Lecture Notes for *Introduction to Computational Geometry* (Com S 418/518)
Yan-Bin Jia, Iowa State University

Interval Trees

Outline:

I. Windowing query

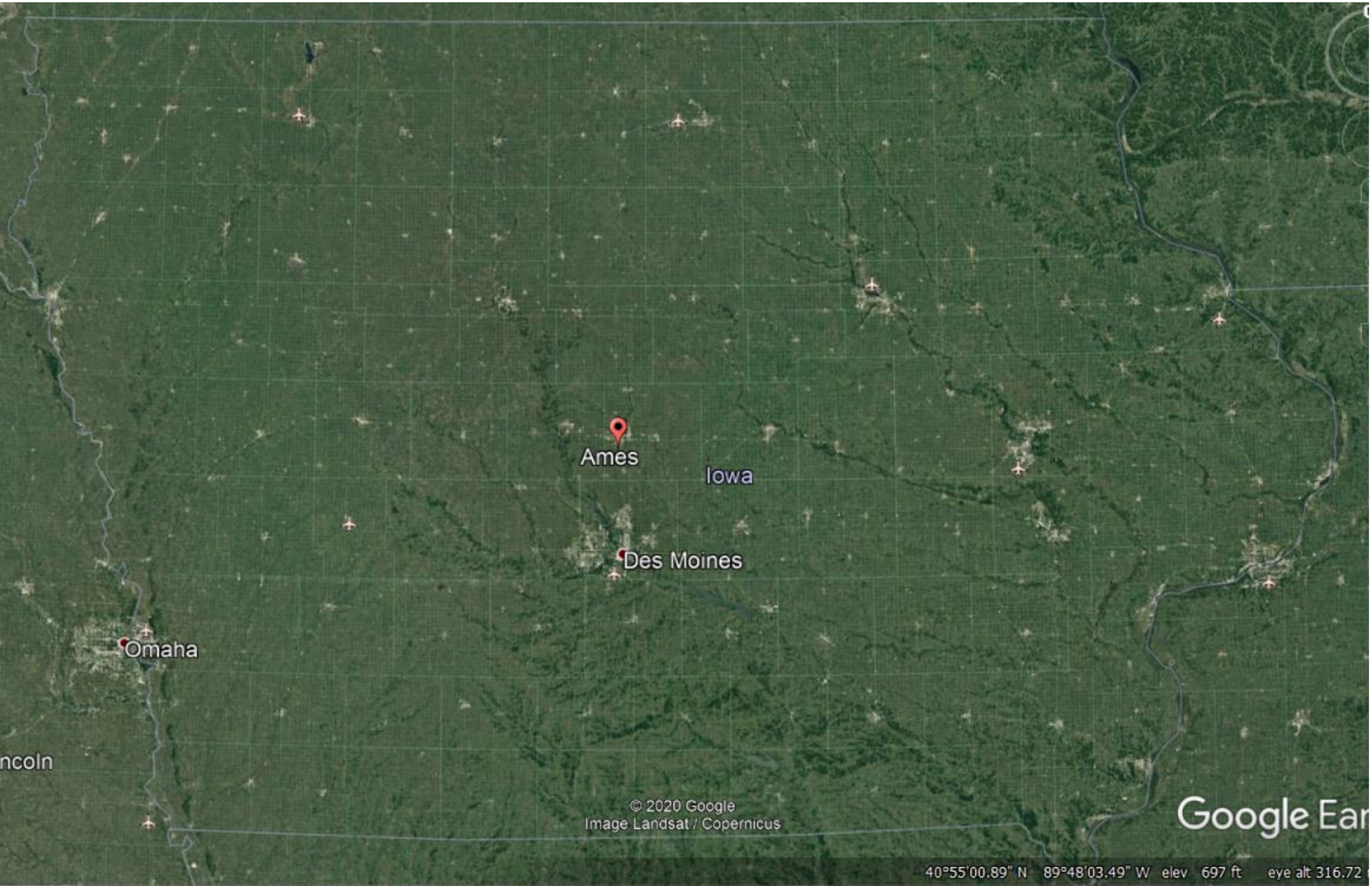
II. Interval tree

III. Construction and query

IV. Query with a line segment

Textbook: *Computational Geometry: Algorithms and Applications (*3rd ed.) by M. de Berg et al., Springer-Verlag, 2008.

I. Windowing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40°55'00.89" N 89°48'03.49" W elev 697 ft eye alt 316.72

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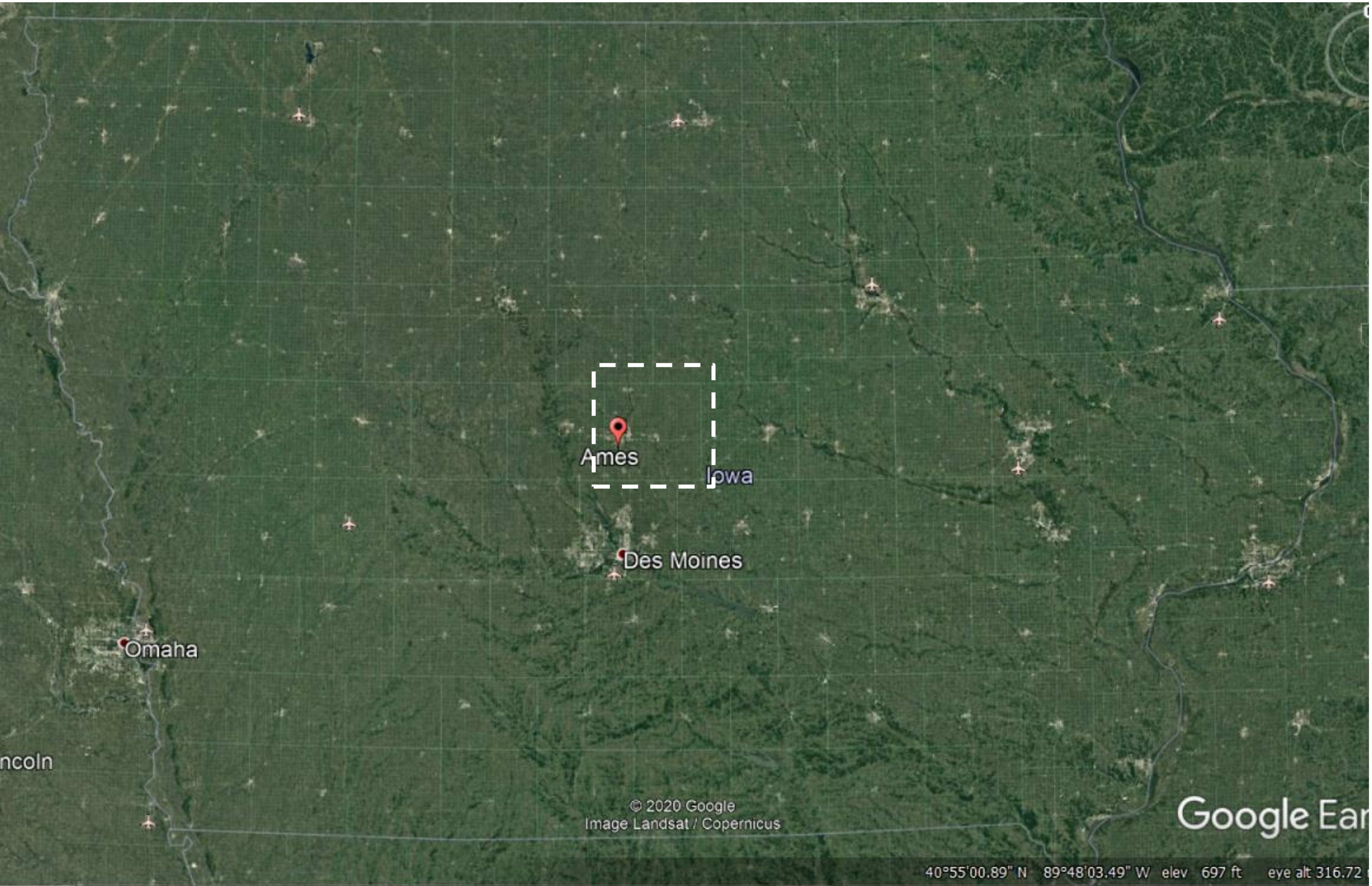
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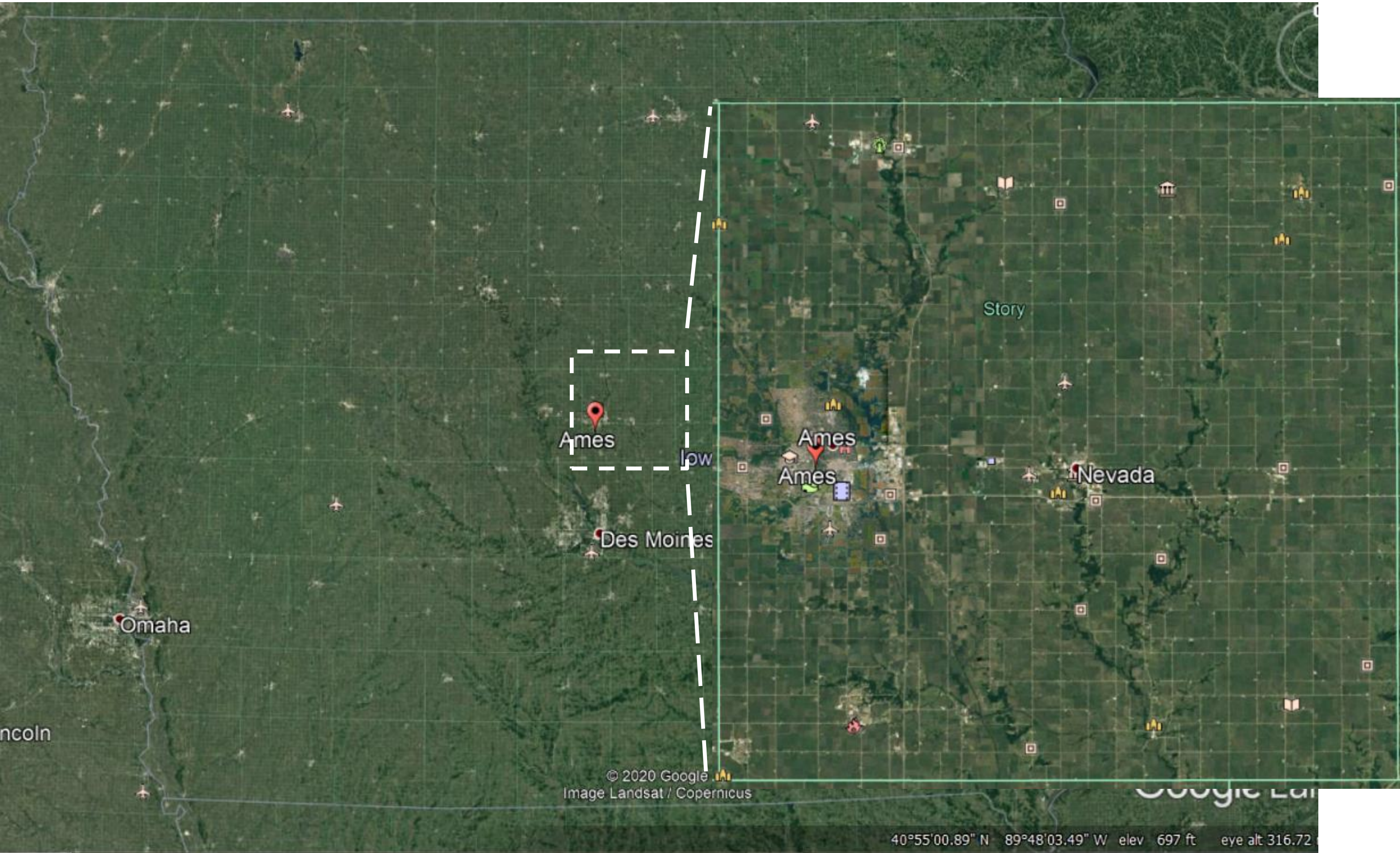

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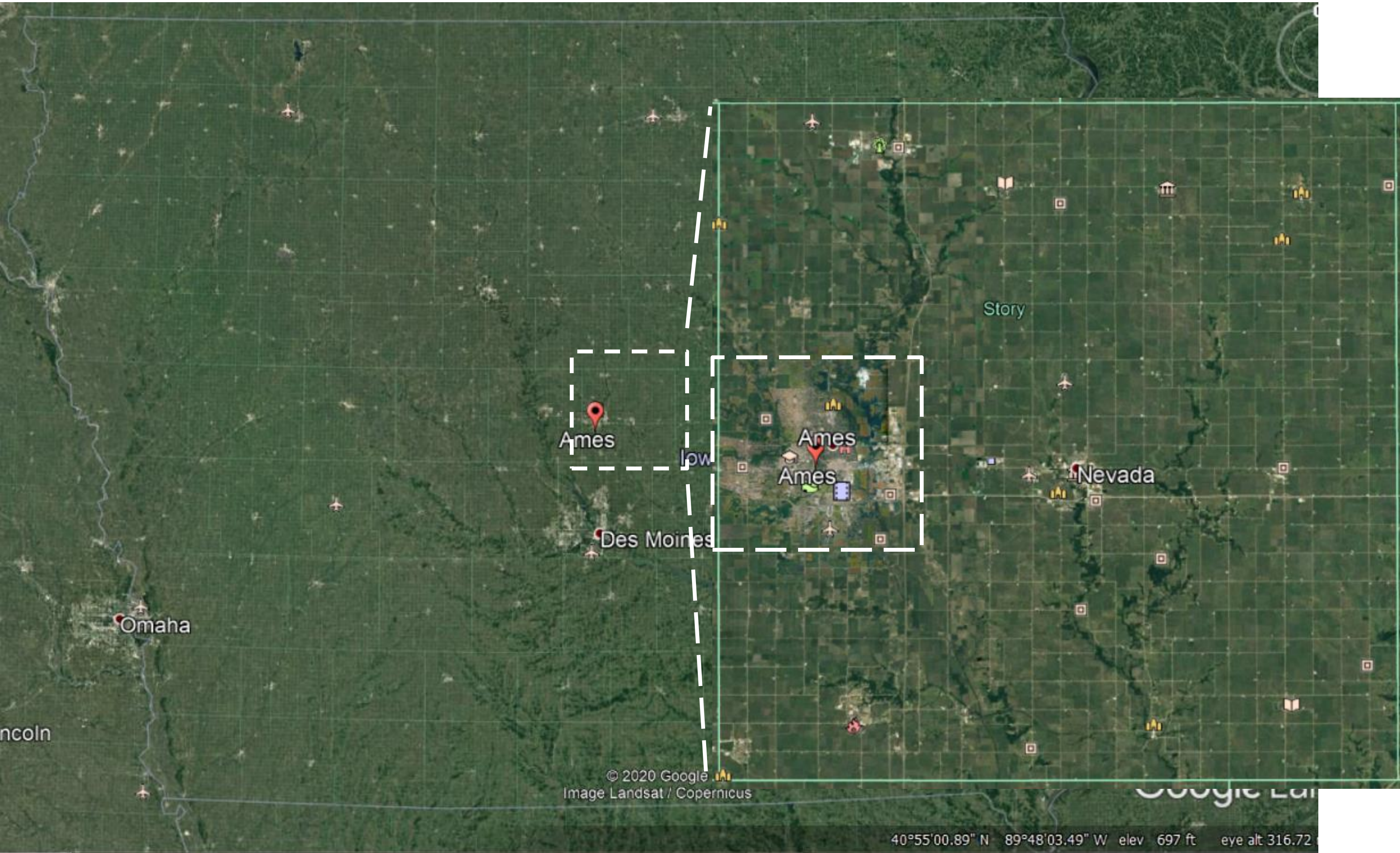

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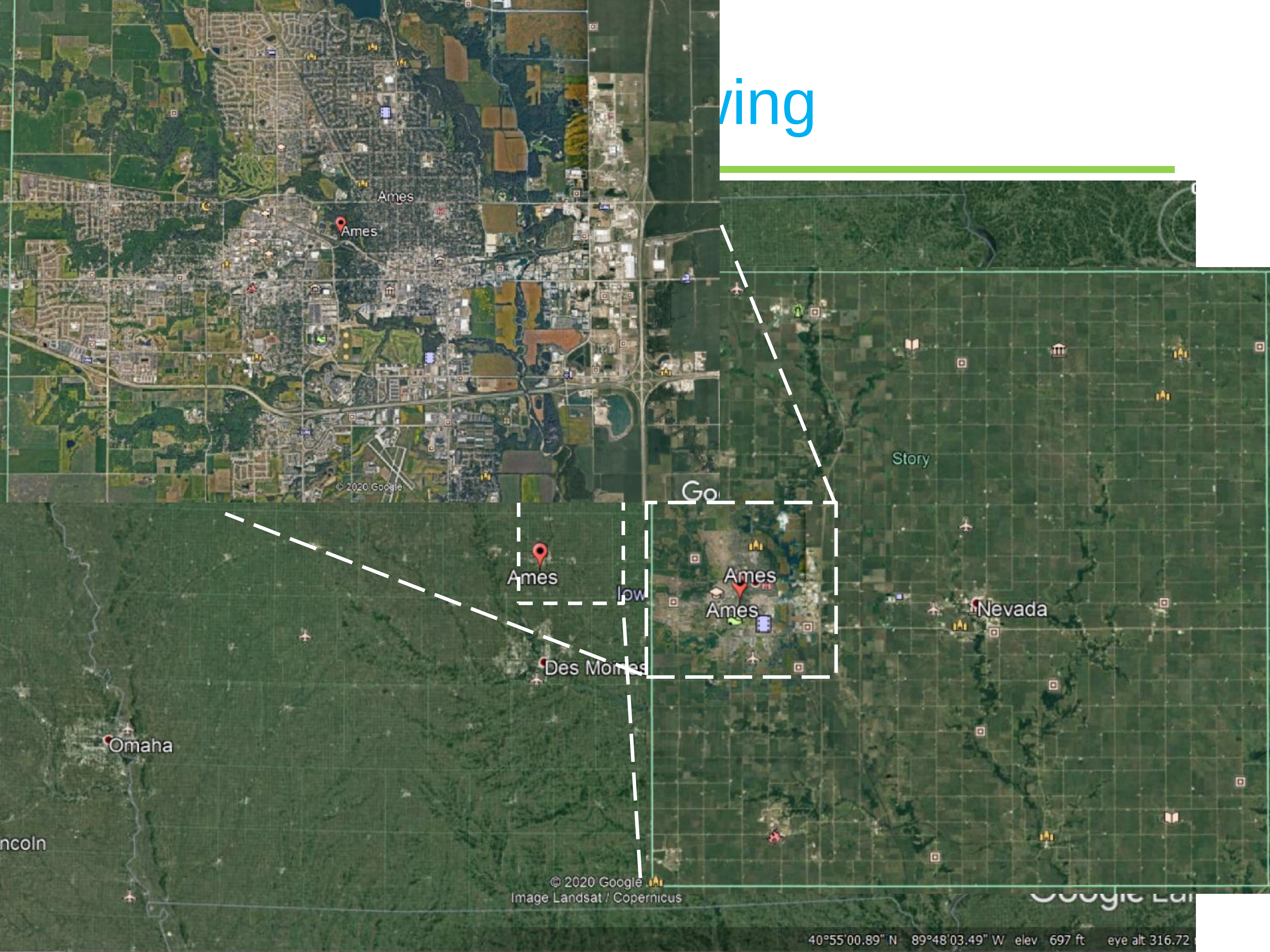
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Determine the part of the map that lie in the window and display them.

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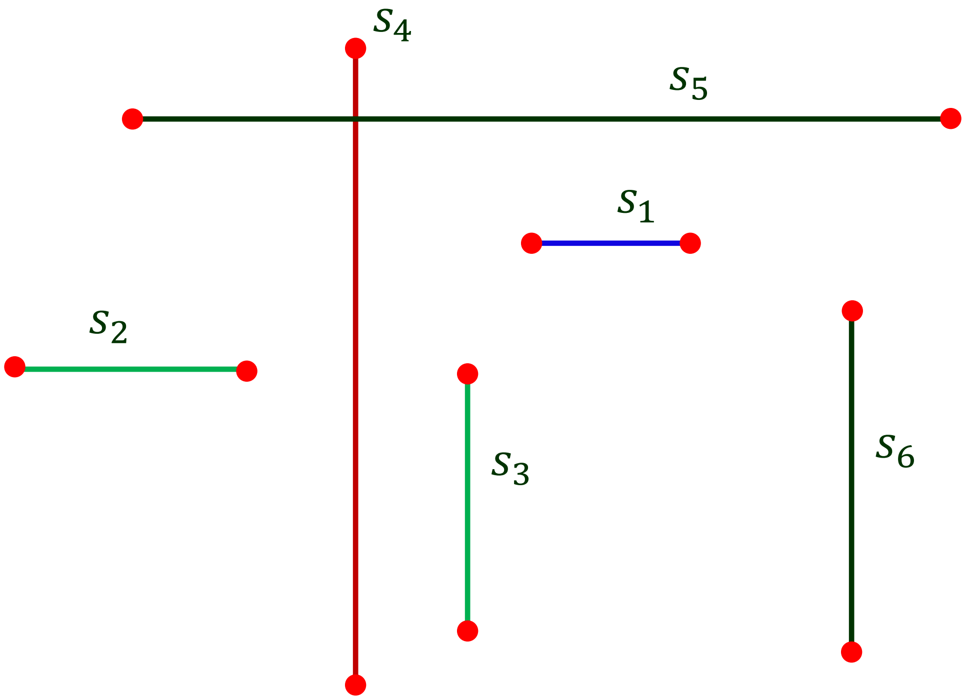
Use some data structure to store the map and allow quick retrieval of its portion inside a window.

vs. Range Query

	Windowing Query	Range Query
data	segments, polygons, curves	points
search space	2D or 3D	k-D

Formulating the Problem

$S = \{s_1, s_2, \ldots, s_n\}$: set of n axis-parallel segments



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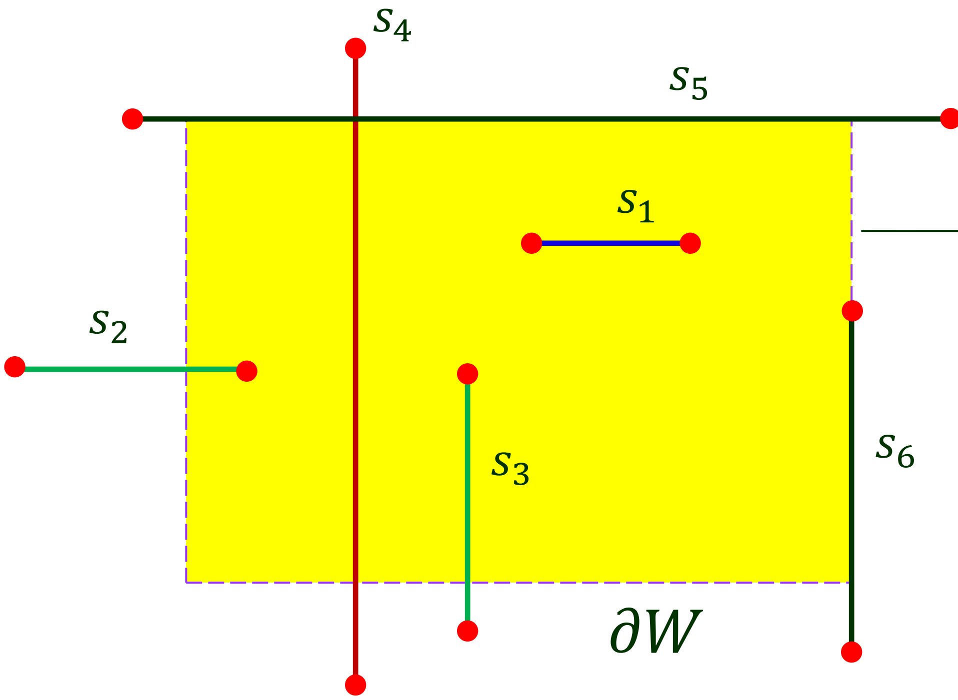

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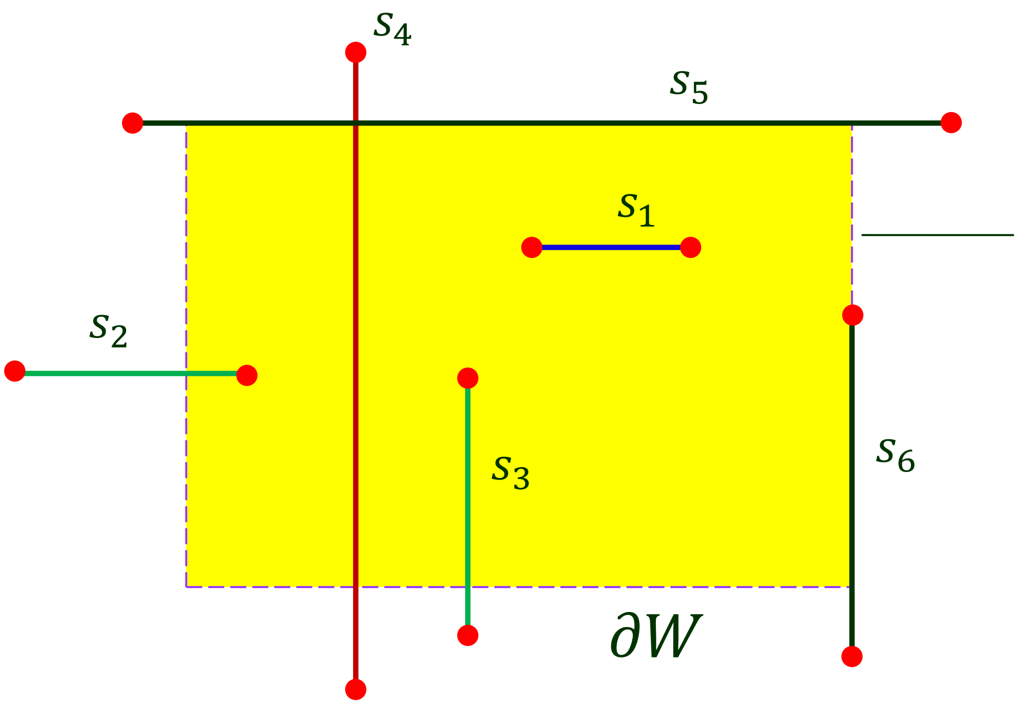

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Possible cases of a segment s:

- s inside W;
- s intersects ∂W once;
- s intersects ∂W twice;
- s overlaps ∂W.

Segments with ≥ 1 Endpoints in W

Range query over $2n$ endpoints

$O(n \log n)$ storage

$O(n \log^2 n + k)$ query time

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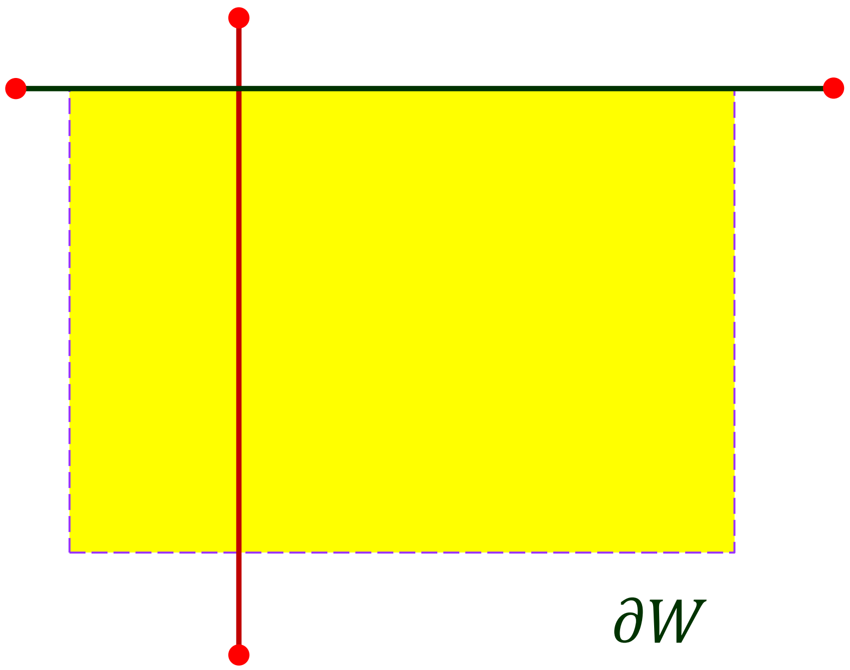
♣ Not to report a segment twice.

Segments with No Endpoint Inside W



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- contains one boundary edge.

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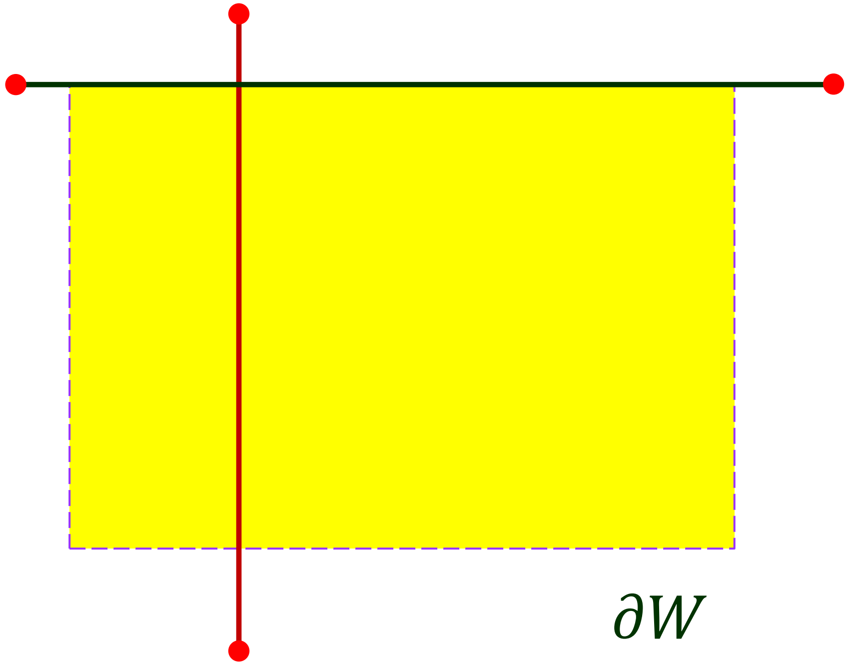

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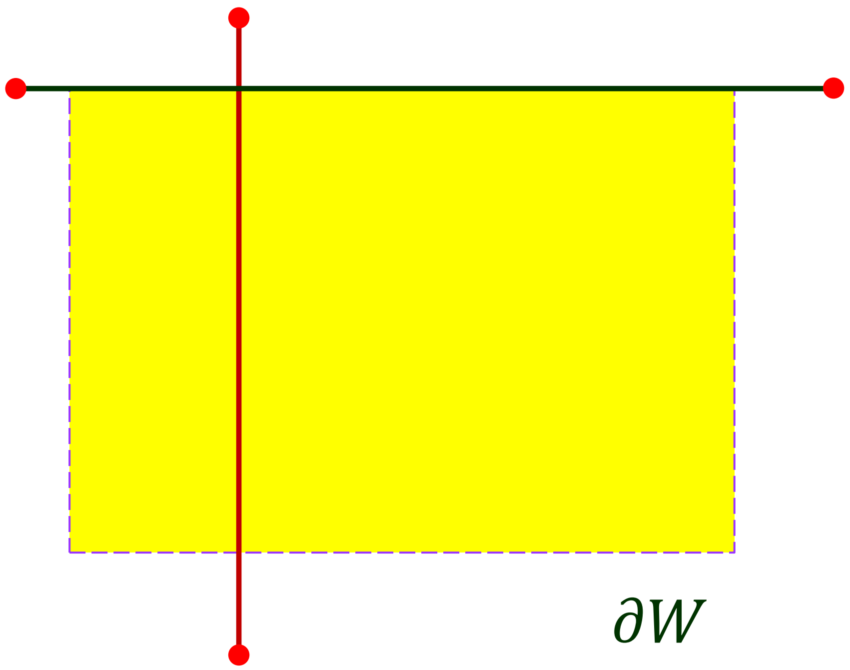

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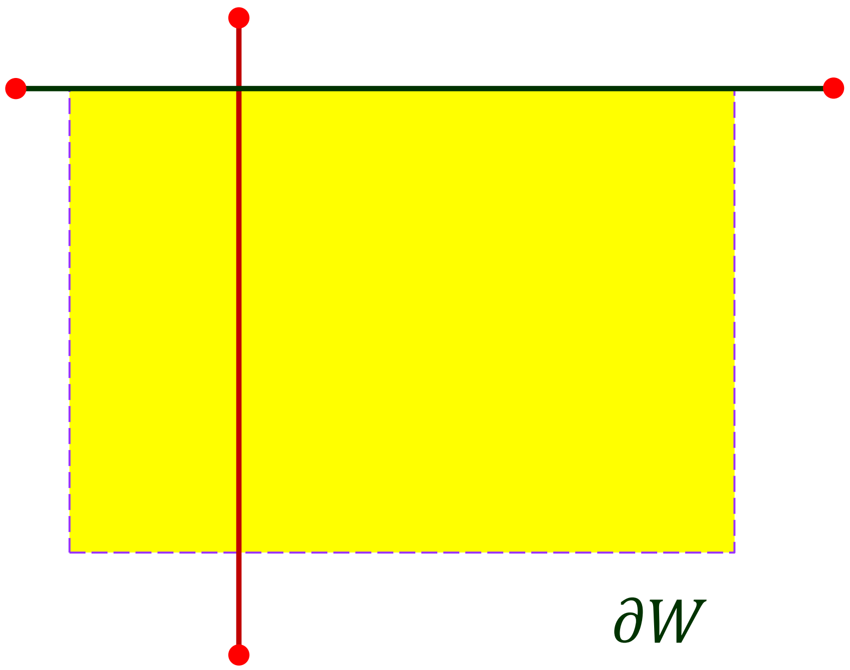

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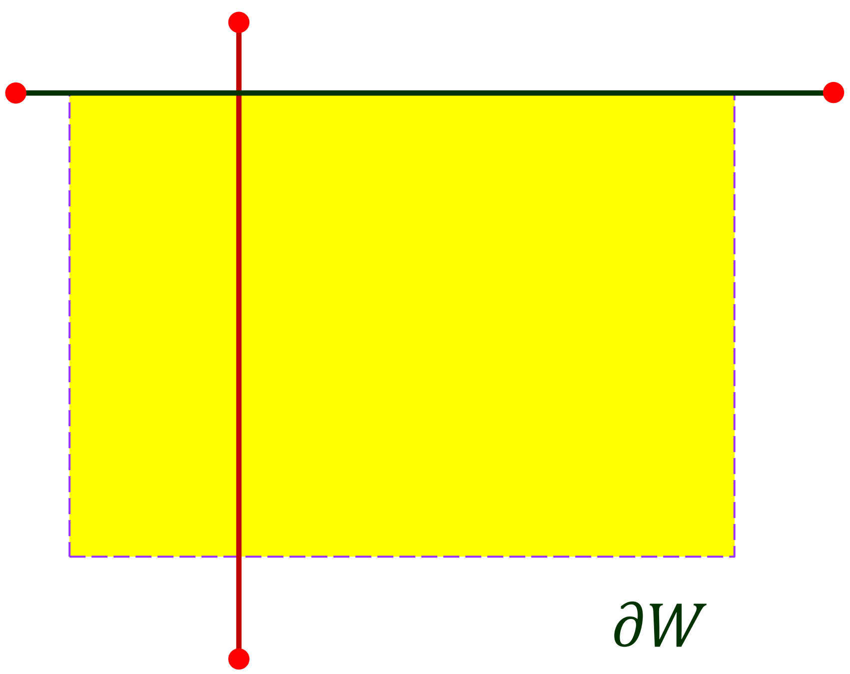

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It's similar to check if it intersects the right boundary edge.

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We first consider the *simpler* version of query when the left boundary edge of W is a *full line*.

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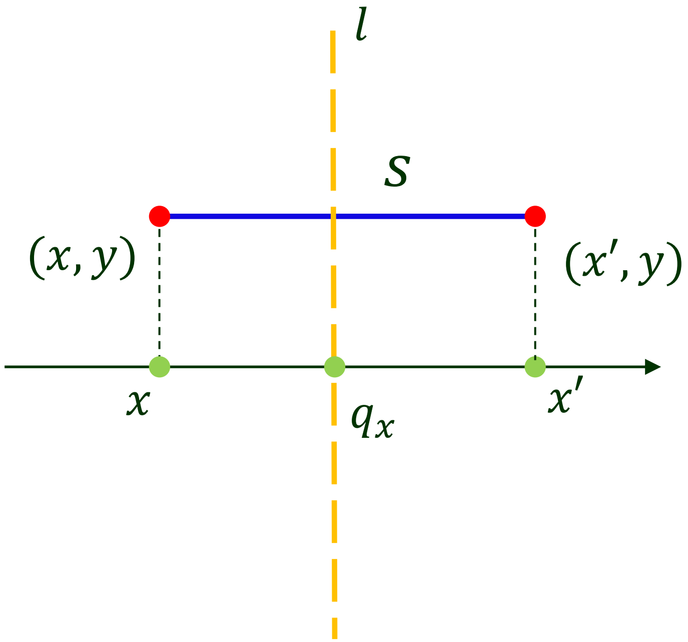

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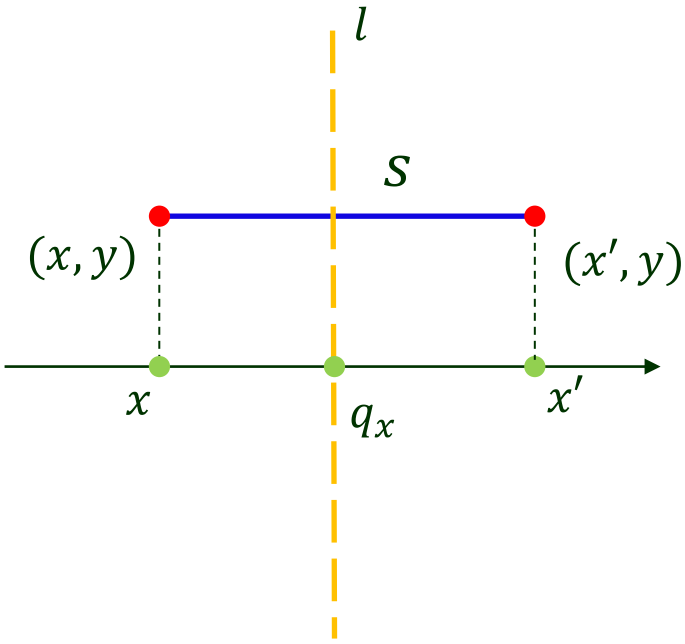

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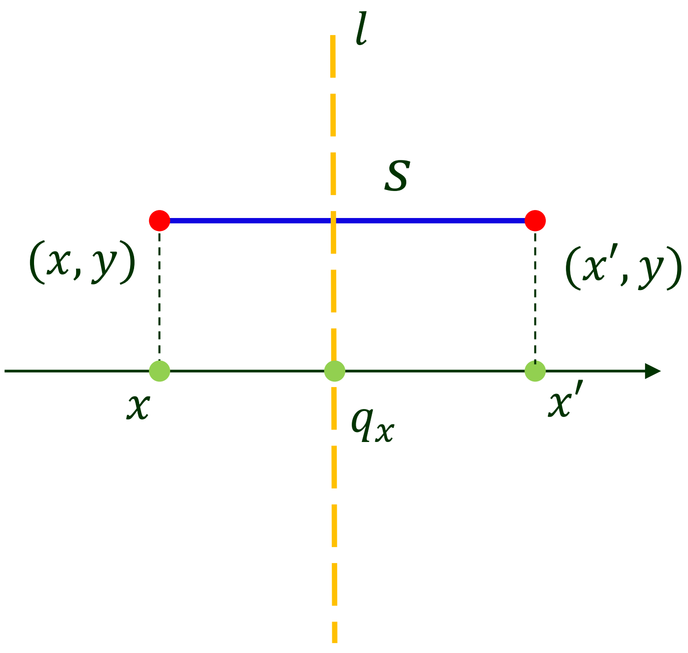
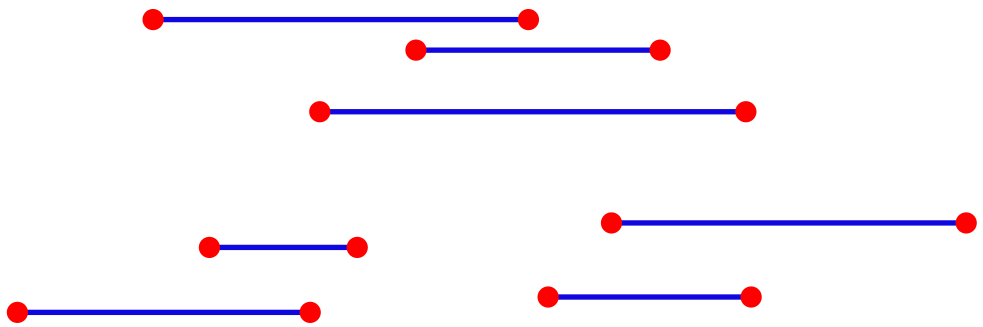


1D problem

Given a set of intervals $I = \{[x_1, y_1], [x_2, y_2] \ldots, [x_n, y_n]\}$, report those that contain the query point q_x.

Partitioning the Interval Set

x_{mid}: median of $2n$ interval endpoints.



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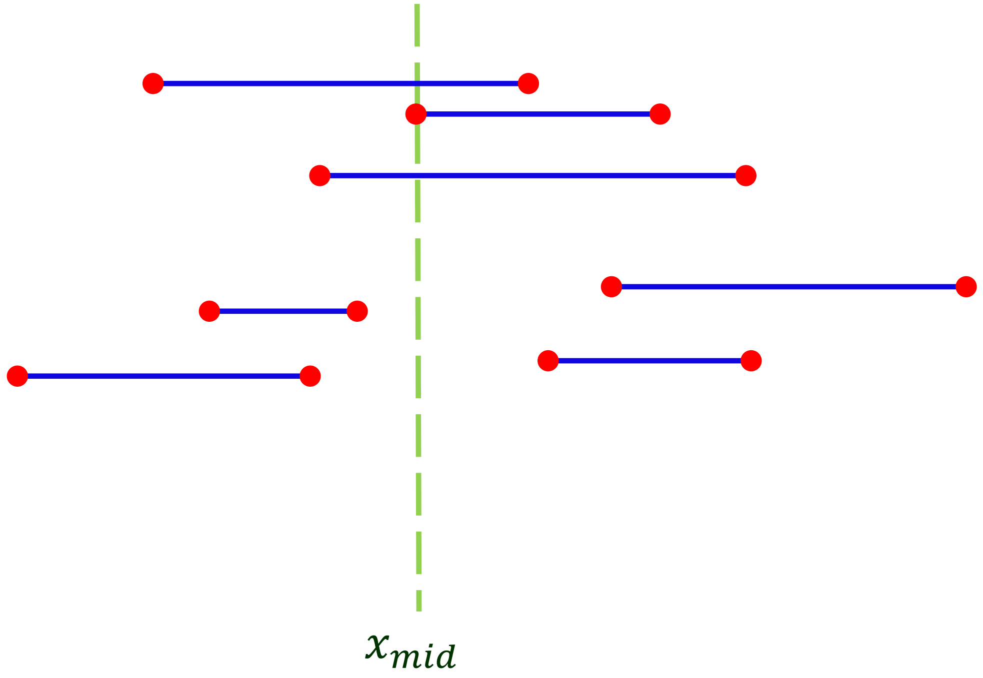

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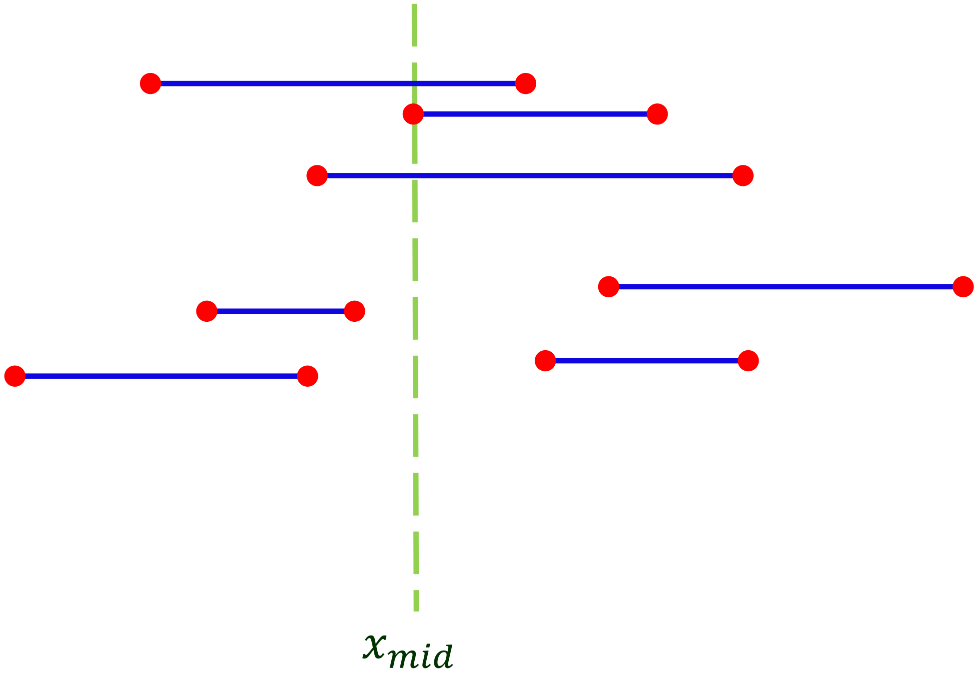

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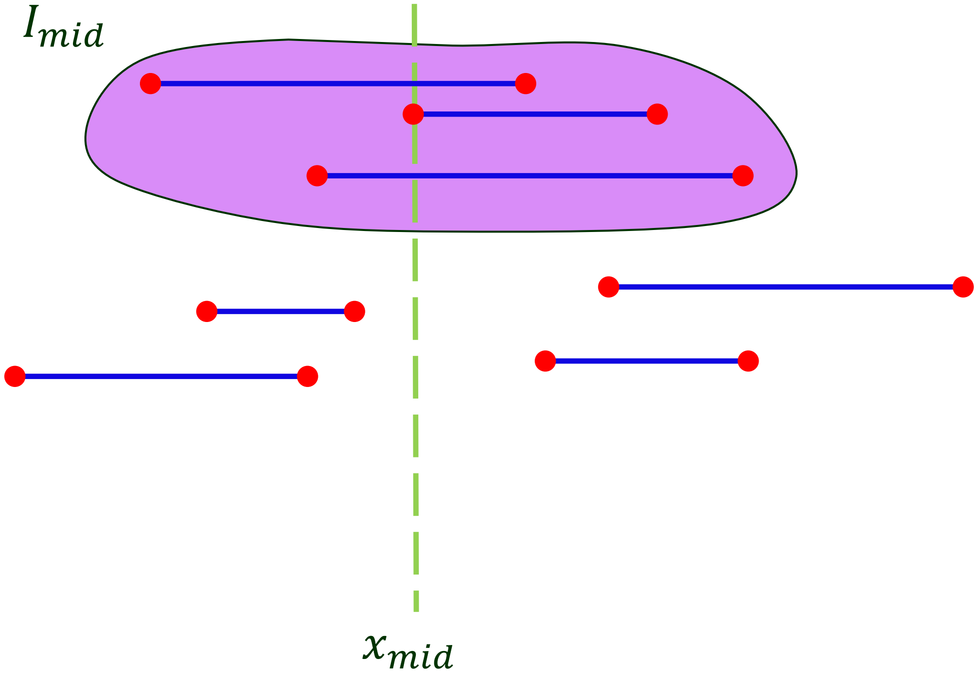

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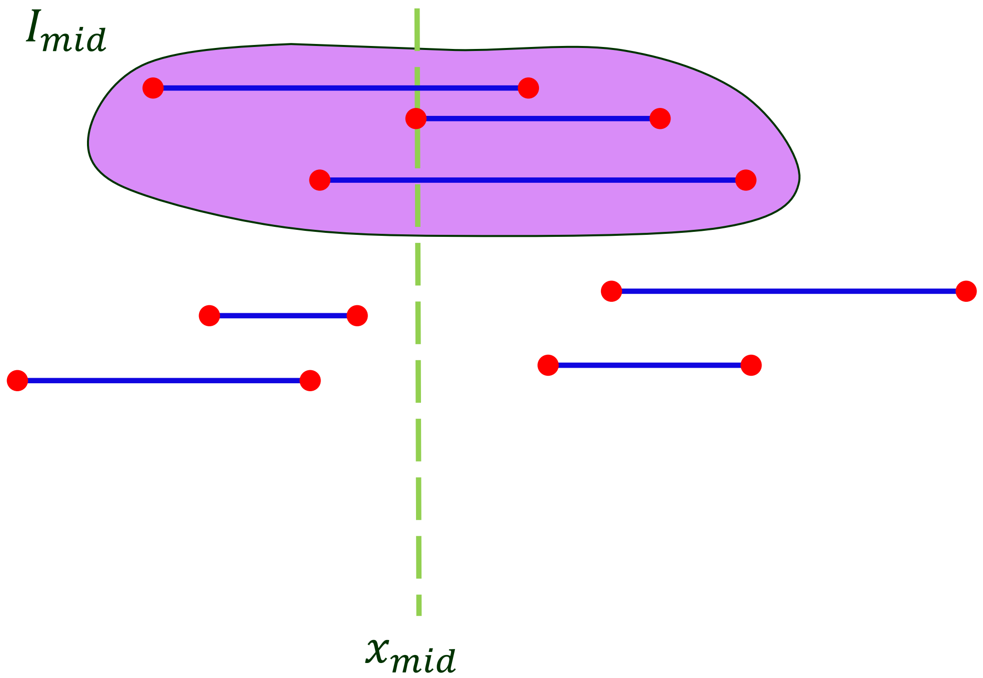

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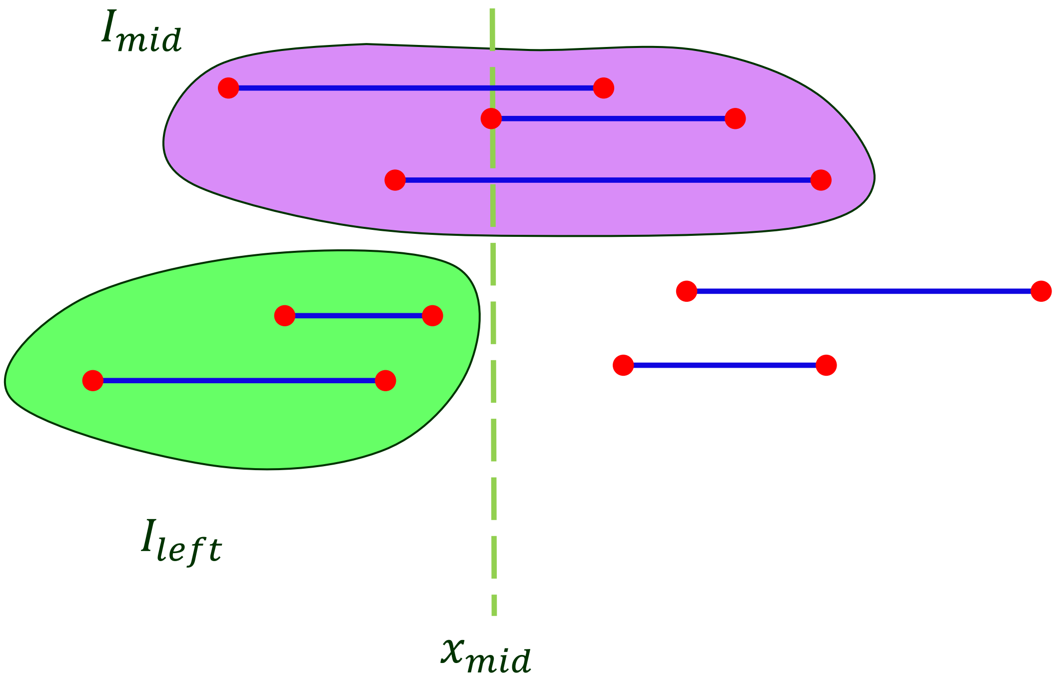

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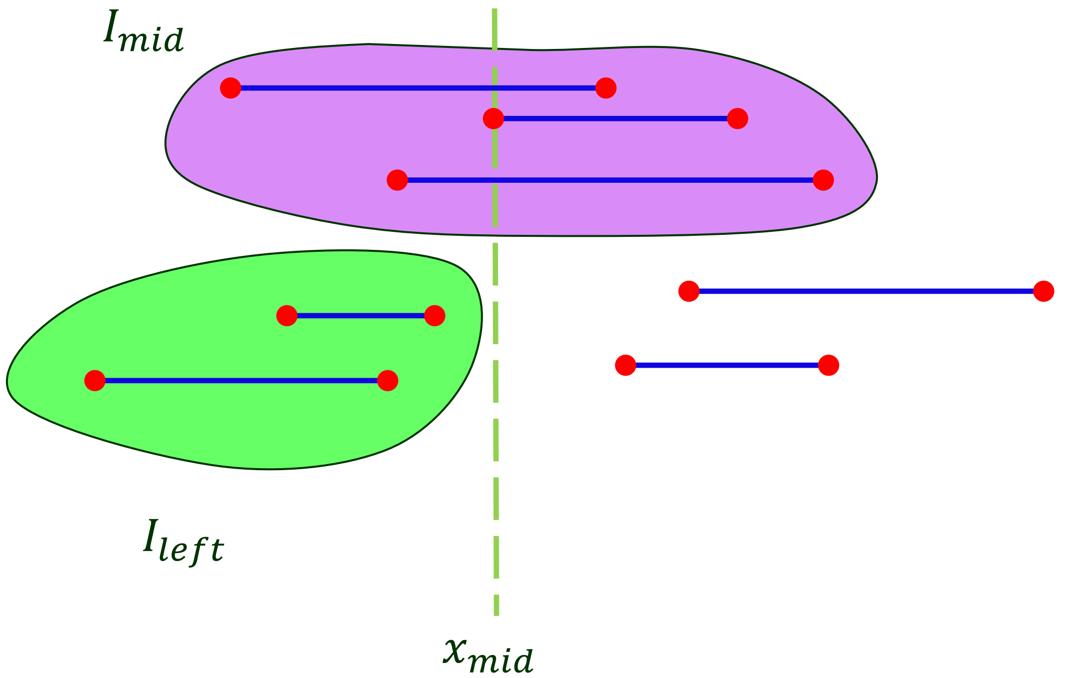

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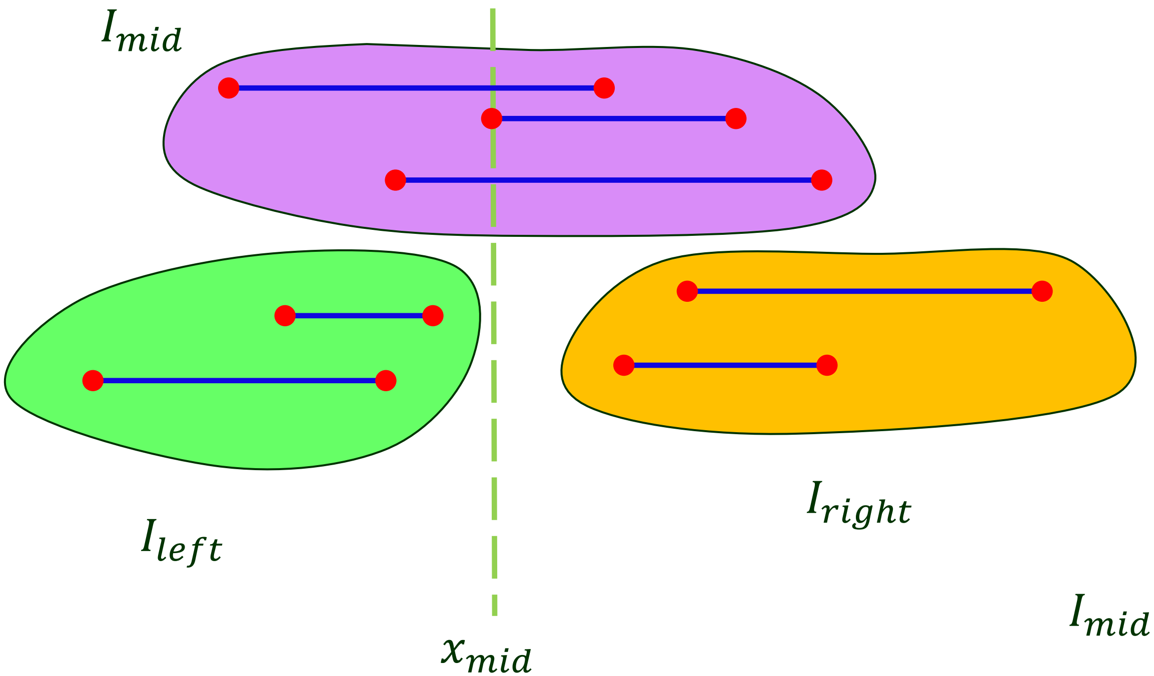

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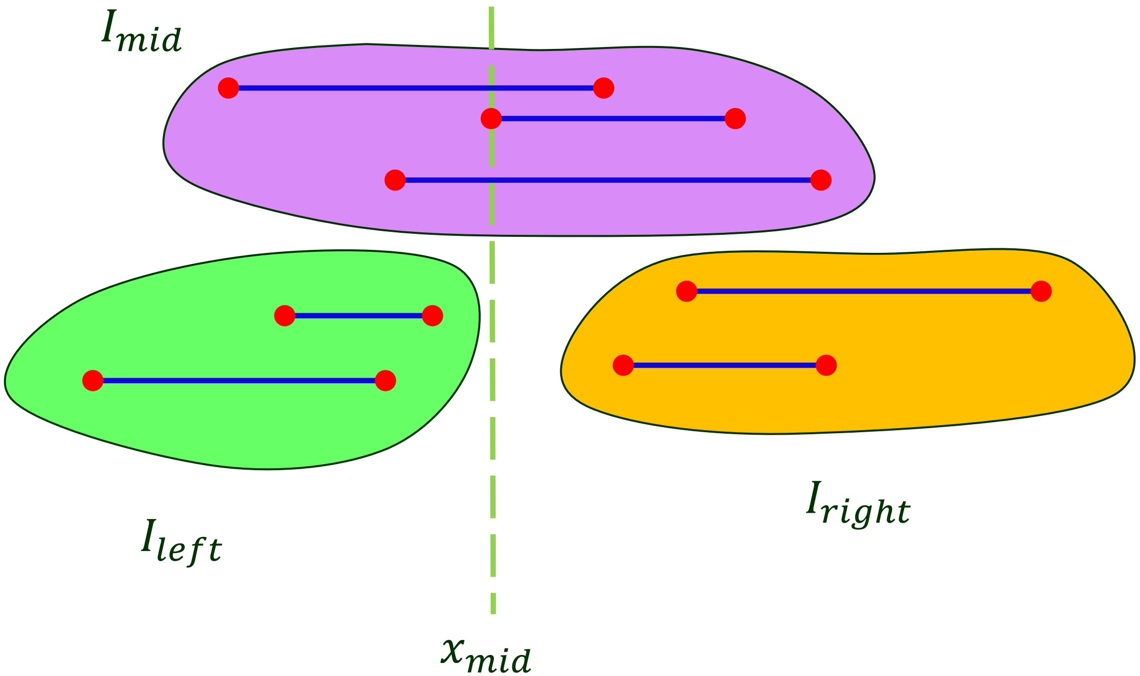
Interval Tree



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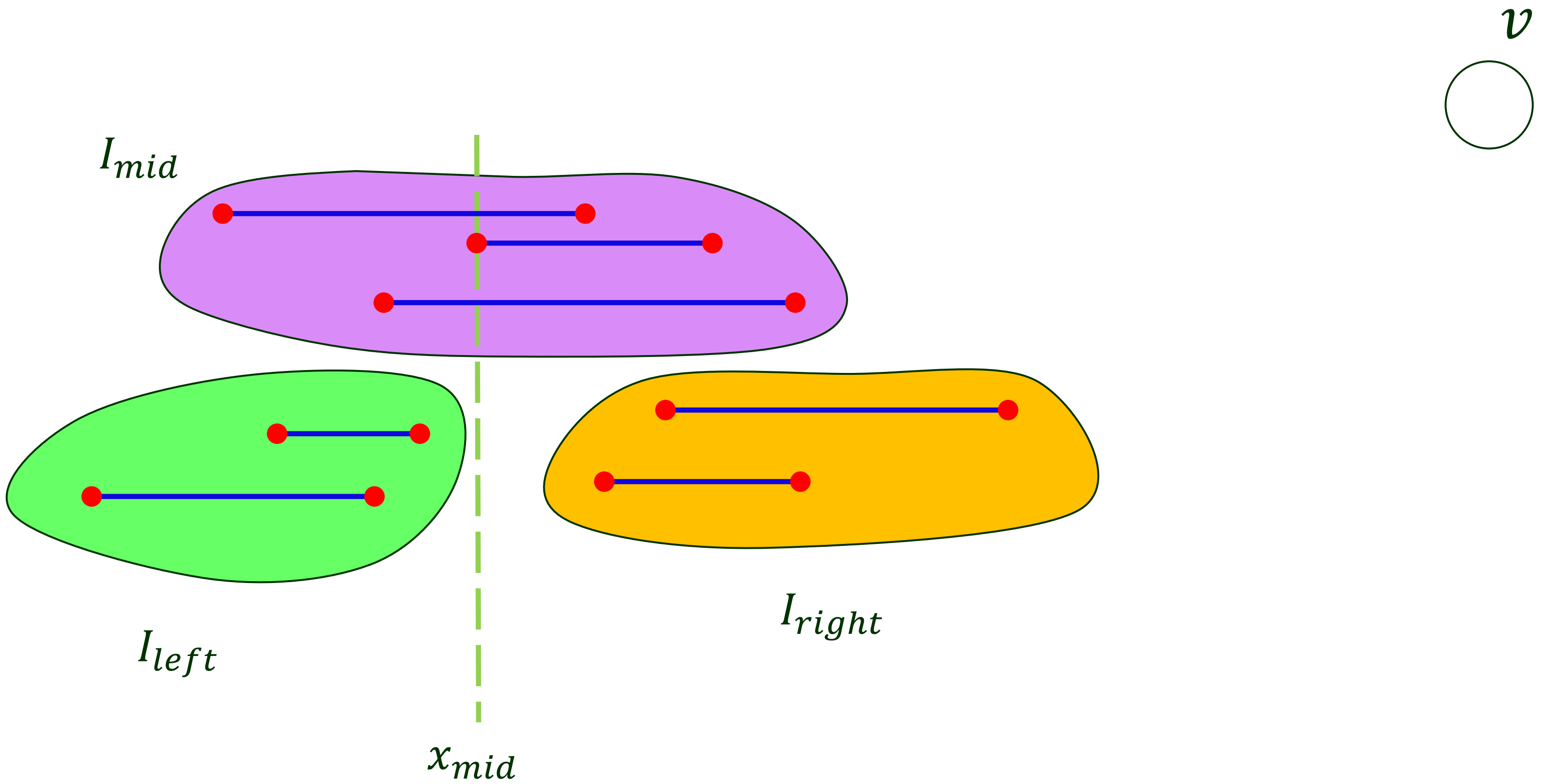

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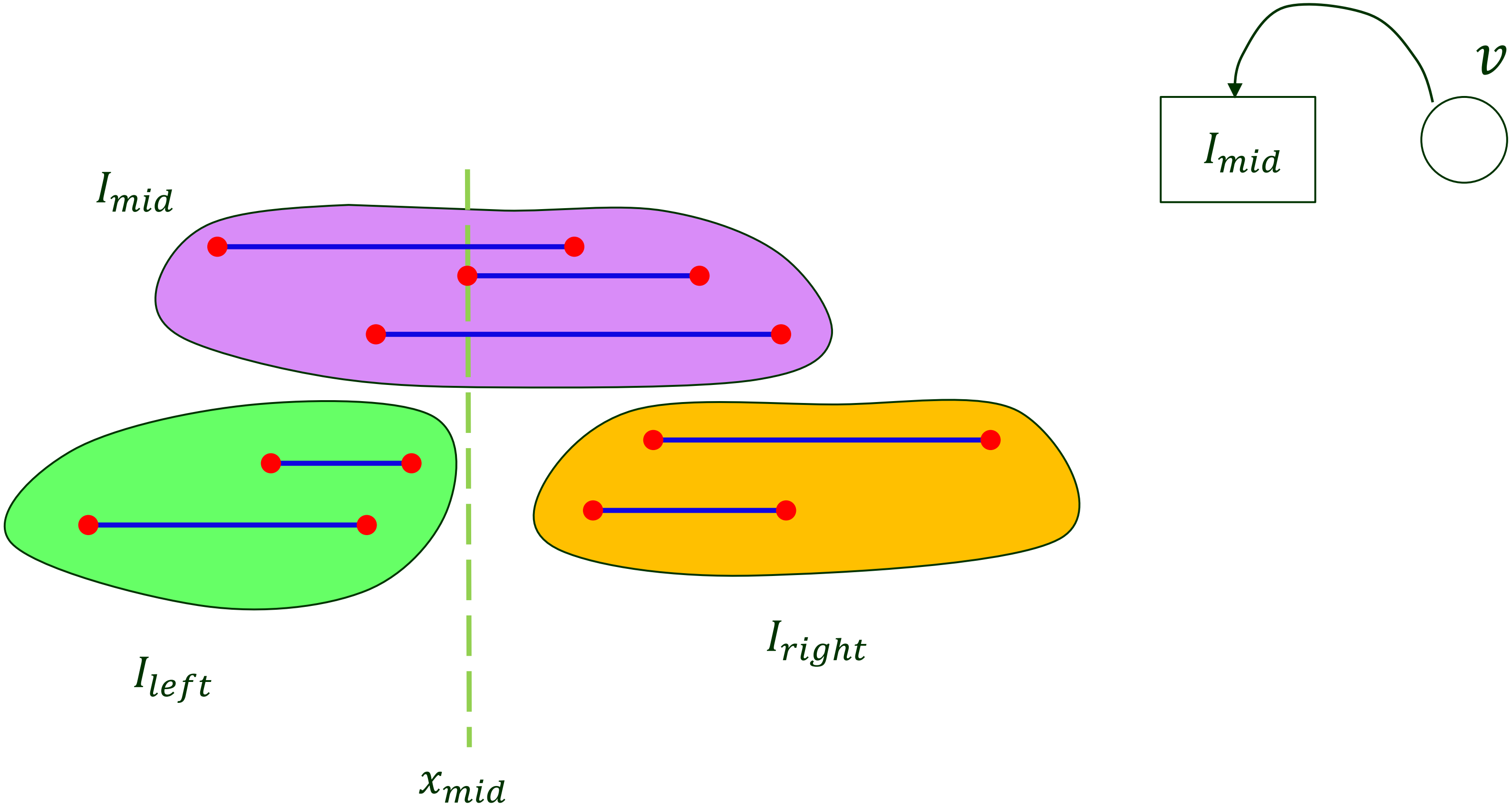

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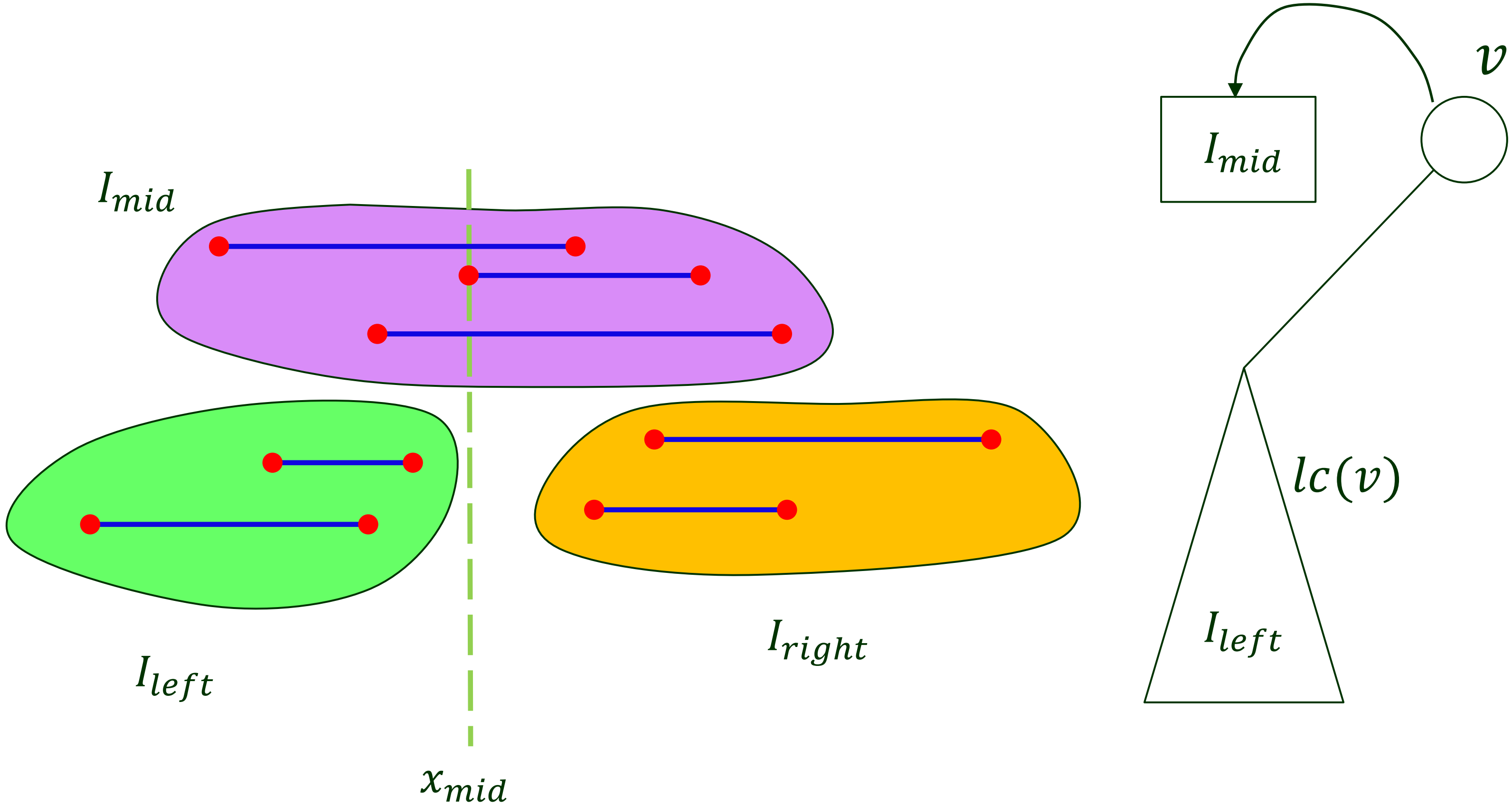

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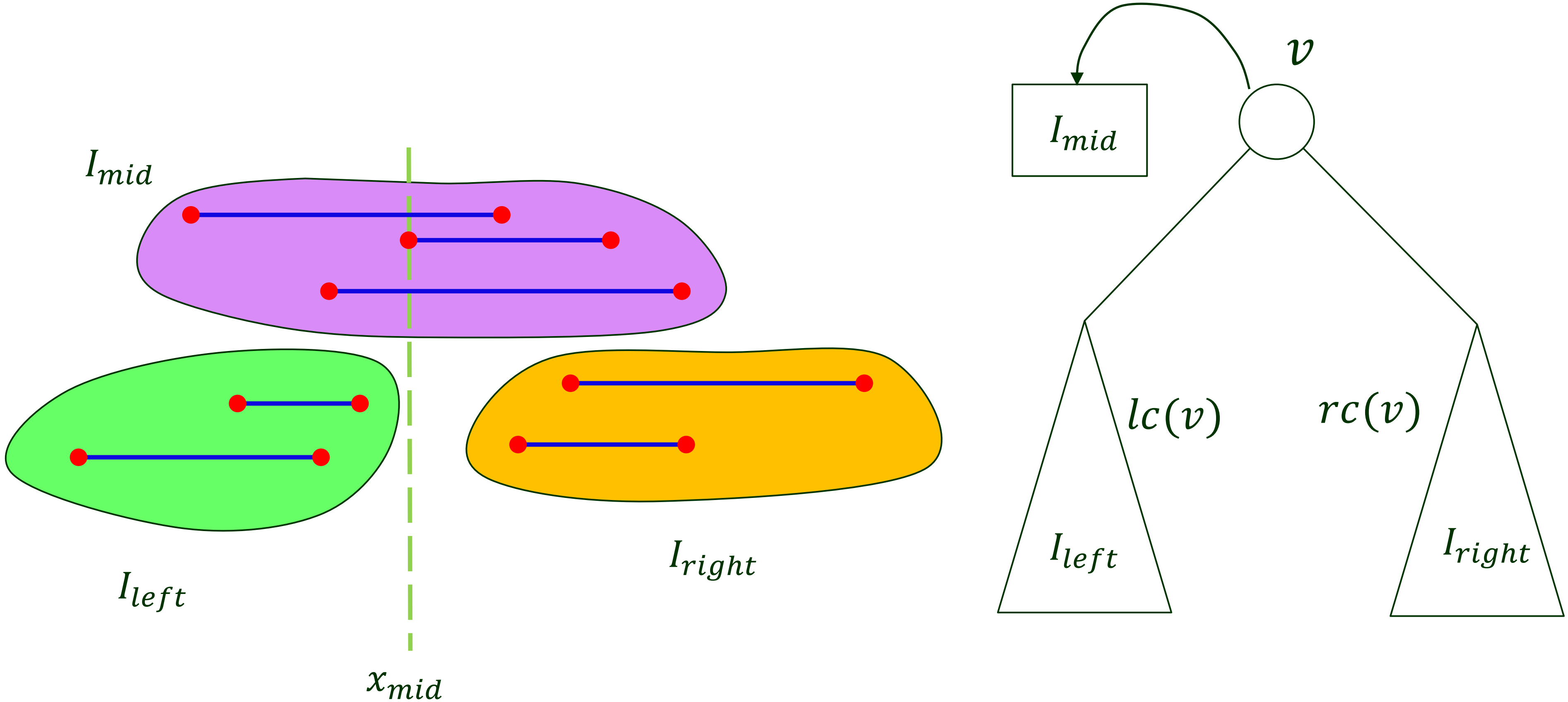

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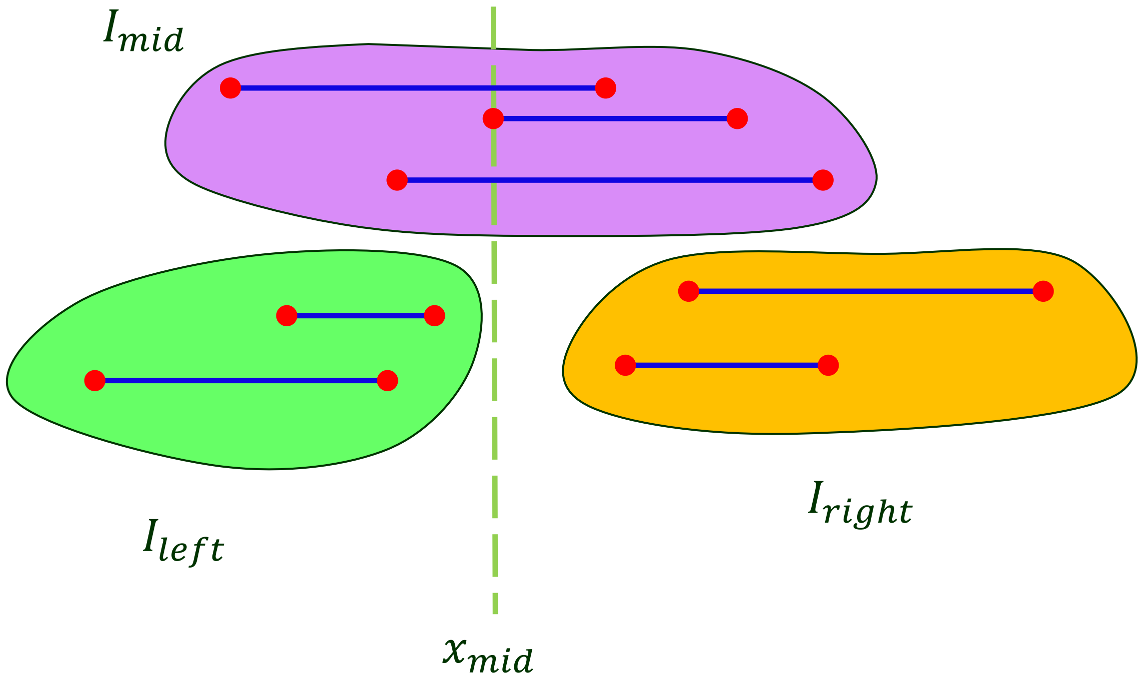

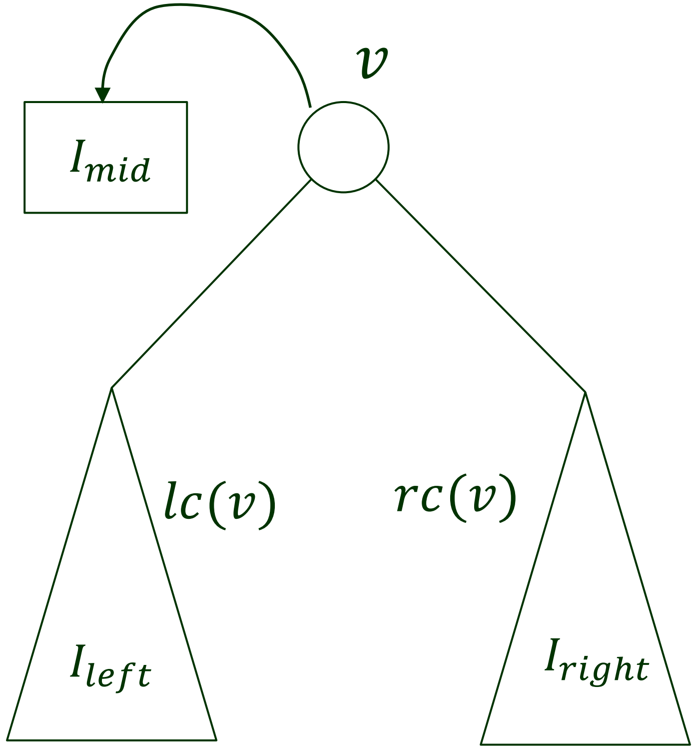

Subtrees $lc(v)$ and $rc(v)$ are recursively defined.

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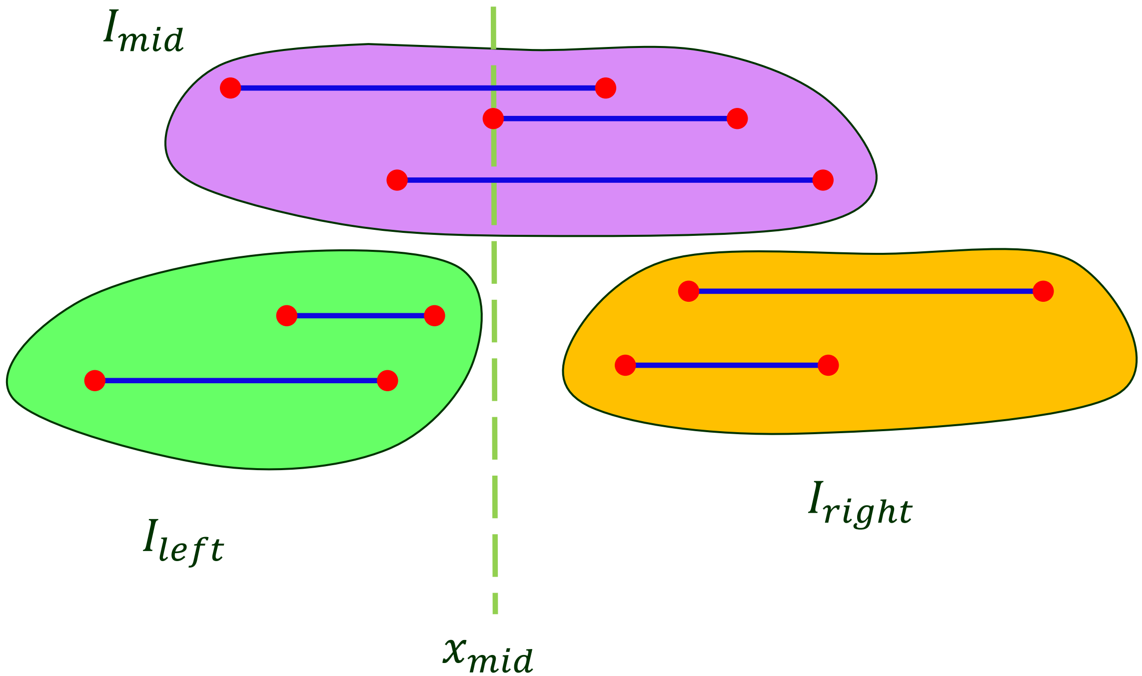

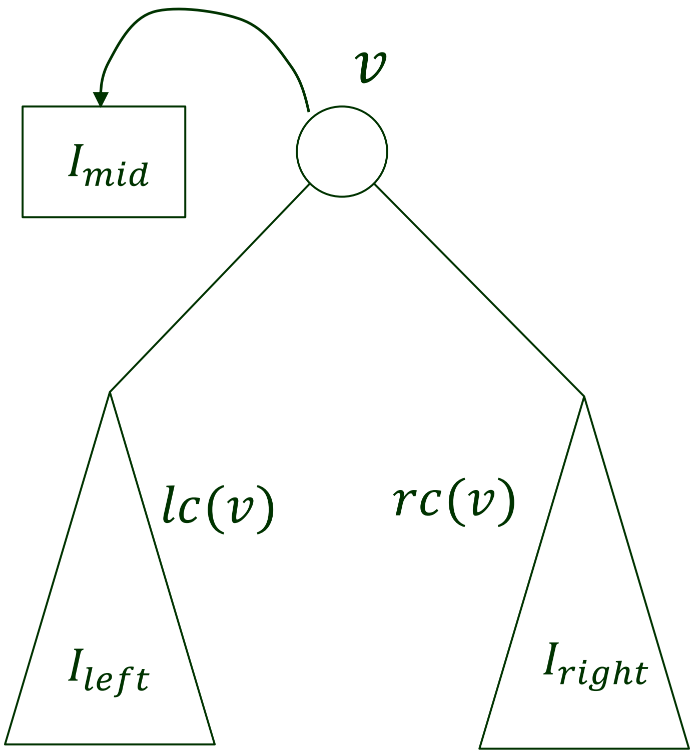

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$O(\log n)$ depth

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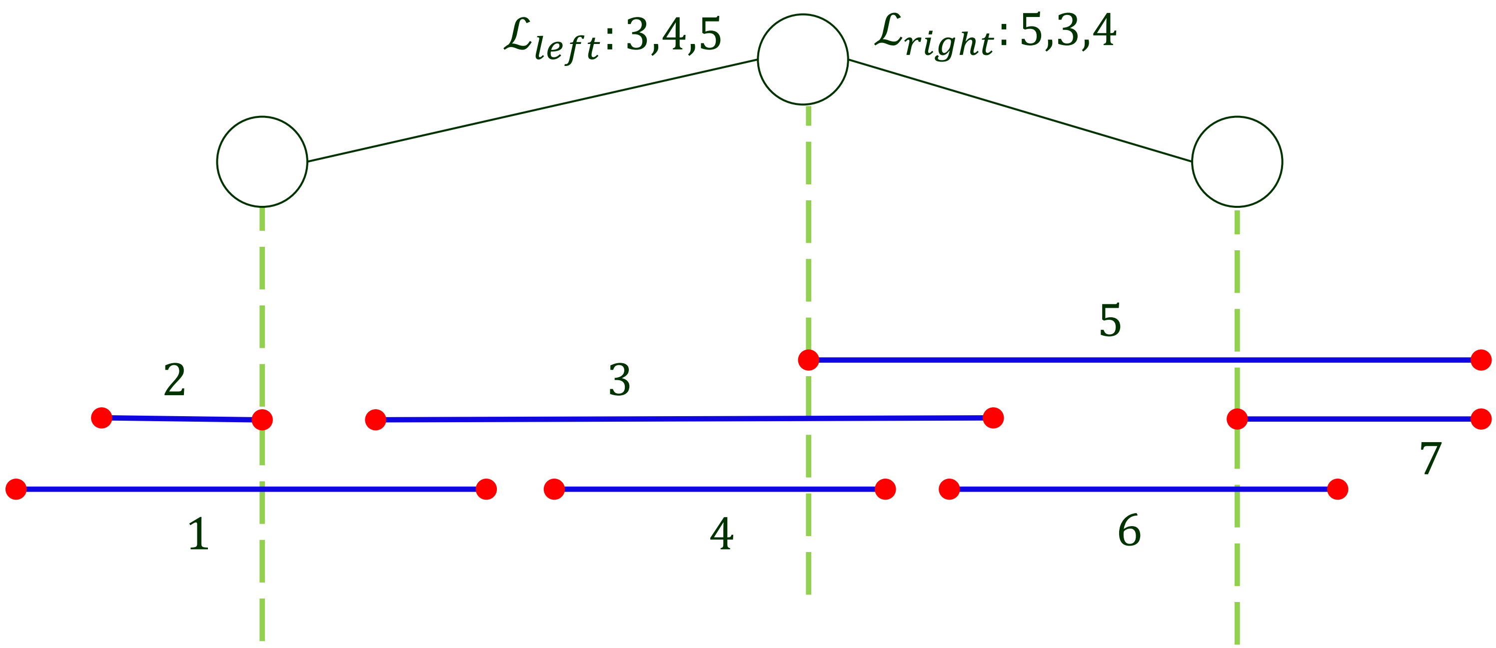

Total Storage

$I_{left}, I_{mid}, I_{right}$ are disjoint.

Every interval appears in only one set I_{mid} (stored once in each of the two sorted lists).

- All associated lists require $O(n)$ storage.
- The tree uses $O(n)$ storage.
- The tree has height $O(\log n)$.

III. Construction of the Interval Tree

ConstructInveralTree(I)

if $I = \emptyset$

    then return an empty leaf

    else compute x_{mid}

        store it in a new node v

        compute I_{mid}, I_{left}, I_{right}

        construct sorted lists $\mathcal{L}_{left}(v)$ and $\mathcal{L}_{right}(v)$

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        $lc(v) \leftarrow$ ConstructIntervalTree(I_{left})

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Presort the points and maintain ordered lists through recursive calls.

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Presort the points and maintain ordered lists through recursive calls.

$\Rightarrow$ $O(n)$ time before recursions on lines 8 and 9

$\Rightarrow$ Total construction time: $O(n \log n)$

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Find all intervals containing q_x.

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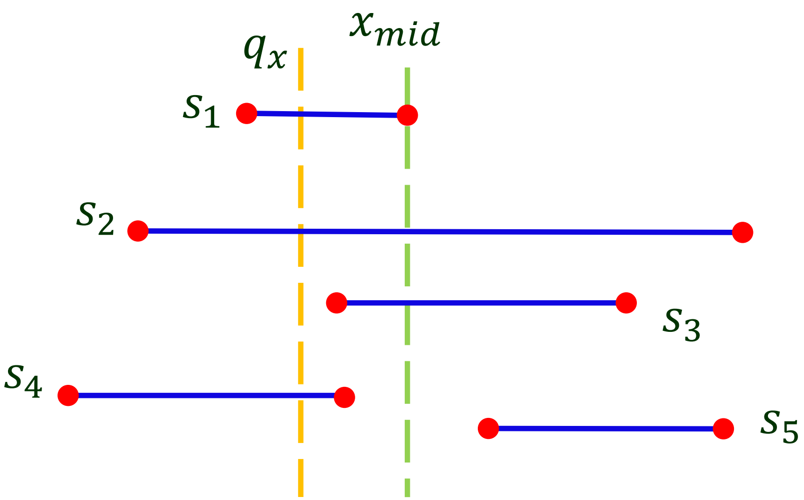

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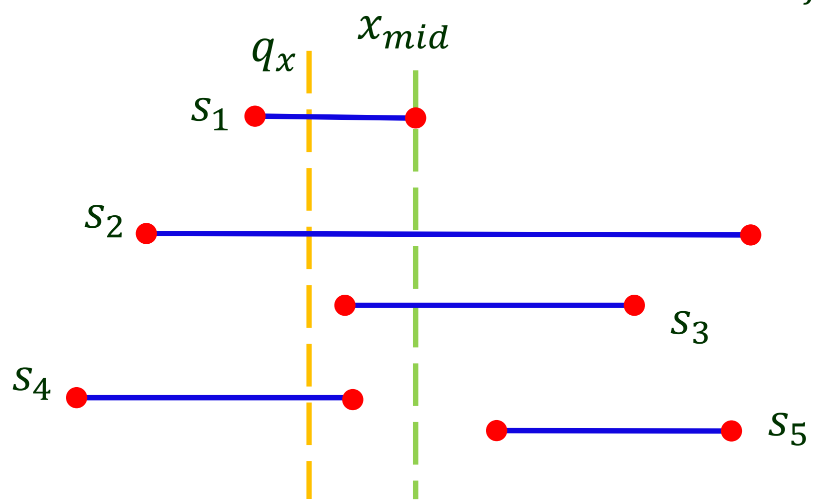
- $q_x = x_{mid}$ ends the query after reporting all intervals in I_{mid}.
- $q_x < x_{mid}(v) \implies$ Search I_{mid} and I_{left}.



Query

Find all intervals containing q_x.

- $q_x = x_{mid}$ ends the query after reporting all intervals in I_{mid}.
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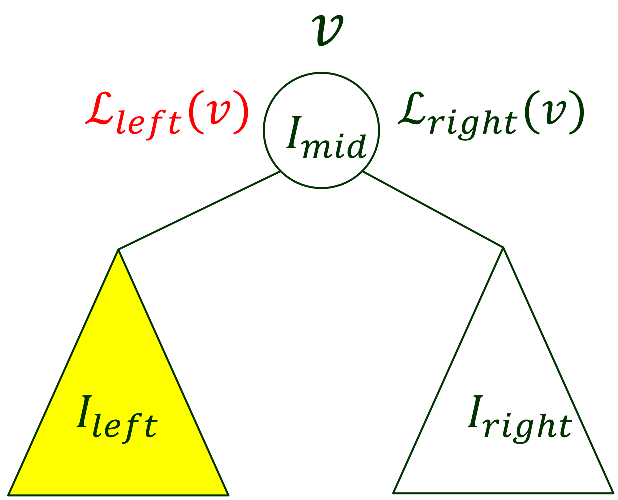

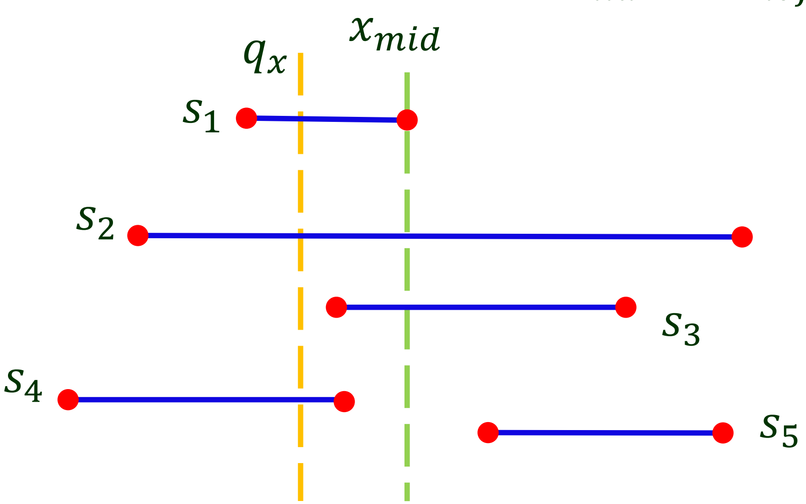

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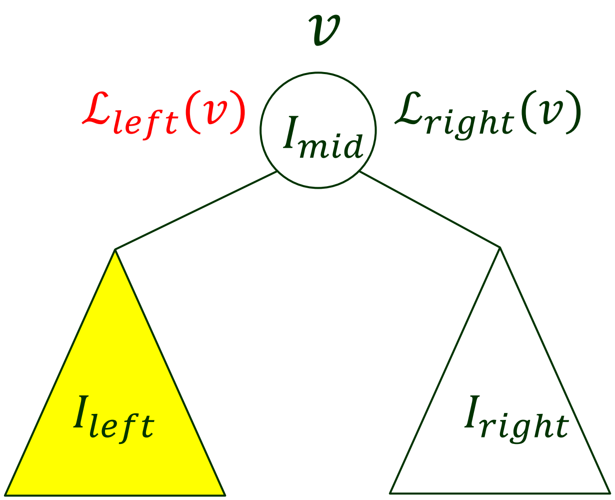

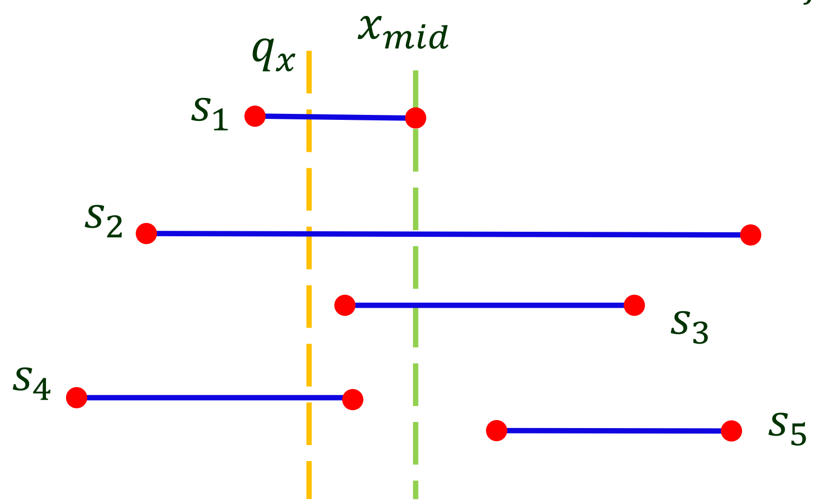

$\mathcal{L}_{left}(v) = \langle s_2, s_1, s_3 \rangle$

Query

Find all intervals containing q_x.

- $q_x = x_{mid}$ ends the query after reporting all intervals in I_{mid}.
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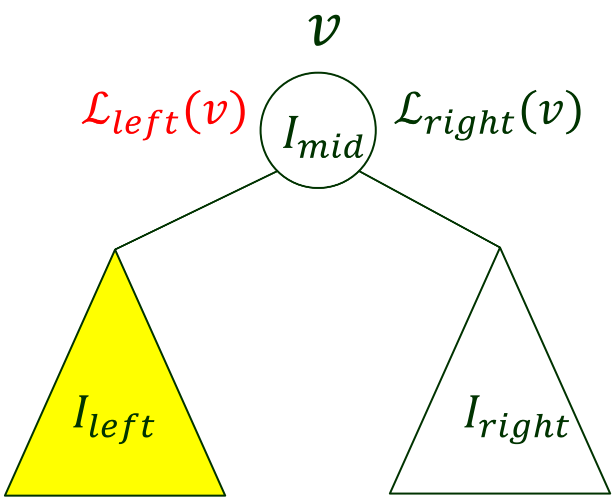
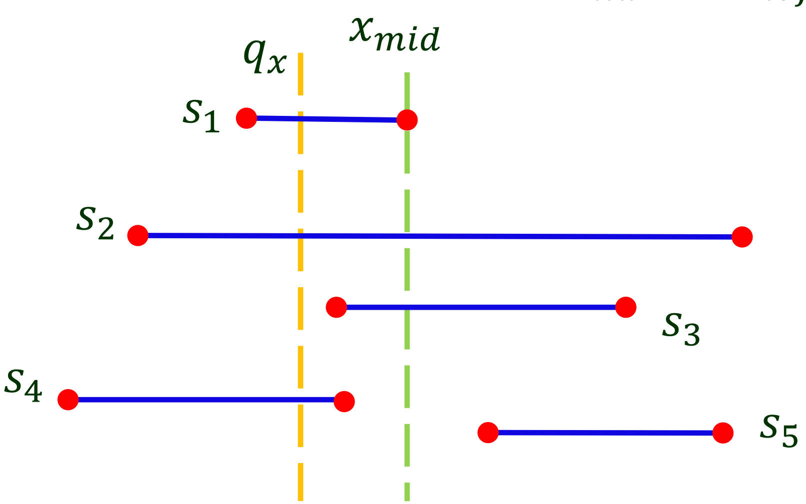




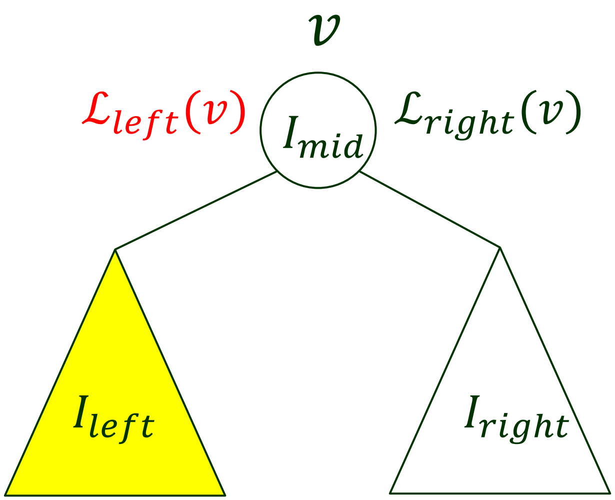
$\mathcal{L}_{left}(v) = \langle s_2, s_1, s_3 \rangle$

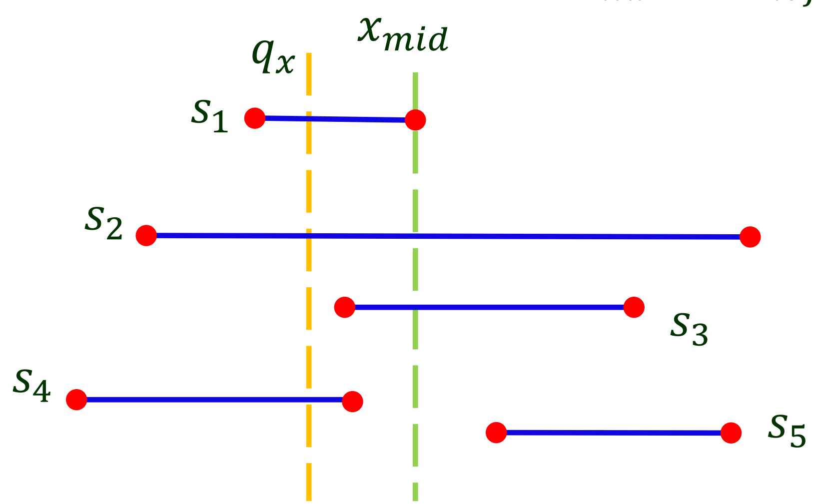
- At v, start at the leftmost endpoint of $\mathcal{L}_{left}$.

Query

Find all intervals containing q_x.

- $q_x = x_{mid}$ ends the query after reporting all intervals in I_{mid}.
- $q_x < x_{mid}(v) \implies$ Search I_{mid} and I_{left}.





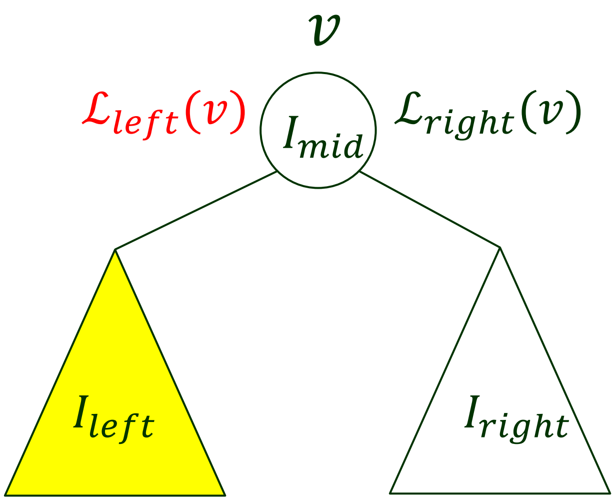
$\mathcal{L}_{left}(v) = \langle s_2, s_1, s_3 \rangle$

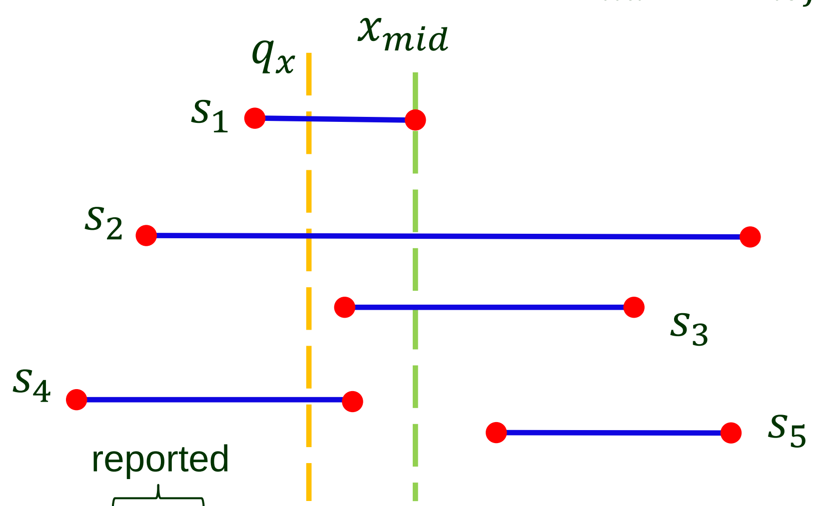
- At v, start at the leftmost endpoint of $\mathcal{L}_{left}$.
- Report all the intervals containing q_x.

Query

Find all intervals containing q_x.

- $q_x = x_{mid}$ ends the query after reporting all intervals in I_{mid}.
- $q_x < x_{mid}(v) \implies$ Search I_{mid} and I_{left}.





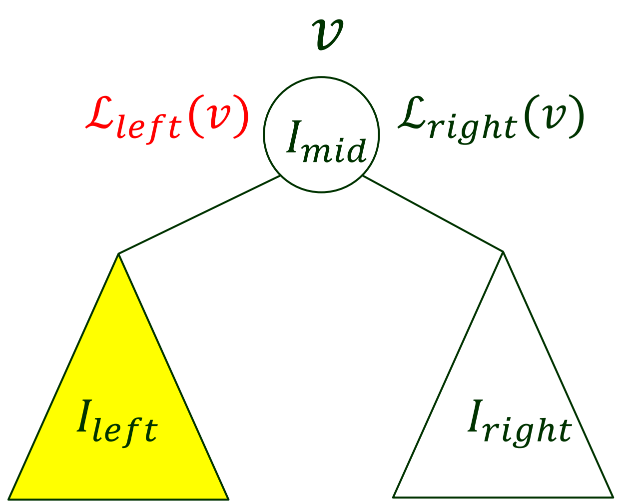
$$\mathcal{L}_{left}(v) = \langle \overbrace{s_2, s_1}^{\text{reported}}, s_3 \rangle$$

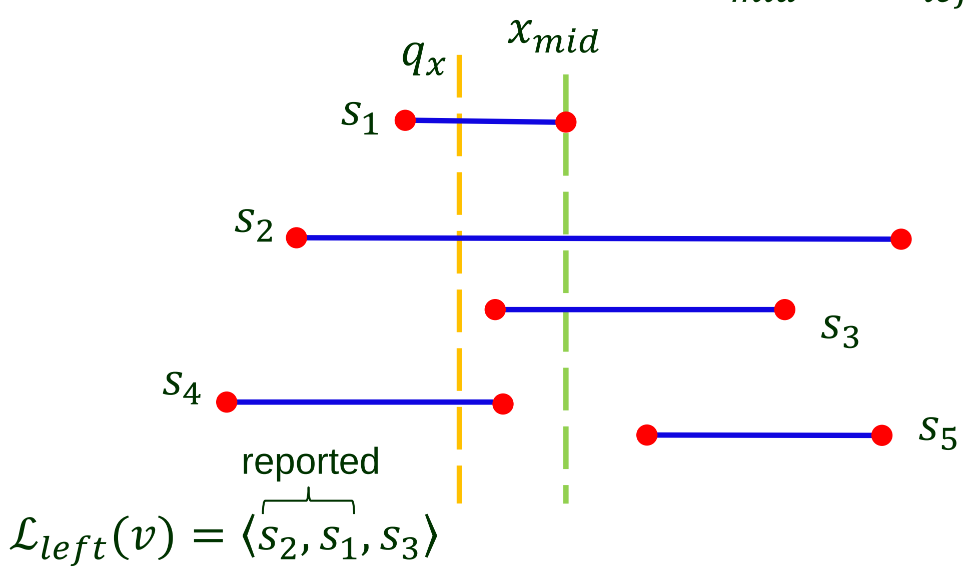
- At v, start at the leftmost endpoint of $\mathcal{L}_{left}$.
- Report all the intervals containing q_x.

Query

Find all intervals containing q_x.

- $q_x = x_{mid}$ ends the query after reporting all intervals in I_{mid}.
- $q_x < x_{mid}(v) \implies$ Search I_{mid} and I_{left}.



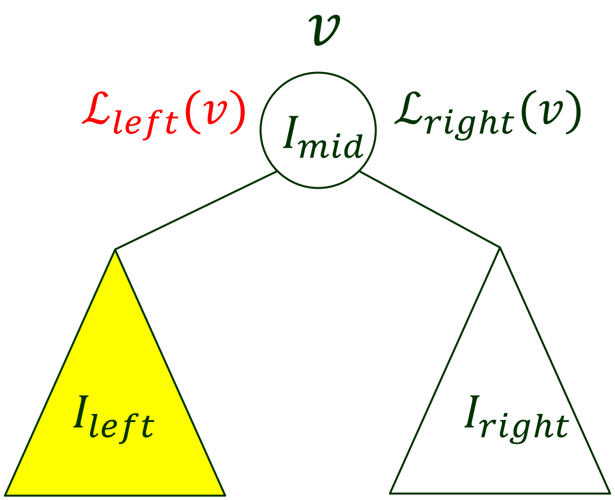


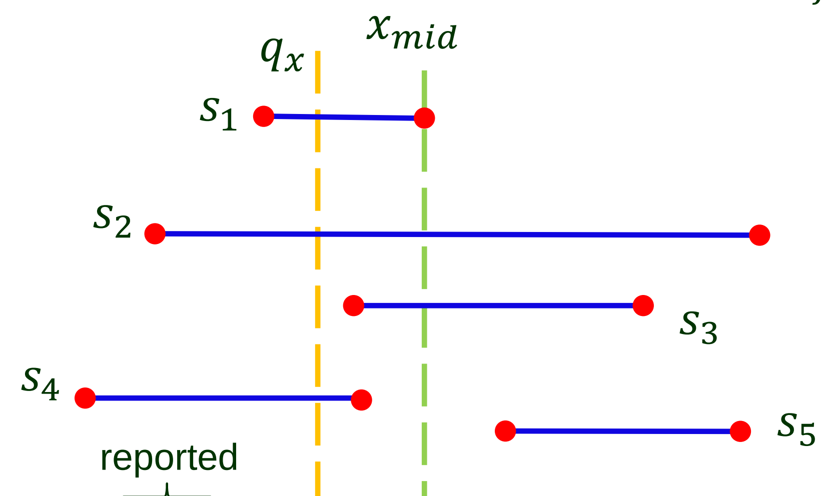
- At v, start at the leftmost endpoint of $\mathcal{L}_{left}$.
- Report all the intervals containing q_x.
- Stop at an interval that does not contain q_x (i.e., its left endpoint is to the right of q_x).

Query

Find all intervals containing q_x.

- $q_x = x_{mid}$ ends the query after reporting all intervals in I_{mid}.
- $q_x < x_{mid}(v) \implies$ Search I_{mid} and I_{left}.



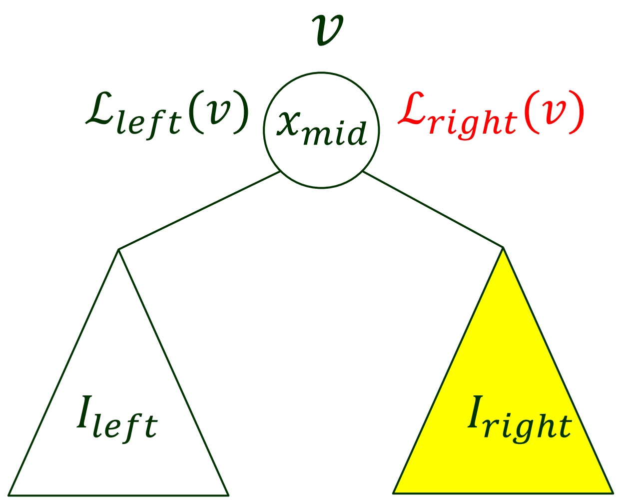


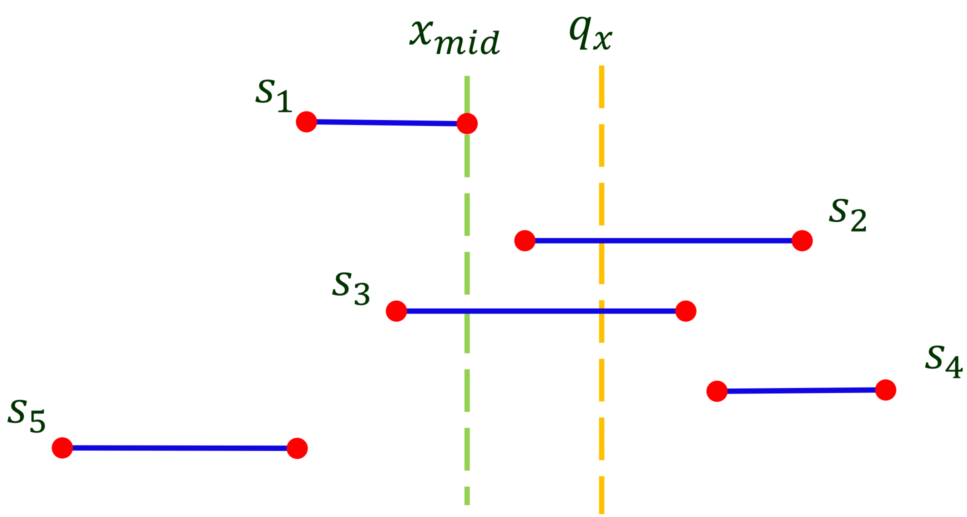
$\mathcal{L}_{left}(v) = \langle \overbrace{s_2, s_1}^{\text{reported}}, s_3 \rangle$

- At v, start at the leftmost endpoint of $\mathcal{L}_{left}$.
- Report all the intervals containing q_x.
- Stop at an interval that does not contain q_x (i.e., its left endpoint is to the right of q_x).
- Query the left subtree.

Query (cont'd)

- $q_x > x_{mid}(v) \quad \Rightarrow \quad$ Search I_{mid} and I_{right}.



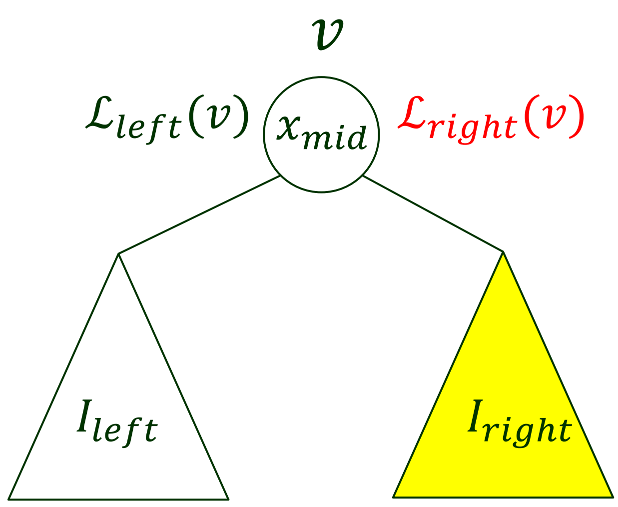


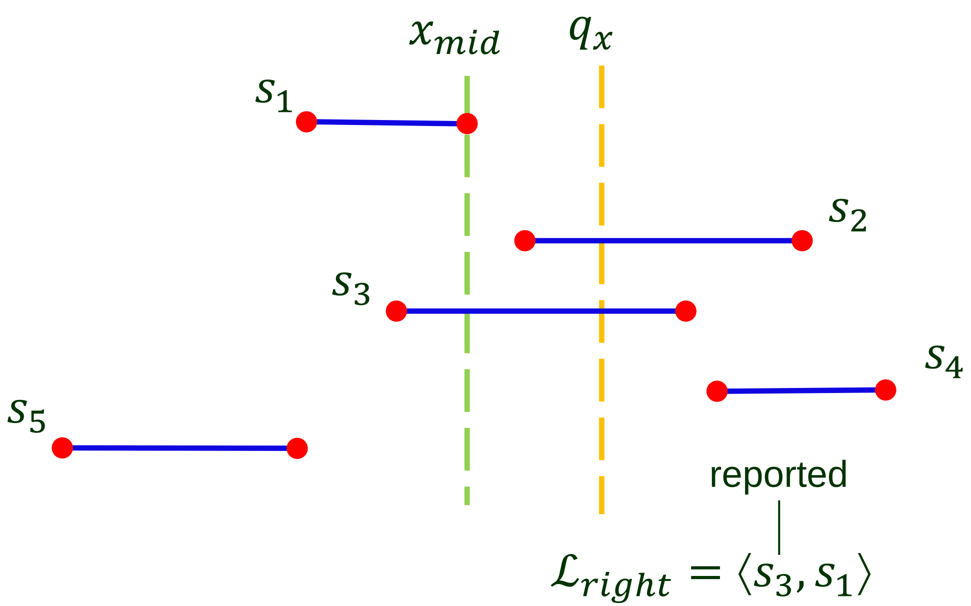
$\mathcal{L}_{right} = \langle s_3, s_1 \rangle$

- At v, start at the rightmost endpoint of $\mathcal{L}_{right}$.
- Report all the intervals containing q_x.
- Stop at an interval that does not contain q_x.
- Query the right subtree.

Query (cont'd)

- $q_x > x_{mid}(v)$ $\Longrightarrow$ Search I_{mid} and I_{right}.

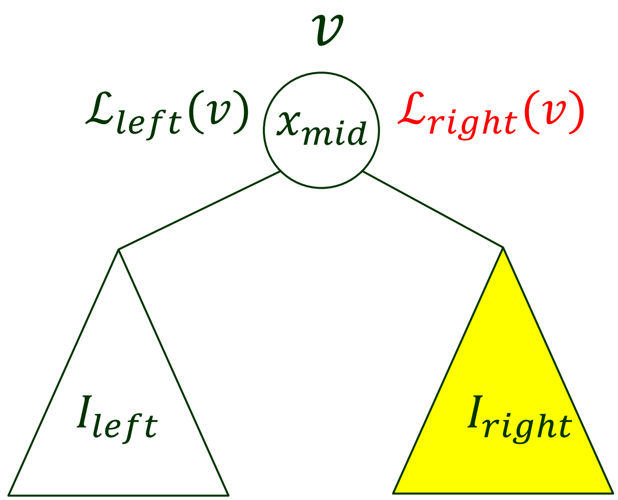


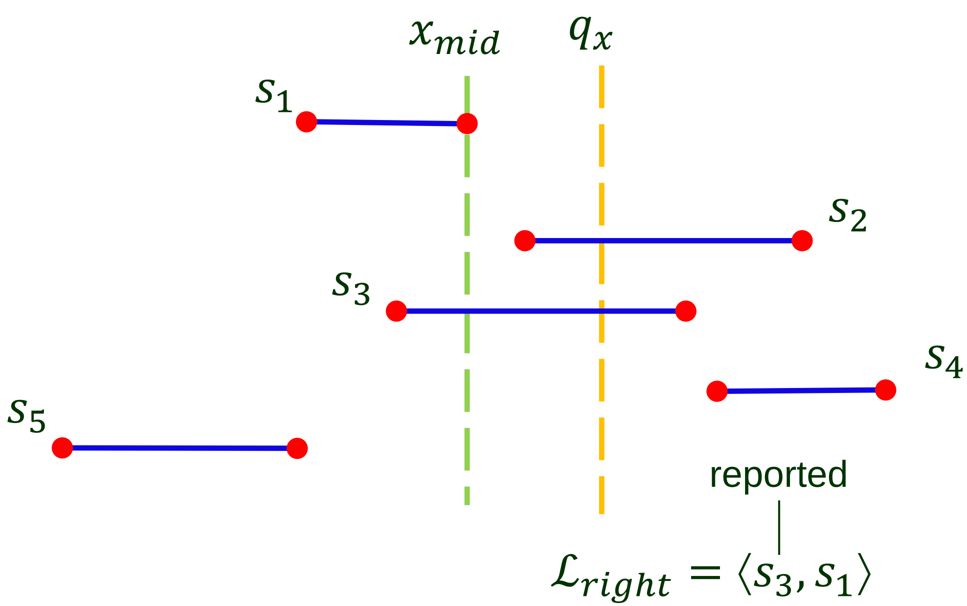


- At v, start at the rightmost endpoint of $\mathcal{L}_{right}$.
- Report all the intervals containing q_x.
- Stop at an interval that does not contain q_x.
- Query the right subtree.

Query (cont'd)

- $q_x > x_{mid}(v)$ $\Longrightarrow$ Search I_{mid} and I_{right}.



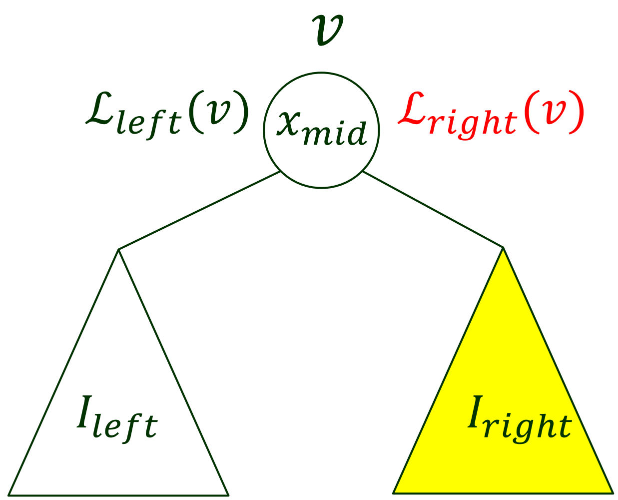


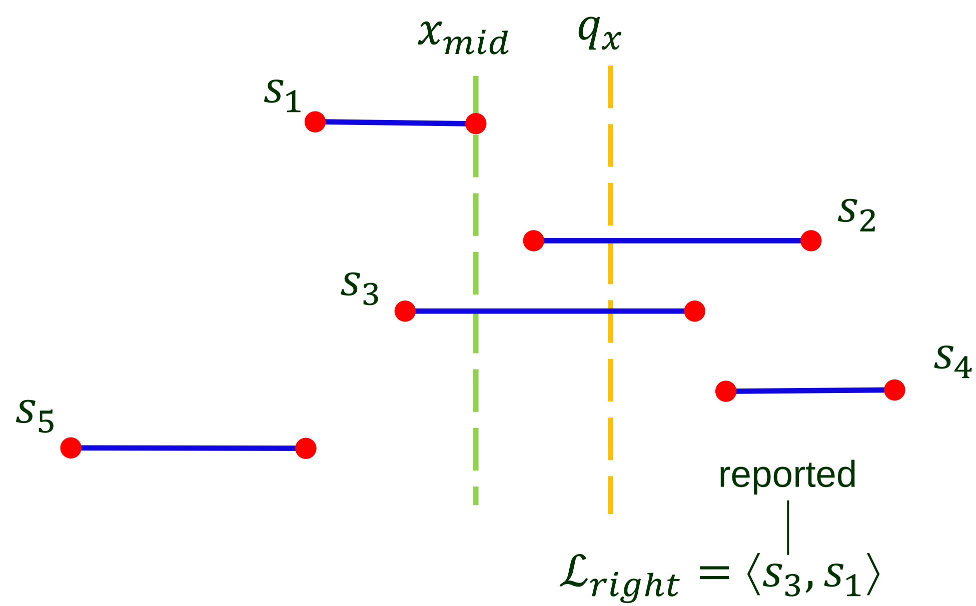
- At v, start at the rightmost endpoint of $\mathcal{L}_{right}$.
- Report all the intervals containing q_x.
- Stop at an interval that does not contain q_x.
- Query the right subtree.

Query time: $O(\log n + k)$

Query (cont'd)

◆ $q_x > x_{mid}(v)$ $\Longrightarrow$ Search I_{mid} and I_{right}.





- At v, start at the rightmost endpoint of $\mathcal{L}_{right}$.
- Report all the intervals containing q_x.
- Stop at an interval that does not contain q_x.
- Query the right subtree.

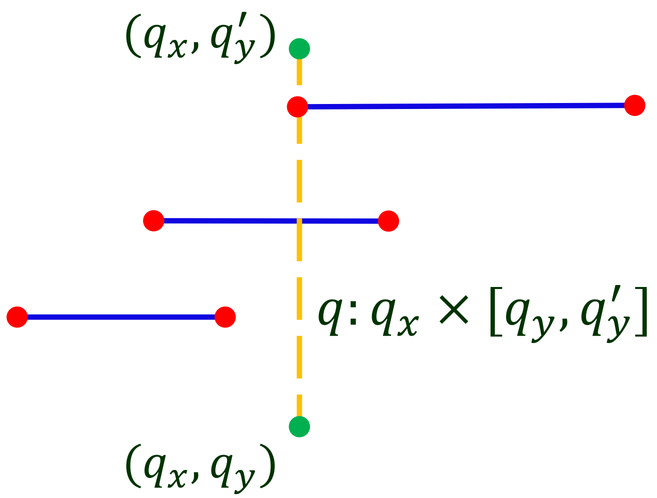
Query time: $O(\log n + k)$

#reported intervals

IV. Query Segment with Finite Length

Query object: a vertical line segment q: $q_x \times [q_y, q'_y]$

$S = \{s_1, s_2, \dots, s_n\}$: set of n horizontal segments



IV. Query Segment with Finite Length

Query object: a vertical line segment q: $q_x \times [q_y, q'_y]$

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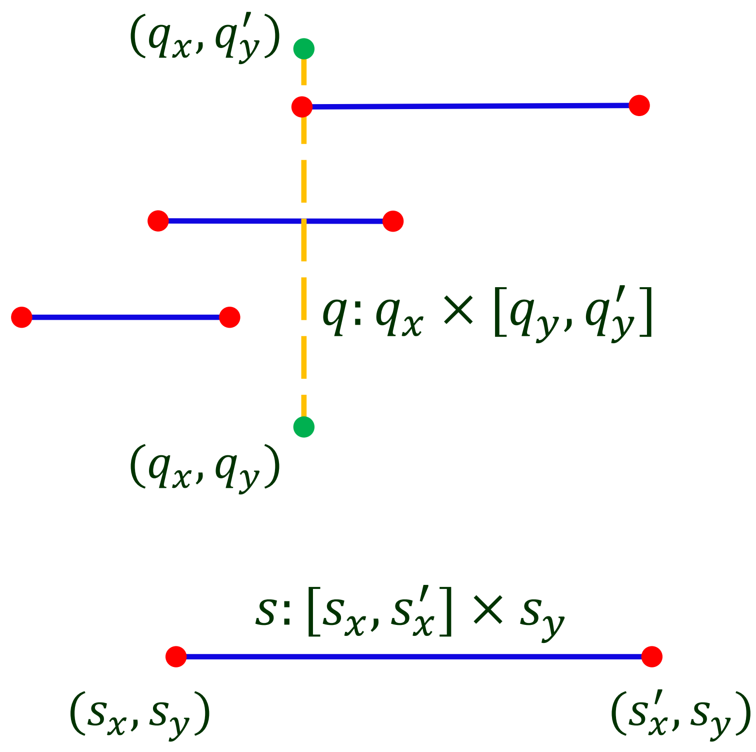
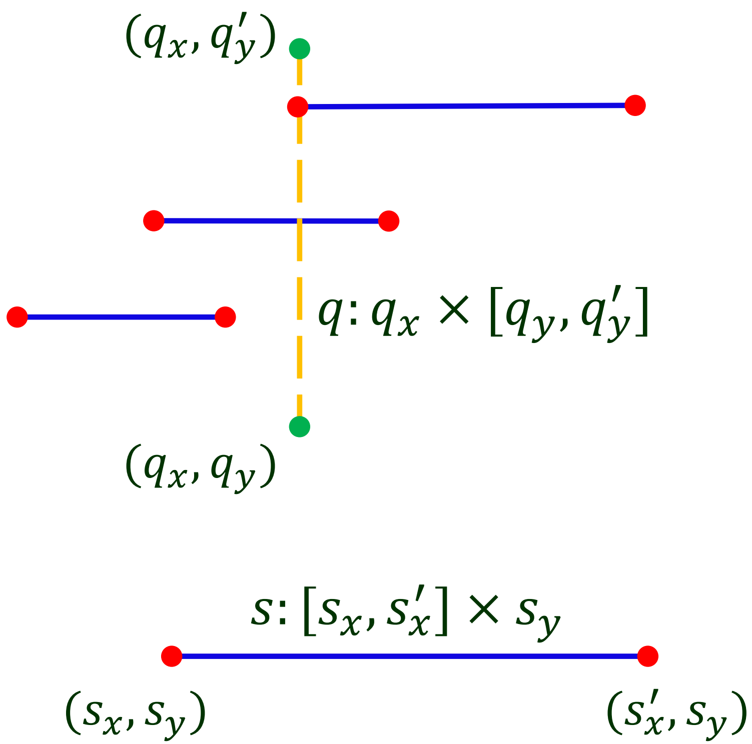


IV. Query Segment with Finite Length

Query object: a vertical line segment q: $q_x \times [q_y, q_y']$

$S = \{s_1, s_2, \dots, s_n\}$: set of n horizontal segments



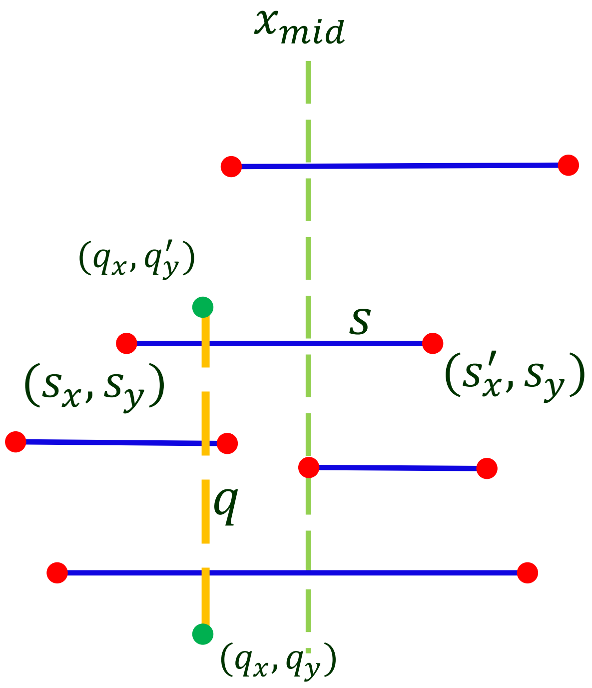
s *intersects* q iff

i) $s_x \le q_x \le s_x'$

ii) $q_y \le s_y \le q_y'$

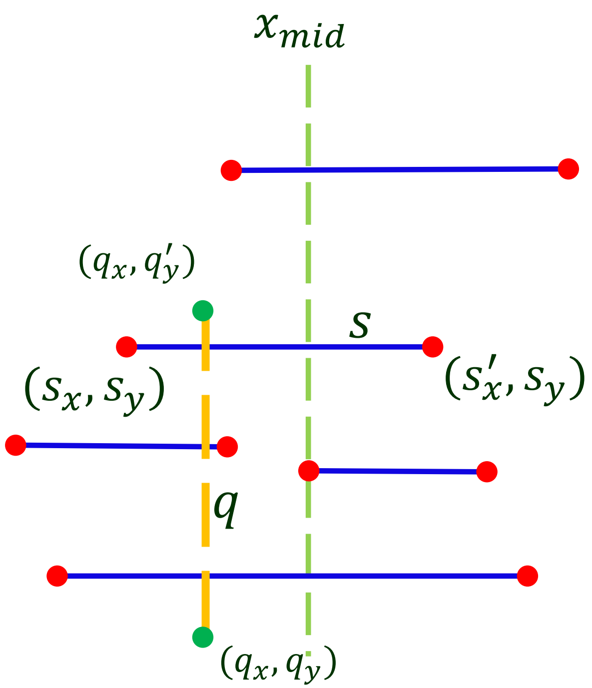
Modified Query

Modify the (earlier described) query with a vertical line object.



Modified Query

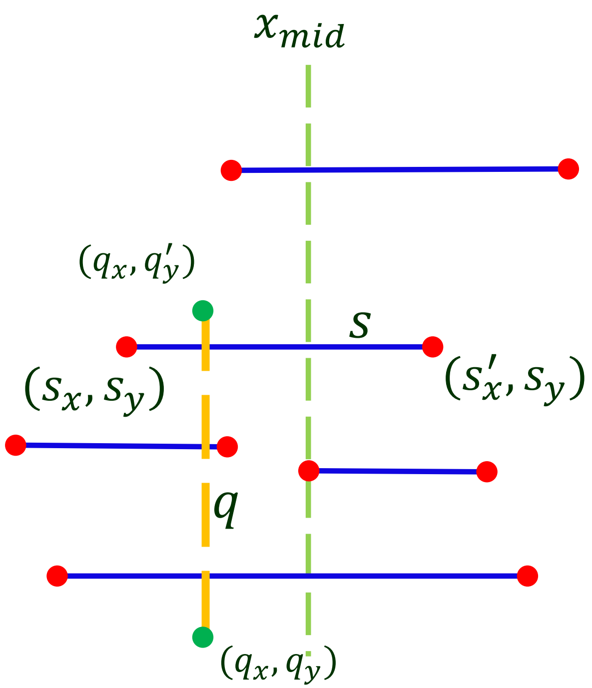
Modify the (earlier described) query with a vertical line object.



- q to the left of x_{mid}, i.e., $q_x \leq x_{mid}$.

Modified Query

Modify the (earlier described) query with a vertical line object.

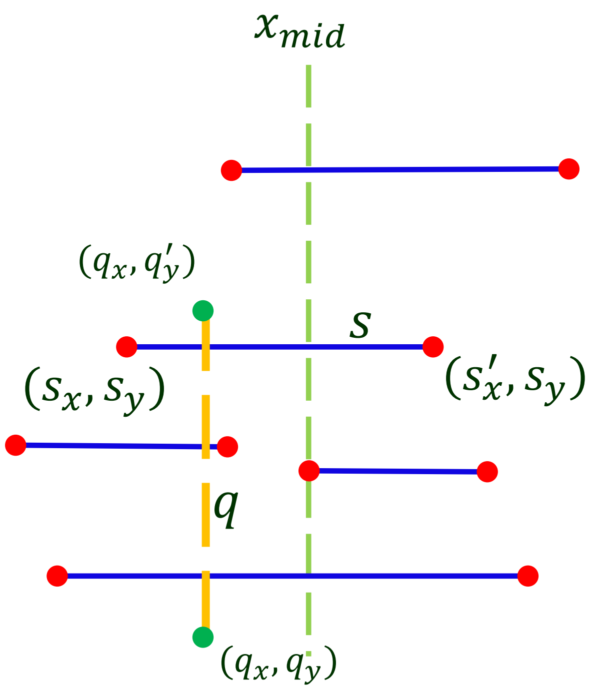


- q to the left of x_{mid}, i.e., $q_x \leq x_{mid}$.
  - Query I_{mid}. We know that every segment $s = [s_x, s'_x] \times s_y \in I_{mid}$ (in the increasing order of left endpoint) satisfies the query iff

$$s_x \leq q_x \leq s'_x \text{ and } q_y \leq s_y \leq q'_y$$

Modified Query

Modify the (earlier described) query with a vertical line object.



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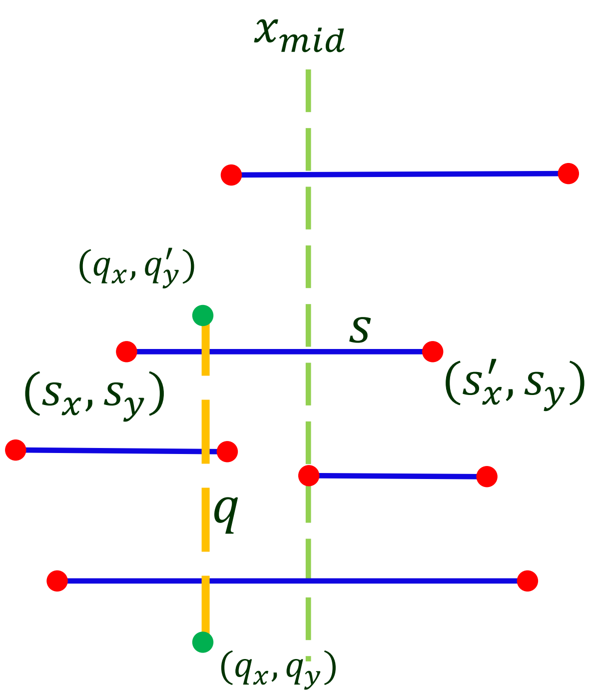
$$x_{mid} \le s'_x$$

Modified Query

Modify the (earlier described) query with a vertical line object.



- q to the left of x_{mid}, i.e., $q_x \leq x_{mid}$.
  - Query I_{mid}. We know that every segment $s = [s_x, s'_x] \times s_y \in I_{mid}$ (in the increasing order of left endpoint) satisfies the query iff

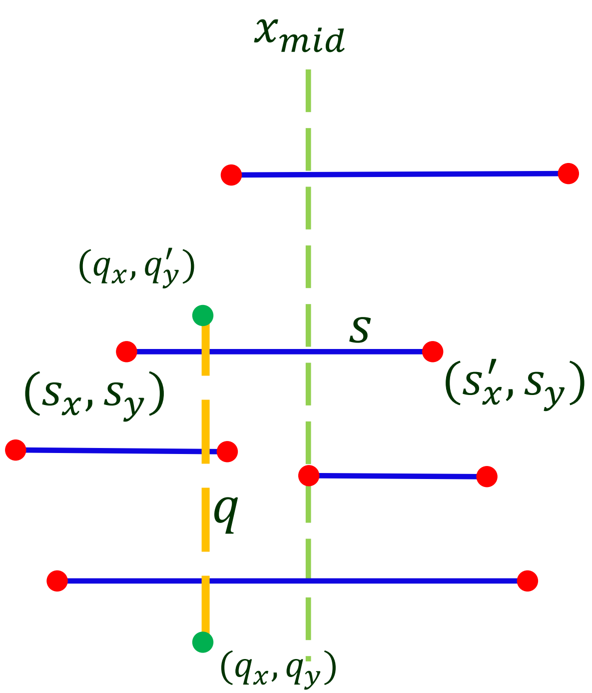
$$s_x \leq q_x \leq s'_x \text{ and } q_y \leq s_y \leq q'_y$$

$x_{mid} \leq s'_x$

$q_x \leq x_{mid}$

Modified Query

Modify the (earlier described) query with a vertical line object.



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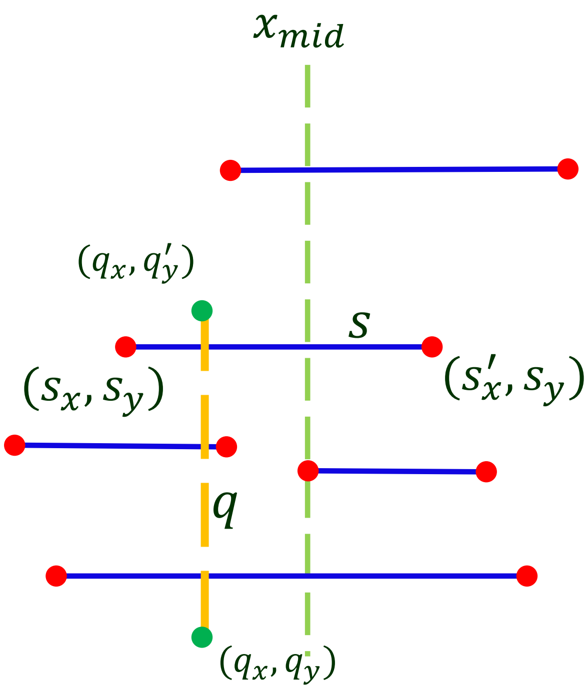
  - Query I_{mid}. We know that every segment $s = [s_x, s'_x] \times s_y \in I_{mid}$ (in the increasing order of left endpoint) satisfies the query iff

$$s_x \le \boxed{q_x \le s'_x} \text{ and } q_y \le s_y \le q'_y$$

$$\left.\begin{array}{l} x_{mid} \le s'_x \\ q_x \le x_{mid} \end{array}\right\}$$

Modified Query

Modify the (earlier described) query with a vertical line object.



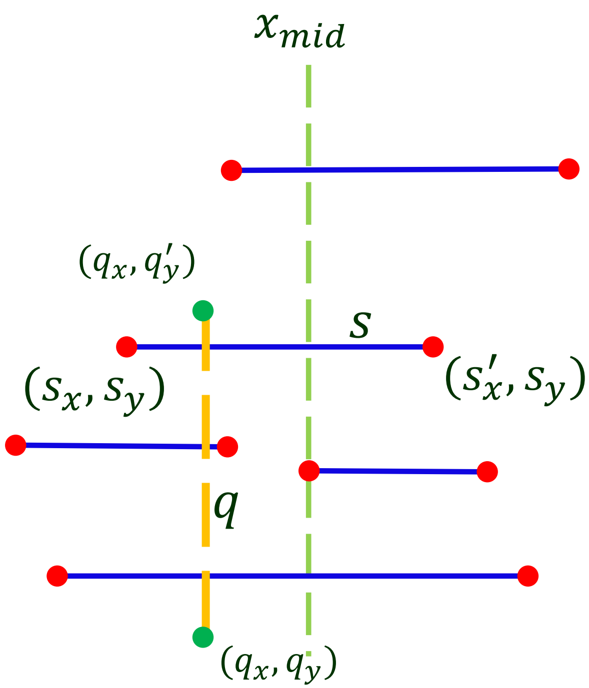
- q to the left of x_{mid}, i.e., $q_x \le x_{mid}$.
  - Query I_{mid}. We know that every segment $s = [s_x, s'_x] \times s_y \in I_{mid}$ (in the increasing order of left endpoint) satisfies the query iff

$$\boxed{s_x \le q_x} \le s'_x \text{ and } \boxed{q_y \le s_y \le q'_y}$$

$$\left.\begin{array}{l} x_{mid} \le s'_x \\ q_x \le x_{mid} \end{array}\right\} \Rightarrow q_x \le s'_x$$

Modified Query

Modify the (earlier described) query with a vertical line object.



- q to the left of x_{mid}, i.e., $q_x \le x_{mid}$.
  - Query I_{mid}. We know that every segment $s = [s_x, s'_x] \times s_y \in I_{mid}$ (in the increasing order of left endpoint) satisfies the query iff

$$\boxed{s_x \le \boxed{q_x}} \le s'_x \text{ and } \boxed{q_y \le s_y \le q'_y}$$

$$\left.\begin{array}{l} x_{mid} \le s'_x \\ q_x \le x_{mid} \end{array}\right\}$$

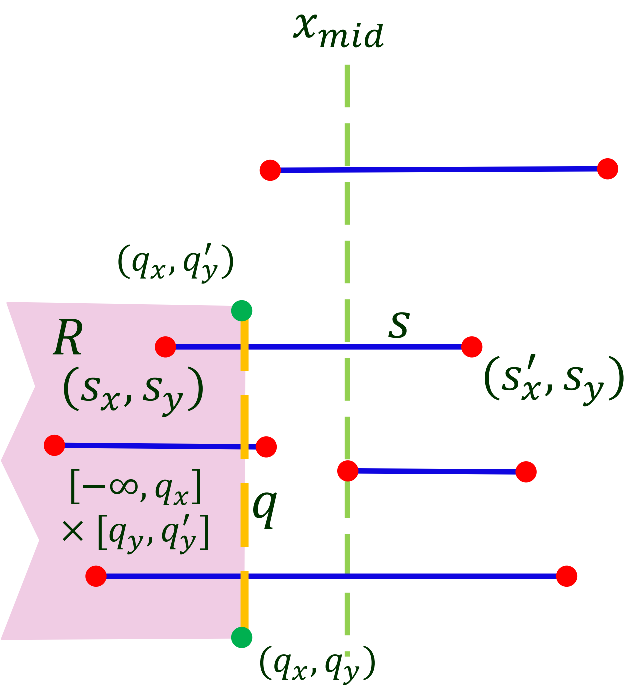
$$\Updownarrow$$

$$(s_x, s_y) \in R\colon [-\infty, q_x] \times [q_y, q'_y]$$

Left endpoint

Modified Query

Modify the (earlier described) query with a vertical line object.



- q to the left of x_{mid}, i.e., $q_x \le x_{mid}$.
  - Query I_{mid}. We know that every segment $s = [s_x, s'_x] \times s_y \in I_{mid}$ (in the increasing order of left endpoint) satisfies the query iff

$$\boxed{s_x \le q_x} \le s'_x \text{ and } \boxed{q_y \le s_y \le q'_y}$$

$$\left.\begin{array}{l} x_{mid} \le s'_x \\ q_x \le x_{mid} \end{array}\right\}$$

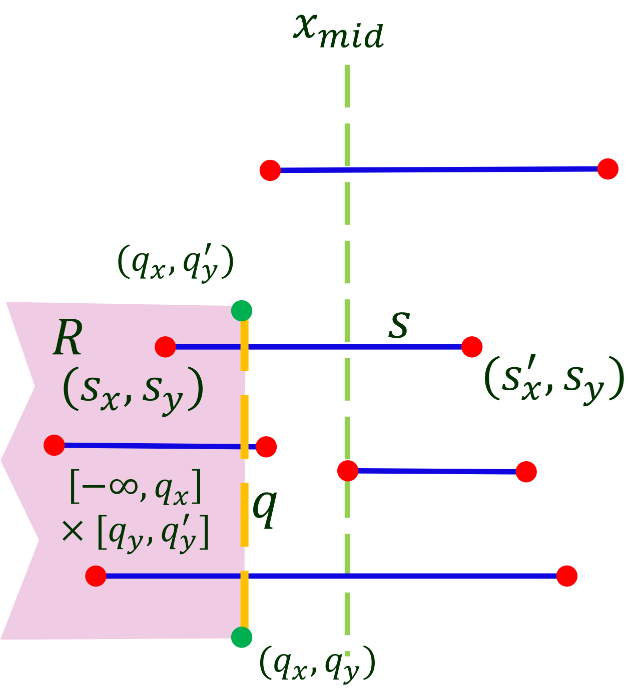
$$\Updownarrow$$

$$(s_x, s_y) \in R: [-\infty, q_x] \times [q_y, q'_y]$$

Left endpoint

Modified Query

Modify the (earlier described) query with a vertical line object.



- q to the left of x_{mid}, i.e., $q_x \le x_{mid}$.
  - Query I_{mid}. We know that every segment $s = [s_x, s'_x] \times s_y \in I_{mid}$ (in the increasing order of left endpoint) satisfies the query iff

$$\boxed{s_x \le \boxed{q_x}} \le s'_x \text{ and } \boxed{q_y \le s_y \le q'_y}$$

$$\left.\begin{array}{l} x_{mid} \le s'_x \\ q_x \le x_{mid} \end{array}\right\} \Rightarrow q_x \le s'_x$$

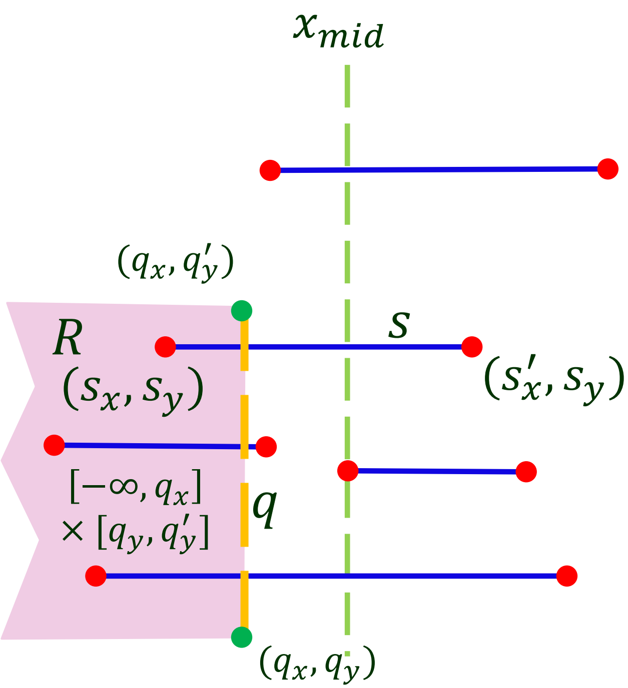
$$\Updownarrow$$

$$\underbrace{\underbrace{(s_x, s_y)}_{\text{Left endpoint}} \in R: [-\infty, q_x] \times [q_y, q'_y]}$$

Rectangular range query – 2D range tree (constructed over left endpoints from I_{mid})

Modified Query

Modify the (earlier described) query with a vertical line object.



- q to the left of x_{mid}, i.e., $q_x \le x_{mid}$.
  - Query I_{mid}. We know that every segment $s = [s_x, s'_x] \times s_y \in I_{mid}$ (in the increasing order of left endpoint) satisfies the query iff

$$\boxed{s_x \le \boxed{q_x}} \le s'_x \text{ and } \boxed{q_y \le s_y \le q'_y}$$

$$\left.\begin{array}{l} x_{mid} \le s'_x \\ q_x \le x_{mid} \end{array}\right\} \Rightarrow q_x \le s'_x$$

$$\Updownarrow$$

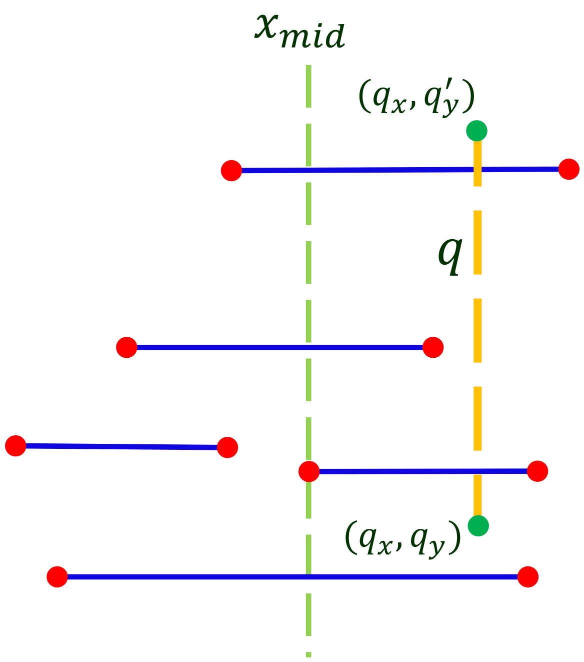
$$\underbrace{\underbrace{(s_x, s_y)}_{\text{Left endpoint}} \in R : [-\infty, q_x] \times [q_y, q'_y]}$$

Rectangular range query – 2D range tree
(constructed over left endpoints from I_{mid})

  - Query the left subtree (storing I_{left}).

Modified Query (cont'd)

- q to the right of x_{mid}, i.e., $q_x > x_{mid}$.

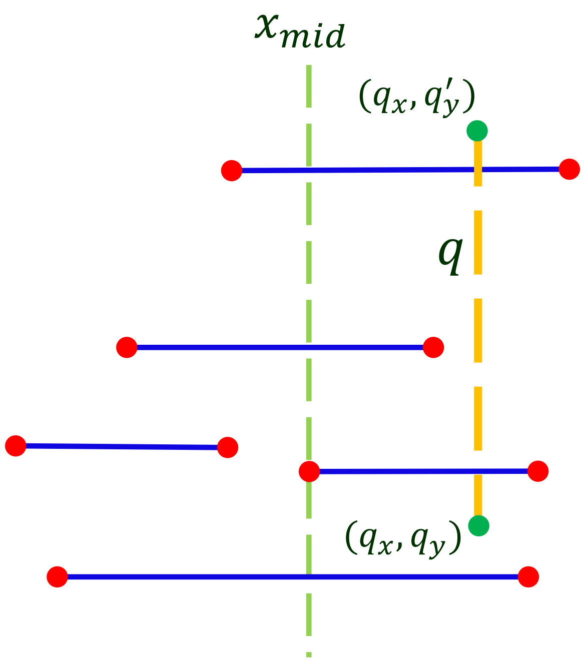


Modified Query (cont'd)

- q to the right of x_{mid}, i.e., $q_x > x_{mid}$.

  - Query I_{mid}. Every segment $s \in I_{mid}$ satisfies the query iff

$$s_x \le q_x \le s'_x \text{ and } q_y \le s_y \le q'_y$$

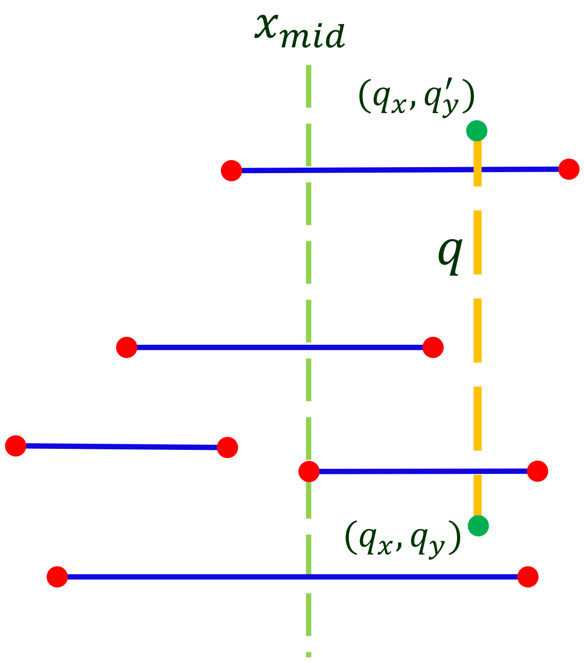


Modified Query (cont'd)

- q to the right of x_{mid}, i.e., $q_x > x_{mid}$.
  - Query I_{mid}. Every segment $s \in I_{mid}$ satisfies the query iff

$$s_x \leq q_x \leq s'_x \text{ and } q_y \leq s_y \leq q'_y$$

$$s_x \leq x_{mid}$$

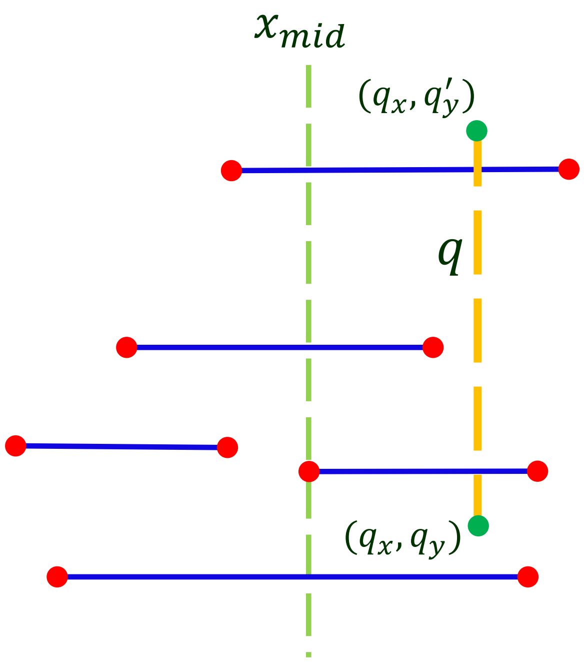


Modified Query (cont'd)

- q to the right of x_{mid}, i.e., $q_x > x_{mid}$.

  - Query I_{mid}. Every segment $s \in I_{mid}$ satisfies the query iff

$$s_x \le q_x \le s'_x \text{ and } q_y \le s_y \le q'_y$$

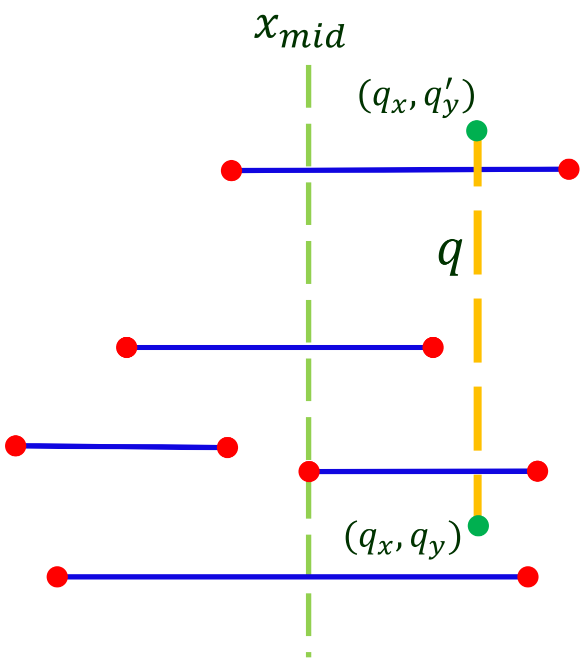


$$s_x \le x_{mid}$$
$$x_{mid} < q_x$$

Modified Query (cont'd)

- q to the right of x_{mid}, i.e., $q_x > x_{mid}$.

  - Query I_{mid}. Every segment $s \in I_{mid}$ satisfies the query iff

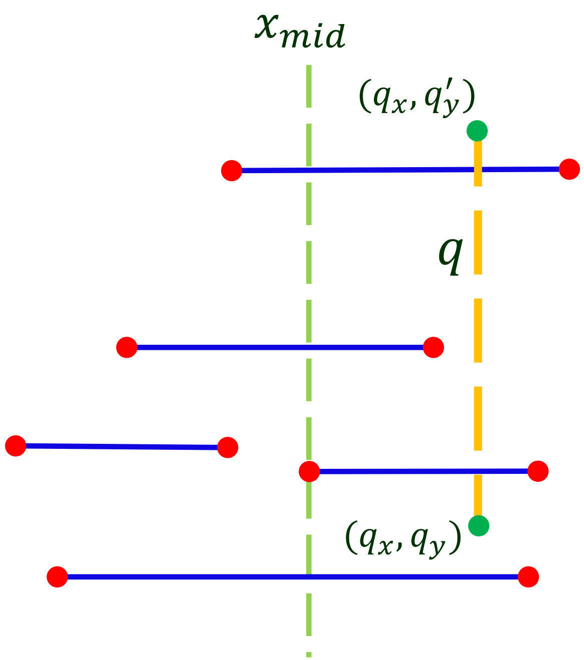


$$\boxed{s_x \le q_x} \le s'_x \text{ and } q_y \le s_y \le q'_y$$

$$\left.\begin{array}{l} s_x \le x_{mid} \\ x_{mid} < q_x \end{array}\right\} \Rightarrow s_x \le q_x$$

Modified Query (cont'd)

- q to the right of x_{mid}, i.e., $q_x > x_{mid}$.
  - Query I_{mid}. Every segment $s \in I_{mid}$ satisfies the query iff



$$\boxed{s_x \le q_x} \le s'_x \text{ and } \boxed{q_y \le s_y \le q'_y}$$

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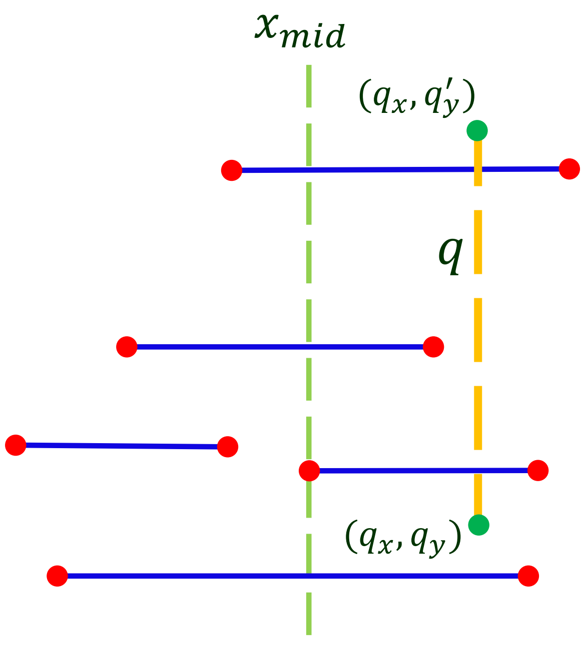


$$s_x \le q_x \le s'_x \text{ and } q_y \le s_y \le q'_y$$

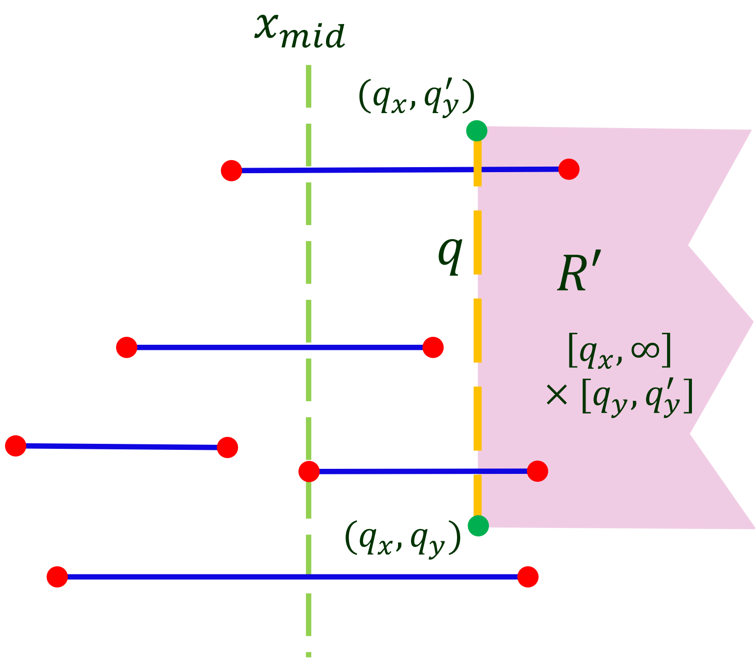
$$\left.\begin{array}{l} s_x \le x_{mid} \\ x_{mid} < q_x \end{array}\right\}$$

$$(s'_x, s_y) \in R' : [q_x, \infty] \times [q_y, q'_y]$$

Right endpoint

Modified Query (cont'd)

- q to the right of x_{mid}, i.e., $q_x > x_{mid}$.



- Query I_{mid}. Every segment $s \in I_{mid}$ satisfies the query iff

$$\left.\begin{array}{l} s_x \le x_{mid} \\ x_{mid} < q_x \end{array}\right\} \Rightarrow s_x \le q_x$$

$$s_x \le q_x \le s'_x \text{ and } q_y \le s_y \le q'_y$$

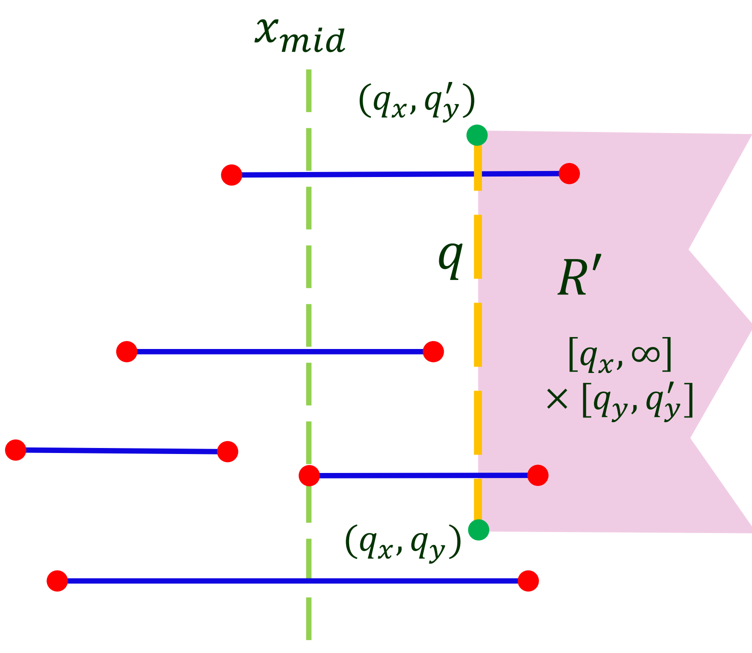
$$\Updownarrow$$

$$(s'_x, s_y) \in R' : [q_x, \infty] \times [q_y, q'_y]$$

Right endpoint

Modified Query (cont'd)

- q to the right of x_{mid}, i.e., $q_x > x_{mid}$.



- Query I_{mid}. Every segment $s \in I_{mid}$ satisfies the query iff

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$$\left.\begin{array}{l} s_x \le x_{mid} \\ x_{mid} < q_x \end{array}\right\} \Rightarrow s_x \le q_x$$

$$\Updownarrow$$

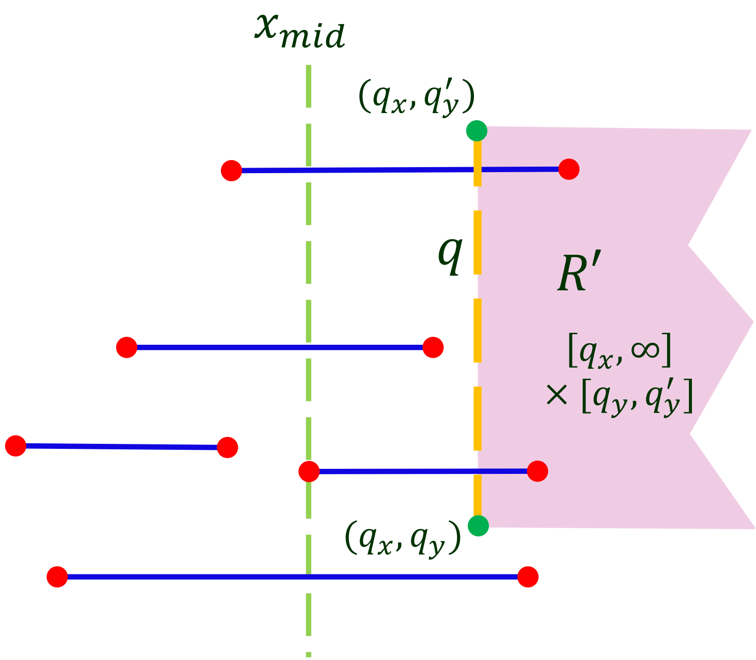
$$(s'_x, s_y) \in R' : [q_x, \infty] \times [q_y, q'_y]$$

Right endpoint

Rectangular range query – 2D range tree (constructed over right endpoints from I_{mid})

Modified Query (cont'd)

- q to the right of x_{mid}, i.e., $q_x > x_{mid}$.



- Query I_{mid}. Every segment $s \in I_{mid}$ satisfies the query iff

$$\boxed{s_x \le q_x} \le s'_x \text{ and } \boxed{q_y \le s_y \le q'_y}$$

$$\left.\begin{array}{l} s_x \le x_{mid} \\ x_{mid} < q_x \end{array}\right\} \Rightarrow s_x \le q_x$$

$$q_x \le s'_x \text{ and } q_y \le s_y \le q'_y \iff (s'_x, s_y) \in R' : [q_x, \infty] \times [q_y, q'_y]$$

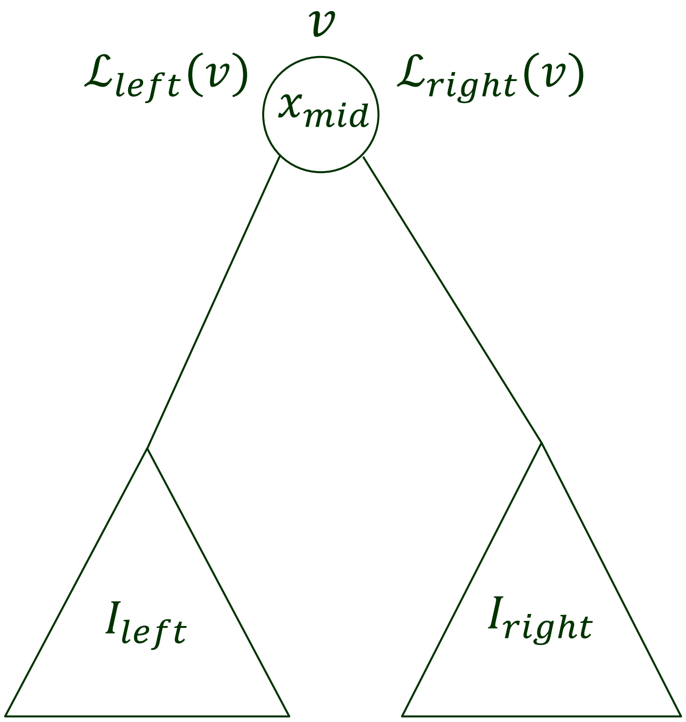
Right endpoint

Rectangular range query – 2D range tree (constructed over right endpoints from I_{mid})

- Query the right subtree (storing I_{right}).

Data Structure

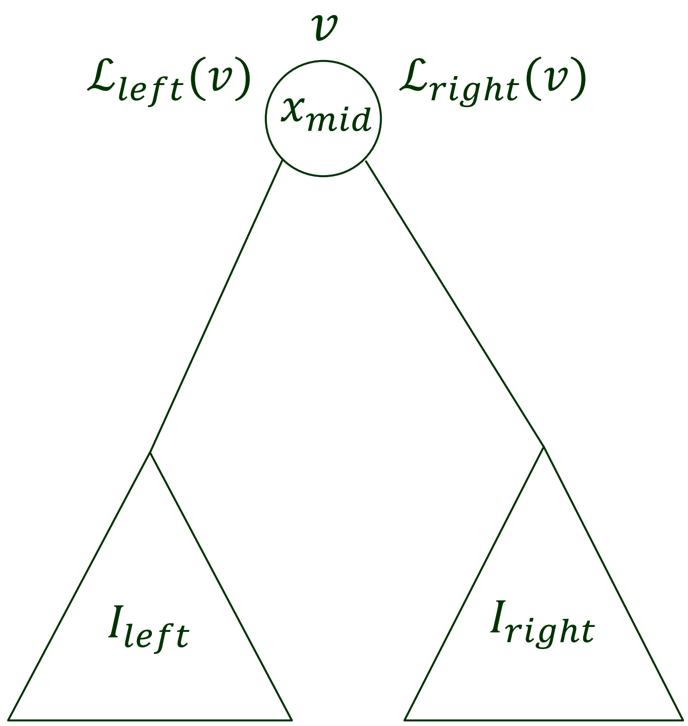
Main structure: interval tree $\mathcal{T}$ on x-coordinates.



At each node v, replace

Data Structure

Main structure: interval tree $\mathcal{T}$ on x-coordinates.



At each node v, replace

- $\mathcal{L}_{left}(v)$ with a range tree $\mathcal{T}_{left}(v)$ on the left endpoints of $I_{mid}(v)$.

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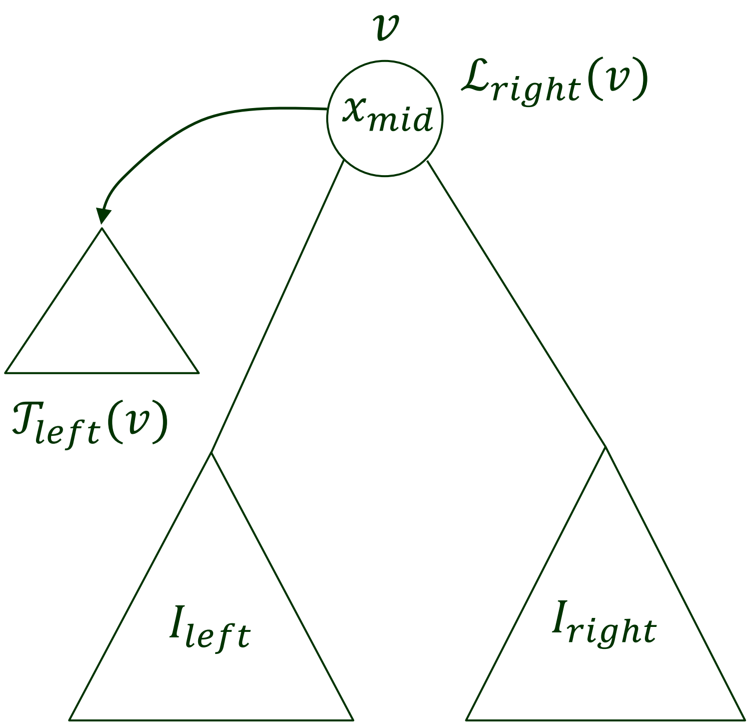

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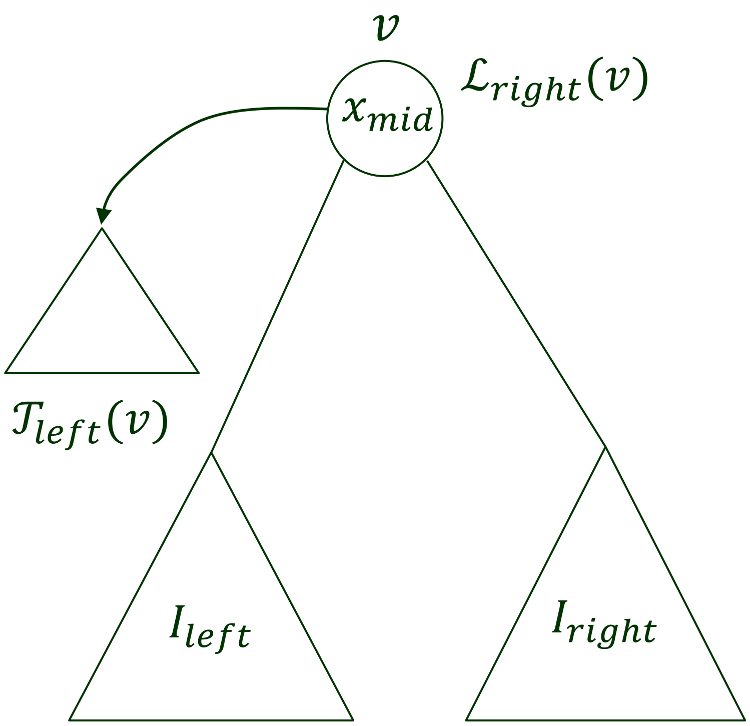

At each node v, replace

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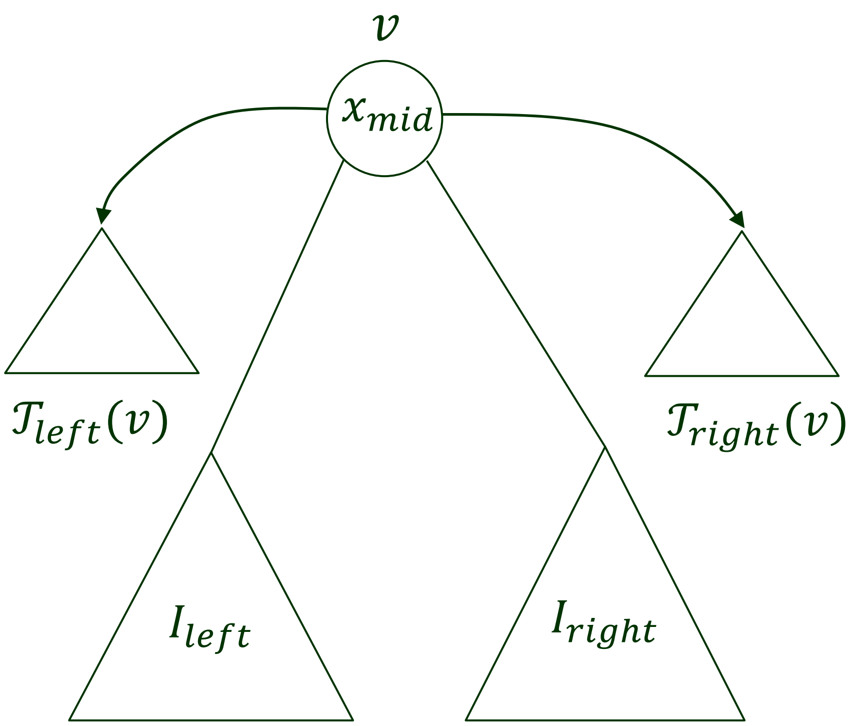
- $\mathcal{L}_{right}(v)$ with a range tree $\mathcal{T}_{right}(v)$ on the right endpoints of $I_{mid}(v)$.

Data Structure

Main structure: interval tree $\mathcal{T}$ on x-coordinates.



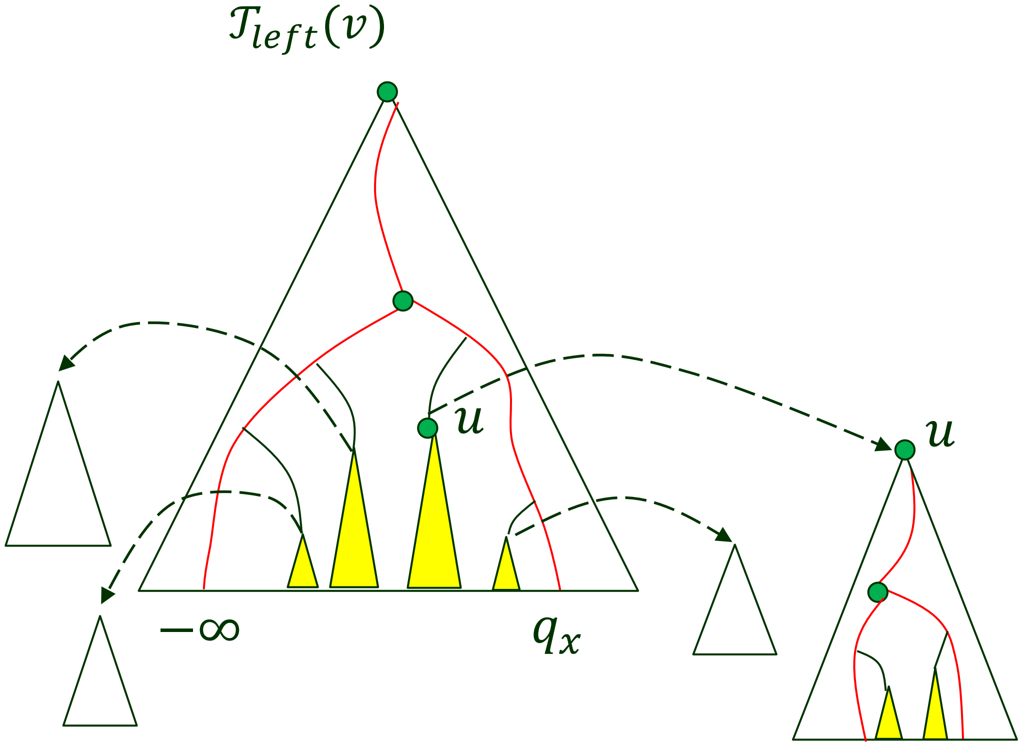
At each node v, replace

- $\mathcal{L}_{left}(v)$ with a range tree $\mathcal{T}_{left}(v)$ on the left endpoints of $I_{mid}(v)$.

- $\mathcal{L}_{right}(v)$ with a range tree $\mathcal{T}_{right}(v)$ on the right endpoints of $I_{mid}(v)$.

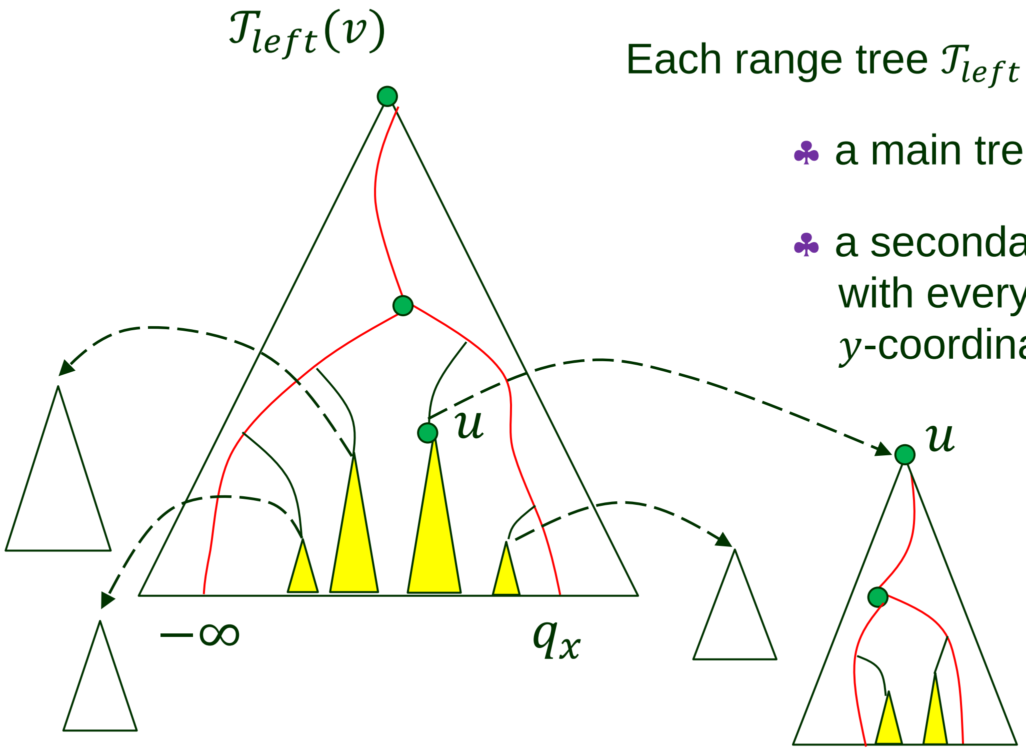
Range Queries

- Query with $[-\infty, q_x] \times [q_y, q'_y]$ performed on $\mathcal{T}_{left}(v)$.



Range Queries

- Query with $[-\infty, q_x] \times [q_y, q_y']$ performed on $\mathcal{T}_{left}(v)$.

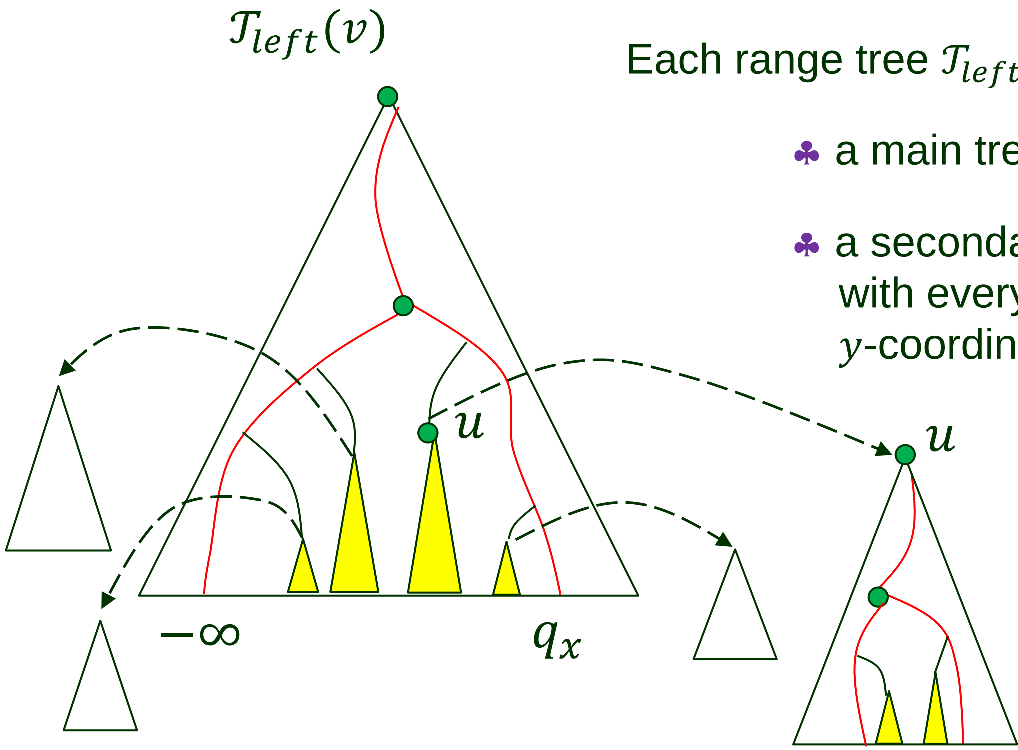


Each range tree $\mathcal{T}_{left}(v)$ (or $\mathcal{T}_{right}(v)$) has

- a main tree on the x-coordinate
- a secondary structure associated with every internal node on the y-coordinate.

Range Queries

- Query with $[-\infty, q_x] \times [q_y, q_y']$ performed on $\mathcal{T}_{left}(v)$.
- Query with $[q_x, \infty] \times [q_y, q_y']$ performed on $\mathcal{T}_{right}(v)$.



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- a main tree on the x-coordinate
- a secondary structure associated with every internal node on the y-coordinate.

Query Time

Perform a query on the range tree $\mathcal{T}_{left}(v)$ instead of traversing the sorted list $\mathcal{L}_{left}(v)$.

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- $O(\log n)$ nodes on the search path in the interval tree.

Query Time

Perform a query on the range tree $\mathcal{T}_{left}(v)$ instead of traversing the sorted list $\mathcal{L}_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.
- at each such node v, $O(\log^2 n + k_v)$ time spent on the range tree $\mathcal{T}_{left}(v)$ or $\mathcal{T}_{right}(v)$.

Query Time

Perform a query on the range tree $\mathcal{T}_{left}(v)$ instead of traversing the sorted list $\mathcal{L}_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.
- at each such node v, $O(\log^2 n + k_v)$ time spent on the range tree $\mathcal{T}_{left}(v)$ or $\mathcal{T}_{right}(v)$.

reported segments
in the range tree

Query Time

Perform a query on the range tree $\mathcal{T}_{left}(v)$ instead of traversing the sorted list $\mathcal{L}_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.
- at each such node v, $O(\log^2 n + k_v)$ time spent on the range tree $\mathcal{T}_{left}(v)$ or $\mathcal{T}_{right}(v)$.

reported segments
in the range tree

Total query time: $O(\log^3 n + k_v \log n)$

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Perform a query on the range tree $\mathcal{T}_{left}(v)$ instead of traversing the sorted list $\mathcal{L}_{left}(v)$.

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\# reported segments in the range tree

Total query time: $O(\log^3 n + k_v \log n)$

reduced to (using fractional cascading)

$O(\log^2 n + k_v)$

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Perform a query on the range tree $\mathcal{T}_{left}(v)$ instead of traversing the sorted list $\mathcal{L}_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.
- at each such node v, $O(\log^2 n + k_v)$ time spent on the range tree $\mathcal{T}_{left}(v)$ or $\mathcal{T}_{right}(v)$.

reported segments
in the range tree

Total query time: $O(\log^3 n + k_v \log n)$

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Storage: $O(n \log n)$

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Perform a query on the range tree $\mathcal{T}_{left}(v)$ instead of traversing the sorted list $\mathcal{L}_{left}(v)$.

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reported segments
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reduced to (using
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Storage: $O(n \log n)$

Construction time: $O(n \log n)$