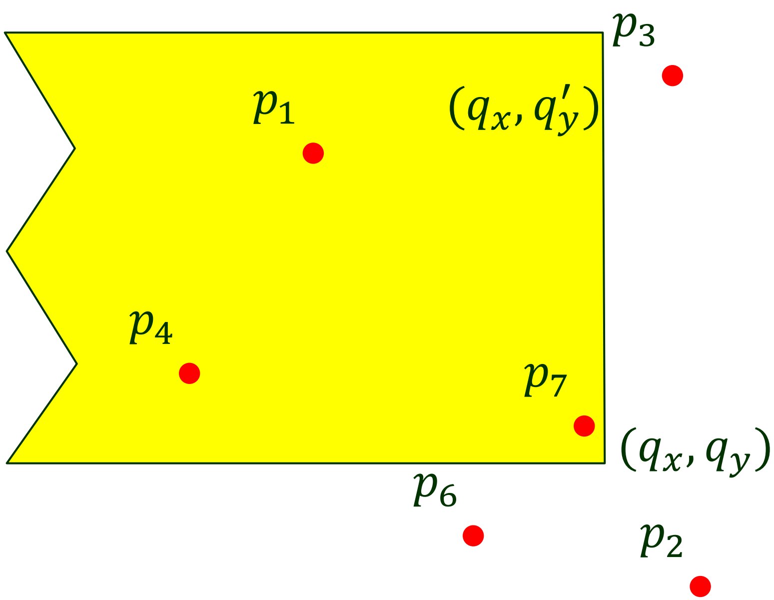
for each node v on $v_{split} \rightsquigarrow \mu'$
    do if the path goes right at v
        then ReportInSubtree($lc(v)$, q_x)

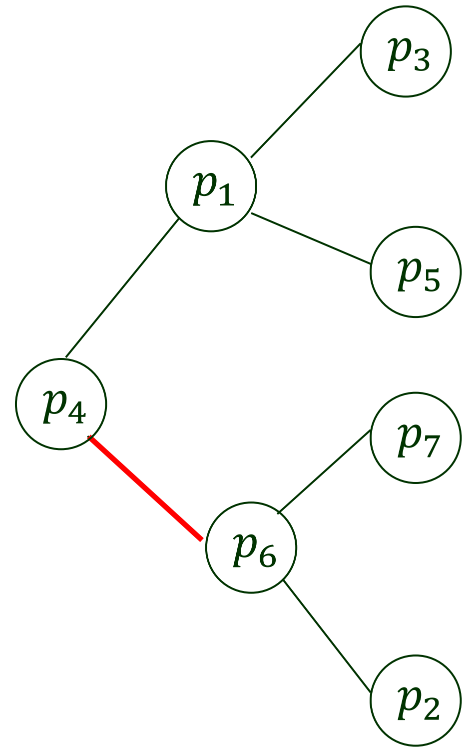
ReportInSubtree(v, q_x)

if $(p(v))_x \leq q_x$
    then report $p(v)$
if v is not a leaf
    then ReportInSubtree($lc(v)$, q_x)
        ReportInSubtree($rc(v)$, q_x)

Example of Execution



$v_{split} = p_4$



QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)

// r is the root of $\mathcal{T}$

search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'

let v_{split} be the node where the two paths split.

for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$

    do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$

        then report $p(v)$

for each node v on $v_{split} \rightsquigarrow \mu$

    do if the path goes left at v

        then ReportInSubtree($rc(v)$, q_x)

for each node v on $v_{split} \rightsquigarrow \mu'$

    do if the path goes right at v

        then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v, q_x)

if $(p(v))_x \leq q_x$

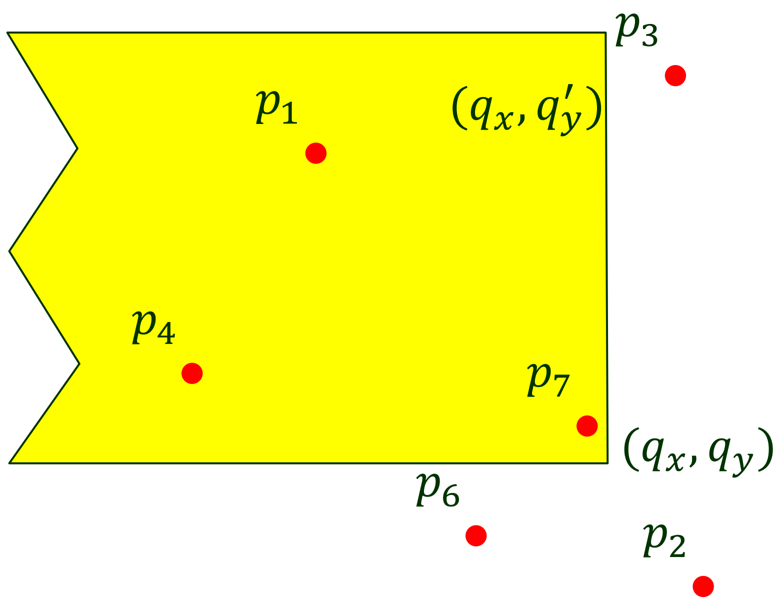
    then report $p(v)$

if v is not a leaf

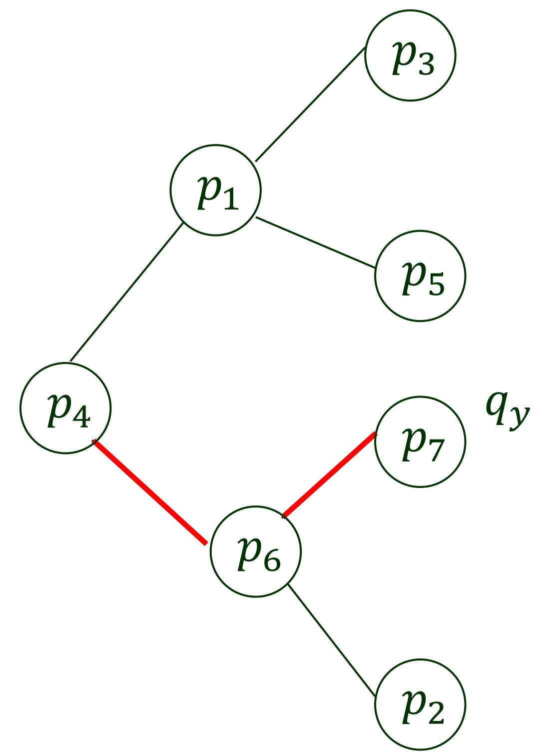
    then ReportInSubtree($lc(v)$, q_x)

        ReportInSubtree($rc(v)$, q_x)

Example of Execution



$v_{split} = p_4$

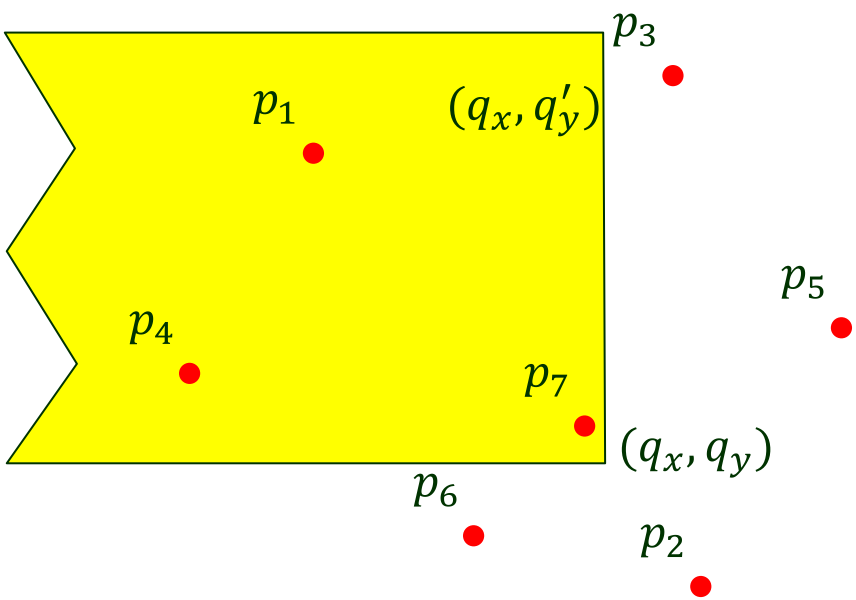


QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)
// r is the root of $\mathcal{T}$
search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
    do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$
        then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
    do if the path goes left at v
        then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
    do if the path goes right at v
        then ReportInSubtree($lc(v)$, q_x)

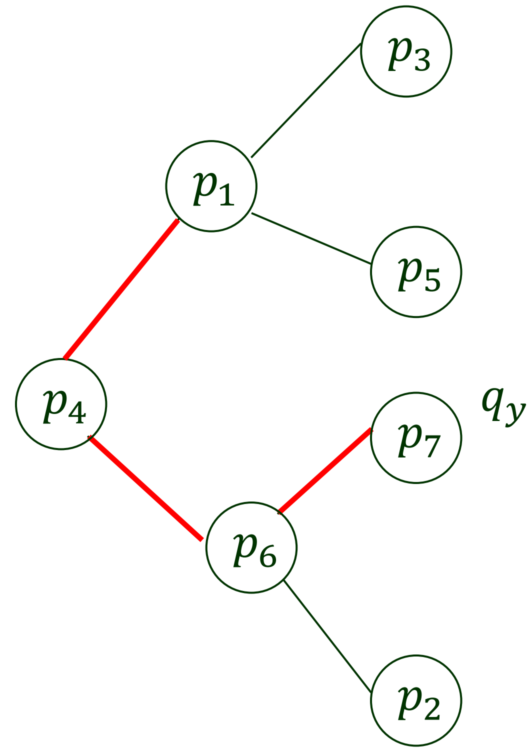
ReportInSubtree(v,q_x)

if $(p(v))_x \le q_x$
    then report $p(v)$
if v is not a leaf
    then ReportInSubtree($lc(v)$, q_x)
        ReportInSubtree($rc(v)$, q_x)

Example of Execution



$v_{split} = p_4$

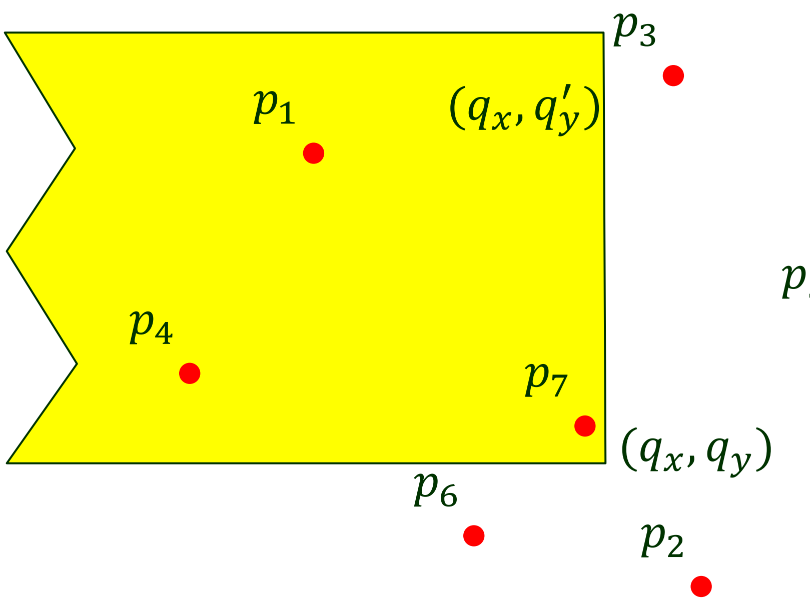


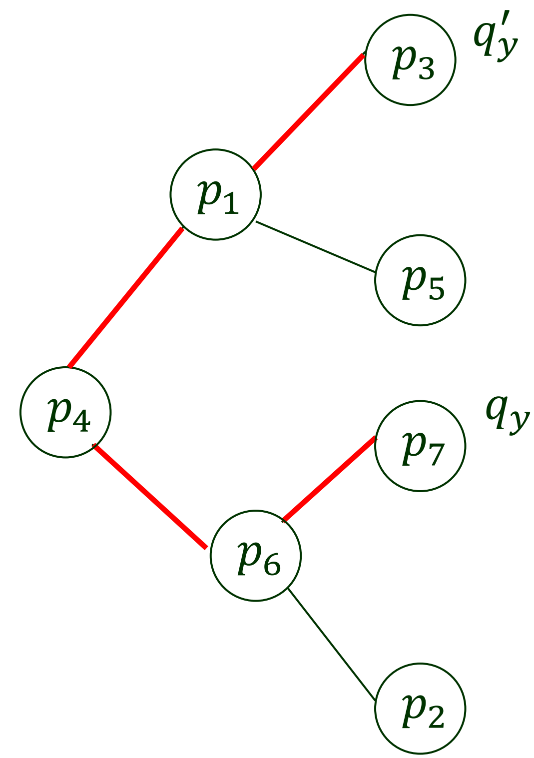
QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)
// r is the root of $\mathcal{T}$
search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
  do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$
    then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
  do if the path goes left at v
    then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
  do if the path goes right at v
    then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v,q_x)

if $(p(v))_x \leq q_x$
  then report $p(v)$
if v is not a leaf
  then ReportInSubtree($lc(v)$, q_x)
    ReportInSubtree($rc(v)$, q_x)

Example of Execution





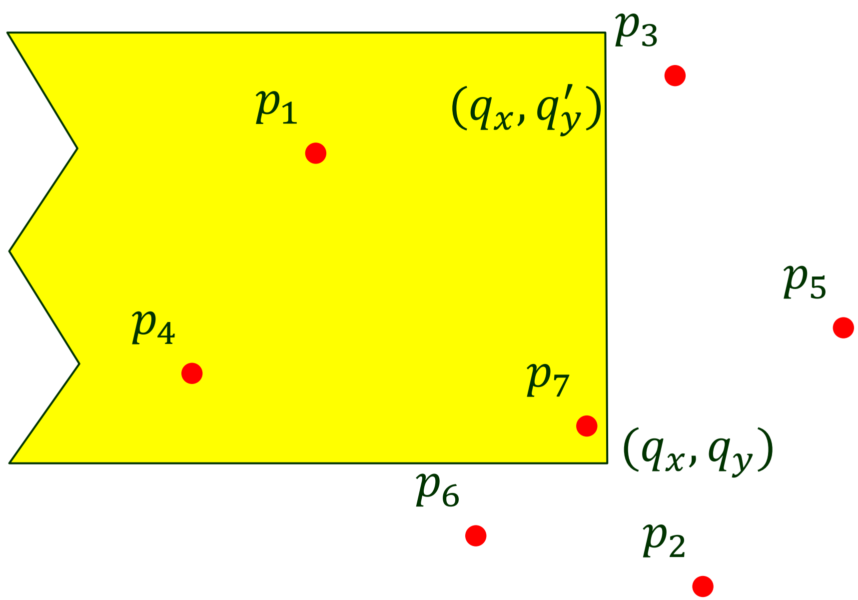
$$v_{split} = p_4$$

```
QueryPrioSearchTree(𝒯, (−∞, q_x] × [q_y, q'_y])
// r is the root of 𝒯
    search with q_y and q'_y in 𝒯, ending at the nodes μ and μ'
    let v_split be the node where the two paths split.
    for each node v on the path r ⇝ μ or v_split ⇝ μ'
        do if p(v) ∈ (−∞, q_x] × [q_y, q'_y]
              then report p(v)
    for each node v on v_split ⇝ μ
        do if the path goes left at v
              then ReportInSubtree(rc(v), q_x)
    for each node v on v_split ⇝ μ'
        do if the path goes right at v
              then ReportInSubtree(lc(v), q_x)
```

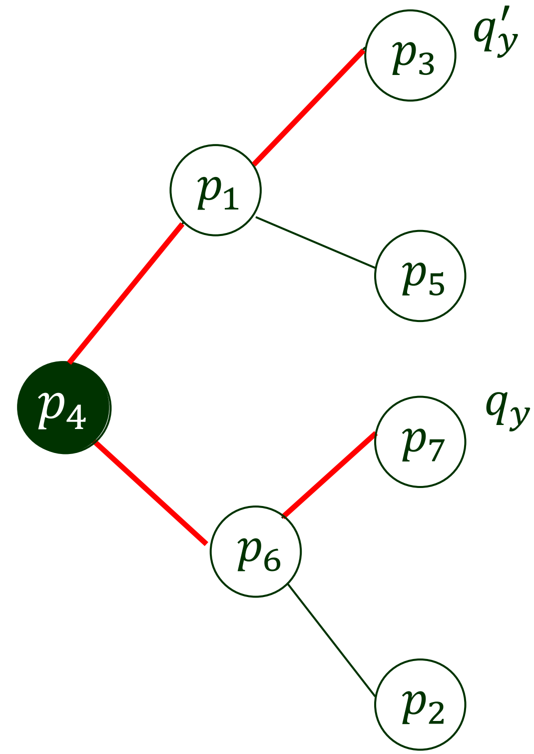
```
ReportInSubtree(v, q_x)

    if (p(v))_x ≤ q_x
        then report p(v)
    if v is not a leaf
        then ReportInSubtree(lc(v), q_x)
             ReportInSubtree(rc(v), q_x)
```

Example of Execution



$v_{split} = p_4$

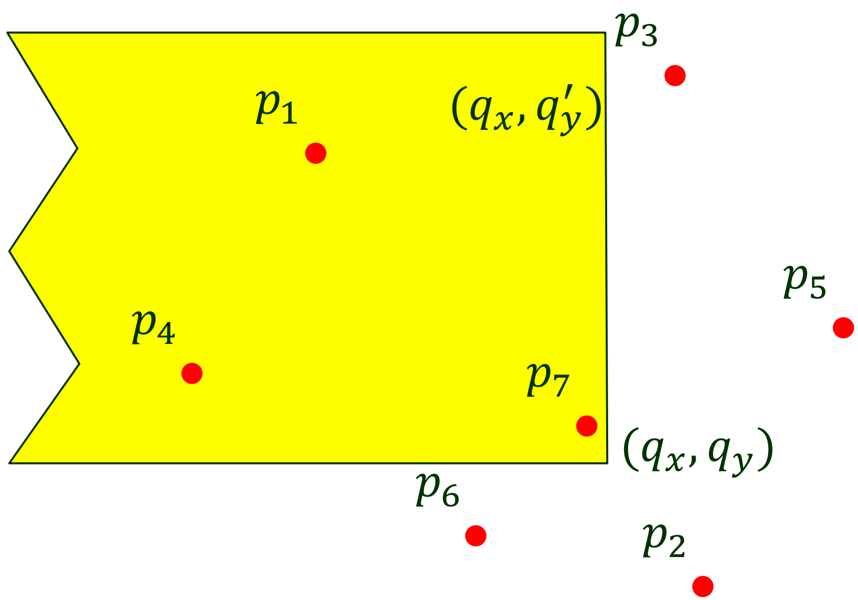


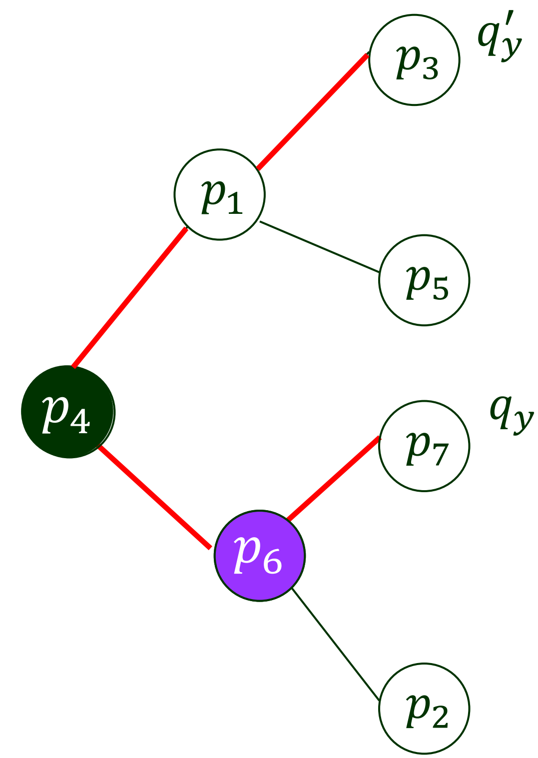
QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)
// r is the root of $\mathcal{T}$
    search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'
    let v_{split} be the node where the two paths split.
    for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
        do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$
            then report $p(v)$
    for each node v on $v_{split} \rightsquigarrow \mu$
        do if the path goes left at v
            then ReportInSubtree($rc(v)$, q_x)
    for each node v on $v_{split} \rightsquigarrow \mu'$
        do if the path goes right at v
            then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v,q_x)

    if $(p(v))_x \le q_x$
        then report $p(v)$
    if v is not a leaf
        then ReportInSubtree($lc(v)$, q_x)
            ReportInSubtree($rc(v)$, q_x)

Example of Execution





$v_{split} = p_4$

QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)

// r is the root of $\mathcal{T}$

search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'

let v_{split} be the node where the two paths split.

for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$

    do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$

        then report $p(v)$

for each node v on $v_{split} \rightsquigarrow \mu$

    do if the path goes left at v

        then ReportInSubtree($rc(v)$, q_x)

for each node v on $v_{split} \rightsquigarrow \mu'$

    do if the path goes right at v

        then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v, q_x)

if $(p(v))_x \le q_x$

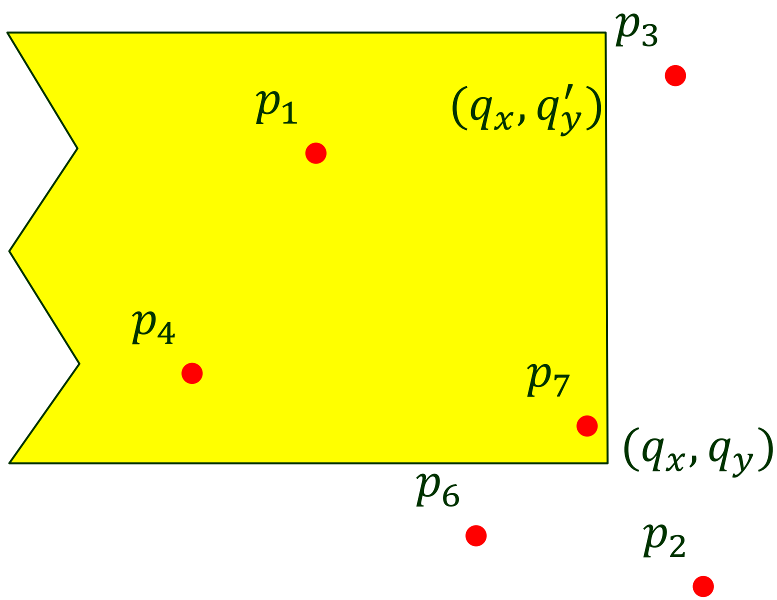
    then report $p(v)$

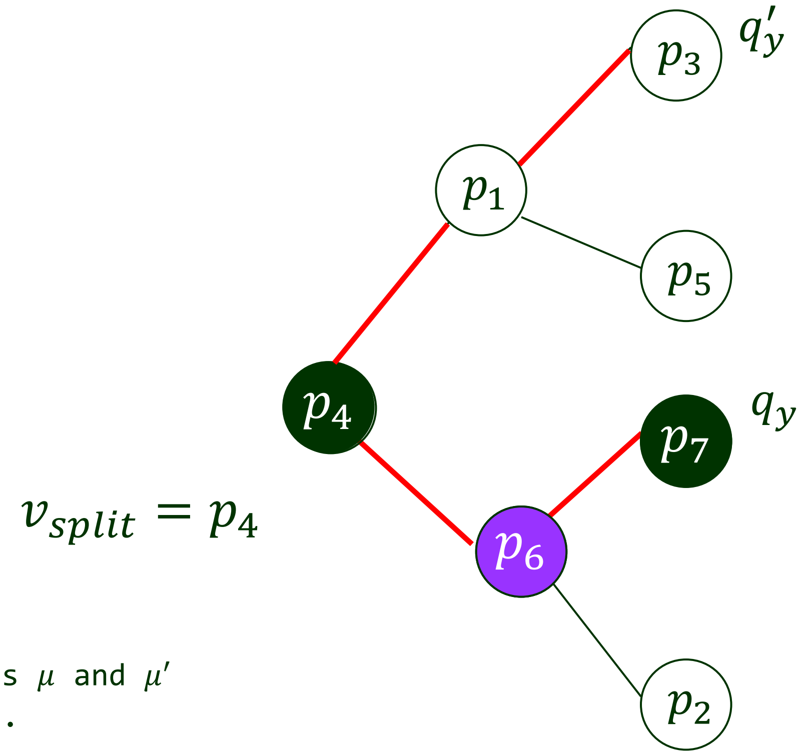
if v is not a leaf

    then ReportInSubtree($lc(v)$, q_x)

        ReportInSubtree($rc(v)$, q_x)

Example of Execution



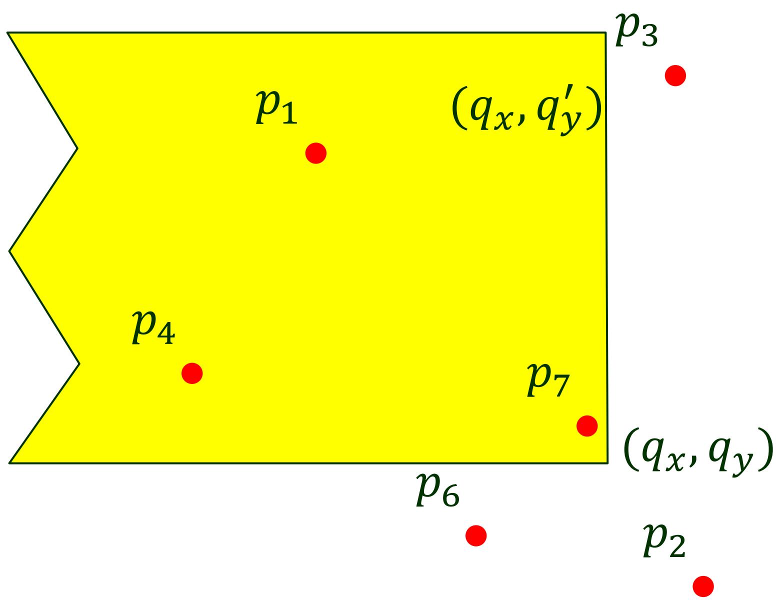


QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)
// r is the root of $\mathcal{T}$
search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
    do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$
        then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
    do if the path goes left at v
        then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
    do if the path goes right at v
        then ReportInSubtree($lc(v)$, q_x)

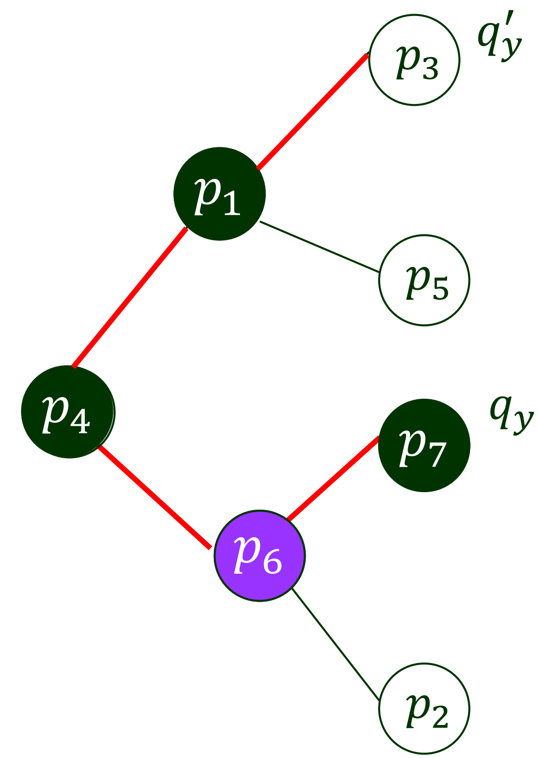
ReportInSubtree(v,q_x)

if $(p(v))_x \le q_x$
    then report $p(v)$
if v is not a leaf
    then ReportInSubtree($lc(v)$, q_x)
        ReportInSubtree($rc(v)$, q_x)

Example of Execution



$v_{split} = p_4$

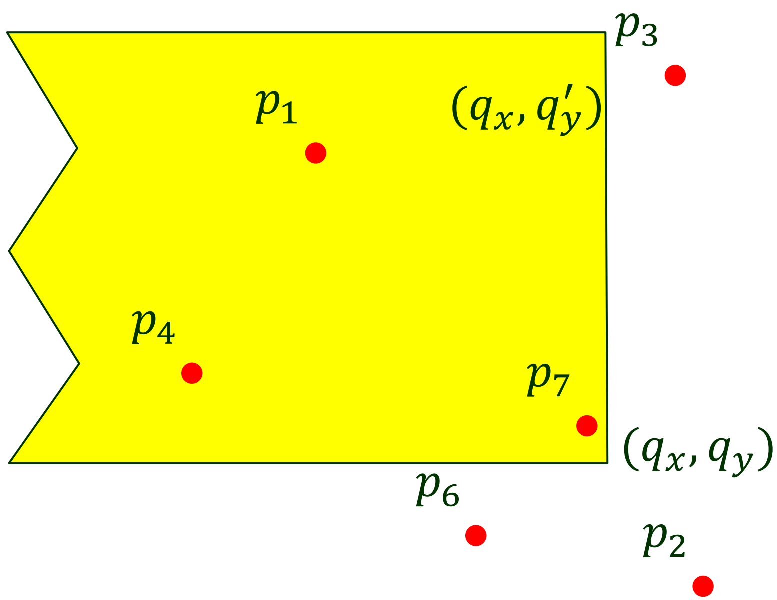


QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q_y']$)
// r is the root of $\mathcal{T}$
search with q_y and q_y' in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
    do if $p(v) \in (-\infty, q_x] \times [q_y, q_y']$
        then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
    do if the path goes left at v
        then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
    do if the path goes right at v
        then ReportInSubtree($lc(v)$, q_x)

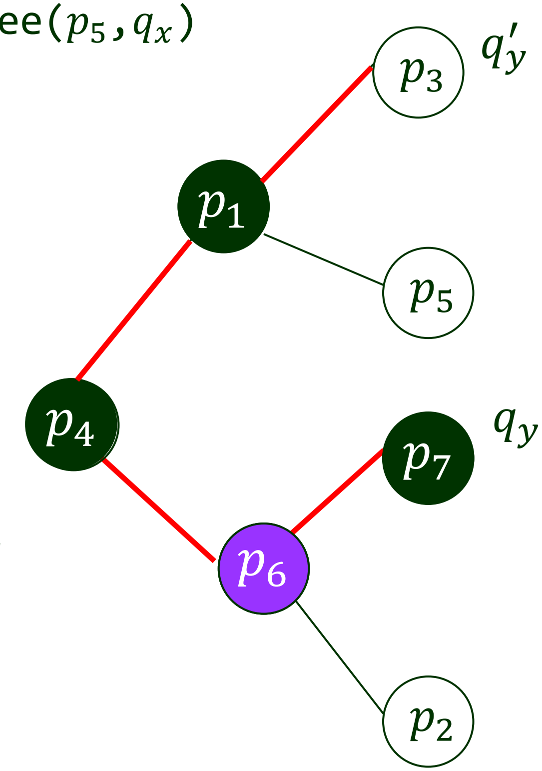
ReportInSubtree(v,q_x)

if $(p(v))_x \leq q_x$
    then report $p(v)$
if v is not a leaf
    then ReportInSubtree($lc(v)$, q_x)
        ReportInSubtree($rc(v)$, q_x)

Example of Execution



ReportInSubtree(p_5,q_x)



$v_{split} = p_4$

QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q_y']$)

// r is the root of $\mathcal{T}$

search with q_y and q_y' in $\mathcal{T}$, ending at the nodes μ and μ'

let v_{split} be the node where the two paths split.

for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$

  do if $p(v) \in (-\infty, q_x] \times [q_y, q_y']$

    then report $p(v)$

for each node v on $v_{split} \rightsquigarrow \mu$

  do if the path goes left at v

    then ReportInSubtree($rc(v)$, q_x)

for each node v on $v_{split} \rightsquigarrow \mu'$

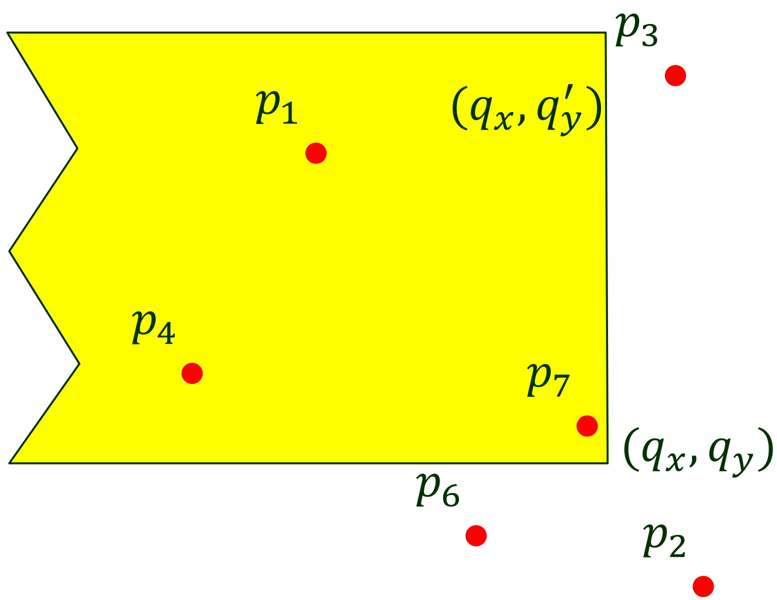
  do if the path goes right at v

    then ReportInSubtree($lc(v)$, q_x)

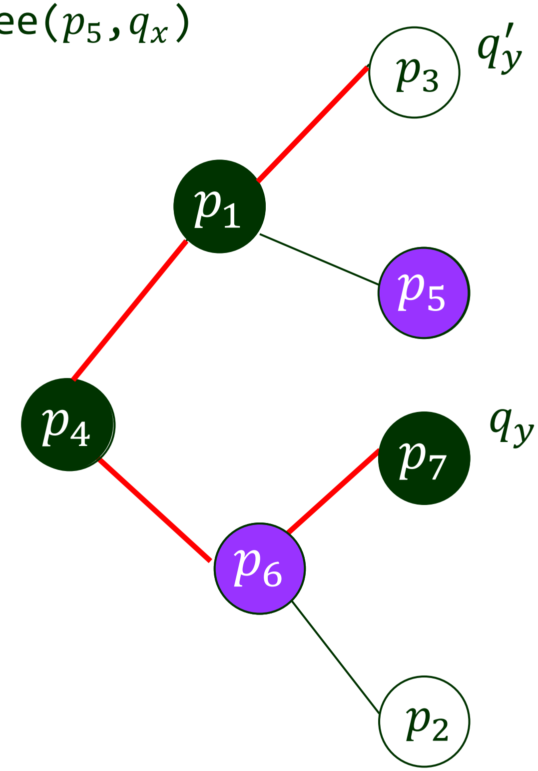
ReportInSubtree(v,q_x)

1. if $(p(v))_x \leq q_x$
2.   then report $p(v)$
3. if v is not a leaf
4.   then ReportInSubtree($lc(v)$, q_x)
5.   ReportInSubtree($rc(v)$, q_x)

Example of Execution



ReportInSubtree(p_5, q_x)



$v_{split} = p_4$

QueryPrioSearchTree$(\mathcal{T}, (-\infty, q_x] \times [q_y, q'_y])$

// r is the root of $\mathcal{T}$

search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'

let v_{split} be the node where the two paths split.

for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$

    do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$

        then report $p(v)$

for each node v on $v_{split} \rightsquigarrow \mu$

    do if the path goes left at v

        then ReportInSubtree$(rc(v), q_x)$

for each node v on $v_{split} \rightsquigarrow \mu'$

    do if the path goes right at v

        then ReportInSubtree$(lc(v), q_x)$

ReportInSubtree(v, q_x)

if $(p(v))_x \leq q_x$

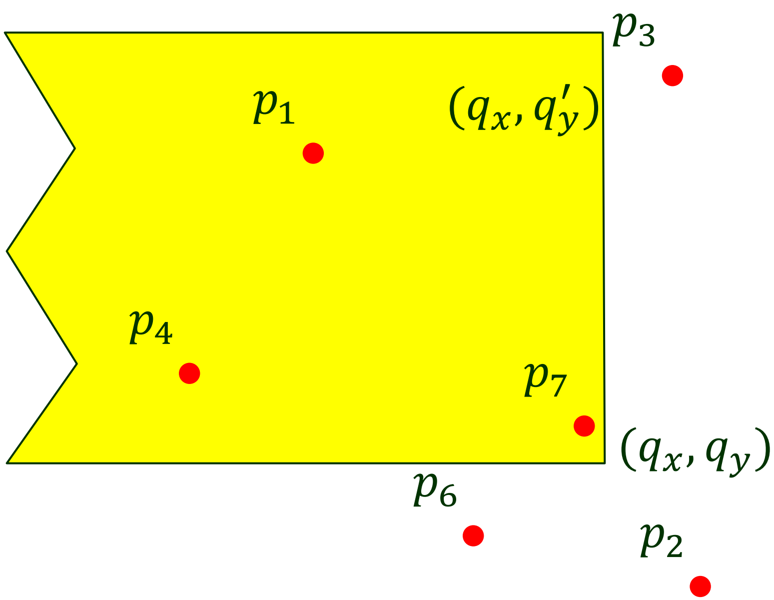
    then report $p(v)$

if v is not a leaf

    then ReportInSubtree$(lc(v), q_x)$

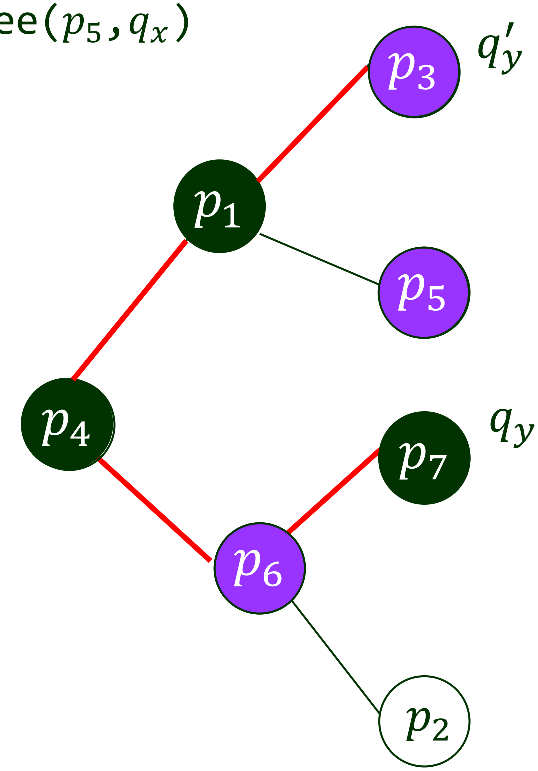
        ReportInSubtree$(rc(v), q_x)$

Example of Execution



ReportInSubtree(p_5,q_x)

$v_{split} = p_4$



QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)

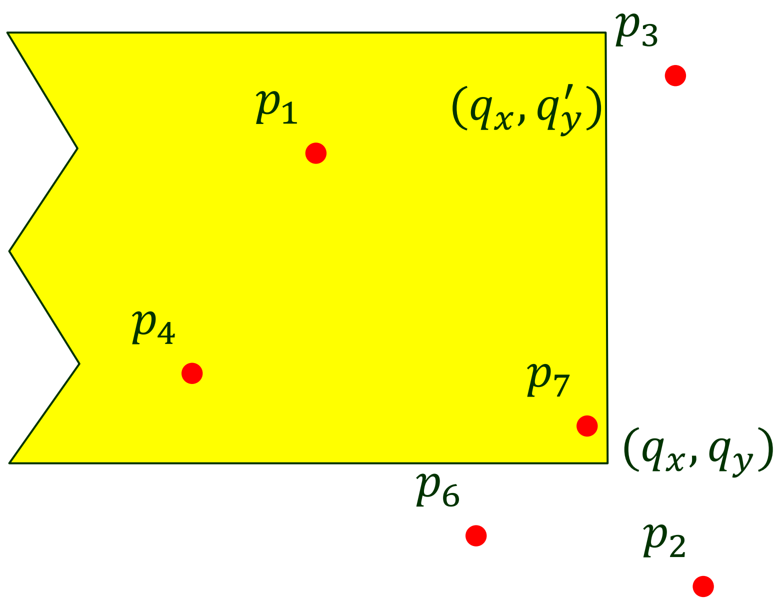
// r is the root of $\mathcal{T}$

search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
  do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$
    then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
  do if the path goes left at v
    then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
  do if the path goes right at v
    then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v,q_x)

if $(p(v))_x \leq q_x$
  then report $p(v)$
if v is not a leaf
  then ReportInSubtree($lc(v)$, q_x)
    ReportInSubtree($rc(v)$, q_x)

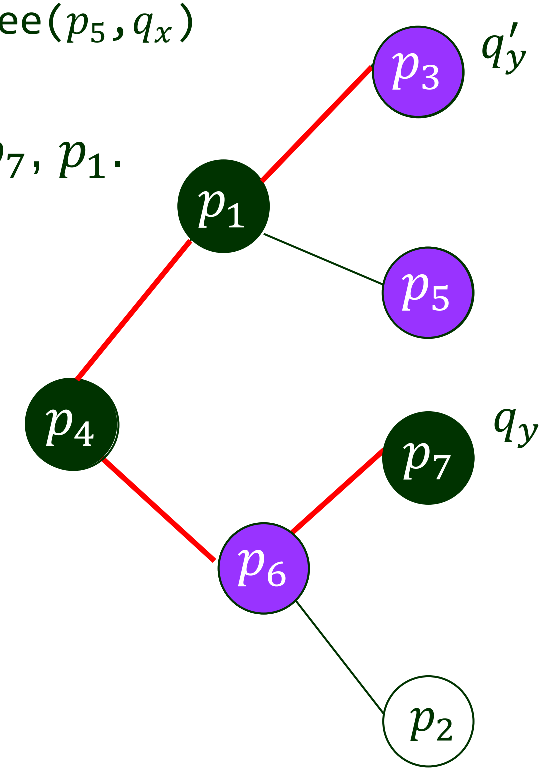
Example of Execution



ReportInSubtree(p_5,q_x)

Returns p_4, p_7, p_1.

$v_{split} = p_4$



QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)
// r is the root of $\mathcal{T}$
search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
    do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$
        then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
    do if the path goes left at v
        then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
    do if the path goes right at v
        then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v,q_x)

if $(p(v))_x \leq q_x$
    then report $p(v)$
if v is not a leaf
    then ReportInSubtree($lc(v)$, q_x)
        ReportInSubtree($rc(v)$, q_x)

Running Time

QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q_y']$)
// r is the root of $\mathcal{T}$
search with q_y and q_y' in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
    do if $p(v) \in (-\infty, q_x] \times [q_y, q_y']$
        then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
    do if the path goes left at v
        then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
    do if the path goes right at v
        then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v,q_x) // $O(k)$

if $\left(p(v)\right)_x \leq q_x$
    then report $p(v)$
    if v is not a leaf
        then ReportInSubtree($lc(v)$, q_x)
             ReportInSubtree($rc(v)$, q_x)

Time cost breaks down to two parts:

Running Time

QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)
// r is the root of $\mathcal{T}$
search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
    do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$
        then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
    do if the path goes left at v
        then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
    do if the path goes right at v
        then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v,q_x) // $O(k)$

if $\left(p(v)\right)_x \le q_x$
    then report $p(v)$
    if v is not a leaf
        then ReportInSubtree($lc(v)$, q_x)
            ReportInSubtree($rc(v)$, q_x)

Time cost breaks down to two parts:

- number of nodes on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$

Running Time

QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q_y']$)
// r is the root of $\mathcal{T}$
search with q_y and q_y' in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
    do if $p(v) \in (-\infty, q_x] \times [q_y, q_y']$
        then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
    do if the path goes left at v
        then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
    do if the path goes right at v
        then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v,q_x) // $O(k)$

if $\big(p(v)\big)_x \leq q_x$
    then report $p(v)$
    if v is not a leaf
        then ReportInSubtree($lc(v)$, q_x)
            ReportInSubtree($rc(v)$, q_x)

Time cost breaks down to two parts:

- number of nodes on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$ $O(\log n)$

Running Time

QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q'_y]$)
// r is the root of $\mathcal{T}$
search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
  do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$
    then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
  do if the path goes left at v
    then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
  do if the path goes right at v
    then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v,q_x) // $O(k)$

if $(p(v))_x \le q_x$
  then report $p(v)$
  if v is not a leaf
    then ReportInSubtree($lc(v)$, q_x)
      ReportInSubtree($rc(v)$, q_x)

Time cost breaks down to two parts:

- number of nodes on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$ $O(\log n)$
- number of recursive calls to ReportInSubtree().

Running Time

QueryPrioSearchTree($\mathcal{T}$, $(-\infty, q_x] \times [q_y, q_y']$)
// r is the root of $\mathcal{T}$
search with q_y and q_y' in $\mathcal{T}$, ending at the nodes μ and μ'
let v_{split} be the node where the two paths split.
for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$
    do if $p(v) \in (-\infty, q_x] \times [q_y, q_y']$
        then report $p(v)$
for each node v on $v_{split} \rightsquigarrow \mu$
    do if the path goes left at v
        then ReportInSubtree($rc(v)$, q_x)
for each node v on $v_{split} \rightsquigarrow \mu'$
    do if the path goes right at v
        then ReportInSubtree($lc(v)$, q_x)

ReportInSubtree(v,q_x) // $O(k)$

if $\big(p(v)\big)_x \leq q_x$
    then report $p(v)$
    if v is not a leaf
        then ReportInSubtree($lc(v)$, q_x)
             ReportInSubtree($rc(v)$, q_x)

Time cost breaks down to two parts:

- number of nodes on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$ $O(\log n)$
- number of recursive calls to ReportInSubtree(). $O(k)$

\# reported points

Running Time

`QueryPrioSearchTree`$(\mathcal{T},\ (-\infty, q_x] \times [q_y, q'_y])$

// r is the root of $\mathcal{T}$

search with q_y and q'_y in $\mathcal{T}$, ending at the nodes μ and μ'

let v_{split} be the node where the two paths split.

for each node v on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$

    do if $p(v) \in (-\infty, q_x] \times [q_y, q'_y]$

        then report $p(v)$

for each node v on $v_{split} \rightsquigarrow \mu$

    do if the path goes left at v

        then `ReportInSubtree`$(rc(v),\ q_x)$

for each node v on $v_{split} \rightsquigarrow \mu'$

    do if the path goes right at v

        then `ReportInSubtree`$(lc(v),\ q_x)$

`ReportInSubtree`(v, q_x) // $O(k)$

if $\big(p(v)\big)_x \le q_x$

    then report $p(v)$

    if v is not a leaf

        then `ReportInSubtree`$(lc(v),\ q_x)$

            `ReportInSubtree`$(rc(v),\ q_x)$

Time cost breaks down to two parts:

- number of nodes on the path $r \rightsquigarrow \mu$ or $v_{split} \rightsquigarrow \mu'$ — $O(\log n)$
- number of recursive calls to `ReportInSubtree()`. — $O(k)$

reported points

$$O(\log n + k)$$

Summary on PST

Min heap over the x-coordinate.

Binary search tree over the y-coordinate.

Storage	$O(n)$
Construction time	$O(n \log n)$
Query time	$O(\log n + k)$