Interval Trees

Outline:

I. Windowing query

II. Interval tree

III. Construction and query

IV. Query with a line segment

I. Windowing
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Windowing Query

Determine the part of the map that lie in the window and display them.
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Applications

♣ Geographic maps

♣ Computer graphics (e.g., flight simulation – part of landscape within sight)

♣ Design of circuit boards – zooming in onto a certain portion
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Brute-force approach: Check every single feature to see if it is inside the window.
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Too slow because the amount of data is huge!
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- Geographic maps
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Brute-force approach: Check every single feature to see if it is inside the window.

Too slow because the amount of data is huge!

Use some data structure to store the map and allow quick retrieval of its portion inside a window.
### vs. Range Query

<table>
<thead>
<tr>
<th></th>
<th>Windowing Query</th>
<th>Range Query</th>
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</thead>
<tbody>
<tr>
<td><strong>data</strong></td>
<td>segments, polygons, curves</td>
<td>points</td>
</tr>
<tr>
<td><strong>search space</strong></td>
<td>2D or 3D</td>
<td>$k$-D</td>
</tr>
</tbody>
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Formulating the Problem

\[ S = \{s_1, s_2, \ldots, s_n\}: \text{set of } n \text{ axis-parallel segments} \]
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Query window: \( W = [x, x'] \times [y, y'] \) with boundary \( \partial W \)
Formulating the Problem

\[ S = \{s_1, s_2, \ldots, s_n\} \]: set of \( n \) axis-parallel segments

Query window: \( W = [x, x'] \times [y, y'] \) with boundary \( \partial W \)

Possible cases of a segment \( s \):
- \( s \) inside \( W \);
- \( s \) intersects \( \partial W \) once;
- \( s \) intersects \( \partial W \) twice;
- \( s \) overlaps \( \partial W \).
Segments with $\geq 1$ Endpoints in $W$

Range query over $2n$ endpoints

$O(n \log n)$ storage

$O(n \log^2 n + k)$ query time
Segments with $\geq 1$ Endpoints in $\mathcal{W}$

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$\downarrow$ fractional cascading

$O(n \log n)$
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Range query over $2n$ endpoints

$O(n \log n)$ storage

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\[\downarrow\]

fractional cascading

$O(n \log n)$

♣ Not to report a segment twice.
Segments with No Endpoint Inside $W$

Segments that pass through $W$ may

- crosses $W$ twice;
- contains one boundary edge.
Segments with No Endpoint Inside $\mathcal{W}$

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  The case of vertical segments is similar.
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Segments that pass through $W$ may

- crosses $W$ twice;
- contains one boundary edge.

- Need only consider horizontal segments. The case of vertical segments is similar.
- Need only know how to efficiently check if such a segment intersects the left boundary edge.
Segments with No Endpoint Inside $W$

Segments that pass through $W$ may

- crosses $W$ twice;
- contains one boundary edge.

- Need only consider horizontal segments.

The case of vertical segments is similar.

- Need only know how to efficiently check if such a segment intersects the left boundary edge.

It’s similar to check if it intersects the right boundary edge.
II. Query Segment as a Full Line

We first consider the \textit{simpler} version of query when the left boundary edge of $W$ is a \textit{full line}.

\[
\begin{align*}
    l: & \quad x = q_x
\end{align*}
\]
II. Query Segment as a Full Line

We first consider the *simpler* version of query when the left boundary edge of $W$ is a *full line*.

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$$l: \ x = q_x$$

Segment $s$: $[x, x'] \times y$ intersects $l$.

$x \leq q_x \leq x'$
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$$l: \ x = q_x$$

Segment $s: [x, x'] \times y$ intersects $l$.

$$x \leq q_x \leq x'$$

1D problem

Given a set of intervals $I = \{[x_1, y_1], [x_2, y_2] \ldots, [x_n, y_n]\}$, report those that contain the query point $q_x$. 
Partitioning the Interval Set

\[ x_{\text{mid}}: \text{median of } 2n \text{ interval endpoints.} \]
Partitioning the Interval Set

$x_{\text{mid}}$: median of $2n$ interval endpoints.
Partitioning the Interval Set

$x_{mid}$: median of $2n$ interval endpoints.

$$I_{mid} = \{[x_j, x'_j] \in I \mid x_j \leq x_{mid} \leq x'_j\}$$
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$I_{\text{mid}} = \{[x_j, x'_j] \in I \mid x_j \leq x_{\text{mid}} \leq x'_j\}$

$I_{\text{left}} = \{[x_j, x'_j] \in I \mid x'_j < x_{\text{mid}}\}$
$x_{mid}$: median of $2n$ interval endpoints.

$I_{mid} = \{ [x_j, x'_j] \in I \mid x_j \leq x_{mid} \leq x'_j \}$

$I_{left} = \{ [x_j, x'_j] \in I \mid x'_j < x_{mid} \}$
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\( x_{mid} \): median of 2n interval endpoints.

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I_{mid} = \{ [x_j, x'_j] \in I \mid x_j \leq x_{mid} \leq x'_j \}
\]

\[
I_{left} = \{ [x_j, x'_j] \in I \mid x'_j < x_{mid} \}
\]

\[
I_{right} = \{ [x_j, x'_j] \in I \mid x_j > x_{mid} \}
\]
Partitioning the Interval Set

\(x_{\text{mid}}\): median of 2n interval endpoints.

\[I_{\text{mid}} = \left\{ [x_j, x'_j] \in I \mid x_j \leq x_{\text{mid}} \leq x'_j \right\}\]

\[I_{\text{left}} = \left\{ [x_j, x'_j] \in I \mid x'_j < x_{\text{mid}} \right\}\]

\[I_{\text{right}} = \left\{ [x_j, x'_j] \in I \mid x_j > x_{\text{mid}} \right\}\]
Interval Tree

$I_{\text{mid}}$

$I_{\text{left}}$

$I_{\text{right}}$

$x_{\text{mid}}$
Interval Tree

$I_{mid}$

$I_{left}$

$I_{right}$

$x_{mid}$
Interval Tree

$I_{mid}$

$I_{left}$

$I_{right}$

$x_{mid}$

$v$
Interval Tree

$I_{mid}$

$I_{left}$

$I_{right}$

$x_{mid}$

$v$

$I_{mid}$

$lc(v)$

$I_{left}$
Interval Tree

$I_{mid}$

$I_{left}$

$x_{mid}$

$I_{right}$

$v$

$I_{mid}$

$lc(v)$

$rc(v)$

$I_{left}$

$I_{right}$
Interval Tree

Subtrees $lc(v)$ and $rc(v)$ are recursively defined.
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$O(\log n)$ depth
Storage of $I_{mid}$

$I_{mid}$ is stored in two lists at $\nu$:
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- $L_{left}(\nu)$: sorted in the increasing order on the left endpoints of the intervals.
Storage of $I_{\text{mid}}$

$I_{\text{mid}}$ is stored in two lists at $v$:

- $L_{\text{left}}(v)$: sorted in the increasing order on the left endpoints of the intervals.
- $L_{\text{right}}(v)$: sorted in the decreasing order on their right endpoints.
Storage of $I_{\text{mid}}$

$I_{\text{mid}}$ is stored in two lists at $v$:

- $\mathcal{L}_{\text{left}}(v)$: sorted in the increasing order on the left endpoints of the intervals.
- $\mathcal{L}_{\text{right}}(v)$: sorted in the decreasing order on their right endpoints.

\[
\begin{align*}
\mathcal{L}_{\text{left}} : & 3,4,5 \\
\mathcal{L}_{\text{right}} : & 5,3,4
\end{align*}
\]
$I_{left}, I_{mid}, I_{right}$ are disjoint.

Every interval appears in only one set $I_{mid}$ (stored once in each of the two sorted lists).

- All associated lists require $O(n)$ storage.
- The tree uses $O(n)$ storage.
- The tree has height $O(\log n)$. 
III. Construction of the Interval Tree

ConstructIntervalTree(I)

1. if $I = \emptyset$
2. then return an empty leaf
3. else compute $x_{mid}$
4. store it in a new node $v$
5. compute $I_{mid}, I_{left}, I_{right}$
6. construct sorted lists $L_{left}(v)$ and $L_{right}(v)$
7. store them at $v$
8. $lc(v) \leftarrow \text{ConstructIntervalTree}(I_{left})$
9. $rc(v) \leftarrow \text{ConstructIntervalTree}(I_{right})$
10. return $v$
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Presort the points and maintain ordered lists through recursive calls.
III. Construction of the Interval Tree

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Presort the points and maintain ordered lists through recursive calls.

\[ \Rightarrow \] $O(n)$ time before recursions on lines 8 and 9
III. Construction of the Interval Tree

ConstructIntervalTree(I)

1. if \( I = \emptyset \)
2. then return an empty leaf
3. else compute \( x_{\text{mid}} \)  // \( O(n) \)
4. store it in a new node \( v \)
5. compute \( I_{\text{mid}}, I_{\text{left}}, I_{\text{right}} \)  // \( O(n) \)
6. construct sorted lists \( L_{\text{left}}(v) \) and \( L_{\text{right}}(v) \)
7. store them at \( v \)  // \( O(n) \)
8. \( lc(v) \leftarrow \) ConstructIntervalTree\( (I_{\text{left}}) \)
9. \( rc(v) \leftarrow \) ConstructIntervalTree\( (I_{\text{right}}) \)
10. return \( v \)

Presort the points and maintain ordered lists through recursive calls.

\[ \implies O(n) \text{ time before recursions on lines 8 and 9} \]

\[ \implies \text{Total construction time: } O(n \log n) \]
Query

Find all intervals containing $q_x$.

- $q_x = x_{mid}$ ends the query after reporting all intervals in $I_{mid}$. 
Query

Find all intervals containing $q_x$.

- $q_x = x_{mid}$ ends the query after reporting all intervals in $I_{mid}$.
- $q_x < x_{mid}(v)$
Query

Find all intervals containing $q_x$.

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- $q_x < x_{mid}(v)$
Query

Find all intervals containing $q_x$.

- $q_x = x_{\text{mid}}$ ends the query after reporting all intervals in $I_{\text{mid}}$.
- $q_x < x_{\text{mid}}(v) \iff$ Search $I_{\text{mid}}$ and $I_{\text{left}}$.

Diagram:

- $q_x$ is the query point.
- $x_{\text{mid}}$ is the midpoint.
- Intervals $I_{\text{left}}$ and $I_{\text{mid}}$ are searched for intervals containing $q_x$.
- Intervals $s_1, s_2, s_3, s_4, s_5$ are shown as examples of intervals.
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- $q_x = x_{mid}$ ends the query after reporting all intervals in $I_{mid}$.
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$$L_{left}(v) = \langle s_2, s_1, s_3 \rangle$$
Find all intervals containing $q_x$.

- $q_x = x_{mid}$ ends the query after reporting all intervals in $I_{mid}$.
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- At $v$, start at the leftmost endpoint of $\mathcal{L}_{left}$. 

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- $q_x < x_{mid}(v)$ $\iff$ Search $I_{mid}$ and $I_{left}$.

At $v$, start at the leftmost endpoint of $L_{left}$.

Report all the intervals containing $q_x$. 
Query

Find all intervals containing $q_x$.

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\[
\mathcal{L}_{\text{left}}(v) = \langle s_2, s_1, s_3 \rangle
\]

- At \( v \), start at the leftmost endpoint of \( \mathcal{L}_{\text{left}} \).
- Report all the intervals containing \( q_x \).
- Stop at an interval that does not contain \( q_x \) (i.e., its left endpoint is to the right of \( q_x \)).
Query

Find all intervals containing \( q_x \).

- \( q_x = x_{mid} \) ends the query after reporting all intervals in \( I_{mid} \).
- \( q_x < x_{mid}(v) \iff \) Search \( I_{mid} \) and \( I_{left} \).

- At \( v \), start at the leftmost endpoint of \( L_{left} \).
- Report all the intervals containing \( q_x \).
- Stop at an interval that does not contain \( q_x \) (i.e., its left endpoint is to the right of \( q_x \)).
- Query the left subtree.
Query (cont’d)

- $q_x > x_{mid}(v)$  $\iff$  Search $I_{mid}$ and $I_{right}$.

- At $v$, start at the rightmost endpoint of $L_{right}$.
- Report all the intervals containing $q_x$.
- Stop at an interval that does not contain $q_x$.
- Query the right subtree.

$L_{right} = \langle s_3, s_1 \rangle$
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Query time: $O(\log n + k)$

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#reported intervals
IV. Query Segment with Finite Length

Query object: a vertical line segment $q: q_x \times [q_y, q'_y]$

$S = \{s_1, s_2, ..., s_n\}$: set of $n$ horizontal segments
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\( s: [s_x, s'_x] \times s_y \)
IV. Query Segment with Finite Length

Query object: a vertical line segment \( q: q_x \times [q_y, q'_y] \)

\( S = \{s_1, s_2, \ldots, s_n\} \): set of \( n \) horizontal segments

\[ (q_x, q_y) \]

\[ s: [s_x, s'_x] \times s_y \]

\[ (s_x, s_y) \]

\[ (s'_x, s_y) \]

\( s \) intersects \( q \) iff

i) \( s_x \leq q_x \leq s'_x \)

ii) \( q_y \leq s_y \leq q'_y \)
Modified Query

Modify the (earlier described) query with a vertical line object.
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Modified Query

Modify the (earlier described) query with a vertical line object.

- $q$ to the left of $x_{mid}$, i.e., $q_x \leq x_{mid}$.

- Query $I_{mid}$. We know that every segment $s = [s_x, s_x'] \times s_y \in I_{mid}$ (in the increasing order of left endpoint) satisfies the query iff

$$s_x \leq q_x \leq s_x' \text{ and } q_y \leq s_y \leq q_y'$$
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$$x_{\text{mid}} \leq s_x' \quad \text{and} \quad q_x \leq x_{\text{mid}}$$
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$$s_x \leq q_x \leq s'_x \quad \text{and} \quad q_y \leq s_y \leq q'_y$$

$$x_{mid} \leq s'_x \quad \text{and} \quad q_x \leq x_{mid}$$

$$\left(s_x, s_y\right) \in R: [-\infty, q_x] \times [q_y, q'_y]$$

Left endpoint
Modified Query

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\left( s_x \leq q_x \leq s'_x \right) \quad \text{and} \quad q_y \leq s_y \leq q'_y
\end{align*}
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\[
\begin{align*}
x_{mid} \leq s'_x \\
q_x \leq x_{mid}
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$$x_{mid} \leq s'_x \quad \text{and} \quad q_x \leq x_{mid}$$

$$\left(s_x, s_y\right) \in R: [\infty, q_x] \times [q_y, q'_y]$$

Left endpoint

Rectangular range query – 2D range tree (constructed over left endpoints from $I_{mid}$)
Modified Query

Modify the (earlier described) query with a vertical line object.

- $q$ to the left of $x_{\text{mid}}$, i.e., $q_x \leq x_{\text{mid}}$.

- Query $I_{\text{mid}}$. We know that every segment $s = [s_x, s_x'] \times s_y \in I_{\text{mid}}$ (in the increasing order of left endpoint) satisfies the query iff
  
  \[
  \begin{align*}
  s_x &\leq q_x \leq s_x' \\
  q_y &\leq s_y \leq q_y'
  \end{align*}
  \]

  \[
  x_{\text{mid}} \leq s_x' \quad \text{and} \quad q_x \leq x_{\text{mid}} \quad \text{Left endpoint}
  \]

  Rectangular range query – 2D range tree (constructed over left endpoints from $I_{\text{mid}}$)

- Query the left subtree (storing $I_{\text{left}}$).
- $q$ to the right of $x_{mid}$, i.e., $q_x > x_{mid}$.
**Modified Query (cont’d)**

- $q$ to the right of $x_{mid}$, i.e., $q_x > x_{mid}$.

- Query $I_{mid}$. Every segment $s \in I_{mid}$ satisfies the query iff

$$s_x \leq q_x \leq s'_x \text{ and } q_y \leq s_y \leq q'_y$$
Modified Query (cont’d)

- $q$ to the right of $x_{mid}$, i.e., $q_x > x_{mid}$.

- Query $I_{mid}$. Every segment $s \in I_{mid}$ satisfies the query iff

$$s_x \leq q_x \leq s'_x \text{ and } q_y \leq s_y \leq q'_y$$

$$s_x \leq x_{mid}$$
• $q$ to the right of $x_{mid}$, i.e., $q_x > x_{mid}$.

• Query $I_{mid}$. Every segment $s \in I_{mid}$ satisfies the query iff

\[
s_x \leq q_x \leq s'_x \quad \text{and} \quad q_y \leq s_y \leq q'_y
\]

\[
s_x \leq x_{mid} \\
x_{mid} < q_x
\]
Modified Query (cont’d)

- $q$ to the right of $x_{mid}$, i.e., $q_x > x_{mid}$.

- Query $I_{mid}$. Every segment $s \in I_{mid}$ satisfies the query iff

\[
\begin{align*}
  s_x &\leq q_x &\leq s'_x &\text{and} & q_y &\leq s_y &\leq q'_y \\
  s_x &\leq x_{mid} & x_{mid} &< q_x
\end{align*}
\]
- $q$ to the right of $x_{mid}$, i.e., $q_x > x_{mid}$.

- Query $I_{mid}$. Every segment $s \in I_{mid}$ satisfies the query iff

  \[ s_x \leq q_x \leq s'_x \quad \text{and} \quad q_y \leq s_y \leq q'_y \]

\[
\begin{align*}
\begin{cases}
  s_x &\leq q_x \leq s'_x \\
x_{mid} &< q_x
\end{cases}
\end{align*}
\]
• Query $I_{mid}$. Every segment $s \in I_{mid}$ satisfies the query iff

\[
\begin{align*}
s_x &\leq q_x \leq s'_x \\
q_y &\leq s_y \leq q'_y
\end{align*}
\]

\[
(s'_x, s'_y) \in R' : [q_x, \infty] \times [q_y, q'_y]
\]

Right endpoint
Modified Query (cont’d)

- $q$ to the right of $x_{mid}$, i.e., $q_x > x_{mid}$.

- Query $I_{mid}$. Every segment $s \in I_{mid}$ satisfies the query iff

$$(s'_x, s'_y) \in R': [q_x, \infty] \times [q_y, q'_y]$$

Right endpoint

\[ s_x \leq q_x \leq s'_x \quad \text{and} \quad q_y \leq s_y \leq q'_y \]
Modified Query (cont’d)

- $q$ to the right of $x_{mid}$, i.e., $q_x > x_{mid}$.

- Query $I_{mid}$. Every segment $s \in I_{mid}$ satisfies the query iff

$$s_x \leq q_x \leq s'_x \quad \text{and} \quad q_y \leq s_y \leq q'_y$$

$$(s'_x, s_y) \in R' : [q_x, \infty] \times [q_y, q'_y]$$

Right endpoint

Rectangular range query – 2D range tree (constructed over right endpoints from $I_{mid}$)
Modified Query (cont’d)

- $q$ to the right of $x_{\text{mid}}$, i.e., $q_x > x_{\text{mid}}$.

- Query $I_{\text{mid}}$. Every segment $s \in I_{\text{mid}}$ satisfies the query iff

\[
s_x \leq q_x \leq s'_x \quad \text{and} \quad q_y \leq s_y \leq q'_y
\]

- Rectangle $R': [q_x, \infty] \times [q_y, q'_y]$ and $(s'_x, s_y) \in R'$

- Query the right subtree (storing $I_{\text{right}}$).

Rectangular range query – 2D range tree (constructed over right endpoints from $I_{\text{mid}}$)
Main structure: interval tree $\mathcal{T}$ on $x$-coordinates.

At each node $v$, replace

\begin{align*}
L_{left}(v) & \\ L_{right}(v) & \\ x_{mid} & \\ L_{left}(v) & \\ L_{right}(v) & \end{align*}
Main structure: interval tree $\mathcal{T}$ on $x$-coordinates.

At each node $v$, replace

- $L_{left}(v)$ with a range tree $\mathcal{T}_{left}(v)$ on the left endpoints of $I_{mid}(v)$.
Data Structure

Main structure: interval tree $\mathcal{T}$ on $x$-coordinates.

At each node $v$, replace

- $\mathcal{L}_{left}(v)$ with a range tree $\mathcal{T}_{left}(v)$ on the left endpoints of $I_{mid}(v)$. 

Diagram:

- $v$
- $x_{mid}$
- $\mathcal{L}_{right}(v)$
- $\mathcal{T}_{left}(v)$
- $I_{left}$
- $I_{right}$
Data Structure

Main structure: interval tree $\mathcal{T}$ on $x$-coordinates.

At each node $v$, replace

- $\mathcal{L}_{right}(v)$ with a range tree $\mathcal{T}_{right}(v)$ on the right endpoints of $I_{mid}(v)$.

- $\mathcal{L}_{left}(v)$ with a range tree $\mathcal{T}_{left}(v)$ on the left endpoints of $I_{mid}(v)$. 

Diagram:

- $x_{mid}$
- $\mathcal{T}_{left}(v)$
- $I_{left}$
- $I_{right}$
- $\mathcal{L}_{right}(v)$
Main structure: interval tree $\mathcal{T}$ on $x$-coordinates.

At each node $v$, replace

- $\mathcal{L}_{\text{left}}(v)$ with a range tree $\mathcal{T}_{\text{left}}(v)$ on the left endpoints of $I_{\text{mid}}(v)$.
- $\mathcal{L}_{\text{right}}(v)$ with a range tree $\mathcal{T}_{\text{right}}(v)$ on the right endpoints of $I_{\text{mid}}(v)$.
Query with $[-\infty, q_x] \times [q_y, q'_y]$ performed on $T_{left}(v)$. 
Range Queries

- Query with $[-\infty, q_x] \times [q_y, q_y']$ performed on $T_{left}(v)$.

Each range tree $T_{left}(v)$ (or $T_{right}(v)$) has

- a main tree on the $x$-coordinate
- a secondary structure associated with every internal node on the $y$-coordinate.
Range Queries

- Query with $[-\infty, q_x] \times [q_y, q'_y]$ performed on $T_{left}(v)$.
- Query with $[q_x, \infty] \times [q_y, q'_y]$ performed on $T_{right}(v)$.

Each range tree $T_{left}(v)$ (or $T_{right}(v)$) has

- a main tree on the $x$-coordinate
- a secondary structure associated with every internal node on the $y$-coordinate.
Perform a query on the range tree $T_{\text{left}}(v)$ instead of traversing the sorted list $L_{\text{left}}(v)$. 
Query Time

Perform a query on the range tree $T_{left}(v)$ instead of traversing the sorted list $L_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.
Query Time

Perform a query on the range tree $T_{left}(v)$ instead of traversing the sorted list $L_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.
- at each such node $v$, $O(\log^2 n + k_v)$ time spent on the range tree $T_{left}(v)$ or $T_{right}(v)$. 
Query Time

Perform a query on the range tree $T_{\text{left}}(v)$ instead of traversing the sorted list $L_{\text{left}}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.
- at each such node $v$, $O(\log^2 n + k_v)$ time spent on the range tree $T_{\text{left}}(v)$ or $T_{\text{right}}(v)$.

# reported segments in the range tree
Query Time

Perform a query on the range tree $T_{left}(v)$ instead of traversing the sorted list $L_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.

- at each such node $v$, $O(\log^2 n + k_v)$ time spent on the range tree $T_{left}(v)$ or $T_{right}(v)$.

  # reported segments
  in the range tree

Total query time: $O(\log^3 n + k_v \log n)$
Query Time

Perform a query on the range tree $T_{left}(v)$ instead of traversing the sorted list $L_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.

- at each such node $v$, $O(\log^2 n + k_v)$ time spent on the range tree $T_{left}(v)$ or $T_{right}(v)$.

Total query time: $O(\log^3 n + k_v \log n)$

reduced to (using fractional cascading)

$O(\log^2 n + k_v)$
Query Time

Perform a query on the range tree $T_{left}(v)$ instead of traversing the sorted list $L_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.
- At each such node $v$, $O(\log^2 n + k_v)$ time spent on the range tree $T_{left}(v)$ or $T_{right}(v)$.

Total query time: $O(\log^3 n + k_v \log n)$

Reduced to (using fractional cascading): $O(\log^2 n + k_v)$

Storage: $O(n \log n)$
Query Time

Perform a query on the range tree $T_{left}(v)$ instead of traversing the sorted list $L_{left}(v)$.

- $O(\log n)$ nodes on the search path in the interval tree.
- At each such node $v$, $O(\log^2 n + k_v)$ time spent on the range tree $T_{left}(v)$ or $T_{right}(v)$.

Total query time: $O(\log^3 n + k_v \log n)$

Reduced to (using fractional cascading): $O(\log^2 n + k_v)$

Storage: $O(n \log n)$

Construction time: $O(n \log n)$