Yan-Bin Jia, Iowa State University

Envelopes & Voronoi Diagrams

Outline:

I. Algorithm for 3D Convex Hulls

II. Review of duality

III. Hull-envelope correspondence

IV. Voronoi diagram as a 3D convex hull problem

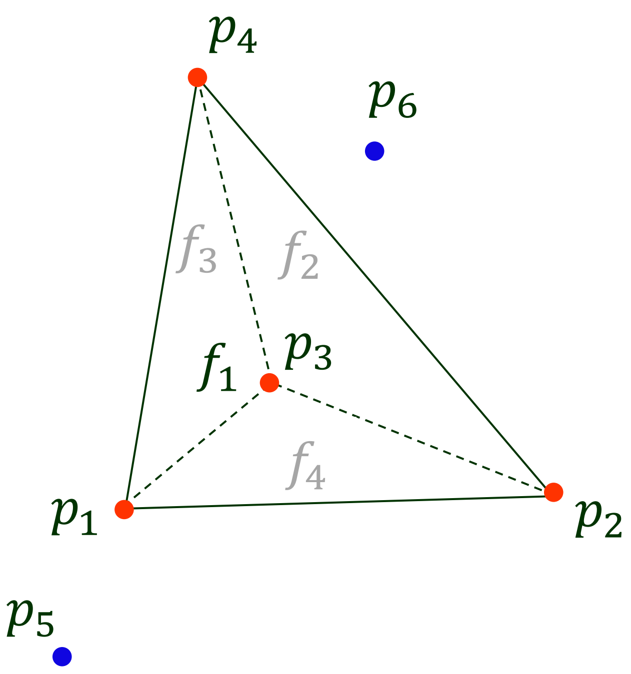
Textbook: *Computational Geometry: Algorithms and Applications (*3rd ed.) by M. de Berg et al., Springer-Verlag, 2008.

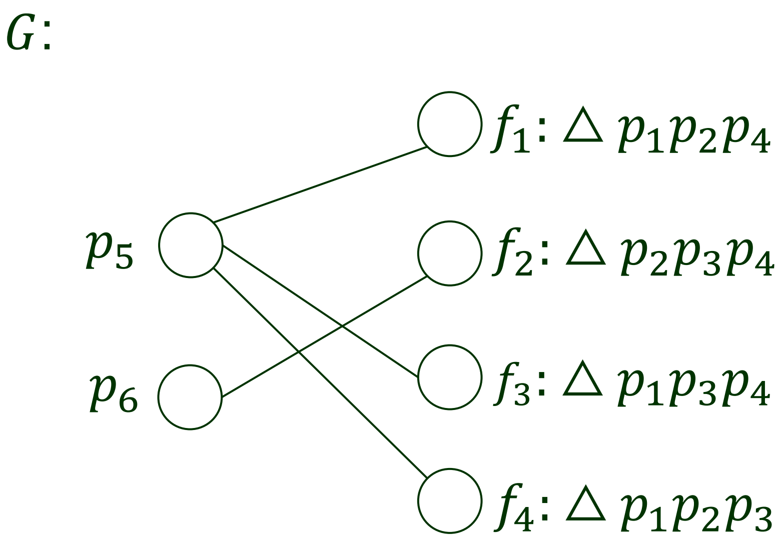
I. Algorithm for 3D Convex Hulls

ConvexHull(P)

1. find $p_1, p_2, p_3, p_4 \in P$ that form a tetrahedron
2. $C \leftarrow CH(\{p_1, p_2, p_3, p_4\})$
3. compute a random permutation $p_5, p_6, \ldots, p_n$
4. initialize the conflict graph G over all facets of C and $p_5, p_6, \ldots, p_n$
5. **for** $r \leftarrow 5$ **to** n
6.   **do** // insert p_r to C
7.   **if** $F_{conflict}(p_r) \neq \emptyset$ // p_r lies outside C
8.   **then**
9.   delete all facets in $F_{conflict}(p_r)$ from C
10.   find the horizon by walking along the boundary of the visible region of p_r
10.   **for** each edge e on the horizon
11.   **do** connect e to p_r to form a triangle f

Execution Example – Initialization

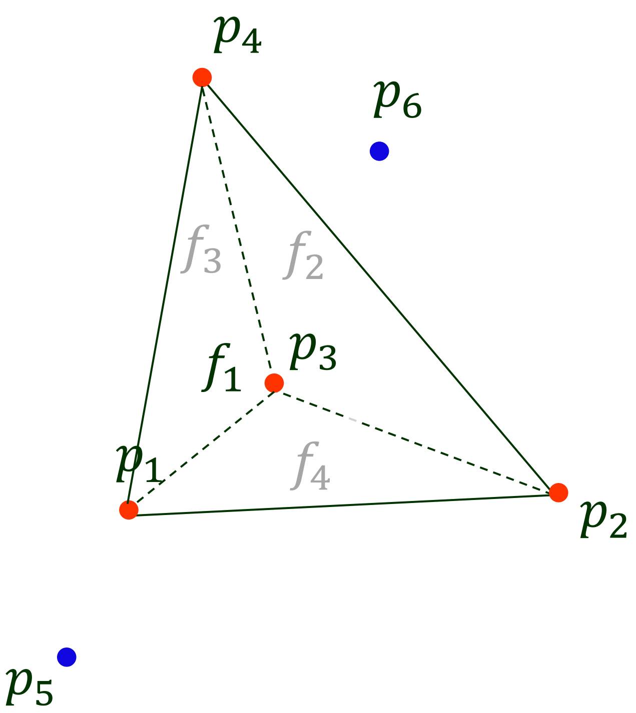




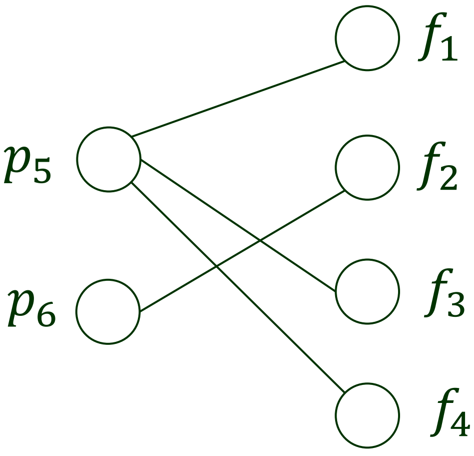
Iteration of Adding p_5

$r = 5$

$F_{conflict}(p_5) = \{f_1, f_3, f_4\}$



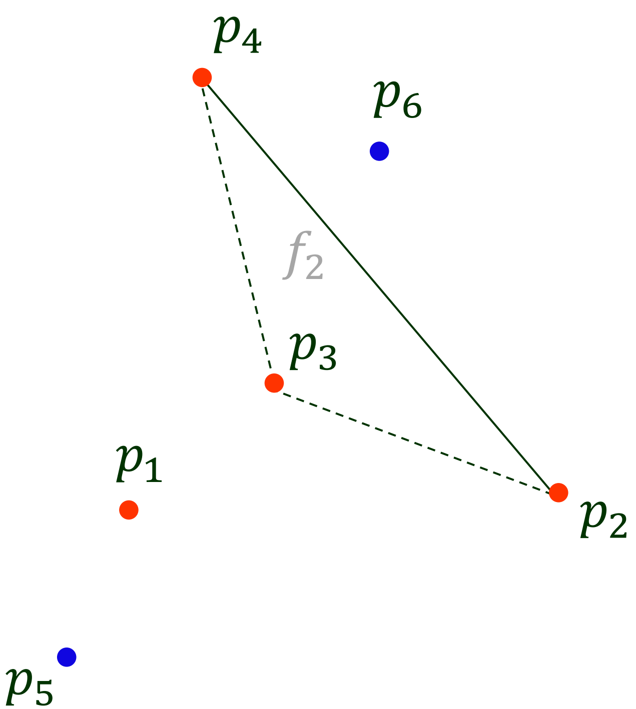
G:



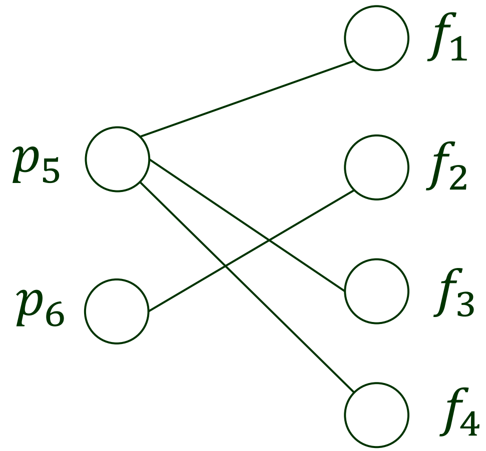
Iteration of Adding p_5

$r = 5$

$F_{conflict}(p_5) = \{f_1, f_3, f_4\}$

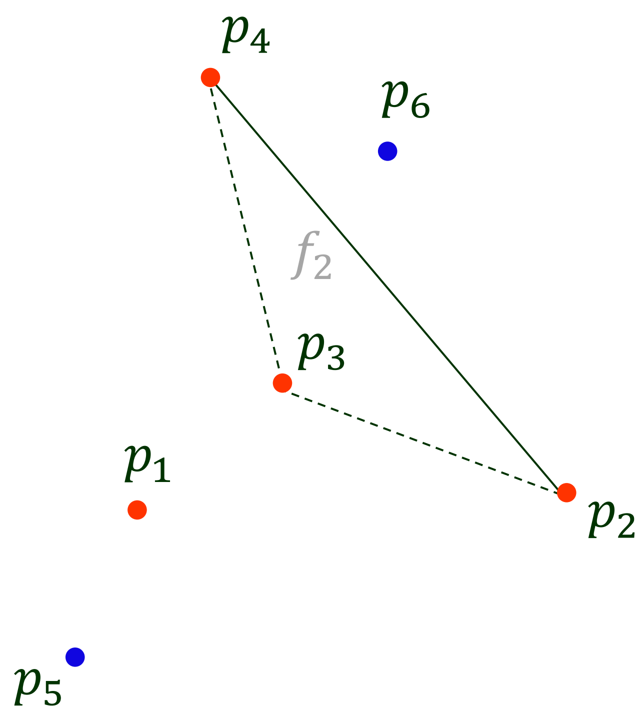


G:



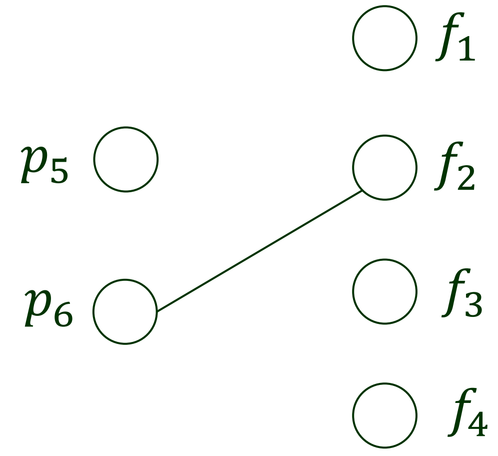
Iteration of Adding p_5

$r = 5$



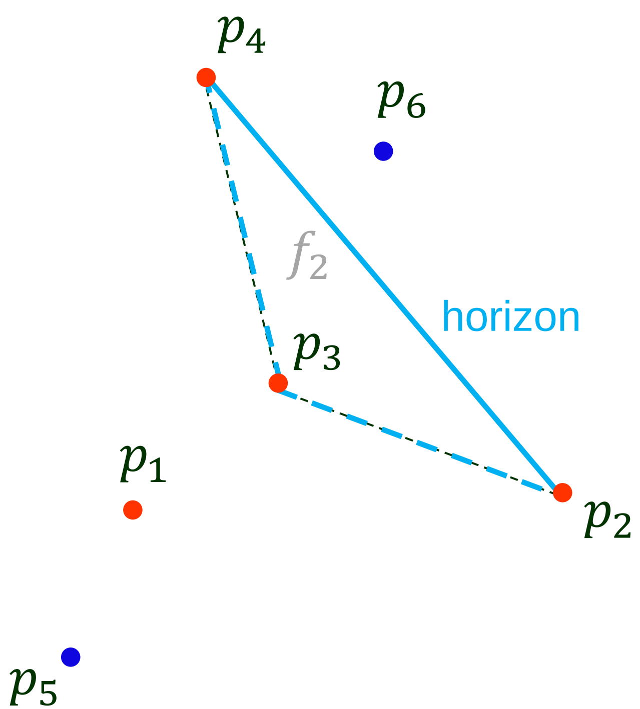
$F_{conflict}(p_5) = \{f_1, f_3, f_4\}$

G:



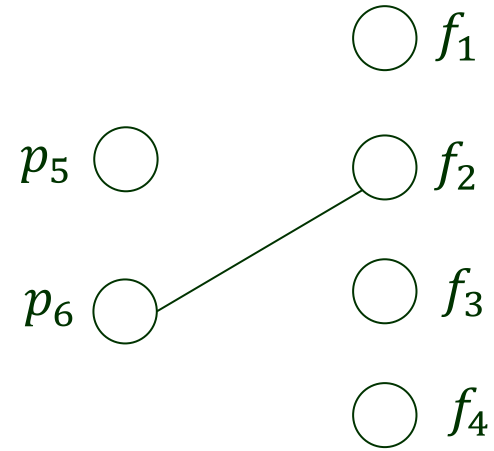
Iteration of Adding p_5

$r = 5$



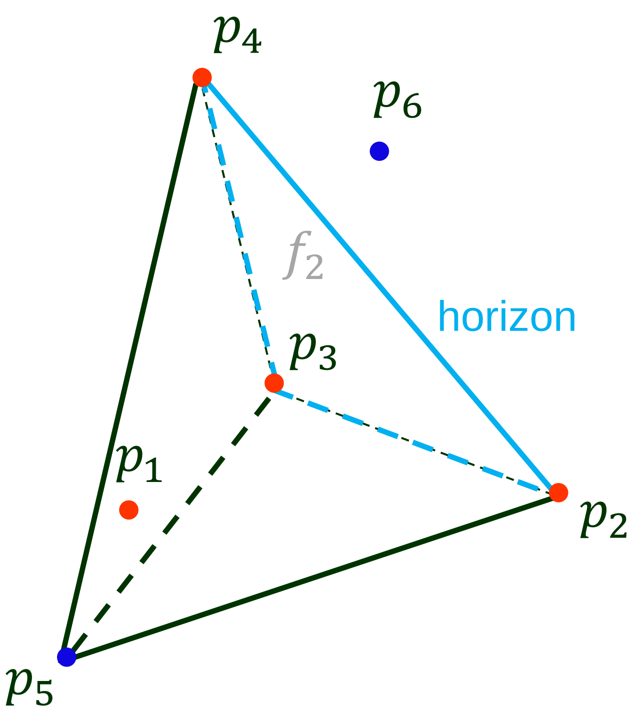
$F_{conflict}(p_5) = \{f_1, f_3, f_4\}$

G:



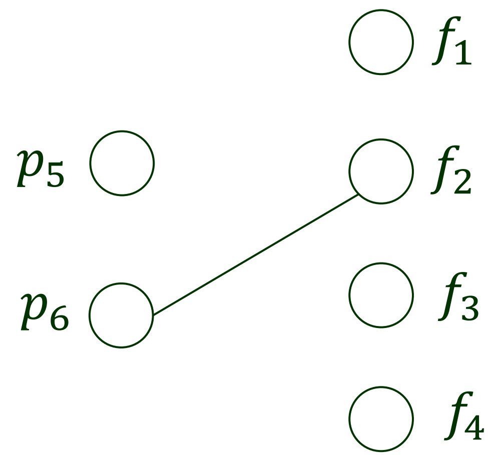
Iteration of Adding p_5

$r = 5$



$F_{conflict}(p_5) = \{f_1, f_3, f_4\}$

G:

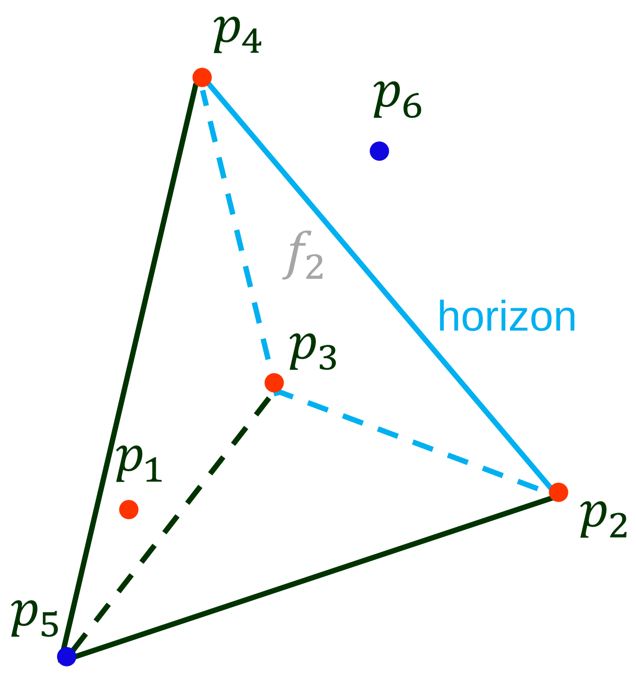


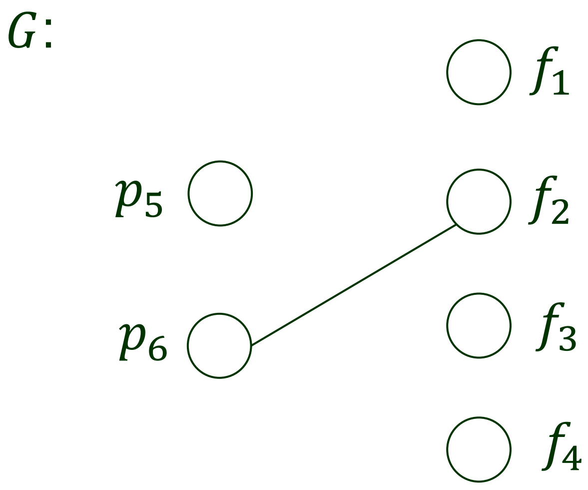
Algorithm (cont'd)

if f is coplanar with its neighbor facet f' along e
then merge f with f' and the merged facet inherits the latter's conflict set
else // determine conflicts for f
  create a node for f in G
  f_1, f_2: facets incident to e
  for all $p \in P_{conflict}(f_1) \cup P_{conflict}(f_2)$
   do
    if f is visible from p
     add $\langle p, f \rangle$ to G // update
     // $P_{conflict}(f)$ and $F_{conflict}(p)$
delete the node corresponding to p_r and
 the nodes corresponding to the facets in
 $F_{conflict}(p_r)$ from G, along with incident arcs
return C

Iteration (cont'd)

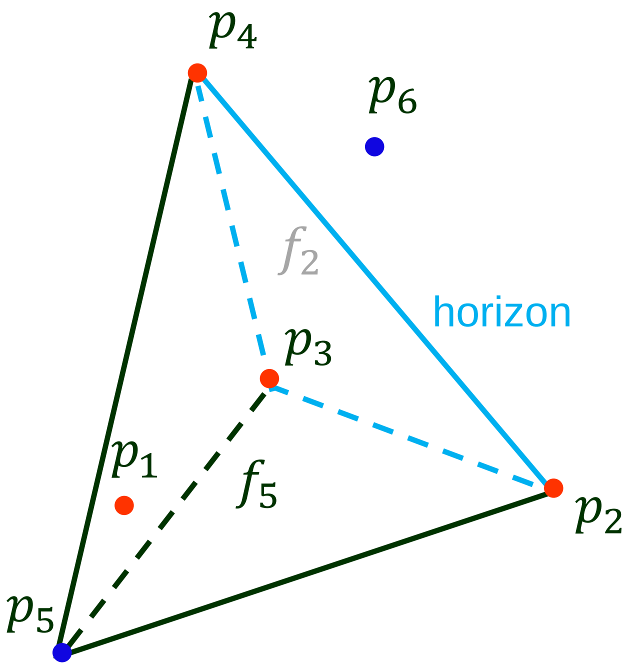
$r = 5$

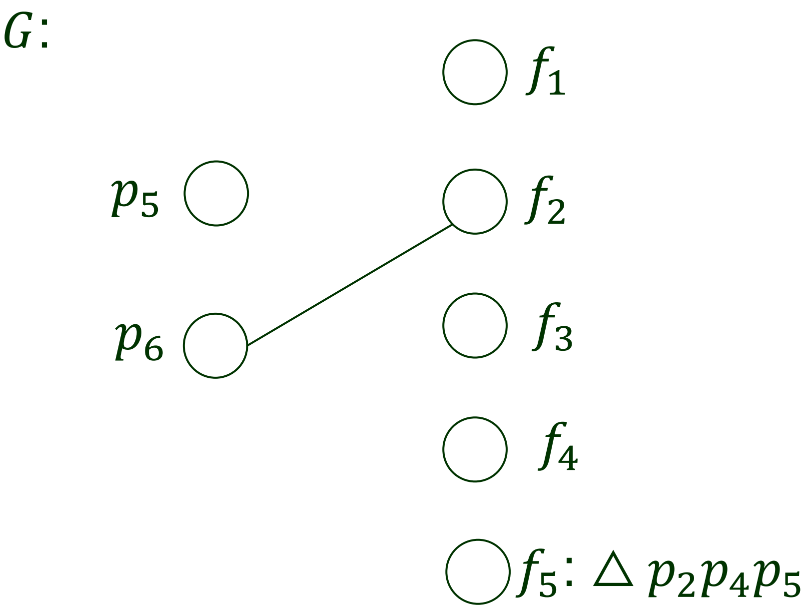




Iteration (cont'd)

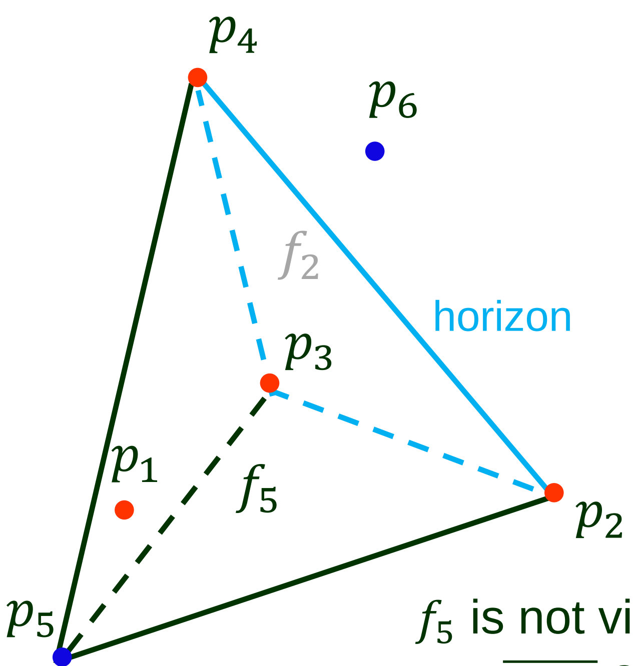
$r = 5$

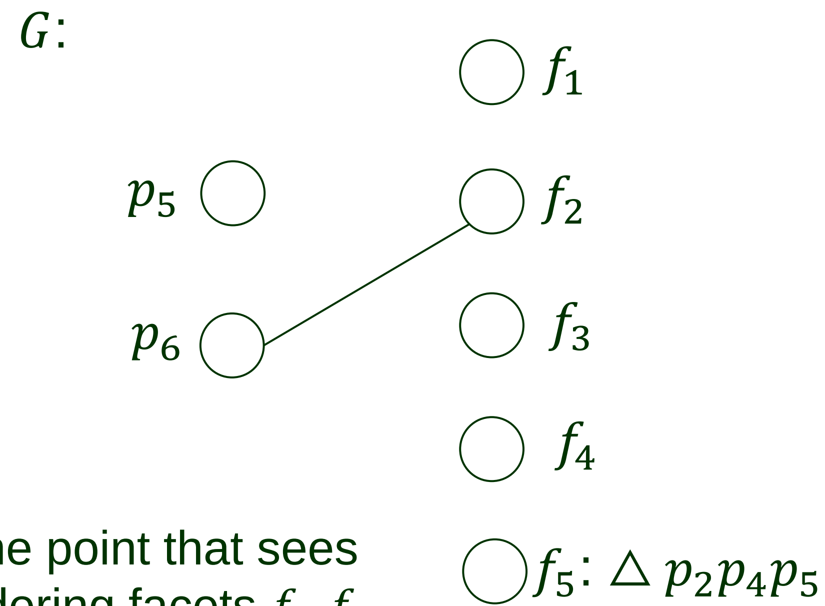




Iteration (cont'd)

$r = 5$

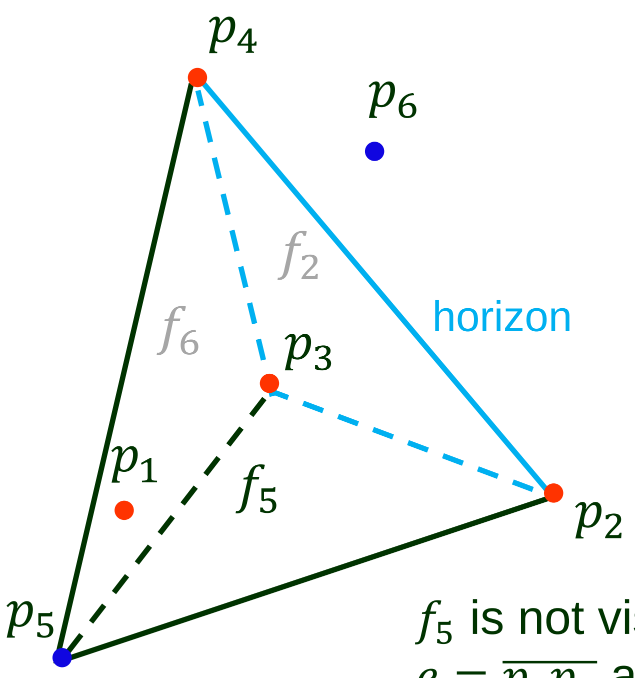




f_5 is not visible from p_6, the point that sees $e = \overline{p_2p_4}$ and its (old) bordering facets f_1, f_2.

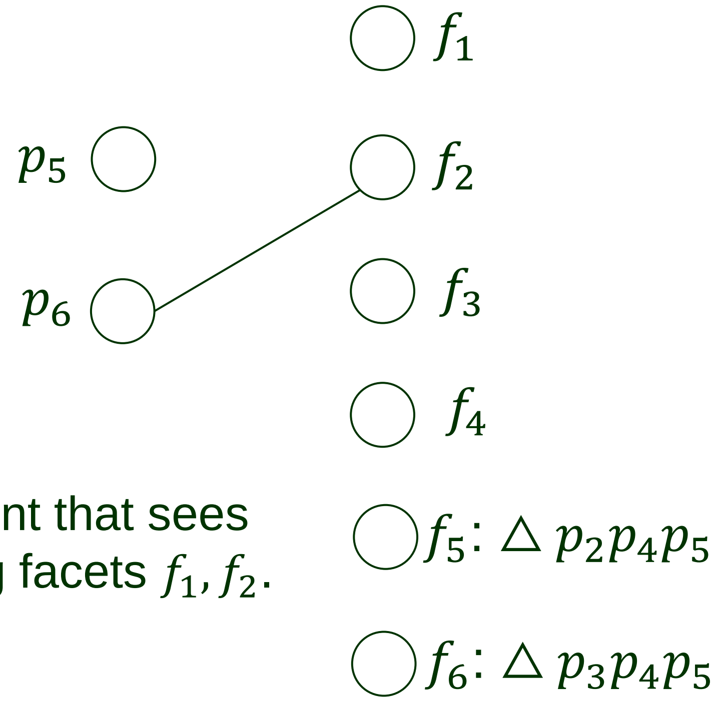
Iteration (cont'd)

$r = 5$



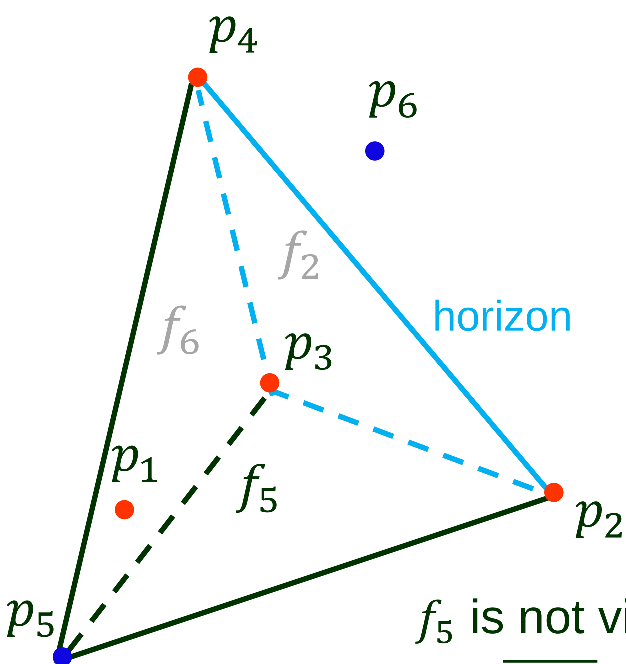
f_5 is not visible from p_6, the point that sees $e = \overline{p_2 p_4}$ and its (old) bordering facets f_1, f_2.

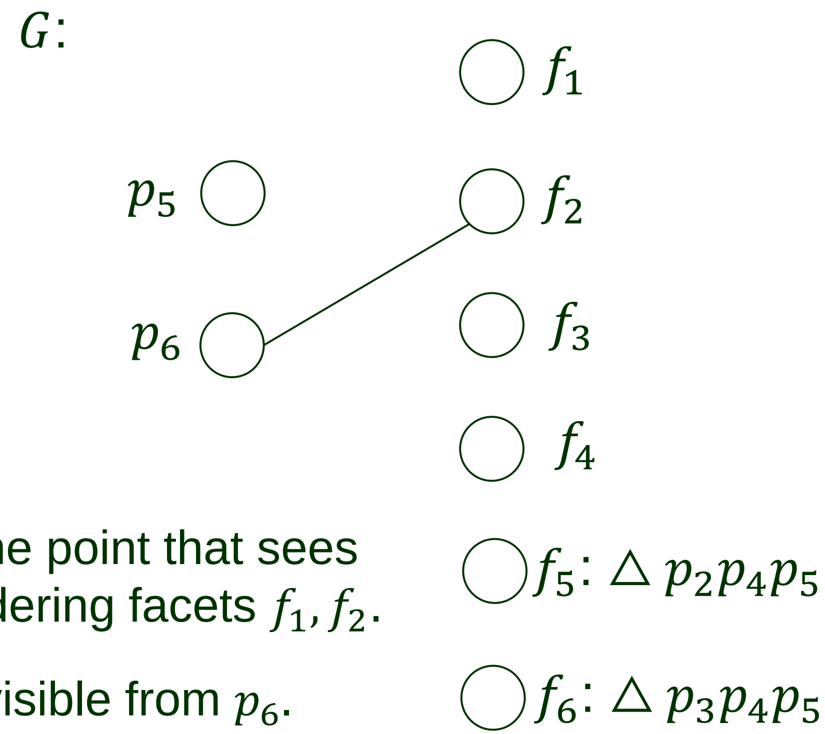
G:



Iteration (cont'd)

$r = 5$



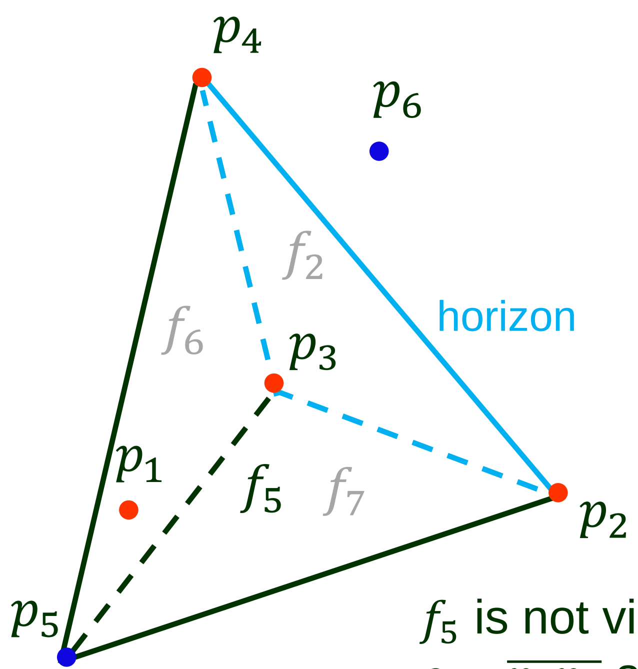


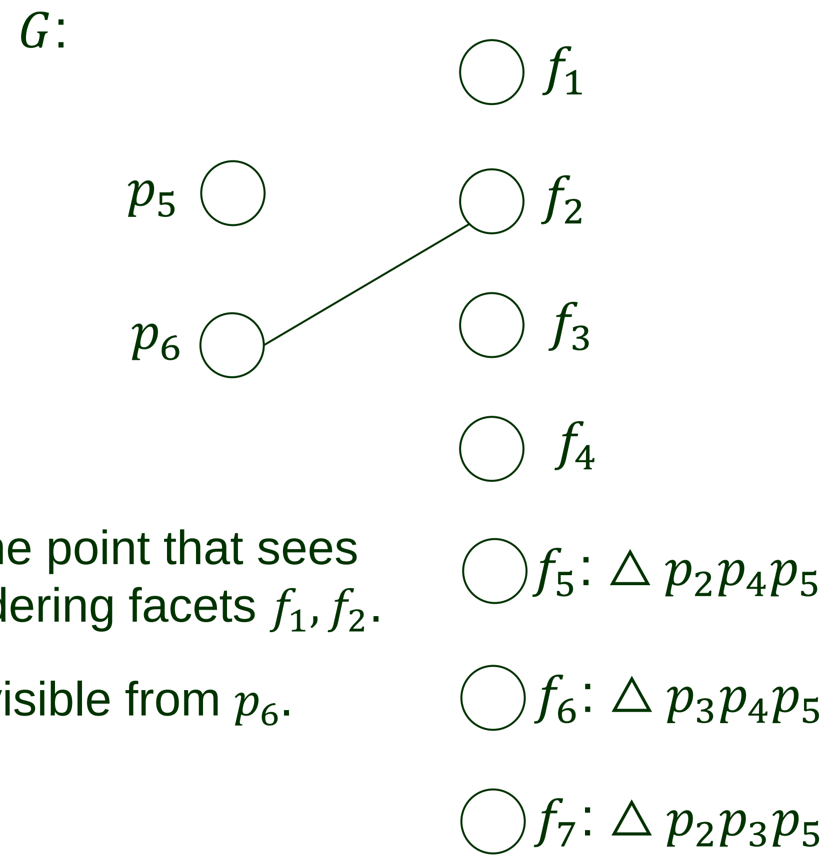
f_5 is not visible from p_6, the point that sees $e = \overline{p_2 p_4}$ and its (old) bordering facets f_1, f_2.

f_6 is not visible from p_6.

Iteration (cont'd)

$r = 5$



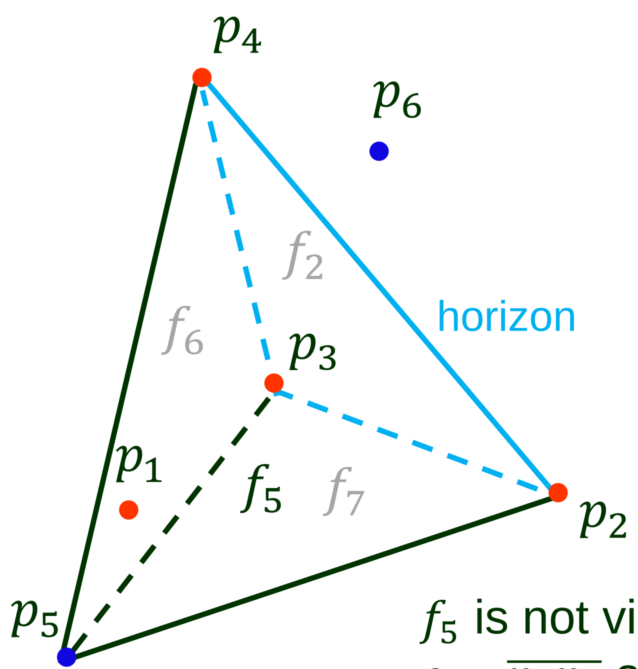


f_5 is not visible from p_6, the point that sees $e = \overline{p_2p_4}$ and its (old) bordering facets f_1, f_2.

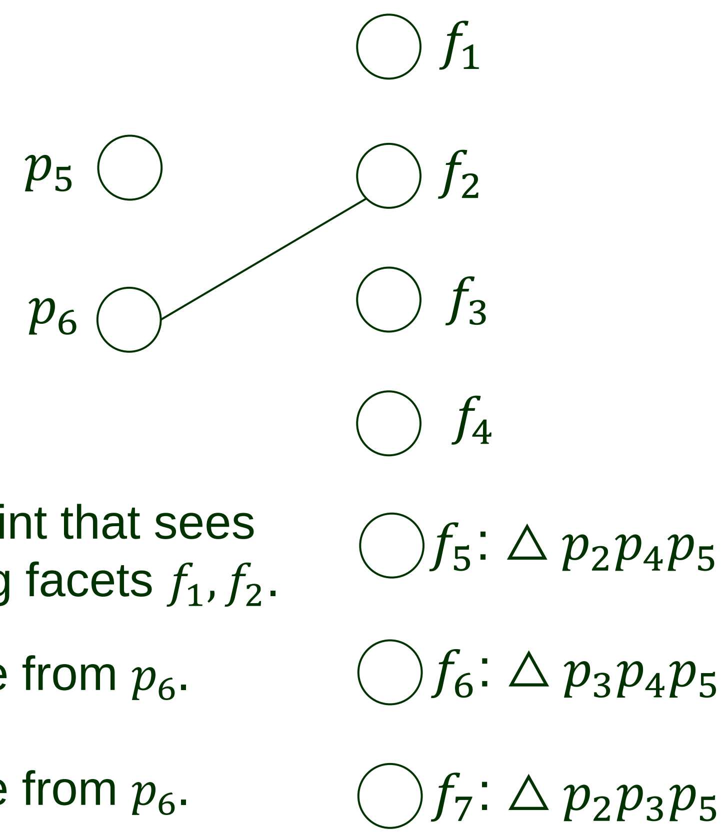
f_6 is not visible from p_6.

Iteration (cont'd)

$r = 5$



G:



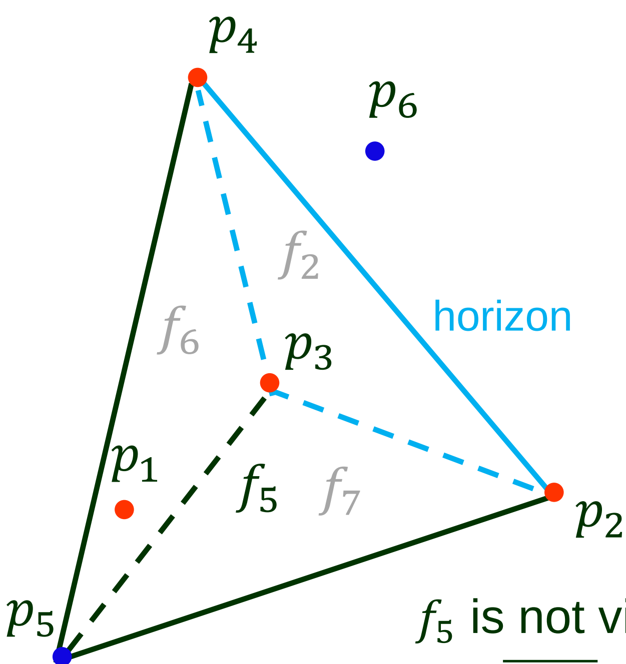
f_5 is not visible from p_6, the point that sees $e = \overline{p_2p_4}$ and its (old) bordering facets f_1, f_2.

f_6 is not visible from p_6.

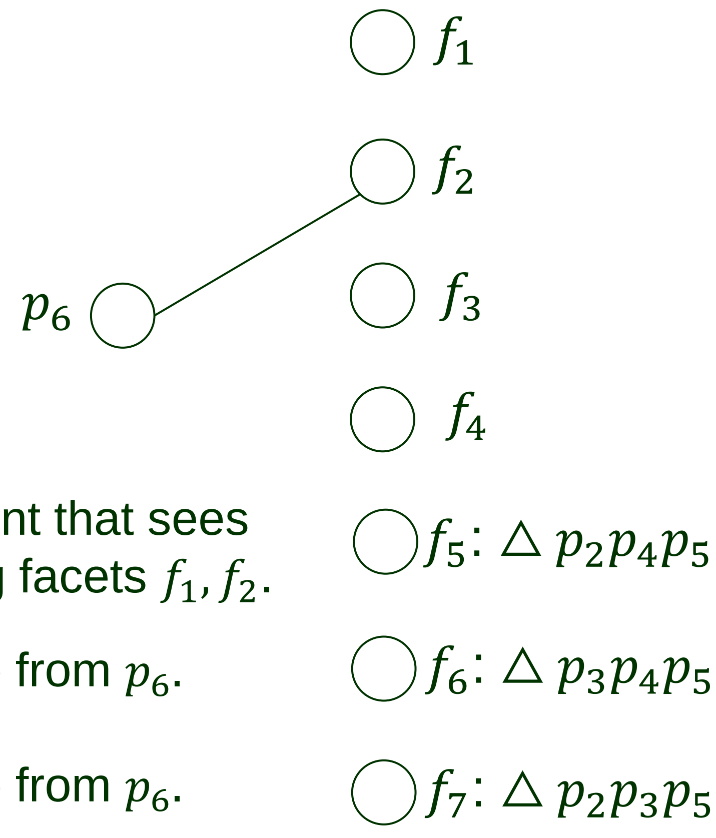
f_7 is not visible from p_6.

Iteration (cont'd)

$r = 5$



G:



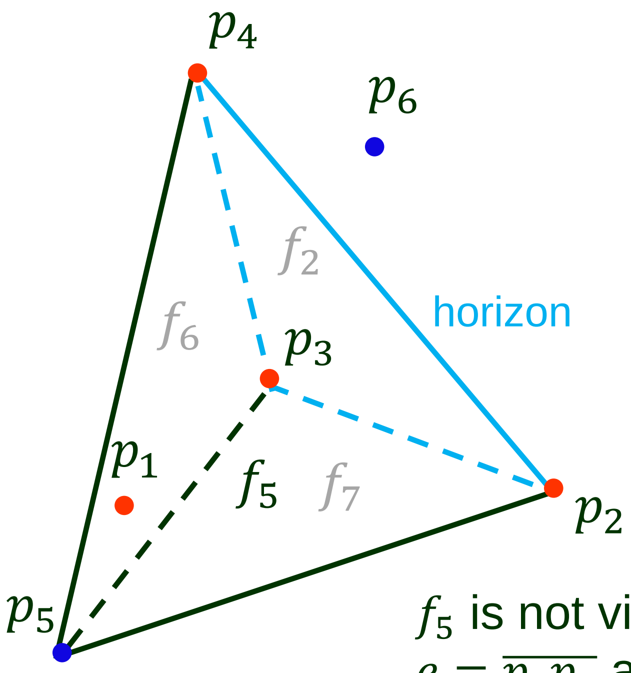
f_5 is not visible from p_6, the point that sees $e = \overline{p_2p_4}$ and its (old) bordering facets f_1, f_2.

f_6 is not visible from p_6.

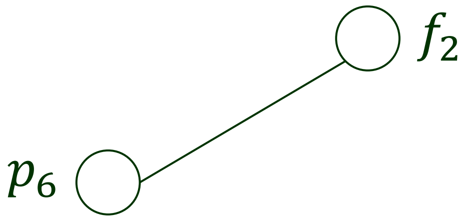
f_7 is not visible from p_6.

Iteration (cont'd)

$r = 5$



G:



f_5 is not visible from p_6, the point that sees $e = \overline{p_2 p_4}$ and its (old) bordering facets f_1, f_2.

f_5: $\triangle\, p_2 p_4 p_5$

f_6 is not visible from p_6.

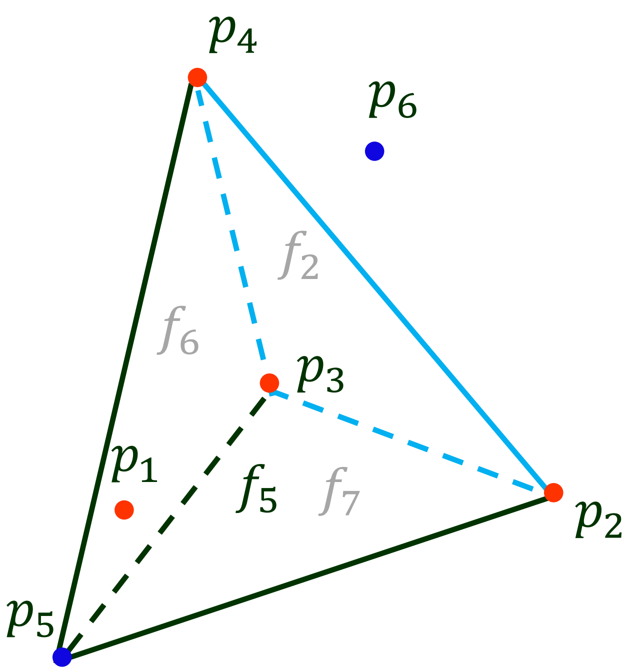
f_6: $\triangle\, p_3 p_4 p_5$

f_7 is not visible from p_6.

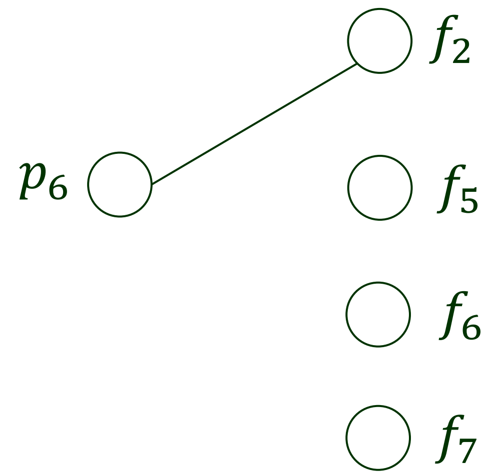
f_7: $\triangle\, p_2 p_3 p_5$

Iteration for p_6

$r = 6$

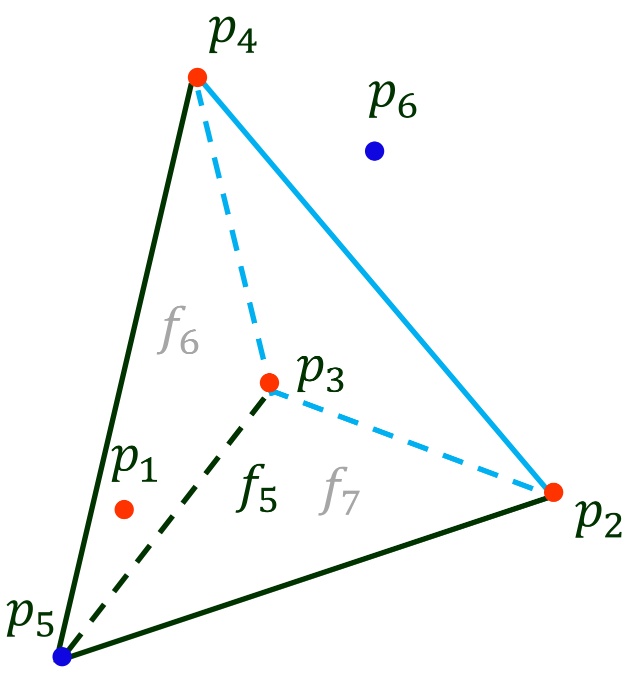


G:

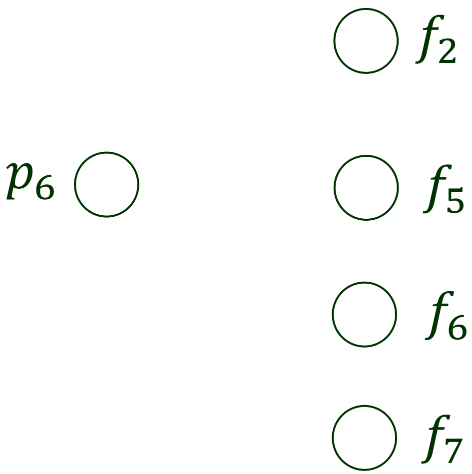


Iteration for p_6

$r = 6$

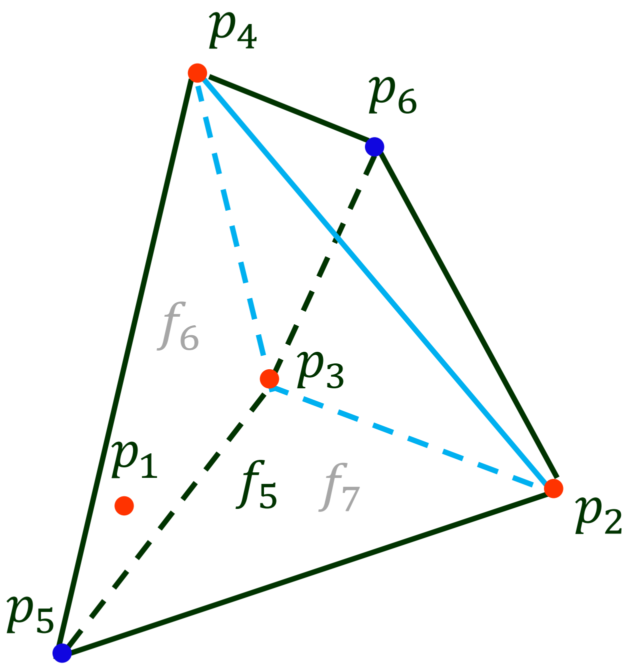


G:

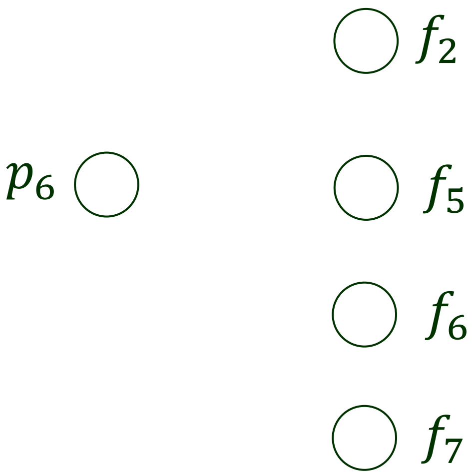


Iteration for p_6

$r = 6$

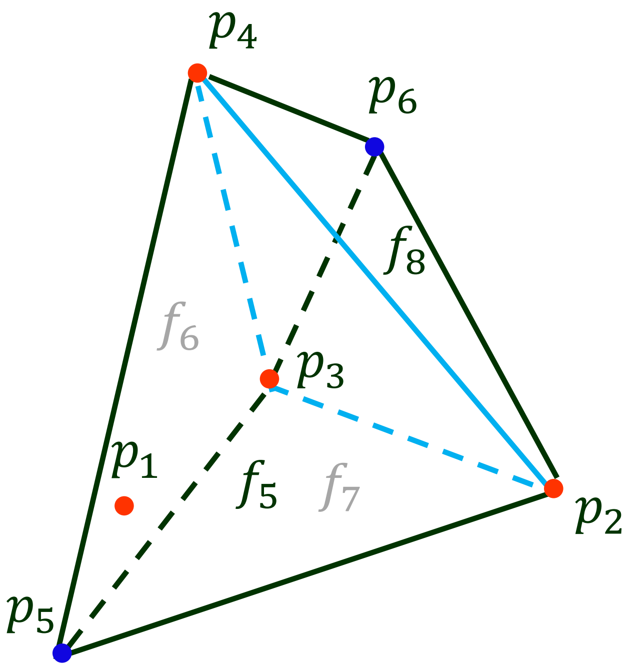


G:

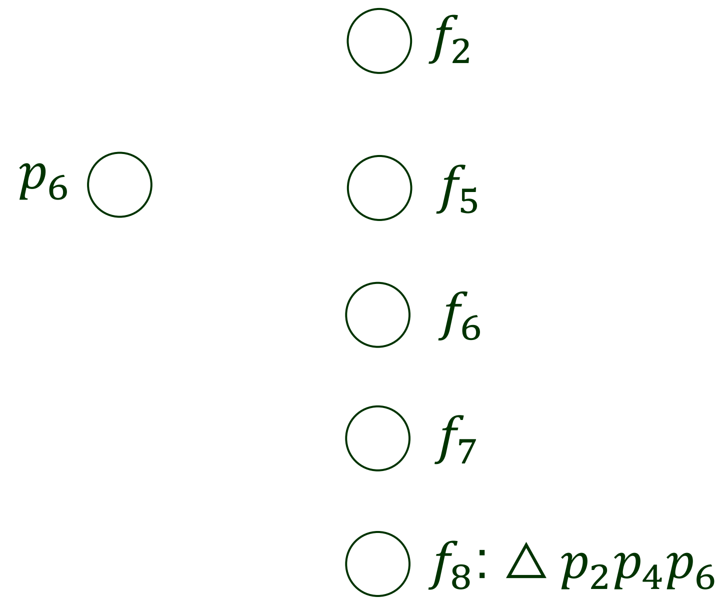


Iteration for p_6

$r = 6$

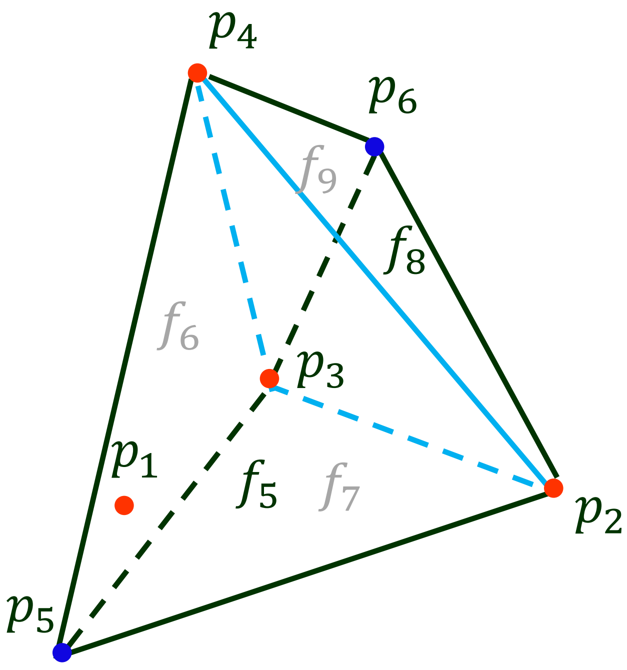


G:

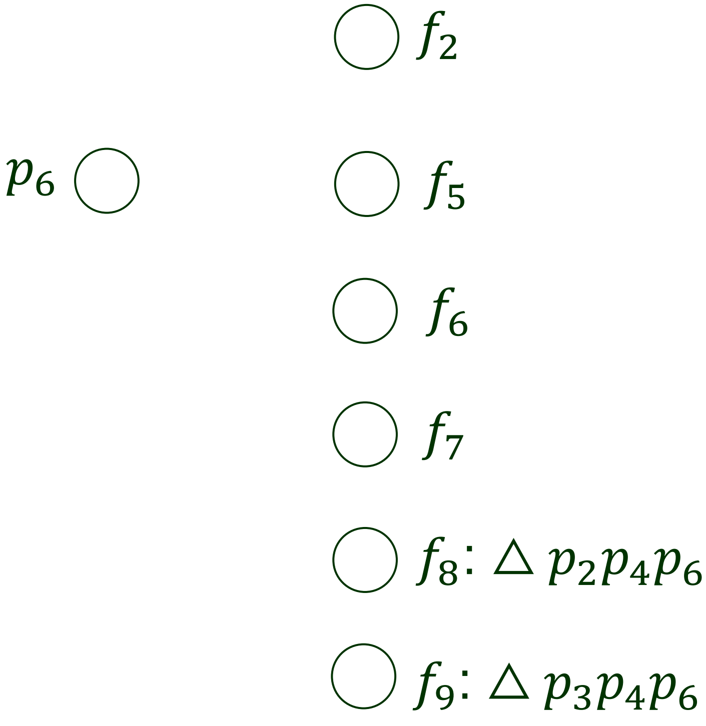


Iteration for p_6

$r = 6$

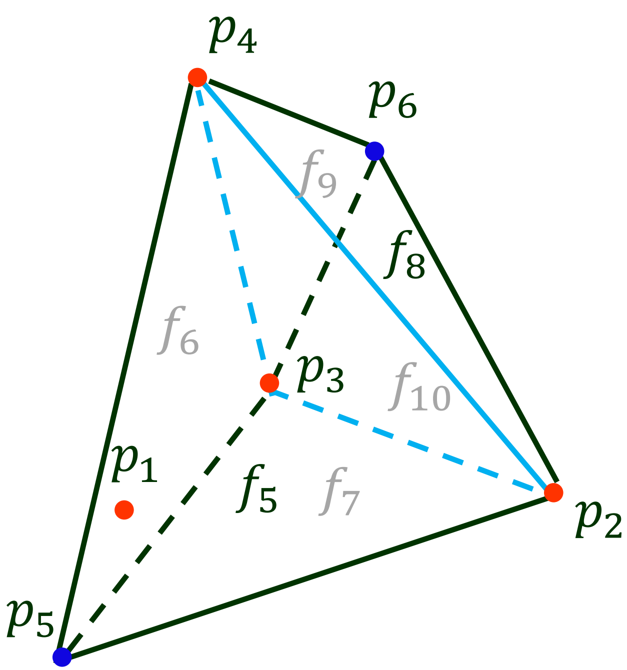


G:



Iteration for p_6

$r = 6$



G:

p_6

f_2

f_5

f_6

f_7

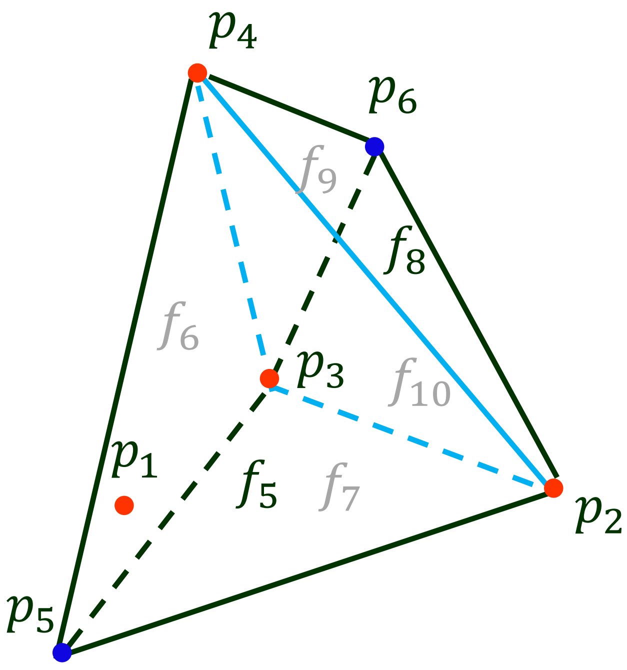
f_8: $\triangle\, p_2 p_4 p_6$

f_9: $\triangle\, p_3 p_4 p_6$

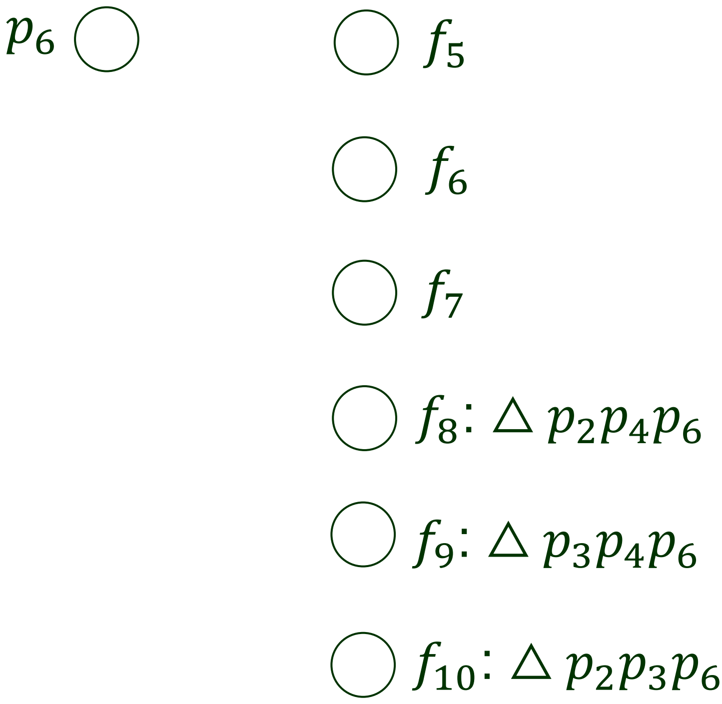
f_{10}: $\triangle\, p_2 p_3 p_6$

Iteration for p_6

$r = 6$



G:



Analysis

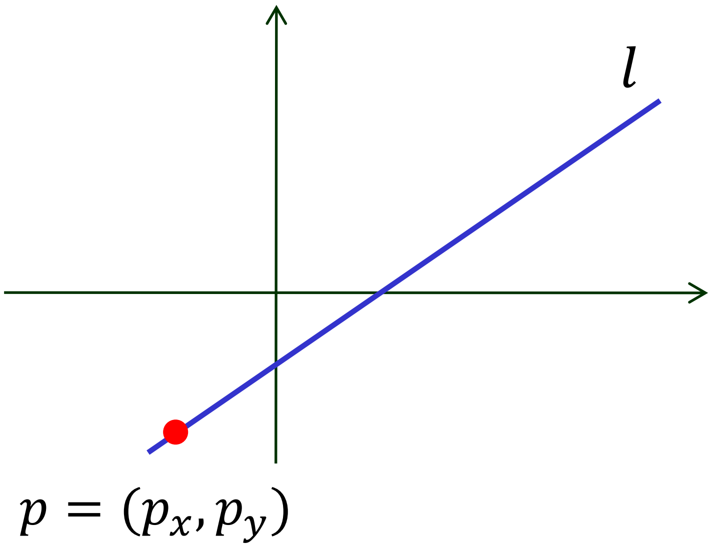
Theorem The randomized incremental algorithm computes the convex hull of n points in 3D in $O(n \log n)$ expected time.

Proof (omitted)

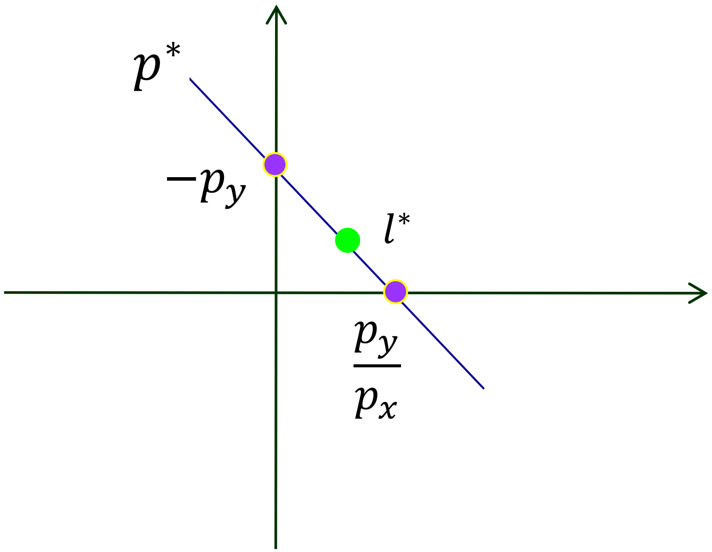
II. Duality: Points $\longleftrightarrow$ (Non-vertical) Lines

Point $p = (p_x, p_y)$ $\Longrightarrow$ Line $p^*: y = p_x x - p_y$

Line $l: y = mx + b$ $\Longrightarrow$ Point $l^* = (m, -b)$



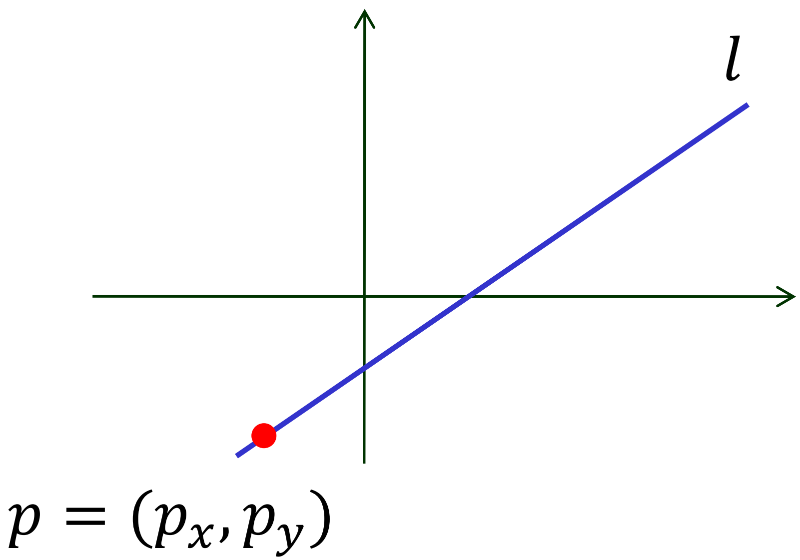
primal plane

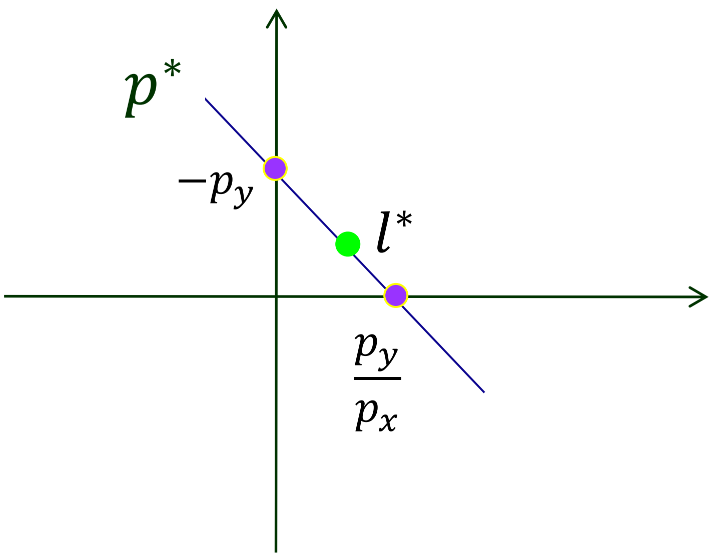


dual plane

Incidence

$$p \in l \Leftrightarrow l^* \in p^*$$





primal plane

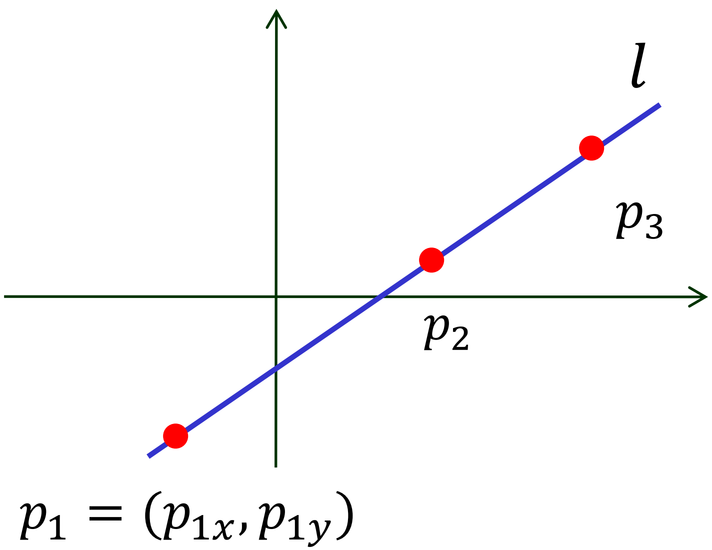
dual plane

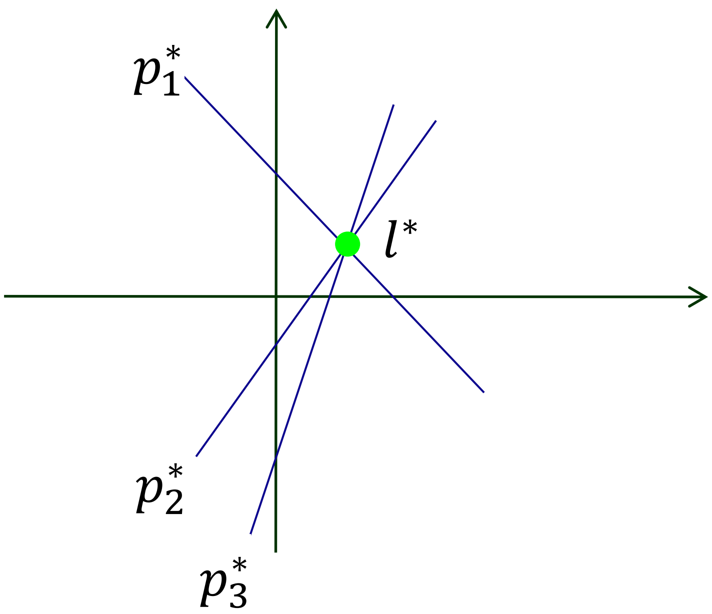
Collinearity ⟷ Concurrency

p_1, p_2, p_3 *collinear* on the line l

$\Updownarrow$

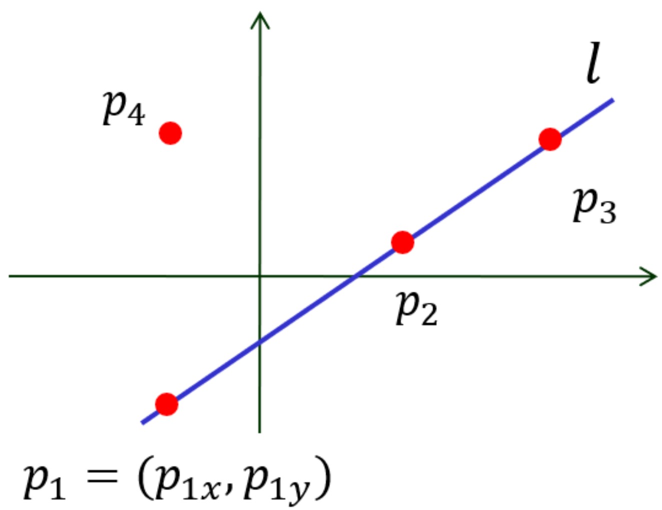
Dual lines p_1^*, p_2^*, p_3^* *concurrent* at the dual point l^*

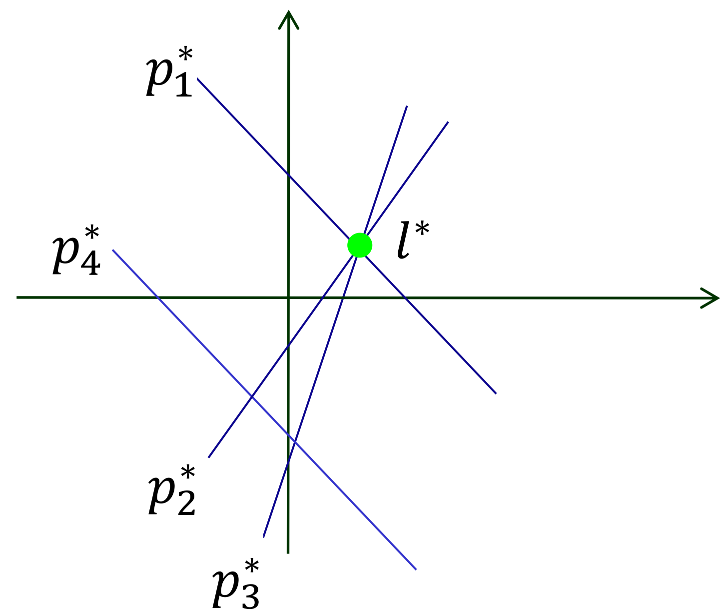




Point-Line Order Preserving

p lies above l iff l^* lies above p^*.

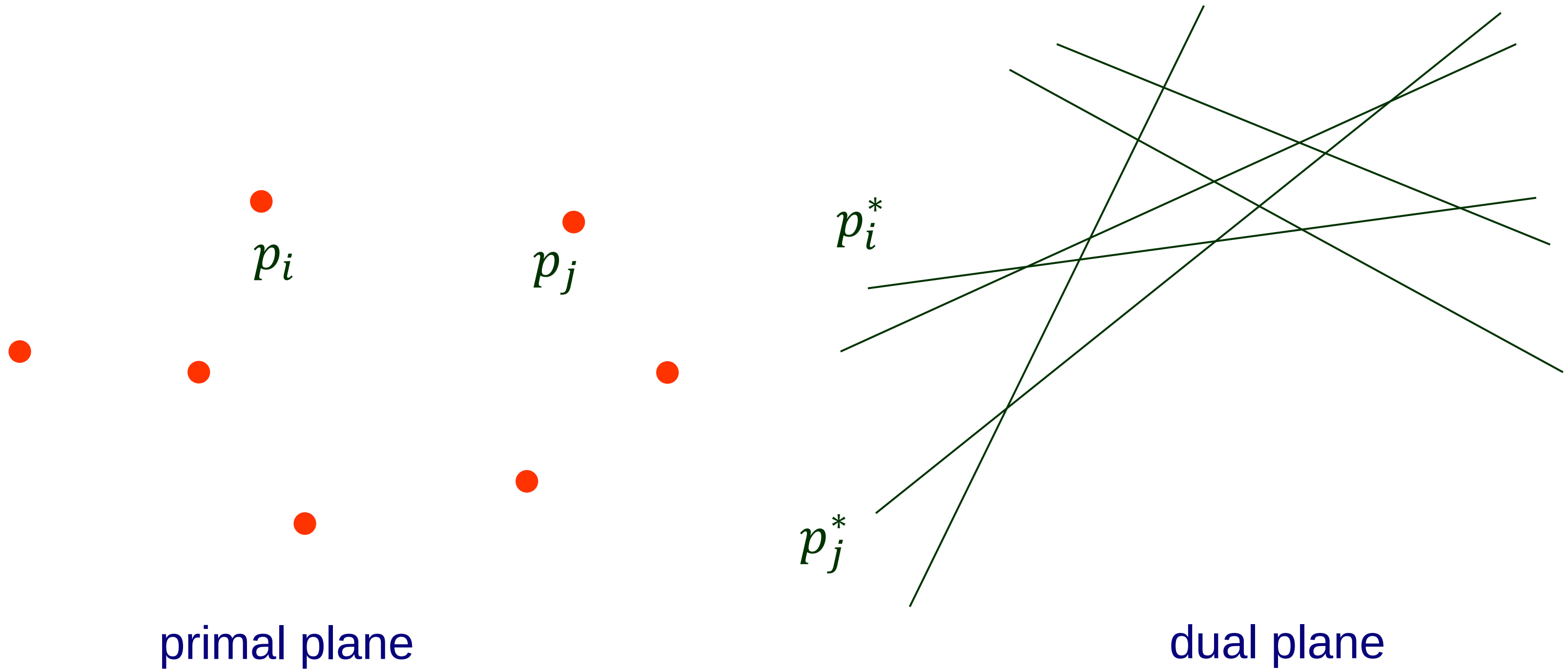




Point Set $\mapsto$ Line Arrangement

P: a set of points in the plane.

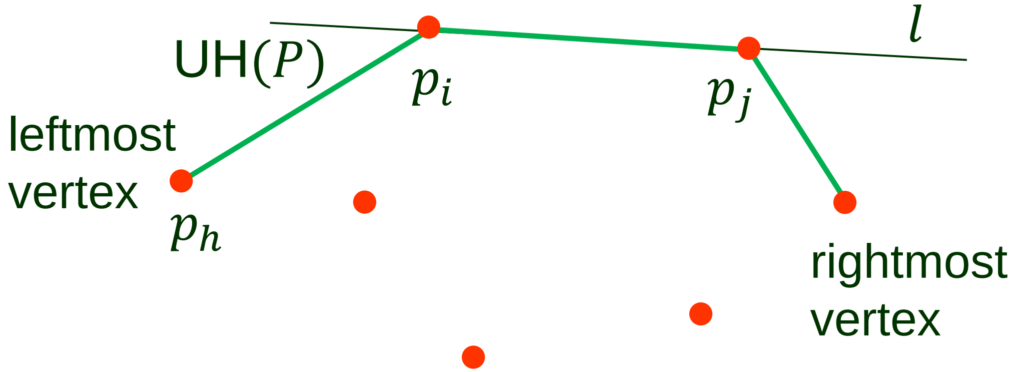
$P^* = \{p^* \mid p \in P\}$: a line arrangement



III. Upper Convex Hull & Lower Envelope

$\mathrm{UH}(P)$: *upper convex hull* of P (part of the boundary from the leftmost vertex to the rightmost one).

l above all points $\Rightarrow$ l^* below their dual lines

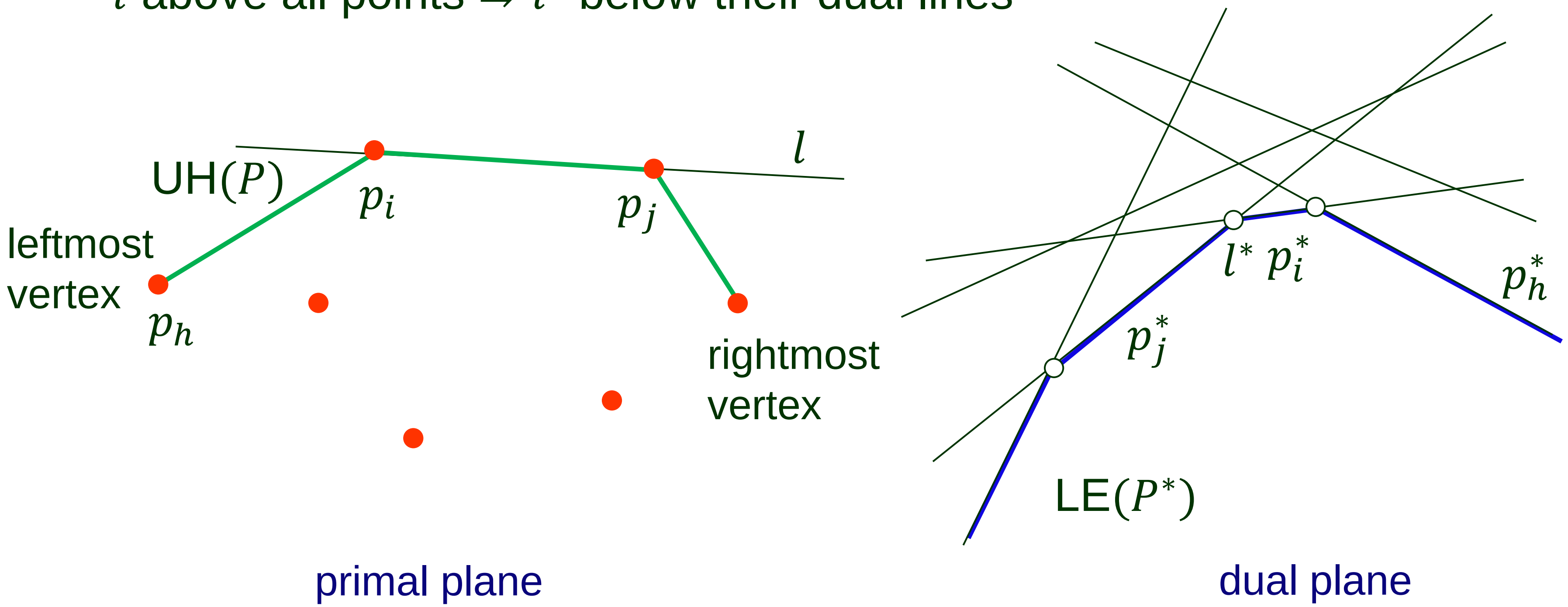


primal plane

III. Upper Convex Hull & Lower Envelope

$\mathrm{UH}(P)$: *upper convex hull* of P (part of the boundary from the leftmost vertex to the rightmost one).

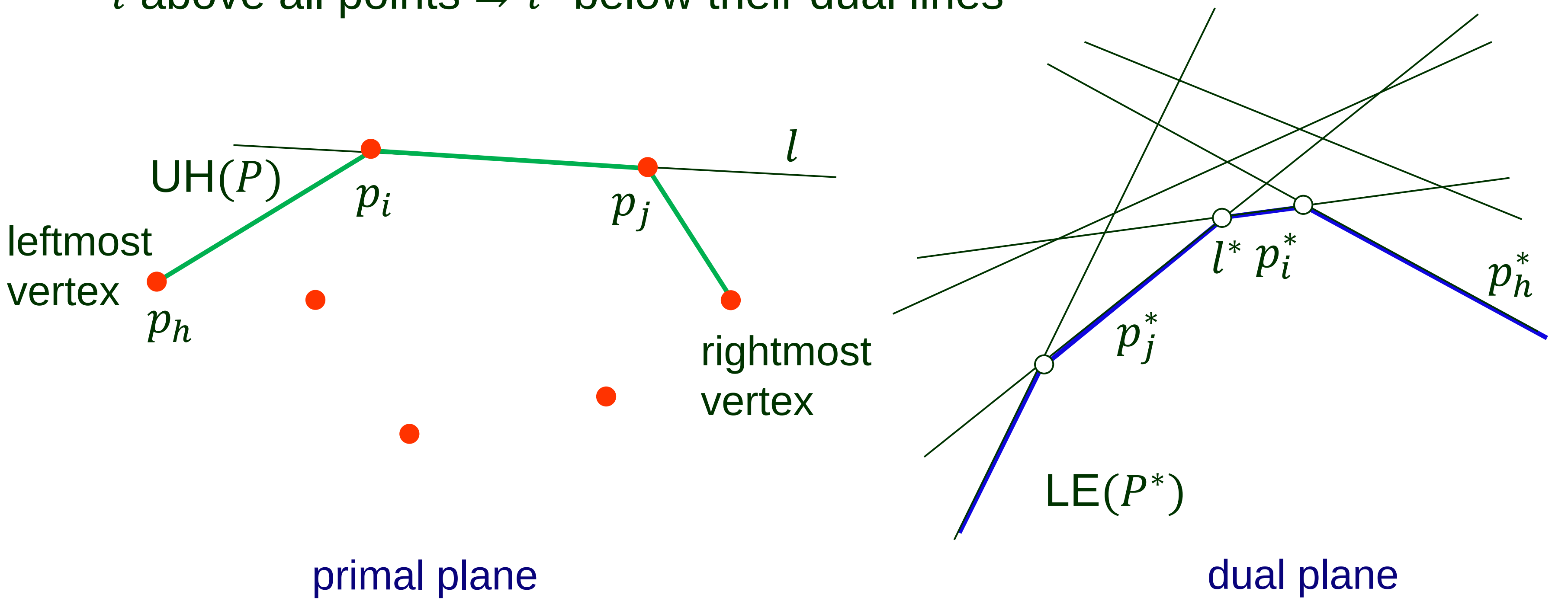
l above all points $\Rightarrow$ l^* below their dual lines



III. Upper Convex Hull & Lower Envelope

$\mathrm{UH}(P)$: *upper convex hull* of P (part of the boundary from the leftmost vertex to the rightmost one).

l above all points $\Rightarrow$ l^* below their dual lines

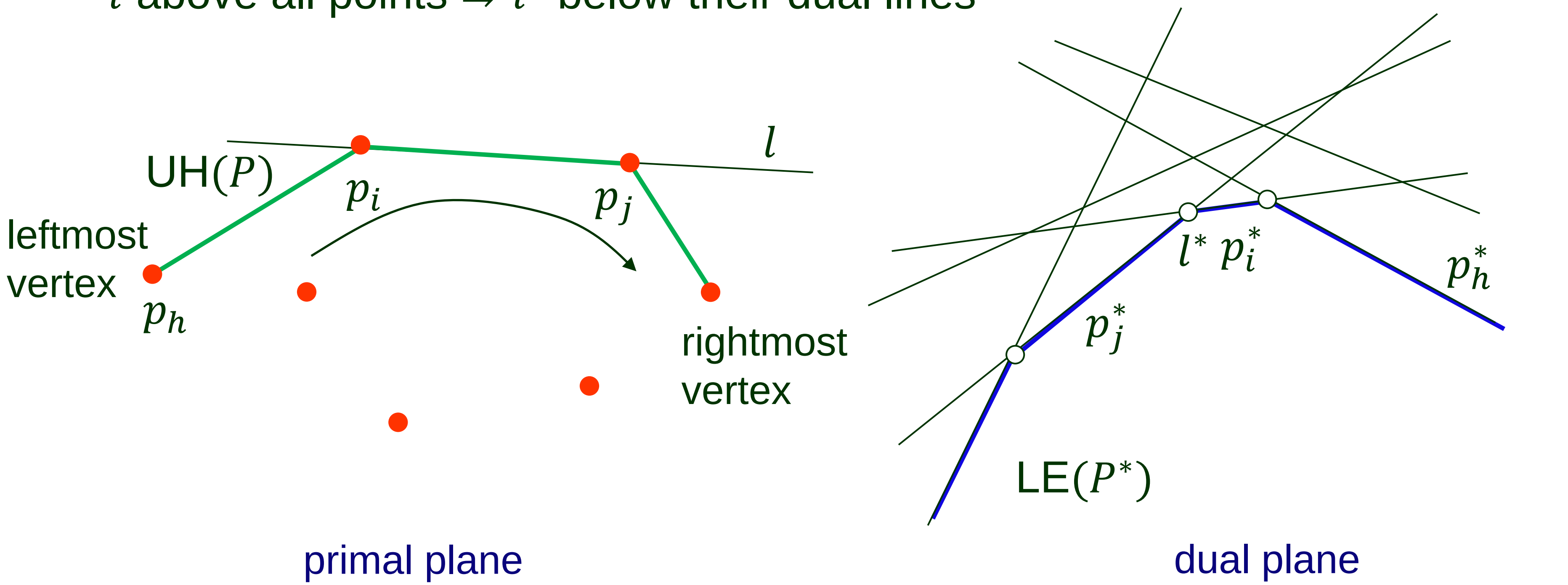


$\mathrm{LE}(P^*)$: *lower envelope* of P^* is the unique bottom cell of the arrangement.

III. Upper Convex Hull & Lower Envelope

$\mathrm{UH}(P)$: *upper convex hull* of P (part of the boundary from the leftmost vertex to the rightmost one).

l above all points $\Rightarrow$ l^* below their dual lines

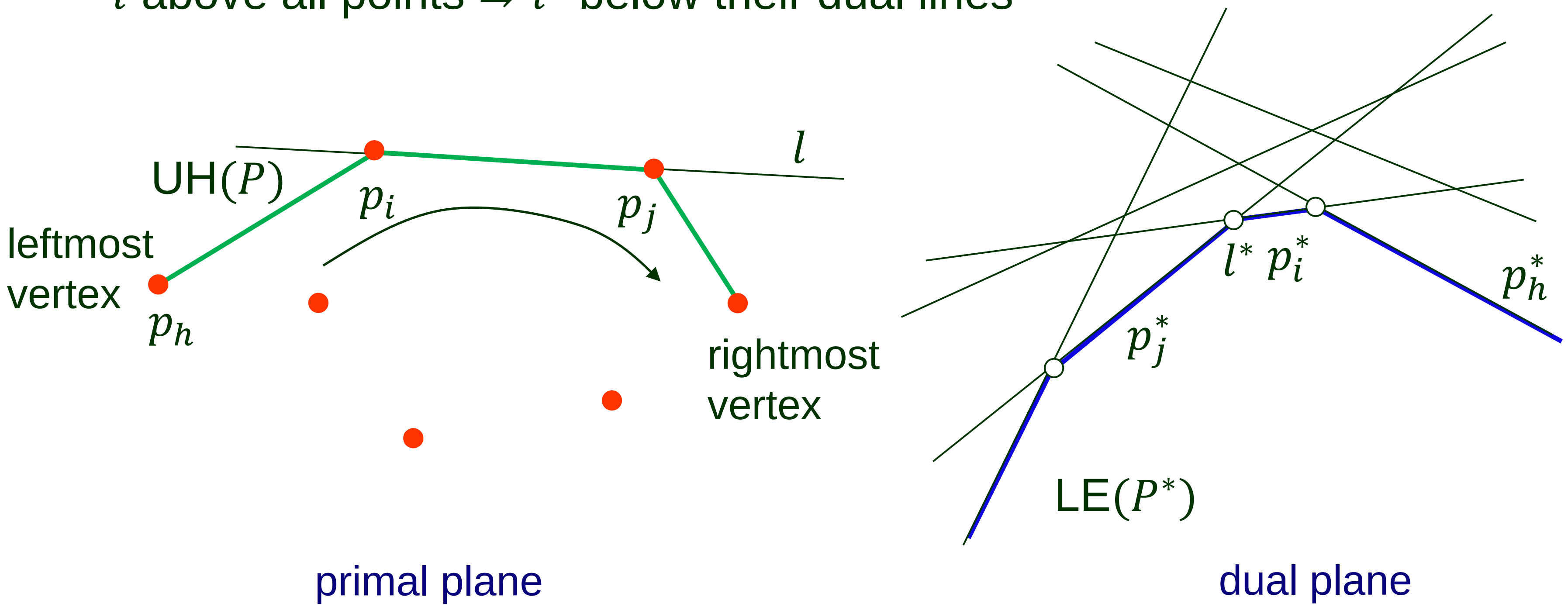


$\mathrm{LE}(P^*)$: *lower envelope* of P^* is the unique bottom cell of the arrangement.

III. Upper Convex Hull & Lower Envelope

$\mathrm{UH}(P)$: *upper convex hull* of P (part of the boundary from the leftmost vertex to the rightmost one).

l above all points $\Rightarrow$ l^* below their dual lines



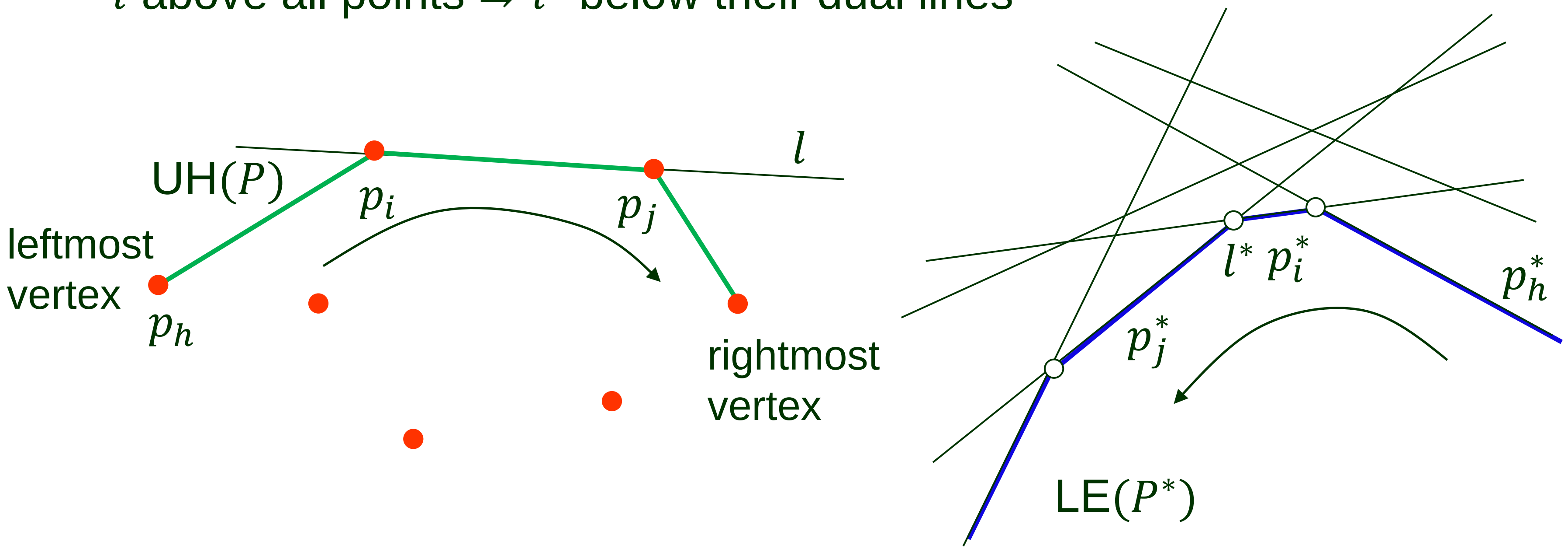
Slope of p_h^* < Slope of p_i^* < Slope of p_j^*

LE(P^*): *lower envelope* of P^* is the unique bottom cell of the arrangement.

III. Upper Convex Hull & Lower Envelope

$\mathrm{UH}(P)$: *upper convex hull* of P (part of the boundary from the leftmost vertex to the rightmost one).

l above all points $\Rightarrow$ l^* below their dual lines

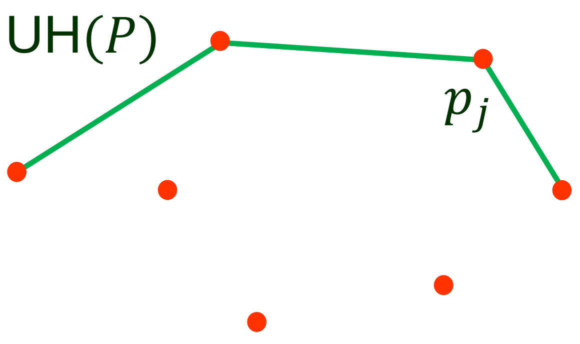


primal plane　　　dual plane

Slope of p_h^* < Slope of p_i^* < Slope of p_j^*

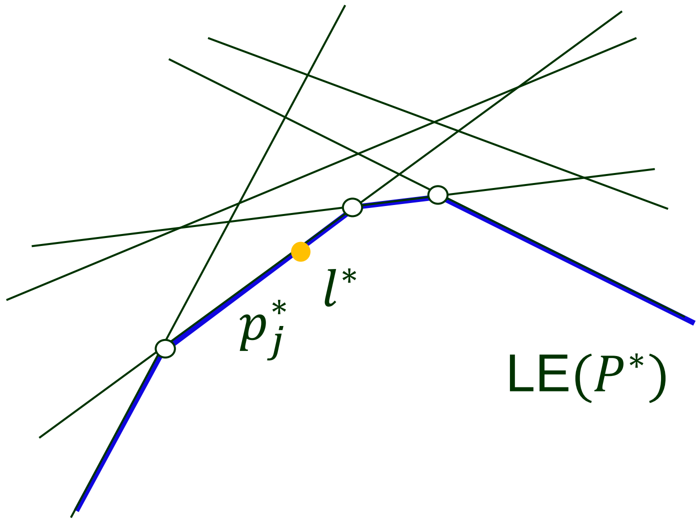
$\mathrm{LE}(P^*)$: *lower envelope* of P^* is the unique bottom cell of the arrangement.

Vertex → Edge



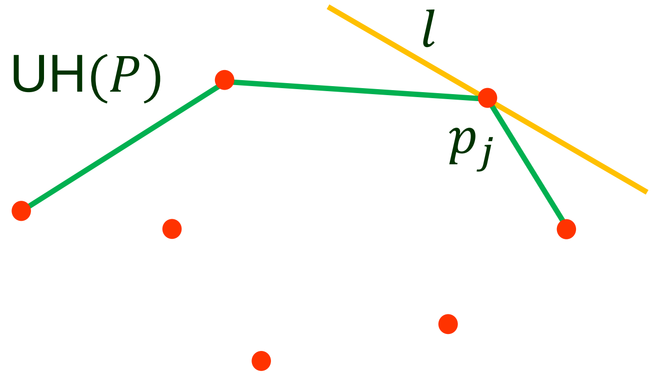
primal plane

p_j is a vertex of UH(P).

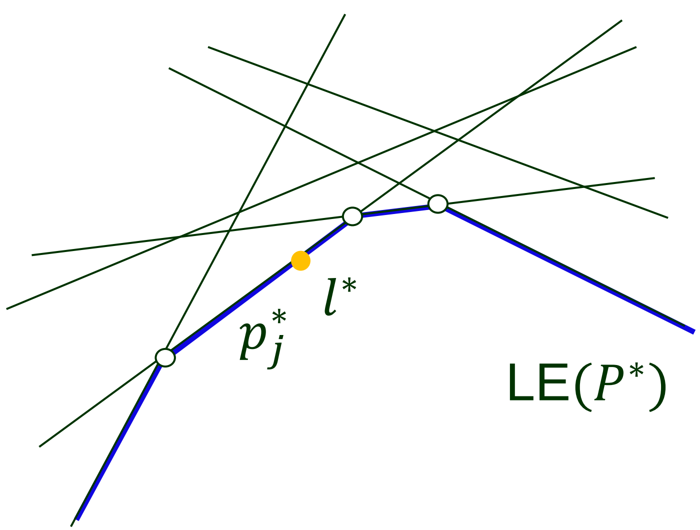


dual plane

Vertex → Edge



primal plane



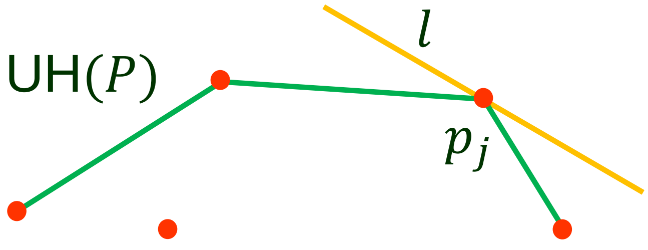
dual plane

p_j is a vertex of UH(P).

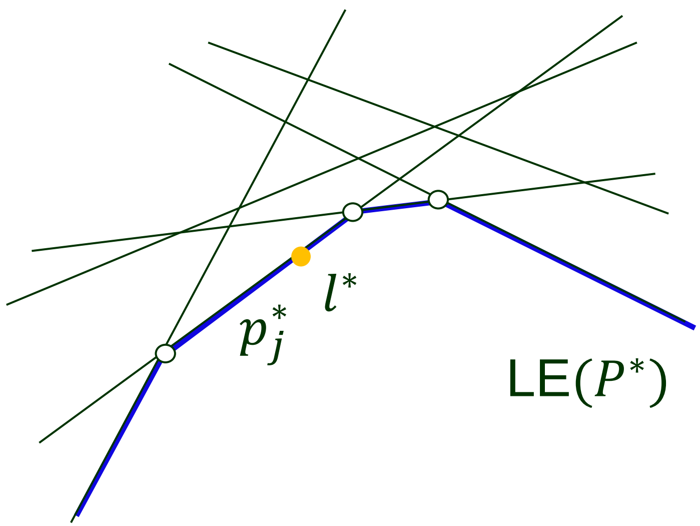
⇕

There is a non-vertical line l through p_j such that all other points are below l.

Vertex → Edge



primal plane



dual plane

p_j is a vertex of UH(P).

$\Updownarrow$

There is a non-vertical line l through p_j such that all other points are below l.

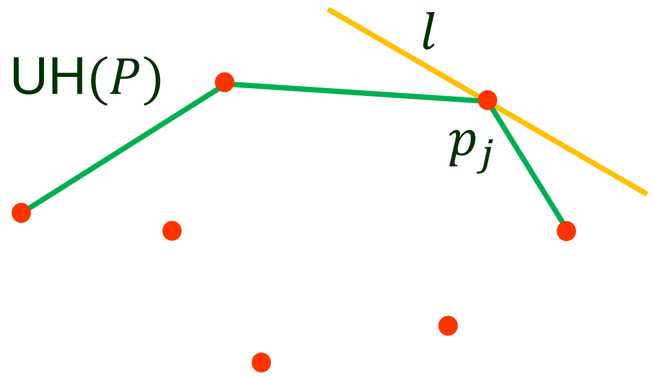
$\Updownarrow$

Its dual point l^* on the line

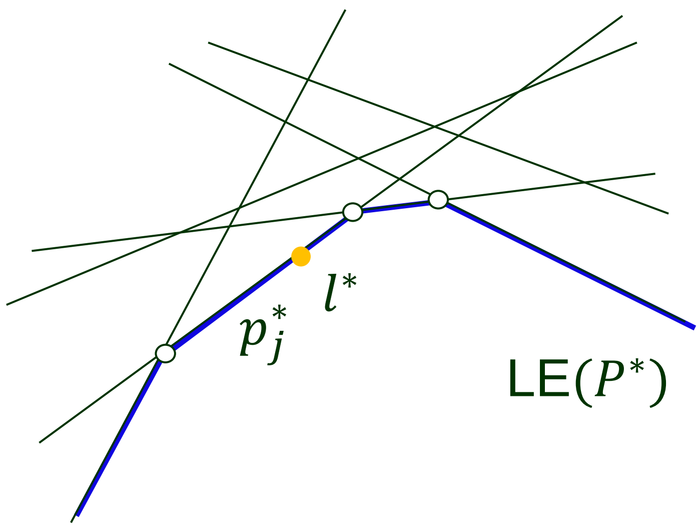
$$p_j^* \in P^* = \{p^* \mid p \in P\}$$

lies below all other lines of P^*.

Vertex → Edge



primal plane



dual plane

p_j is a vertex of UH(P).

$$\Updownarrow$$

There is a non-vertical line l through p_j such that all other points are below l.

$$\Updownarrow$$

Its dual point l^* on the line

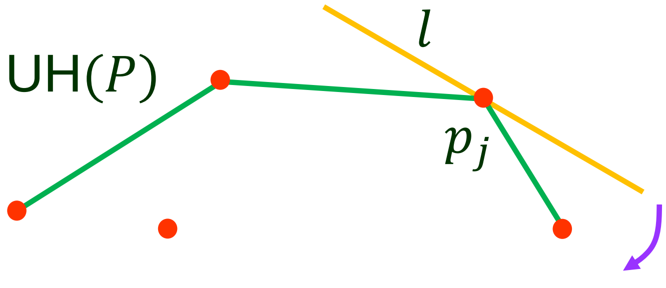
$$p_j^* \in P^* = \{p^* \mid p \in P\}$$

lies below all other lines of P^*.

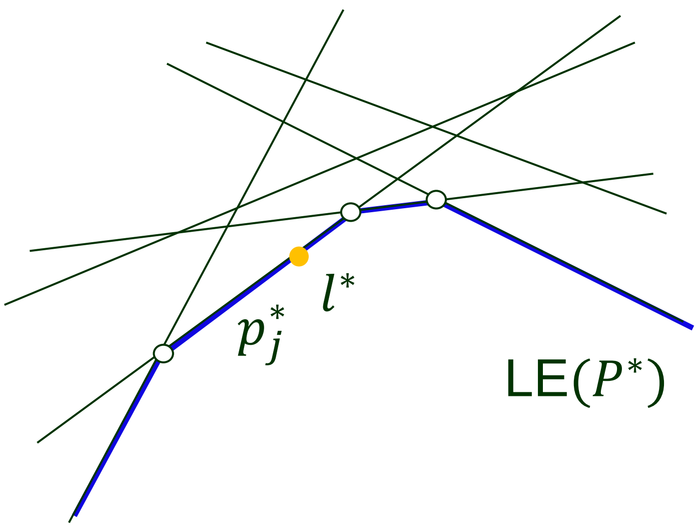
$$\Updownarrow$$

$l^* \in p_j^*$ is on the boundary of the bottom cell; i.e., p_j^* contributes an edge to LE(P^*).

Vertex → Edge



primal plane



dual plane

p_j is a vertex of UH(P).

⇕

There is a non-vertical line l through p_j such that all other points are below l.

⇕

Its dual point l^* on the line

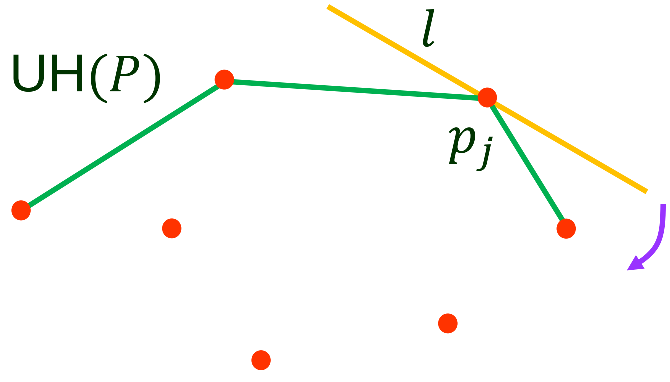
$$p_j^* \in P^* = \{p^* \mid p \in P\}$$

lies below all other lines of P^*.

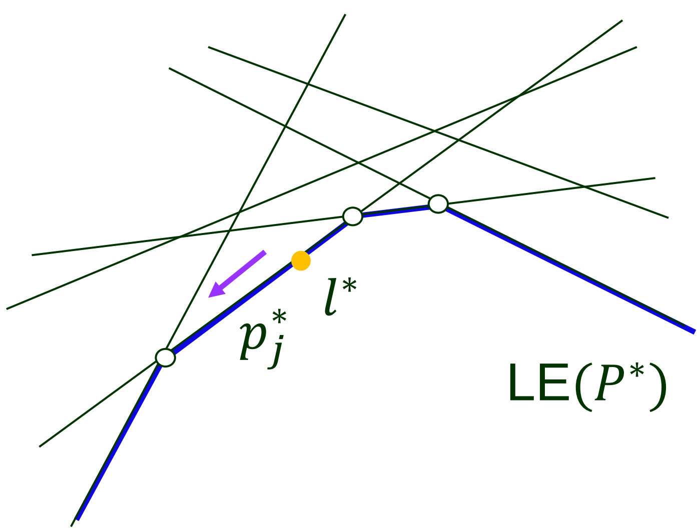
⇕

$l^* \in p_j^*$ is on the boundary of the bottom cell; i.e., p_j^* contributes an edge to LE(P^*).

Vertex → Edge



primal plane



dual plane

p_j is a vertex of UH(P).

$\Updownarrow$

There is a non-vertical line l through p_j such that all other points are below l.

$\Updownarrow$

Its dual point l^* on the line

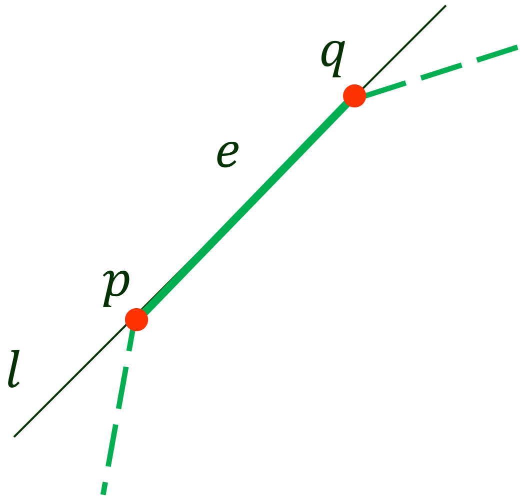
$$p_j^* \in P^* = \{p^* \mid p \in P\}$$

lies below all other lines of P^*.

$\Updownarrow$

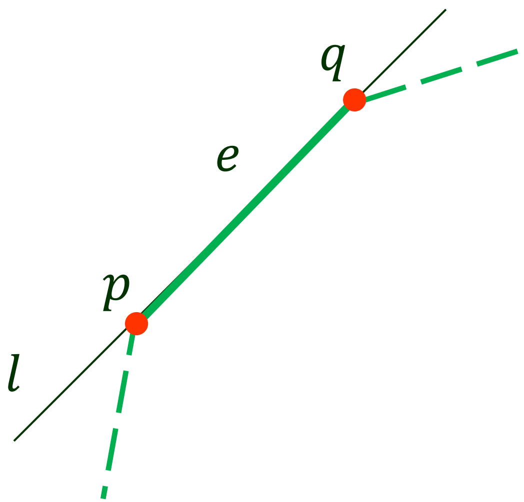
$l^* \in p_j^*$ is on the boundary of the bottom cell; i.e., p_j^* contributes an edge to LE(P^*).

Edge → Vertex

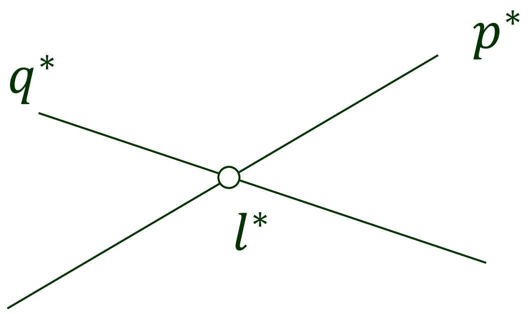


$p, q \in P$ define an edge e in UH(P).

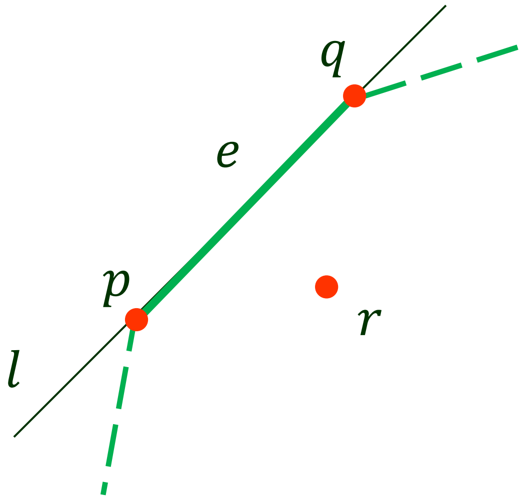
Edge → Vertex



$p, q \in P$ define an edge e in UH(P).



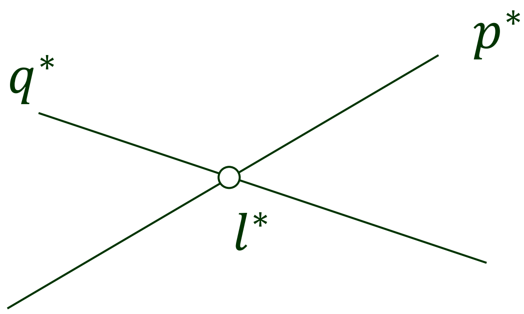
Edge → Vertex



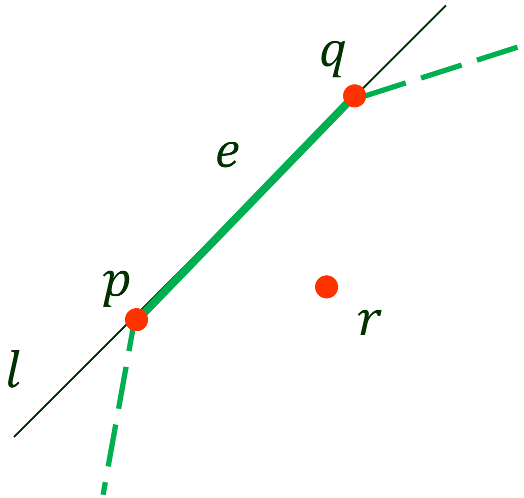
$p, q \in P$ define an edge e in UH(P).

$\Updownarrow$

All the points $r \in P \setminus e$ lie below the line l through p and q.



Edge → Vertex



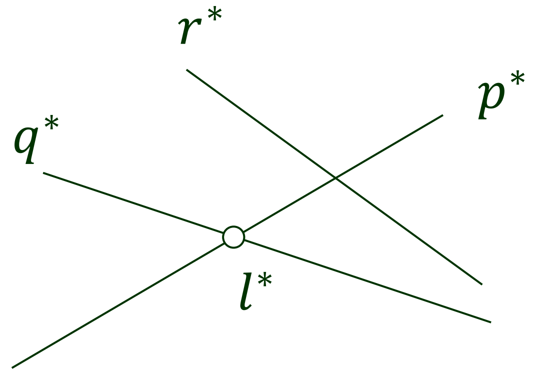
$p, q \in P$ define an edge e in UH(P).

$$\Updownarrow$$

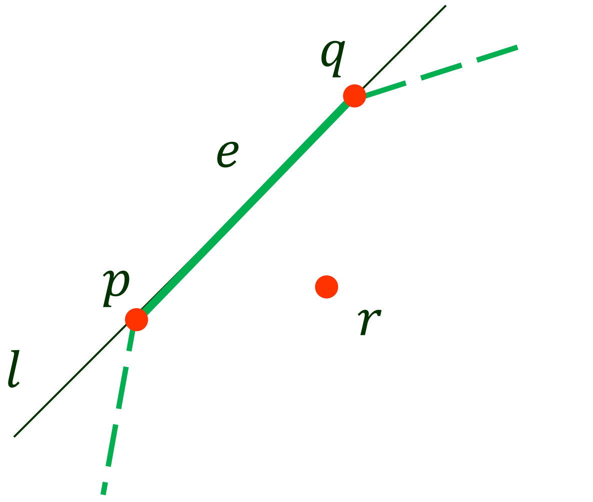
All the points $r \in P \setminus e$ lie below the line l through p and q.

$$\Updownarrow$$

All the lines r^*, $r \in P \setminus e$ lie above l^*.



Edge → Vertex

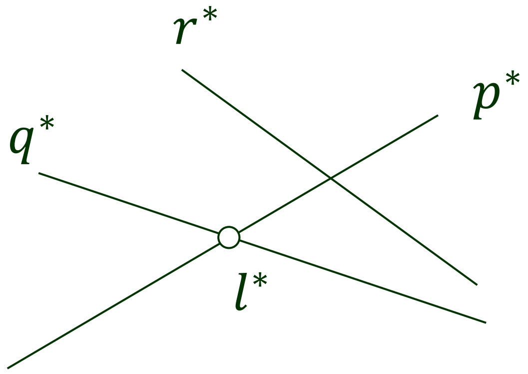


$p, q \in P$ define an edge e in UH(P).

$\Updownarrow$

All the points $r \in P \setminus e$ lie below the line l through p and q.

$\Updownarrow$



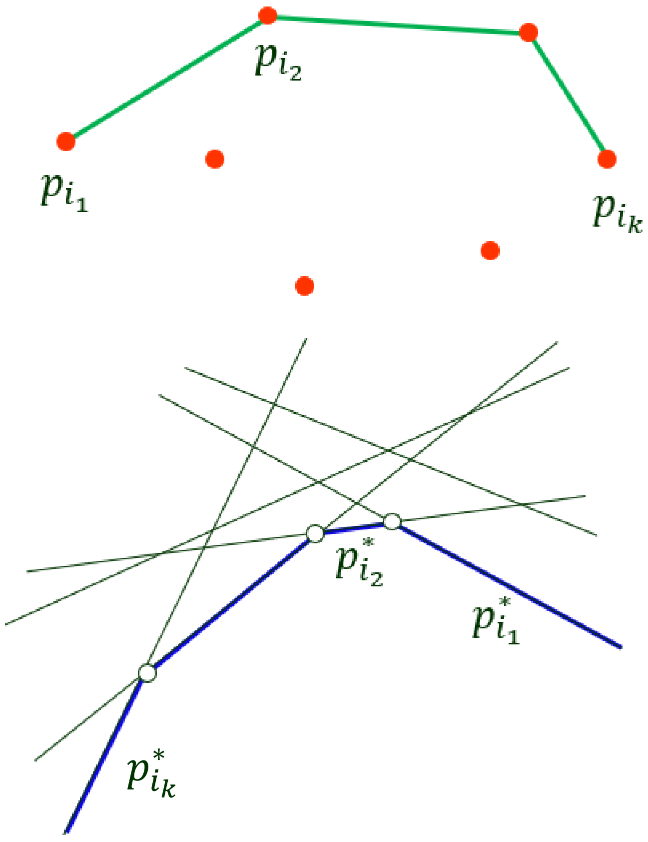
All the lines r^*, $r \in P \setminus e$ lie above l^*.

$\Updownarrow$

l^* is a vertex of LE(P^*).

Order Reversal

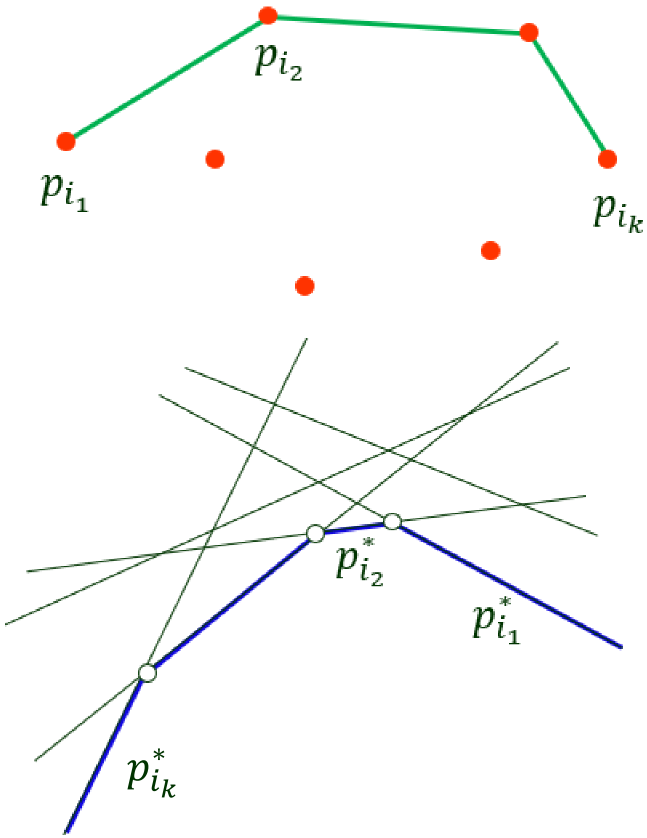
$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH(P).



Order Reversal

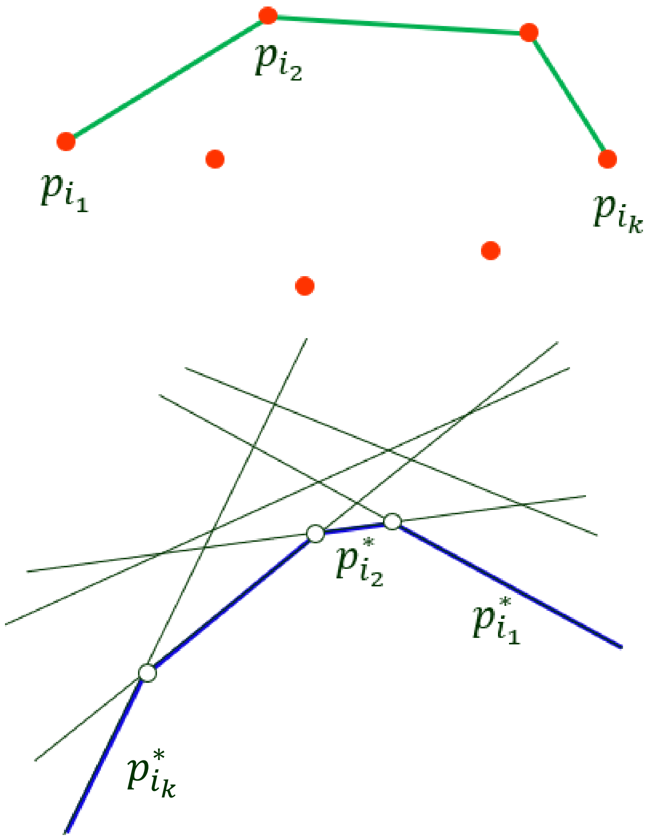
$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH(P).

$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$



Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH(P).



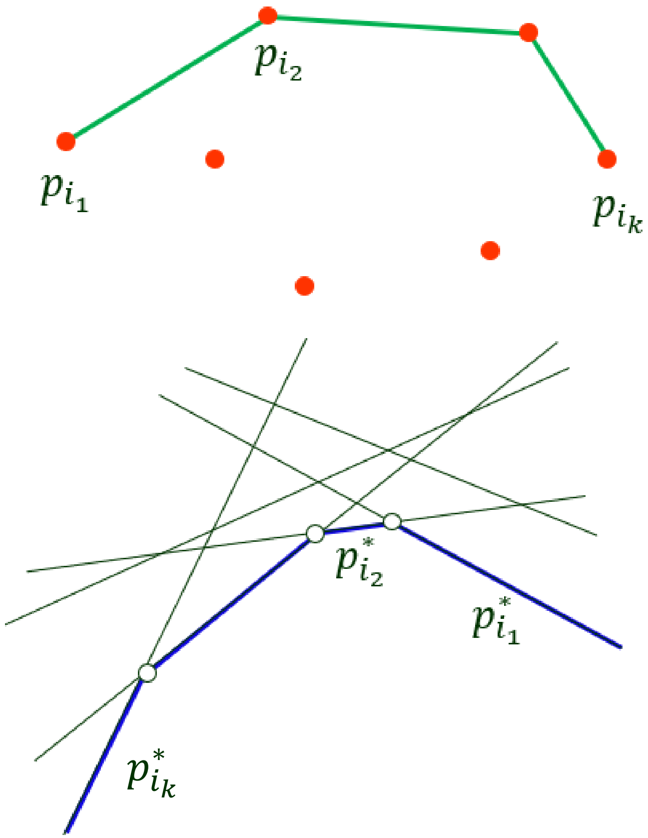
$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$

$\Downarrow$ left-to-right

$x_{i_s} < x_{i_t}$

Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH(P).



$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$

$\Downarrow$ left-to-right

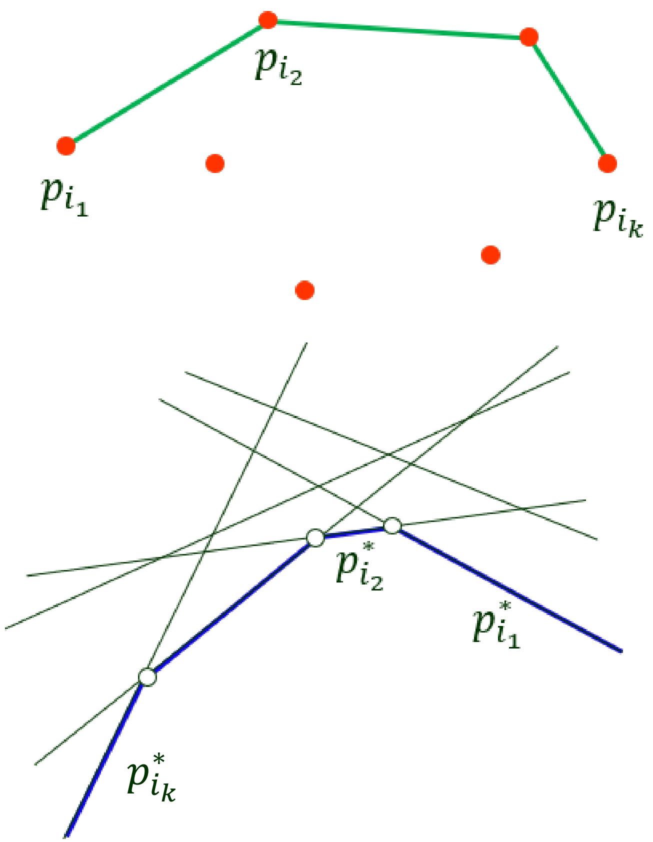
$x_{i_s} < x_{i_t}$

$\Downarrow$

The dual line $p^*_{i_s}: y = x_{i_s}x - y_{i_s}$ has a smaller slope than the dual line $p^*_{i_t}: y = x_{i_t}x - y_{i_t}$

Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH(P).



$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$

$\Downarrow$ left-to-right

$x_{i_s} < x_{i_t}$

$\Downarrow$

The dual line $p^*_{i_s}: y = x_{i_s}x - y_{i_s}$ has a smaller slope than the dual line $p^*_{i_t}: y = x_{i_t}x - y_{i_t}$

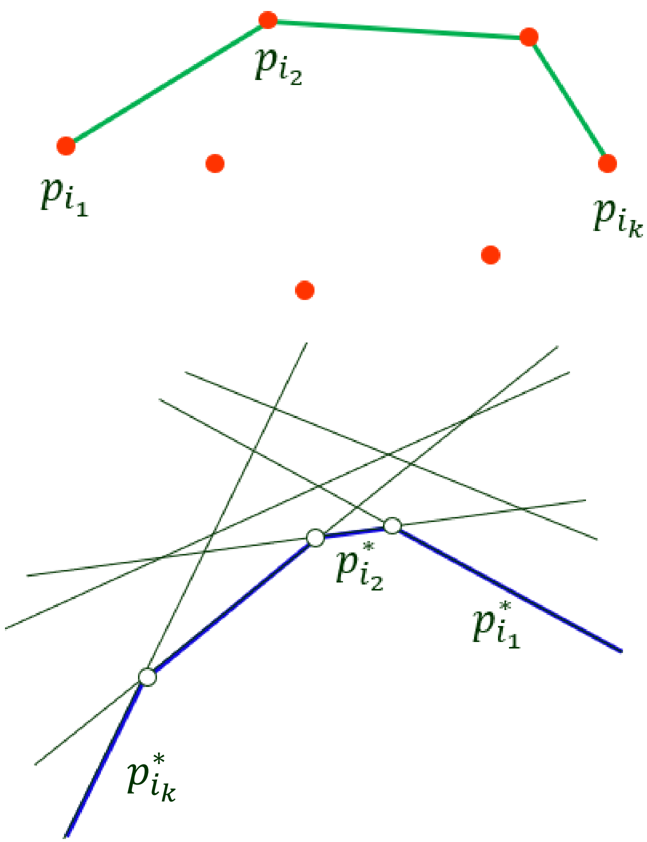
$\Downarrow$

On the lower envelope, the segment of $p^*_{i_s}$ is before that of $p^*_{i_t}$ in the right-to-left order.

Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH(P).

$p^*_{i_1}, p^*_{i_2}, \ldots, p^*_{i_k}$: right-to-left order of edges on LE(P^*).



$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$

$\Downarrow$ left-to-right

$x_{i_s} < x_{i_t}$

$\Downarrow$

The dual line $p^*_{i_s}: y = x_{i_s}x - y_{i_s}$ has a smaller slope than the dual line $p^*_{i_t}: y = x_{i_t}x - y_{i_t}$

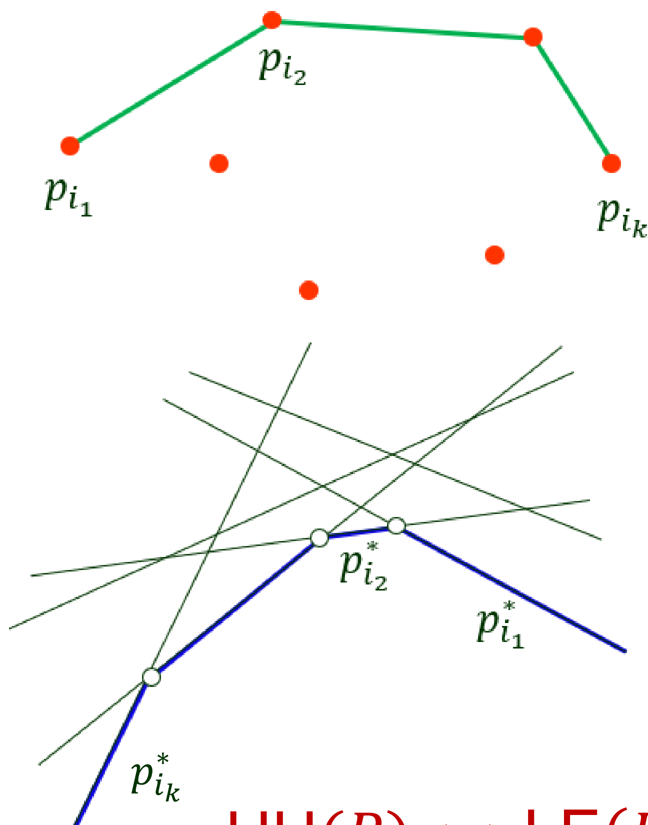
$\Downarrow$

On the lower envelope, the segment of $p^*_{i_s}$ is before that of $p^*_{i_t}$ in the right-to-left order.

Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH(P).

$p^*_{i_1}, p^*_{i_2}, \ldots, p^*_{i_k}$: right-to-left order of edges on LE(P^*).



UH$(P) \leftrightarrow$ LE(P^*)

$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$

$\Downarrow$ left-to-right

$x_{i_s} < x_{i_t}$

$\Downarrow$

The dual line $p^*_{i_s}: y = x_{i_s}x - y_{i_s}$ has a smaller slope than the dual line $p^*_{i_t}: y = x_{i_t}x - y_{i_t}$

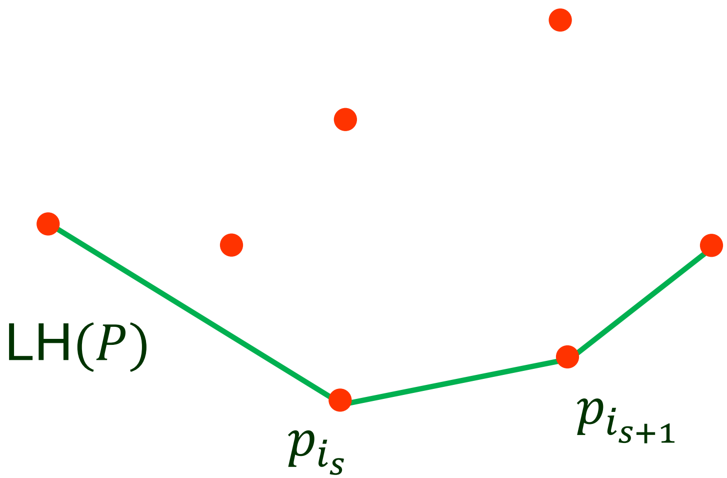
$\Downarrow$

On the lower envelope, the segment of $p^*_{i_s}$ is before that of $p^*_{i_t}$ in the right-to-left order.

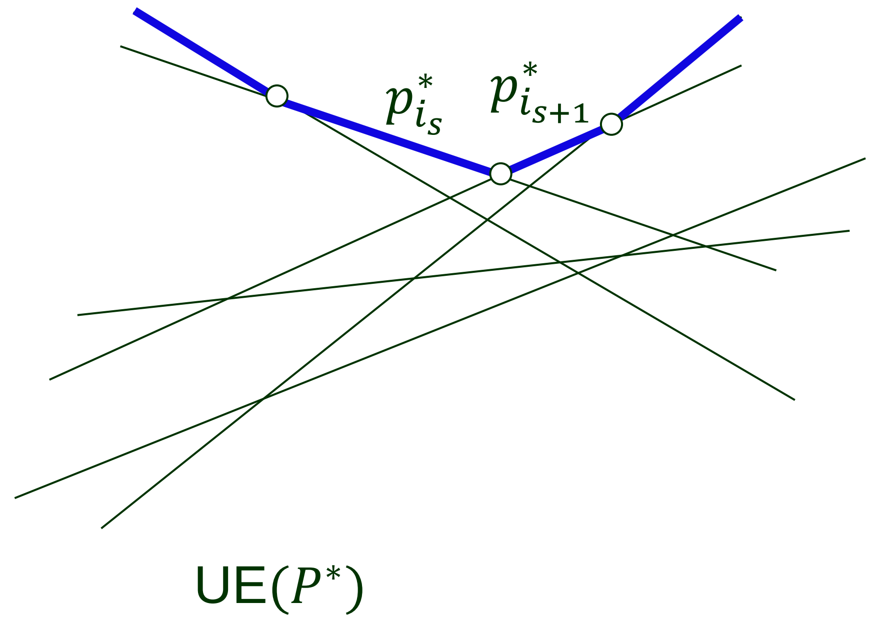
Lower Convex Hull & Upper Envelope

LH(P): *lower convex hull* of P

UE(P^*): *upper envelope* of P^*



primal plane

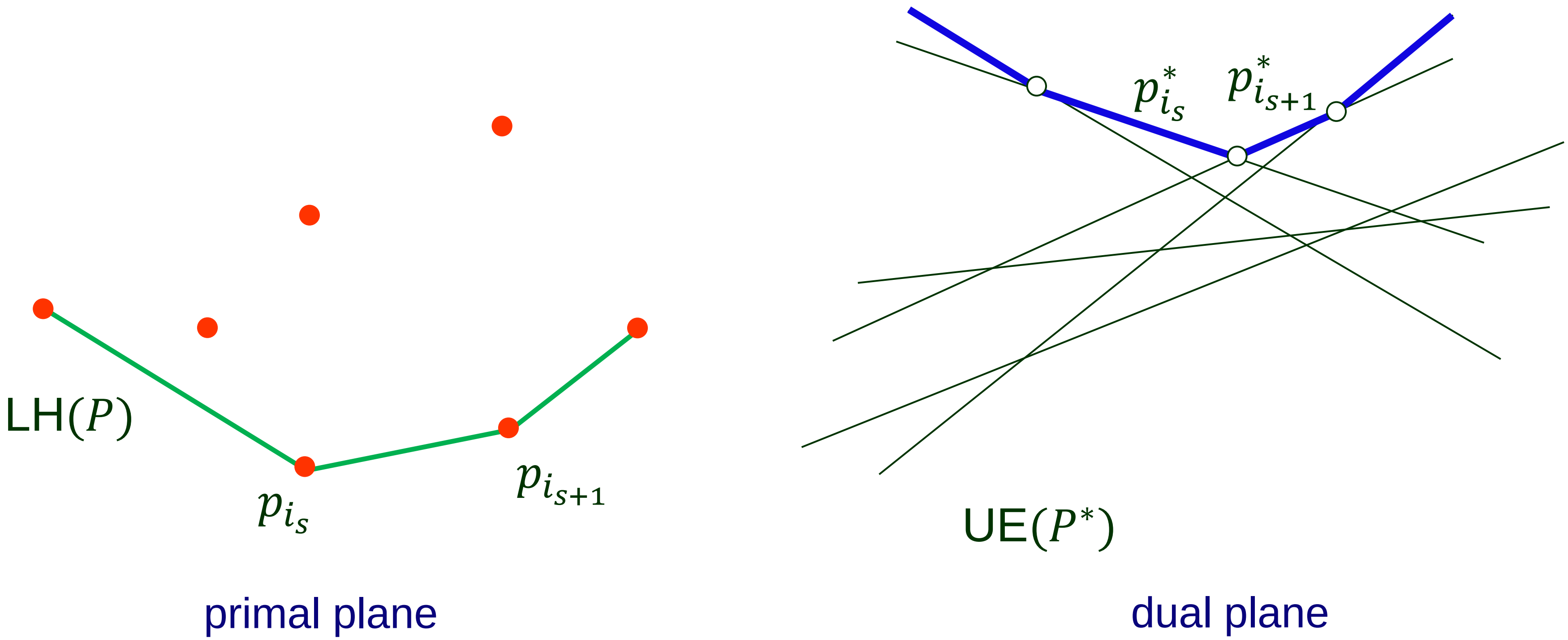


dual plane

Lower Convex Hull & Upper Envelope

LH(P): *lower convex hull* of P

UE(P^*): *upper envelope* of P^*

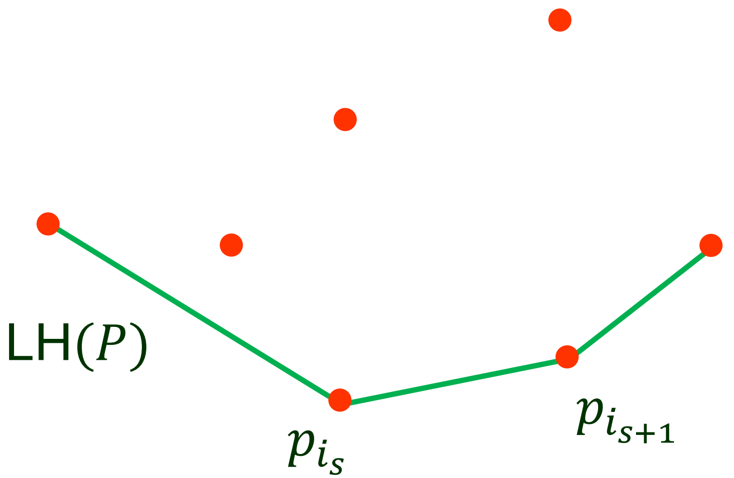


$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on LH(P).

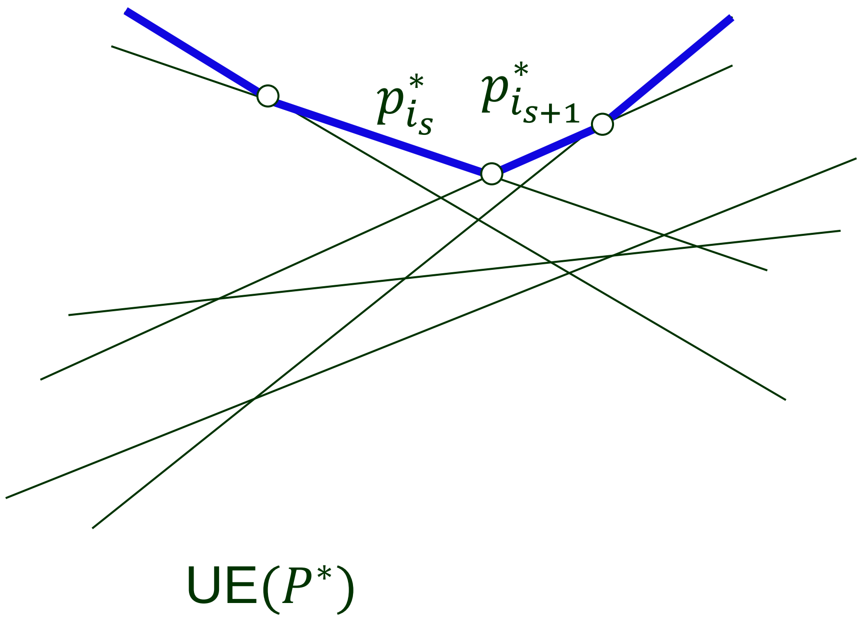
Lower Convex Hull & Upper Envelope

LH(P): *lower convex hull* of P

UE(P^*): *upper envelope* of P^*



primal plane



dual plane

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on LH(P).

$p^*_{i_1}, p^*_{i_2}, \ldots, p^*_{i_k}$: left-to-right order of edges on UE(P^*).

Hull-Envelope Correspondences

$$\mathrm{UH}(P) \longleftrightarrow \mathrm{LE}(P^*)$$

By symmetry,

$$\mathrm{LH}(P) \longleftrightarrow \mathrm{UE}(P^*)$$

Hull-Envelope Correspondences

$$\text{UH}(P) \longleftrightarrow \text{LE}(P^*)$$

By symmetry,

$$\text{LH}(P) \longleftrightarrow \text{UE}(P^*)$$

- The above relationships generalize to convex hulls and intersection of half-spaces in 3D.

Hull-Envelope Correspondences

$$\mathrm{UH}(P) \longleftrightarrow \mathrm{LE}(P^*)$$

By symmetry,

$$\mathrm{LH}(P) \longleftrightarrow \mathrm{UE}(P^*)$$

- The above relationships generalize to convex hulls and intersection of half-spaces in 3D.
- Convex hulls and intersections of half-planes (or half-spaces) are essentially dual concepts.

Hull-Envelope Correspondences

$$\text{UH}(P) \longleftrightarrow \text{LE}(P^*)$$

By symmetry,

$$\text{LH}(P) \longleftrightarrow \text{UE}(P^*)$$

- The above relationships generalize to convex hulls and intersection of half-spaces in 3D.
- Convex hulls and intersections of half-planes (or half-spaces) are essentially dual concepts.

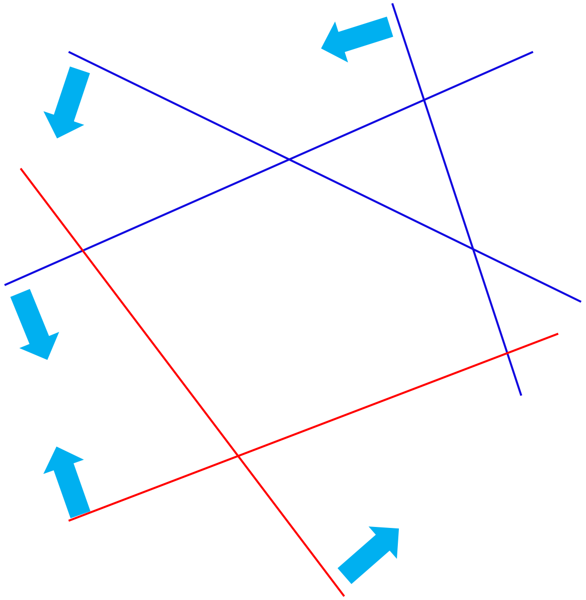
Computing an upper (lower) convex hull

$$\Updownarrow$$

Intersecting lower (upper) half-planes

Algorithm for Half-Plane Intersection

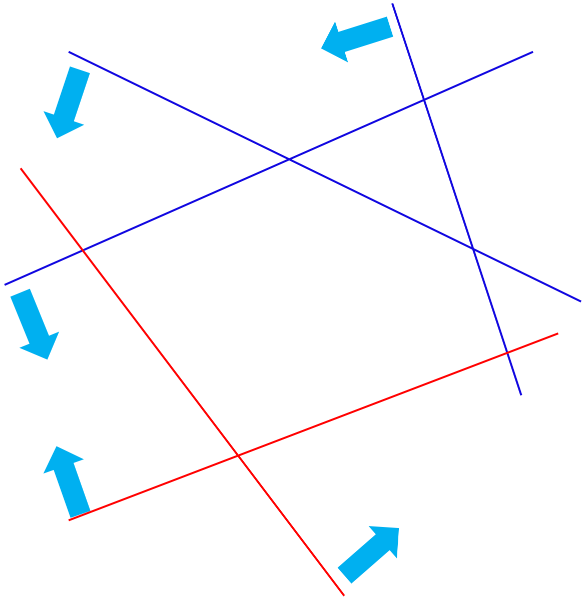
H: a set of half-planes



Idea: Dualize a convex hull algorithm.

Algorithm for Half-Plane Intersection

H: a set of half-planes

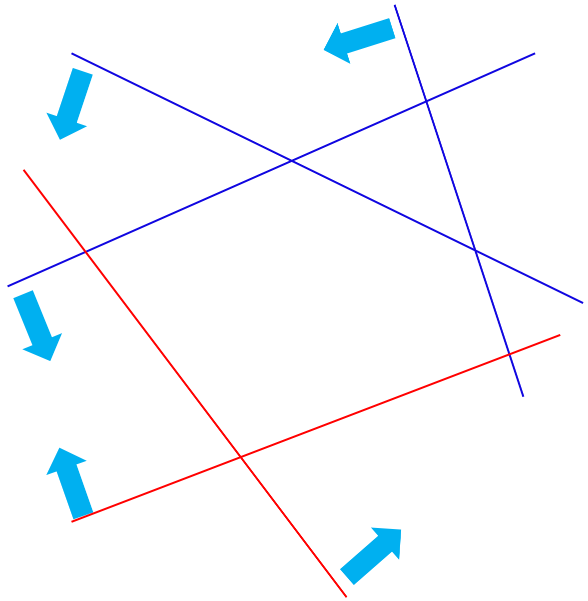


Idea: Dualize a convex hull algorithm.

a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes.

Algorithm for Half-Plane Intersection

H: a set of half-planes

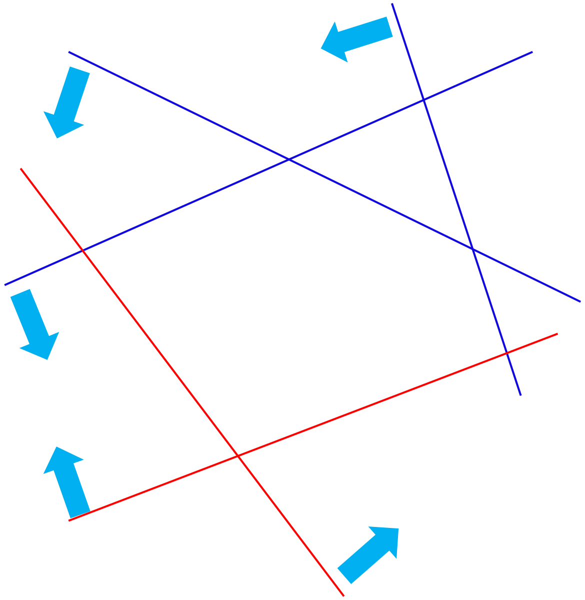


Idea: Dualize a convex hull algorithm.

a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes. $O(n)$

Algorithm for Half-Plane Intersection

H: a set of half-planes



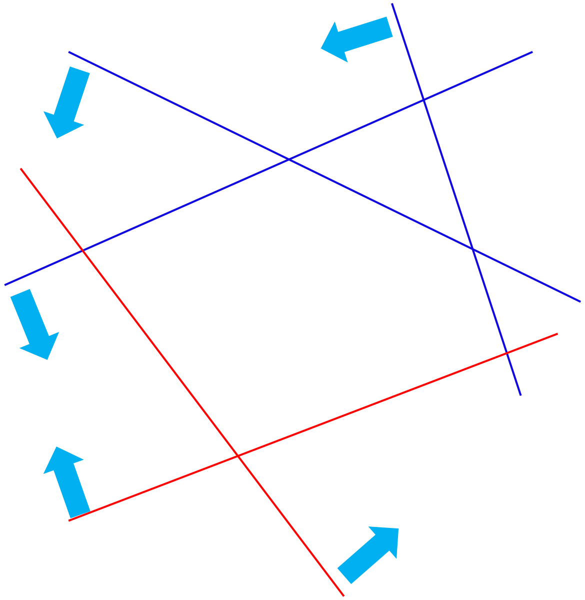
Idea: Dualize a convex hull algorithm.

a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes. $O(n)$

b) Compute $\cap\, H_+$ by constructing the lower convex hull of H_+^*.

Algorithm for Half-Plane Intersection

H: a set of half-planes



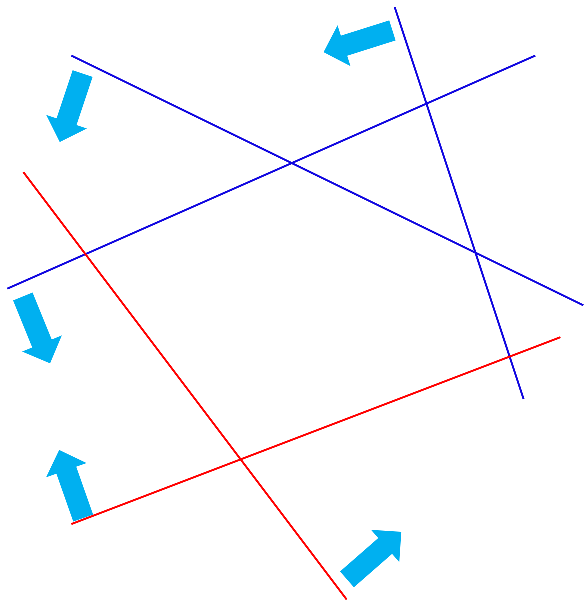
Idea: Dualize a convex hull algorithm.

a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes. $O(n)$

b) Compute $\cap\, H_+$ by constructing the lower convex hull of H_+^*. $O(n \log n)$

Algorithm for Half-Plane Intersection

H: a set of half-planes



Idea: Dualize a convex hull algorithm.

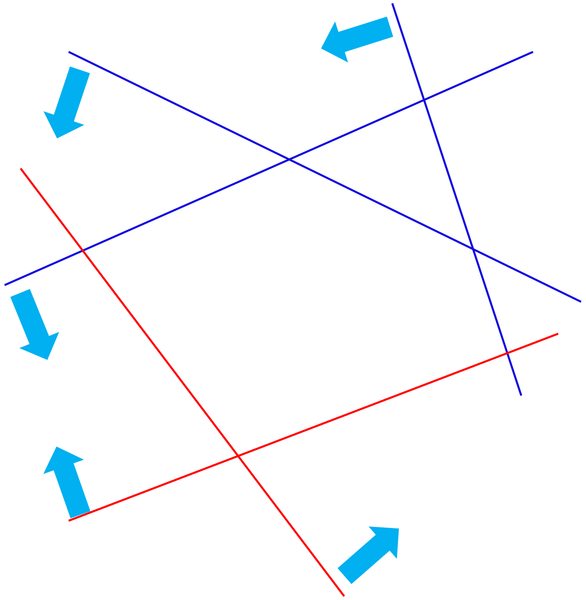
a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes. $O(n)$

b) Compute $\cap H_+$ by constructing the lower convex hull of H_+^*. $O(n \log n)$

c) Compute $\cap H_-$ by constructing the upper convex hull of H_-^*. $O(n \log n)$

Algorithm for Half-Plane Intersection

H: a set of half-planes



Idea: Dualize a convex hull algorithm.

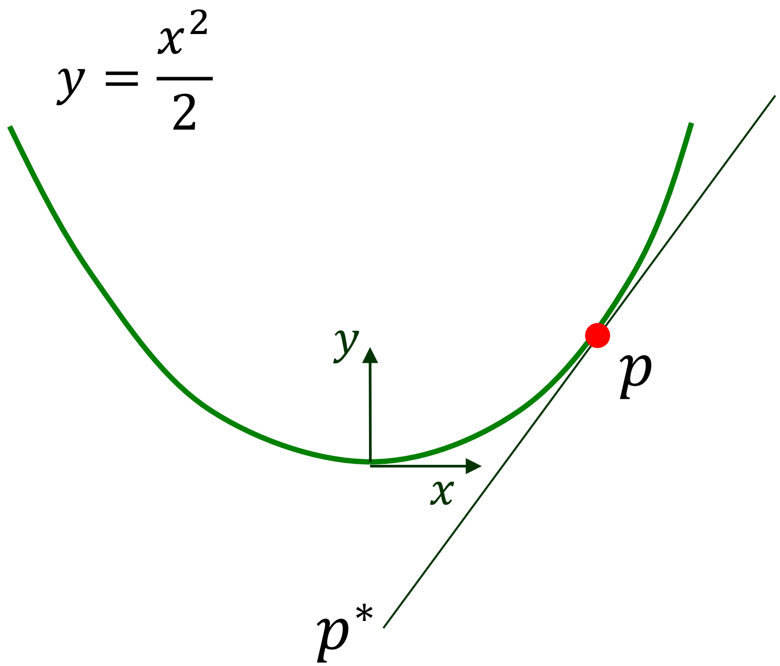
a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes. $O(n)$

b) Compute $\cap\, H_+$ by constructing the lower convex hull of H_+^*. $O(n \log n)$

c) Compute $\cap\, H_-$ by constructing the upper convex hull of H_-^*. $O(n \log n)$

d) Intersect H_+ and H_-. $O(n)$

IV. Review: Duality with a Parabola

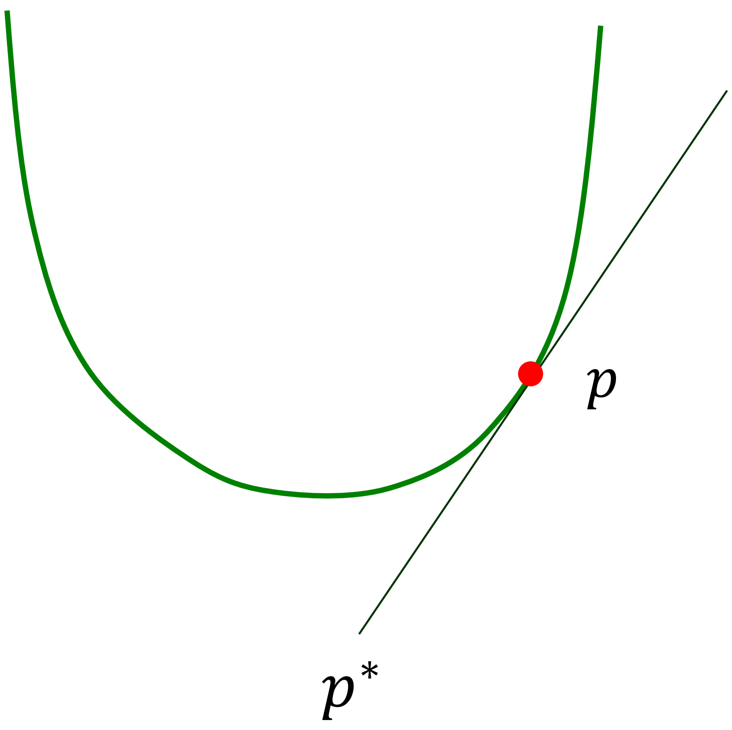


- Dual p^* of p on the parabola is the tangent line at p.

Point Not on a Parabola

$$y = \frac{x^2}{2}$$

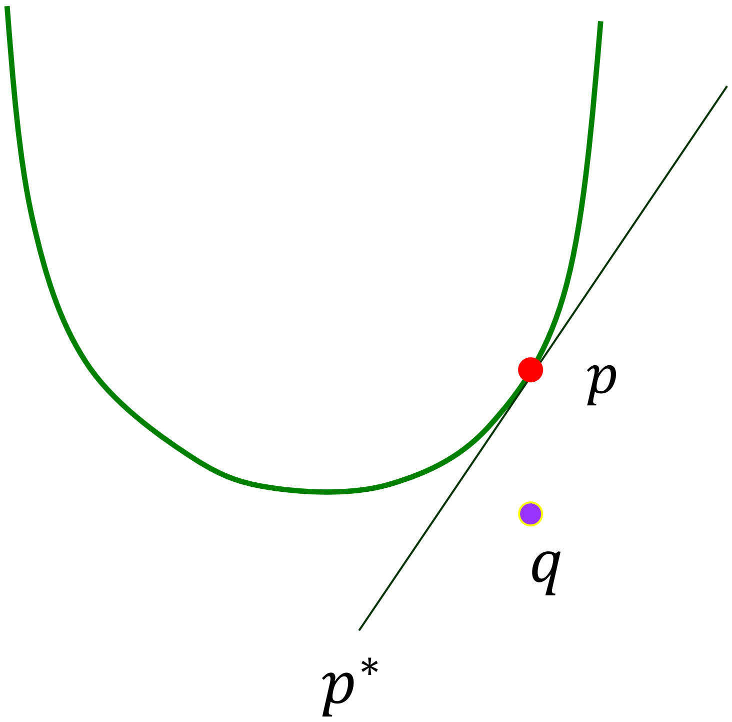
$$p = (p_x, p_y)$$



Point Not on a Parabola

$$y = \frac{x^2}{2}$$

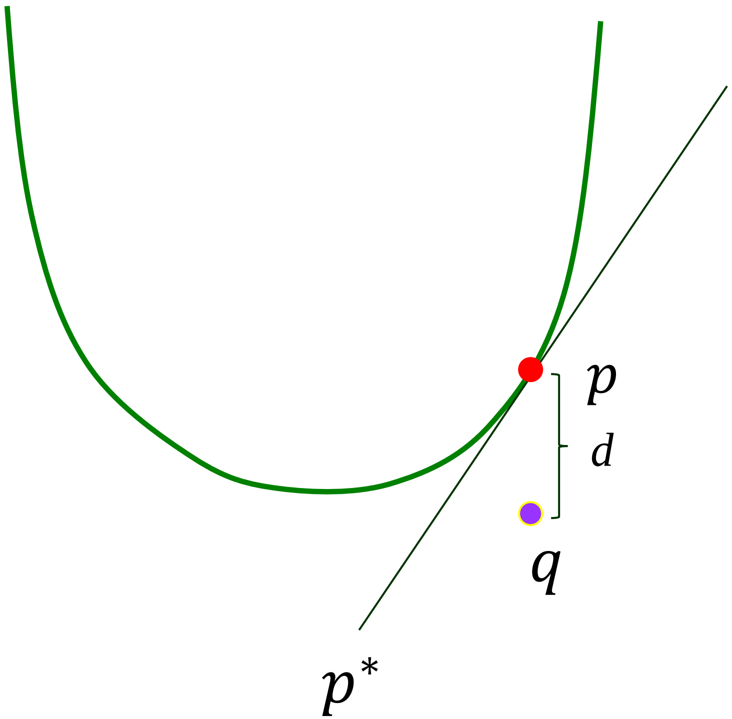
$$p = (p_x, p_y)$$



Point Not on a Parabola

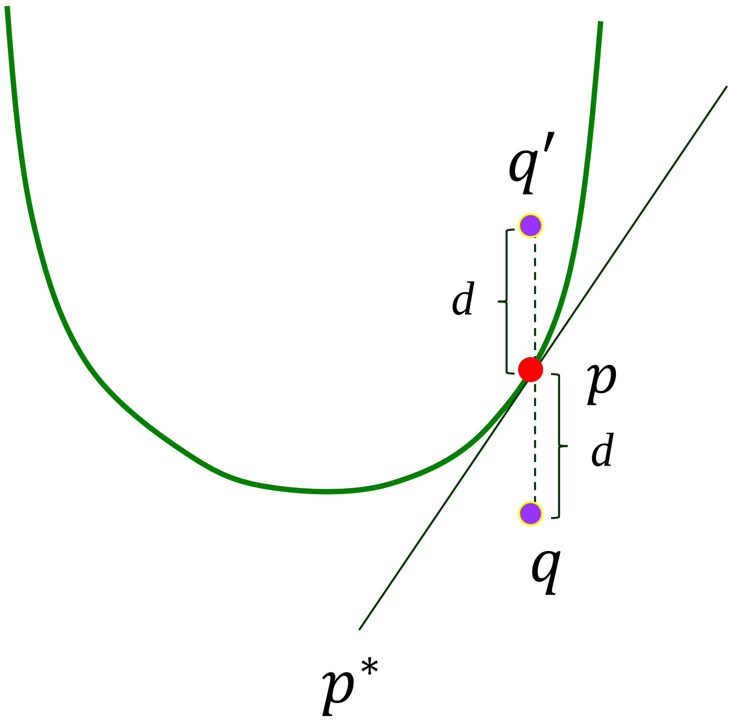
$$y = \frac{x^2}{2}$$

$$p = (p_x, p_y)$$



Point Not on a Parabola

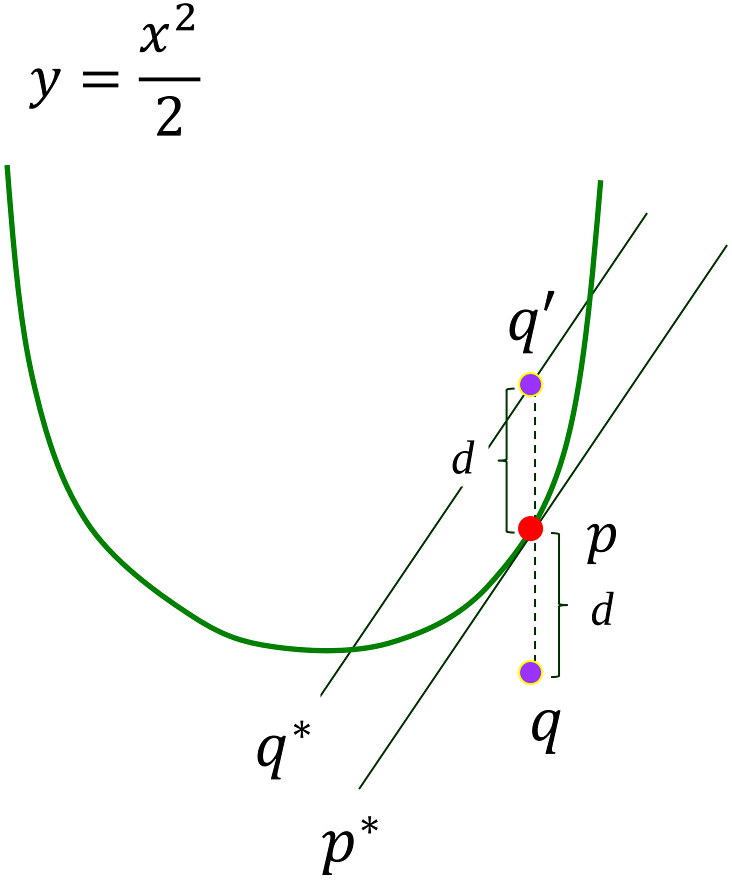
$$y = \frac{x^2}{2}$$



$$p = (p_x, p_y)$$

$$q = (p_x, p_y - d)$$

Point Not on a Parabola

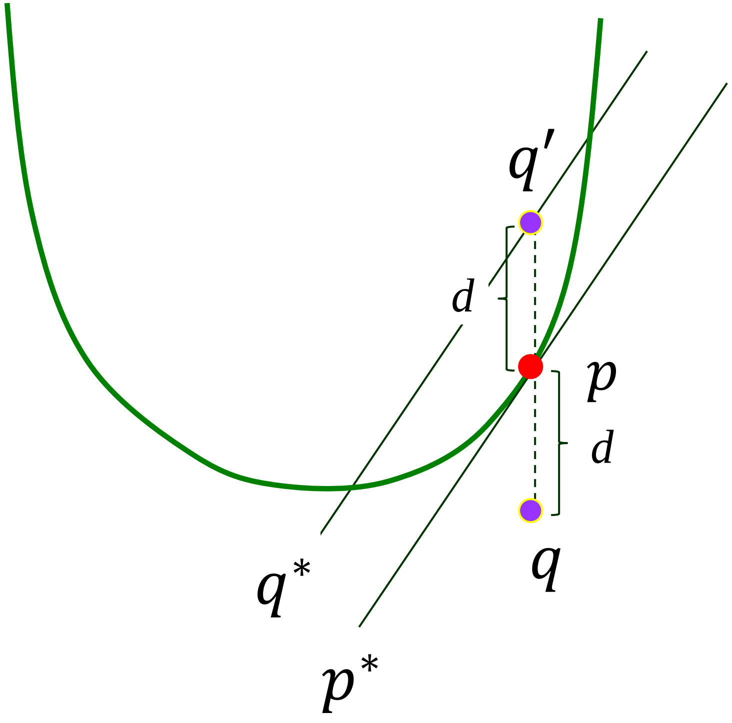


$p = (p_x, p_y)$

$q = (p_x, p_y - d)$

Point Not on a Parabola

$$y = \frac{x^2}{2}$$



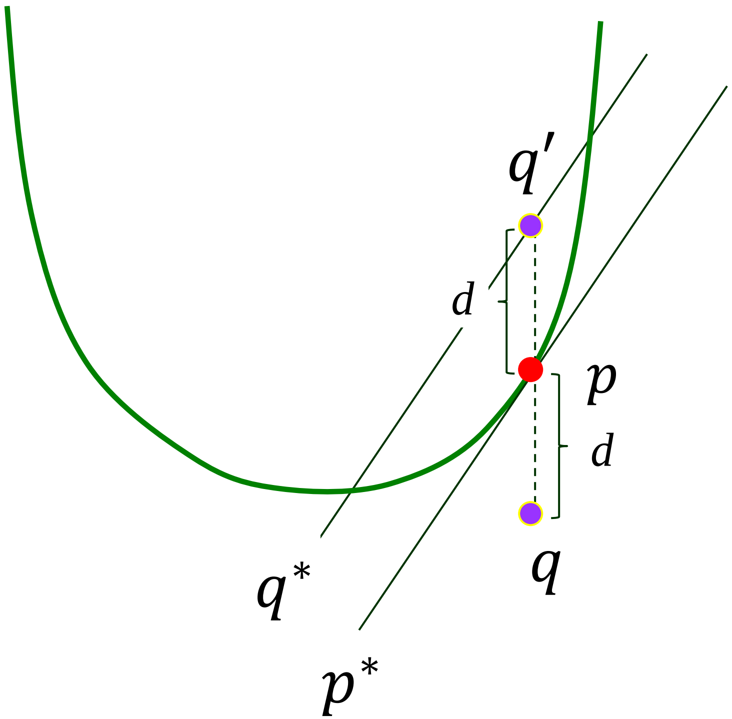
$$p = (p_x, p_y)$$

$$q = (p_x, p_y - d)$$

$$q' = (p_x, p_y + d)$$

Point Not on a Parabola

$$y = \frac{x^2}{2}$$



$$p = (p_x, p_y)$$

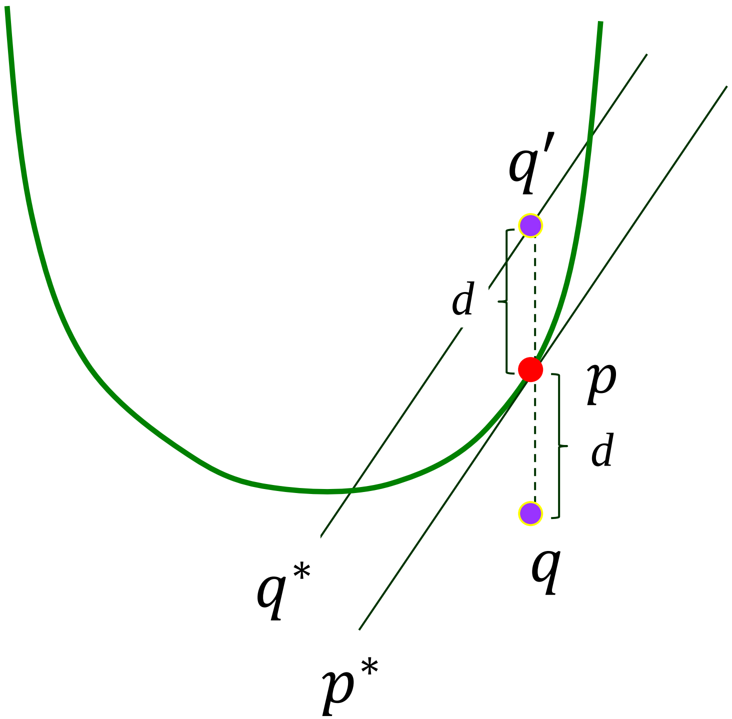
$$q = (p_x, p_y - d)$$

$$q' = (p_x, p_y + d)$$

$$q' - p = p - q$$

Point Not on a Parabola

$$y = \frac{x^2}{2}$$



$$p = (p_x, p_y)$$

$$q = (p_x, p_y - d)$$

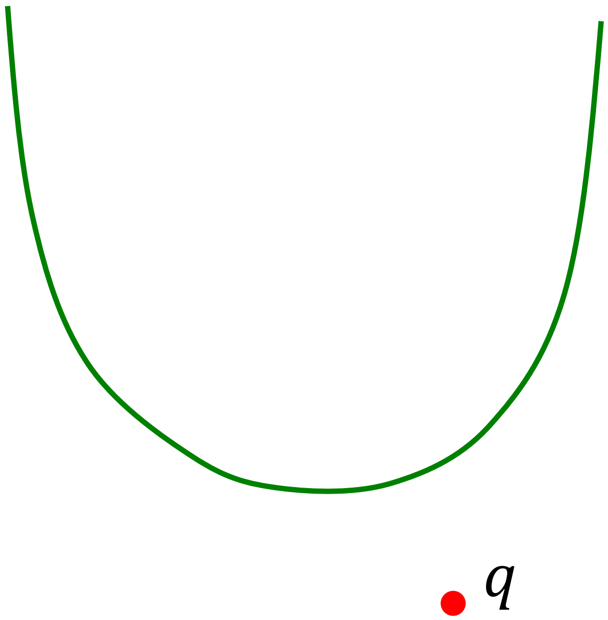
$$q' = (p_x, p_y + d)$$

$$q' - p = p - q$$

- The dual line $q^* \parallel p^*$ and it passes through q'.

More on Duality

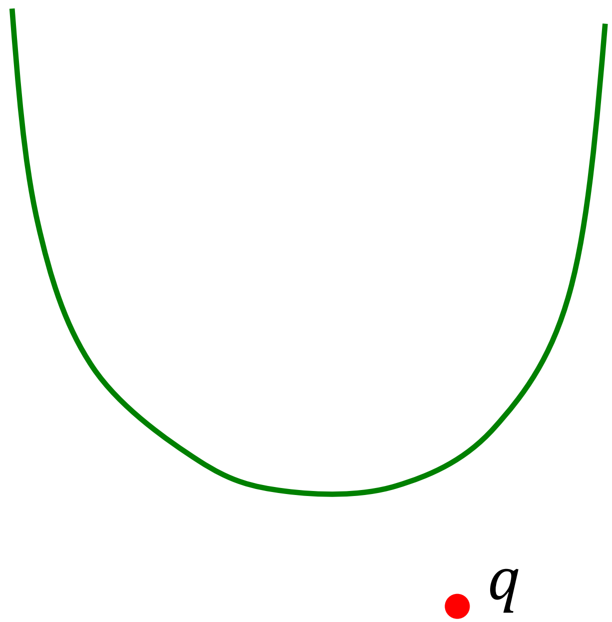
$$y = \frac{x^2}{2}$$



Construct the dual line q^* of q without measuring distances:

More on Duality

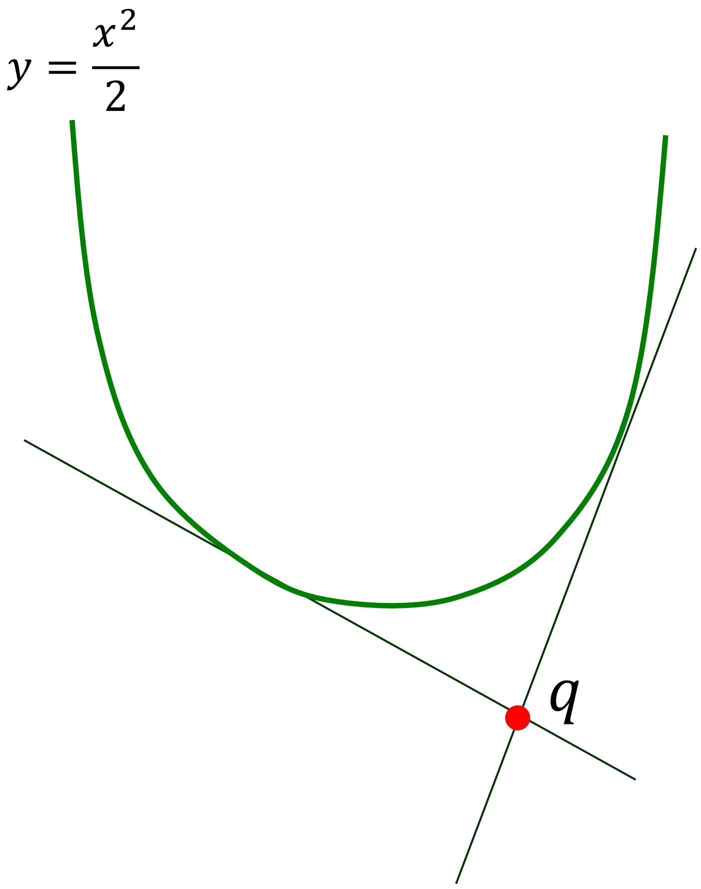
$$y = \frac{x^2}{2}$$



Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

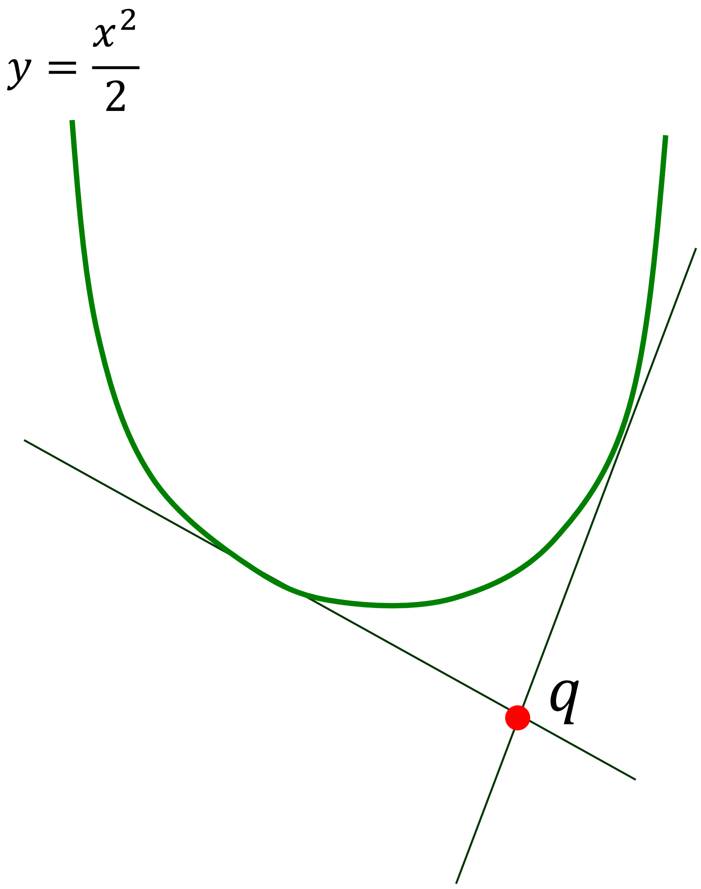
More on Duality



Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

More on Duality

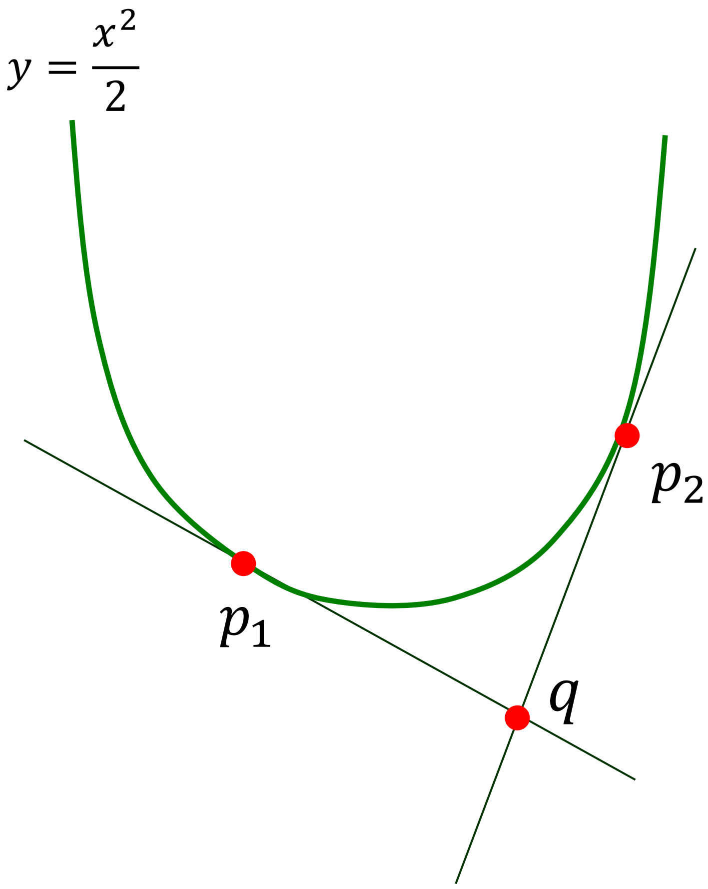


Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

More on Duality

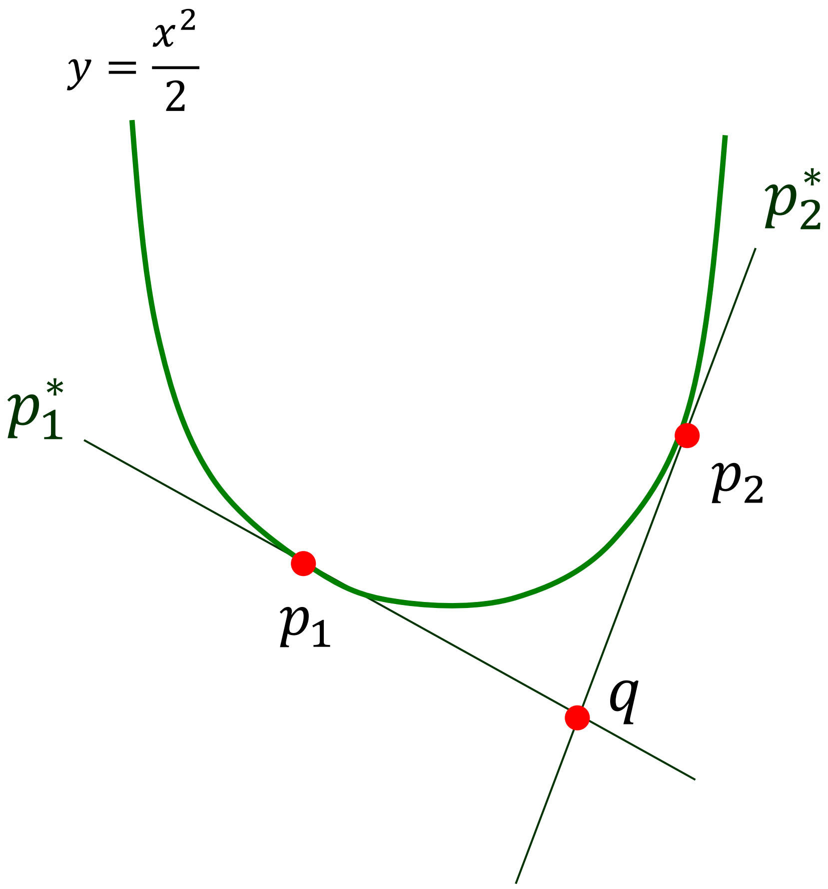


Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

More on Duality



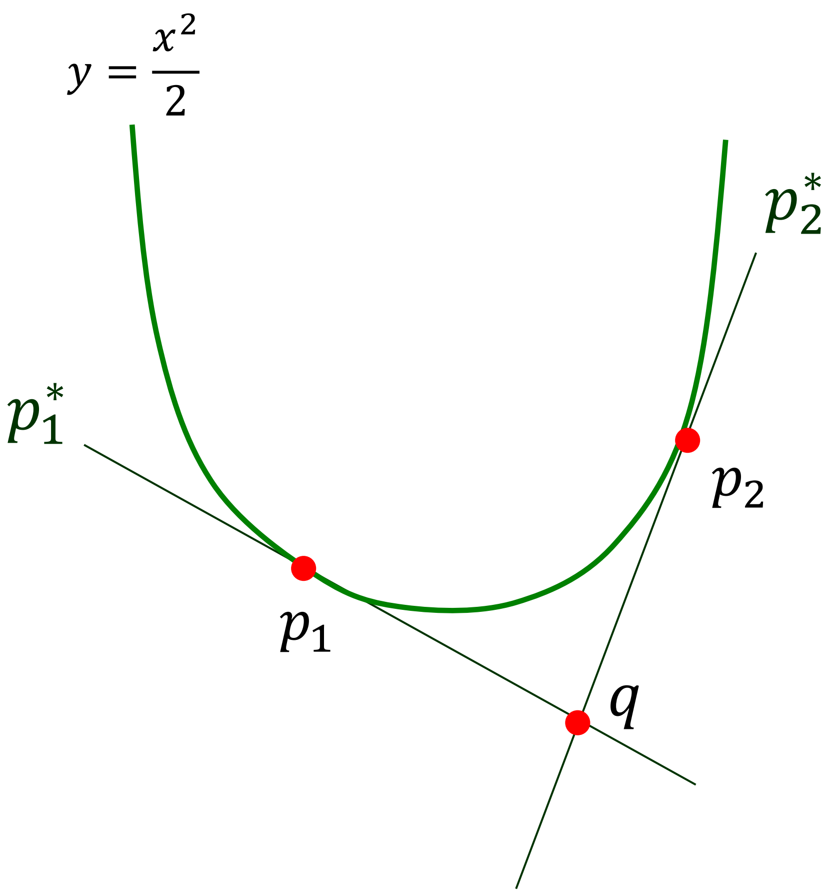
Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

The two tangent lines are p_1^* and p_2^*.

More on Duality



Construct the dual line q^* of q without measuring distances:

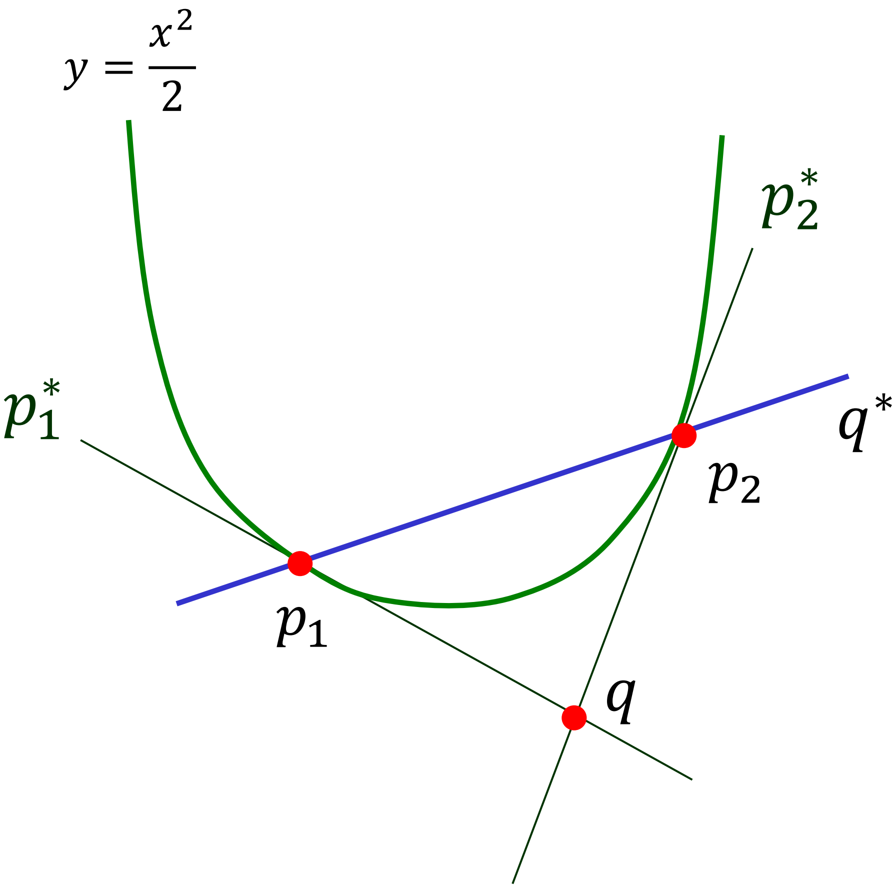
1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

3) q^* is the line through p_1 and p_2.

The two tangent lines are p_1^* and p_2^*.

More on Duality



Construct the dual line q^* of q without measuring distances:

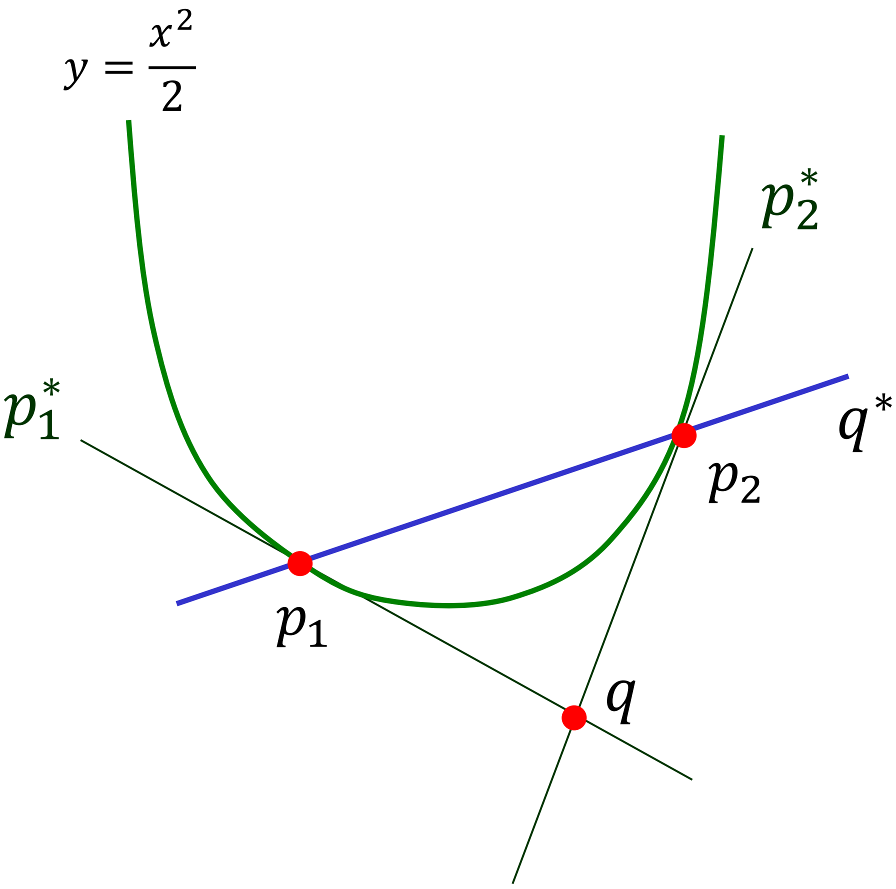
1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

3) q^* is the line through p_1 and p_2.

The two tangent lines are p_1^* and p_2^*.

More on Duality



Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

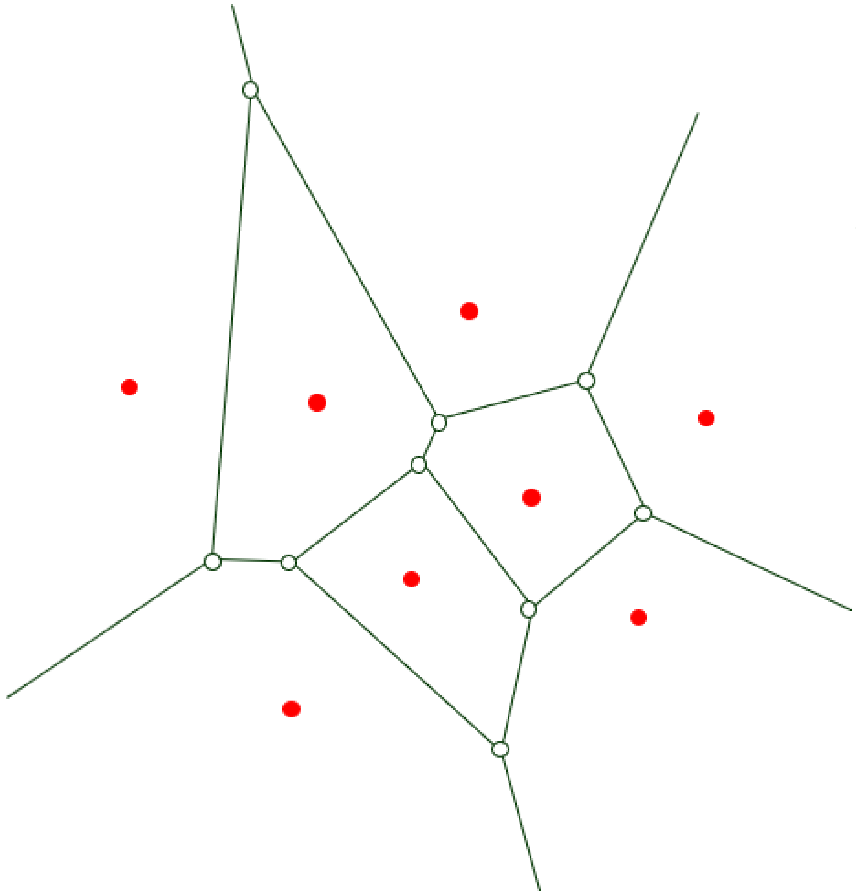
3) q^* is the line through p_1 and p_2.

The two tangent lines are p_1^* and p_2^*.

p_1^* and p_2^* intersect at $q \Leftrightarrow q^*$ passes through p_1 and p_2.

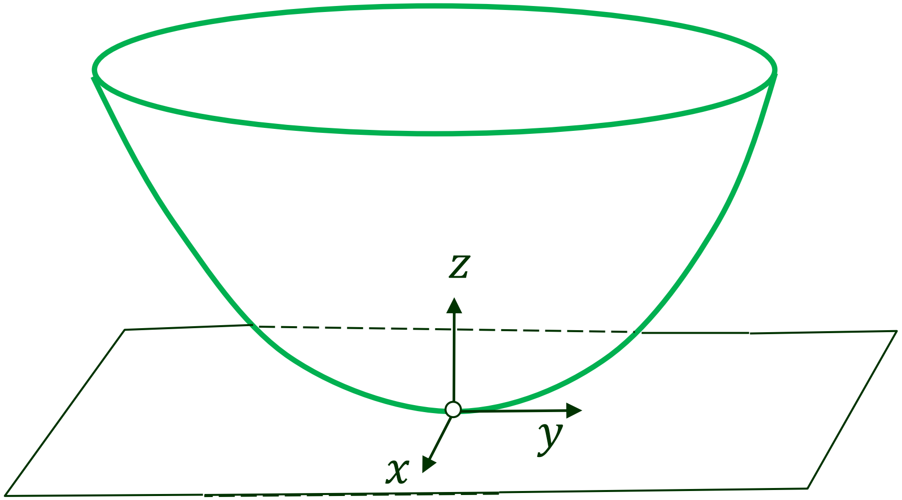
Voronoi Diagram Revisited

P: a set of n sites.



Unit Paraboloid

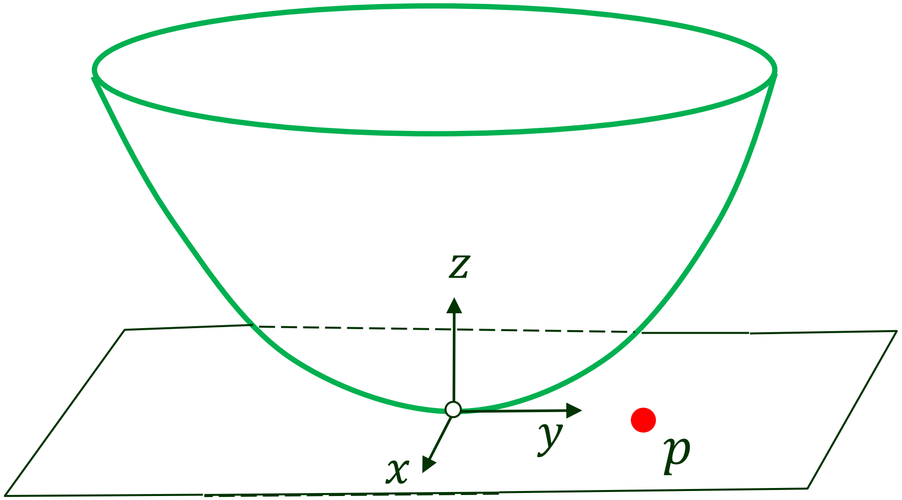
$U: z = x^2 + y^2$



Unit Paraboloid

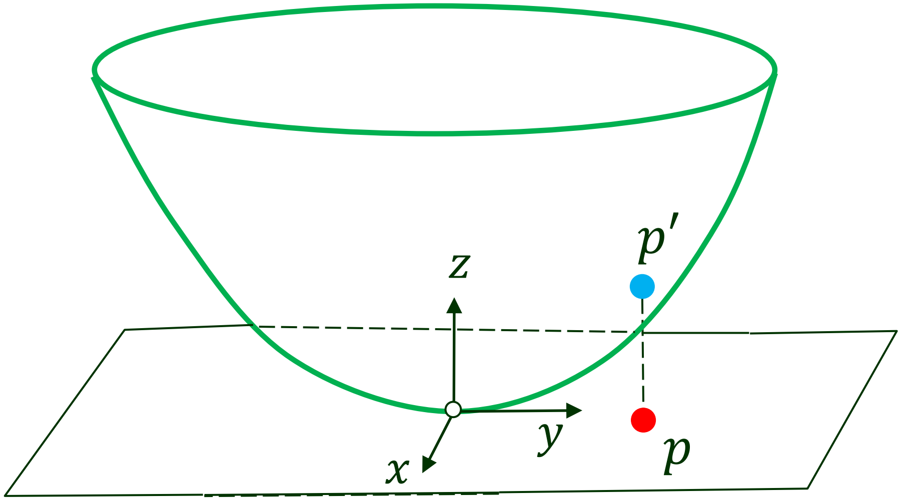
$p = (p_x, p_y, 0)$

$U: z = x^2 + y^2$



Unit Paraboloid

$U: z = x^2 + y^2$



$p = (p_x, p_y, 0)$

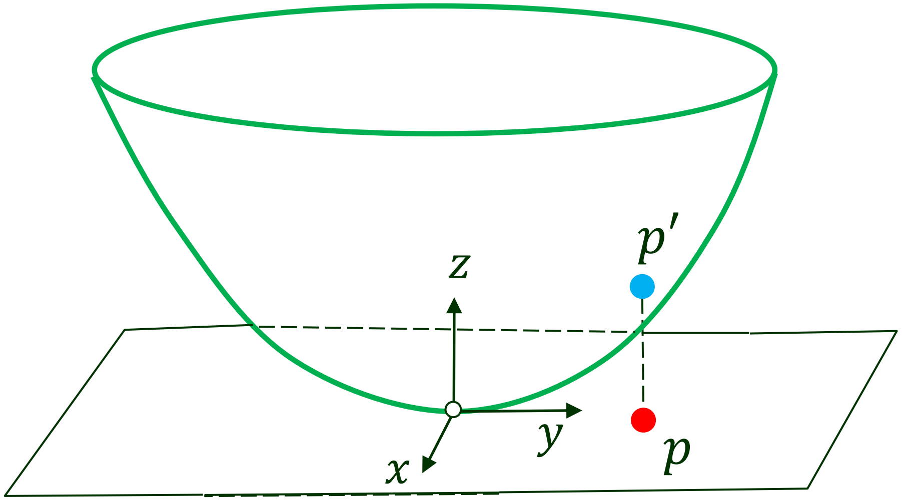
Projection of p onto U:

$p' = (p_x, p_y, p_x^2 + p_y^2)$:

Unit Paraboloid

$U\colon z = x^2 + y^2$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



$p = (p_x, p_y, 0)$

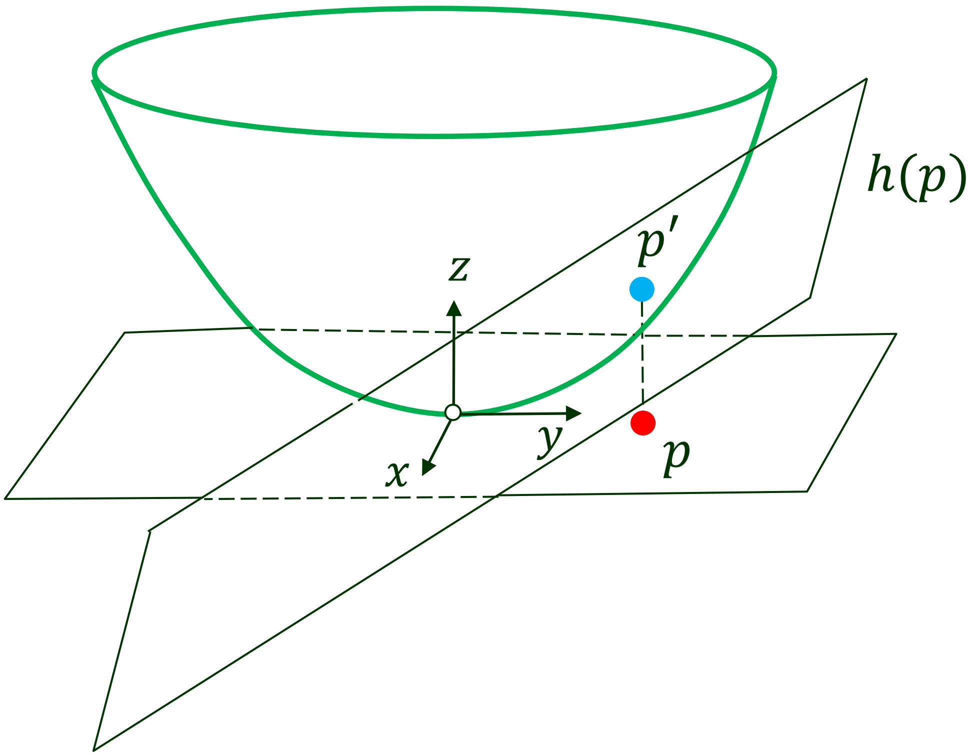
Projection of p onto U:

$p' = (p_x, p_y, p_x^2 + p_y^2)$:

Unit Paraboloid

$$U: z = x^2 + y^2$$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



$$p = (p_x, p_y, 0)$$

Projection of p onto U:

$$p' = (p_x, p_y, p_x^2 + p_y^2):$$

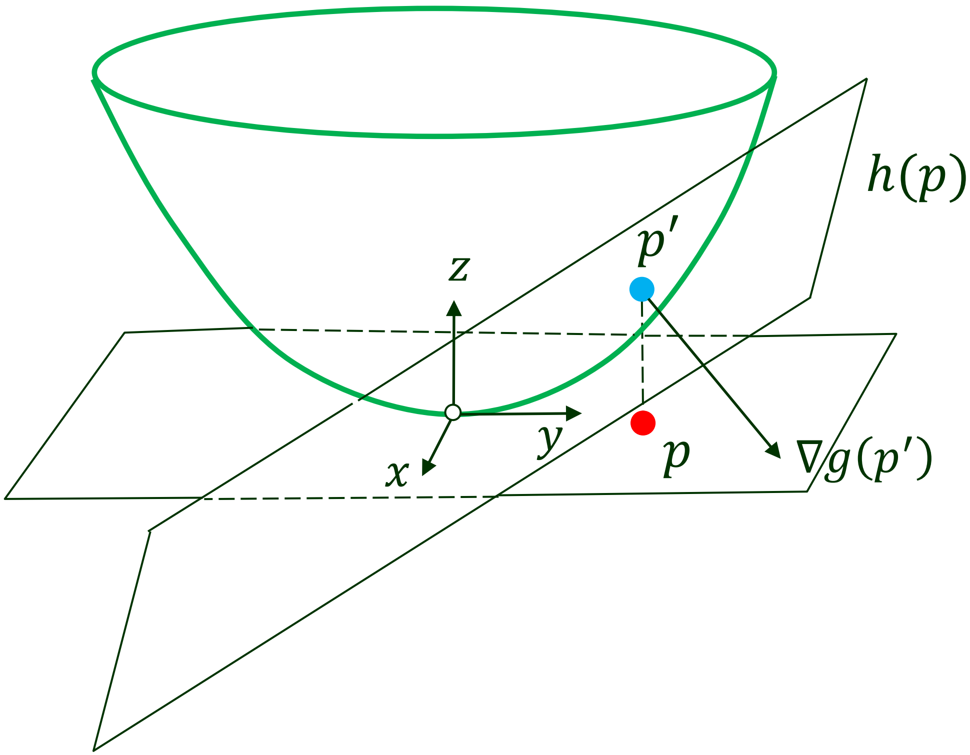
Tangent plane $h(p)$ to U through p' has normal:

$$\nabla g(p') = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)\Big|_{p'} = (2p_x, 2p_y, -1)$$

Unit Paraboloid

$$U: z = x^2 + y^2$$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



$$p = (p_x, p_y, 0)$$

Projection of p onto U:

$$p' = (p_x, p_y, p_x^2 + p_y^2):$$

Tangent plane $h(p)$ to U through p' has normal:

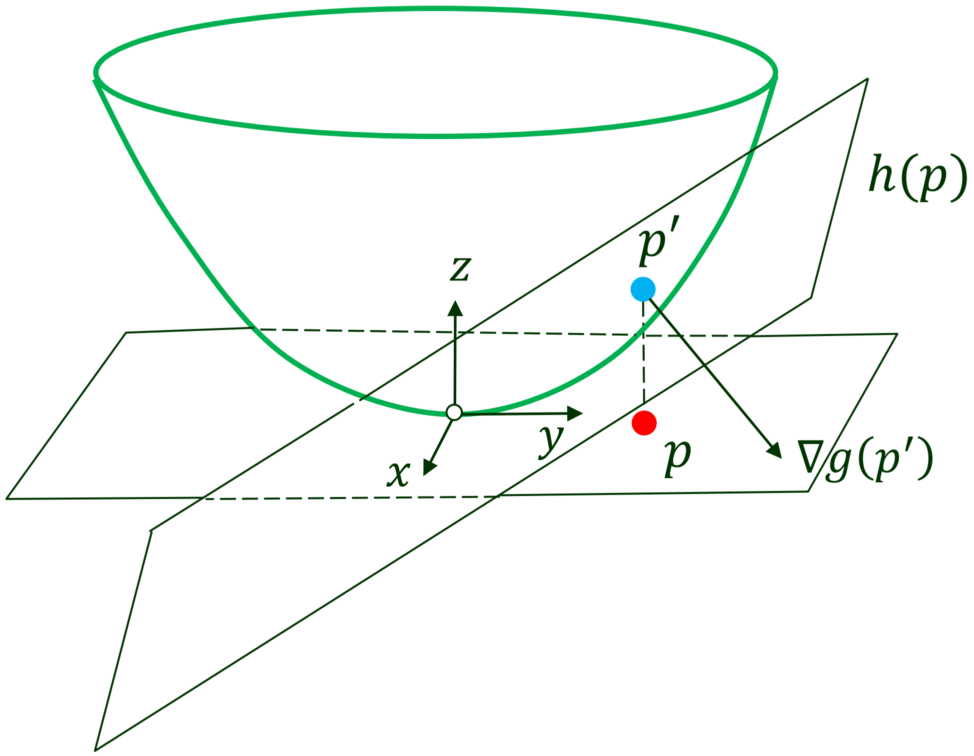
$$\nabla g(p') = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) \Big|_{p'} = (2p_x, 2p_y, -1)$$

Unit Paraboloid

$p = (p_x, p_y, 0)$

$U: z = x^2 + y^2$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



Projection of p onto U:

$p' = (p_x, p_y, p_x^2 + p_y^2)$:

Tangent plane $h(p)$ to U through p' has normal:

$$\nabla g(p') = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)\Big|_{p'} = (2p_x, 2p_y, -1)$$

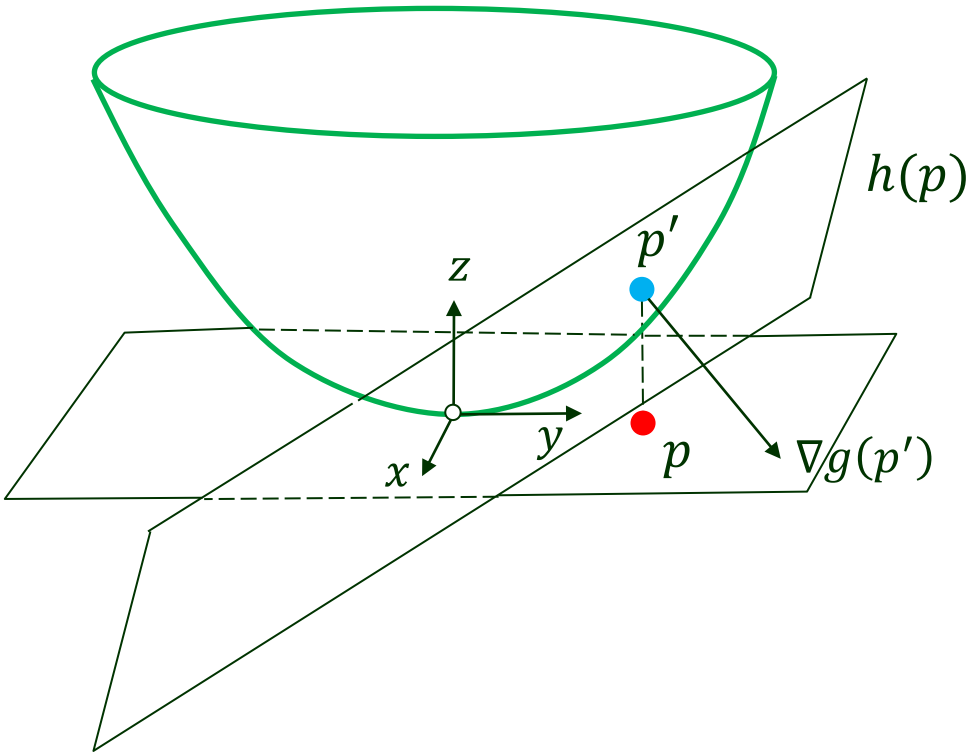
$$\Downarrow$$

$$\big((x, y, z) - p'\big) \cdot \nabla g(p') = 0$$

Unit Paraboloid

$U: z = x^2 + y^2$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



$p = (p_x, p_y, 0)$

Projection of p onto U:

$p' = (p_x, p_y, p_x^2 + p_y^2)$:

Tangent plane $h(p)$ to U through p' has normal:

$$\nabla g(p') = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)\Bigg|_{p'} = (2p_x, 2p_y, -1)$$

$$\Downarrow$$

$$\left((x, y, z) - p'\right) \cdot \nabla g(p') = 0$$

$$\Downarrow$$

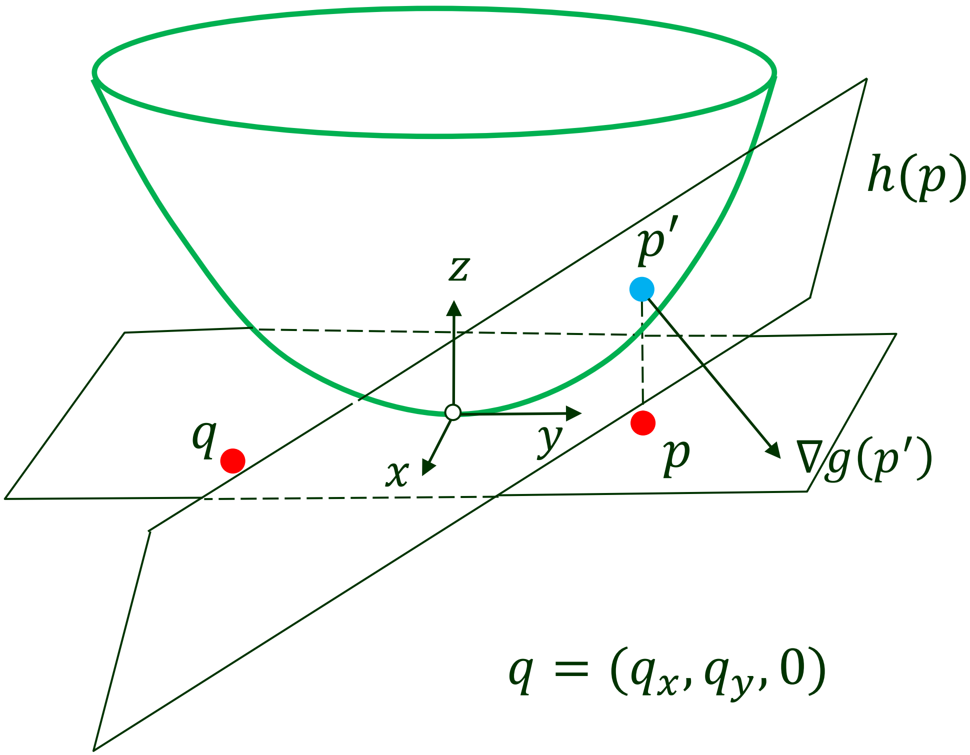
$$h(p):\ z = 2p_x x + 2p_y y - (p_x^2 + p_y^2)$$

Unit Paraboloid

$$p = (p_x, p_y, 0)$$

$$U: z = x^2 + y^2$$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



$$q = (q_x, q_y, 0)$$

Projection of p onto U:

$$p' = (p_x, p_y, p_x^2 + p_y^2):$$

Tangent plane $h(p)$ to U through p' has normal:

$$\nabla g(p') = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)\Bigg|_{p'} = (2p_x, 2p_y, -1)$$

$$\Downarrow$$

$$\big((x, y, z) - p'\big) \cdot \nabla g(p') = 0$$

$$\Downarrow$$

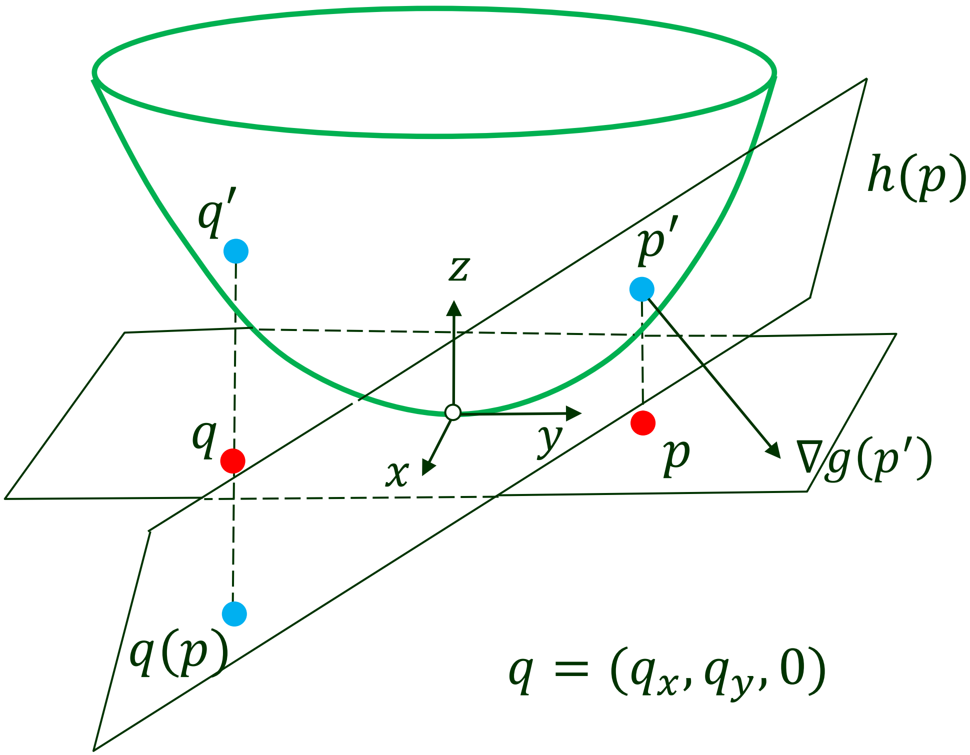
$$h(p):\ z = 2p_x x + 2p_y y - (p_x^2 + p_y^2)$$

Unit Paraboloid

$p = (p_x, p_y, 0)$

$U: z = x^2 + y^2$

i.e. $g(x,y,z) \equiv x^2 + y^2 - z = 0$



$q = (q_x, q_y, 0)$

Projection of p onto U:

$p' = (p_x, p_y, p_x^2 + p_y^2)$:

Tangent plane $h(p)$ to U through p' has normal:

$$\nabla g(p') = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)\Big|_{p'} = (2p_x, 2p_y, -1)$$

$$\Downarrow$$

$$\big((x,y,z) - p'\big) \cdot \nabla g(p') = 0$$

$$\Downarrow$$

$$h(p):\ z = 2p_x x + 2p_y y - (p_x^2 + p_y^2)$$

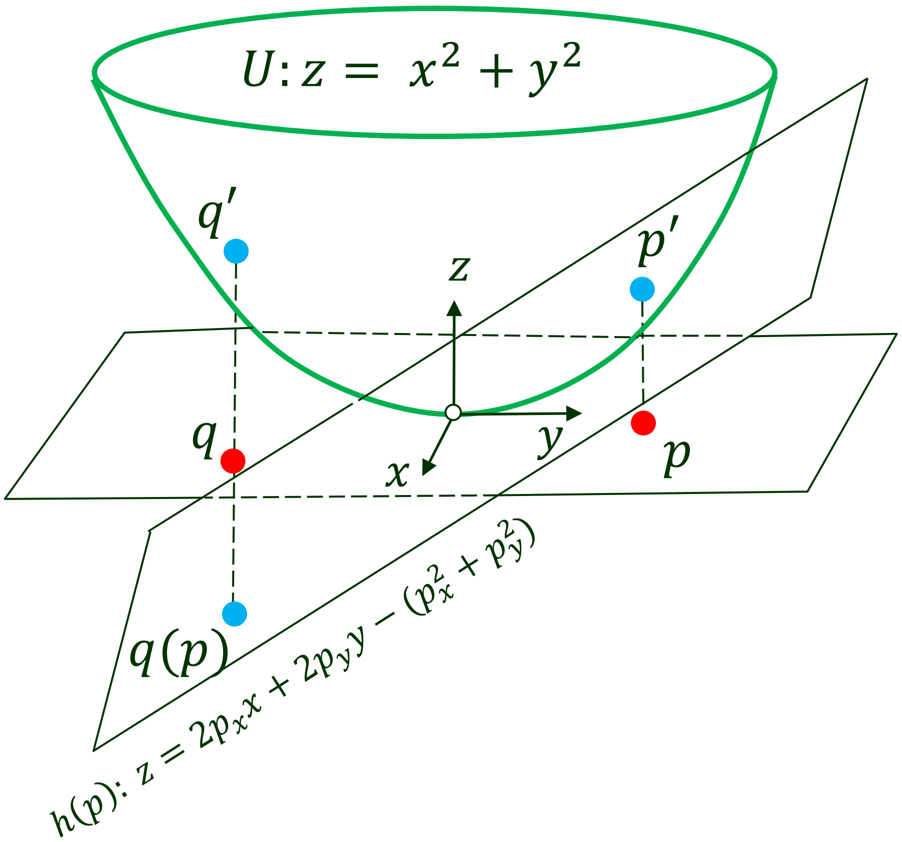
Vertical line through q intersects

- U at $q' = (q_x, q_y, q_x^2 + q_y^2)$
- $h(p)$ at $q(p)$.

Distance Encoded in Tangent Plane

$d(p, q)$: distance between two points p and q

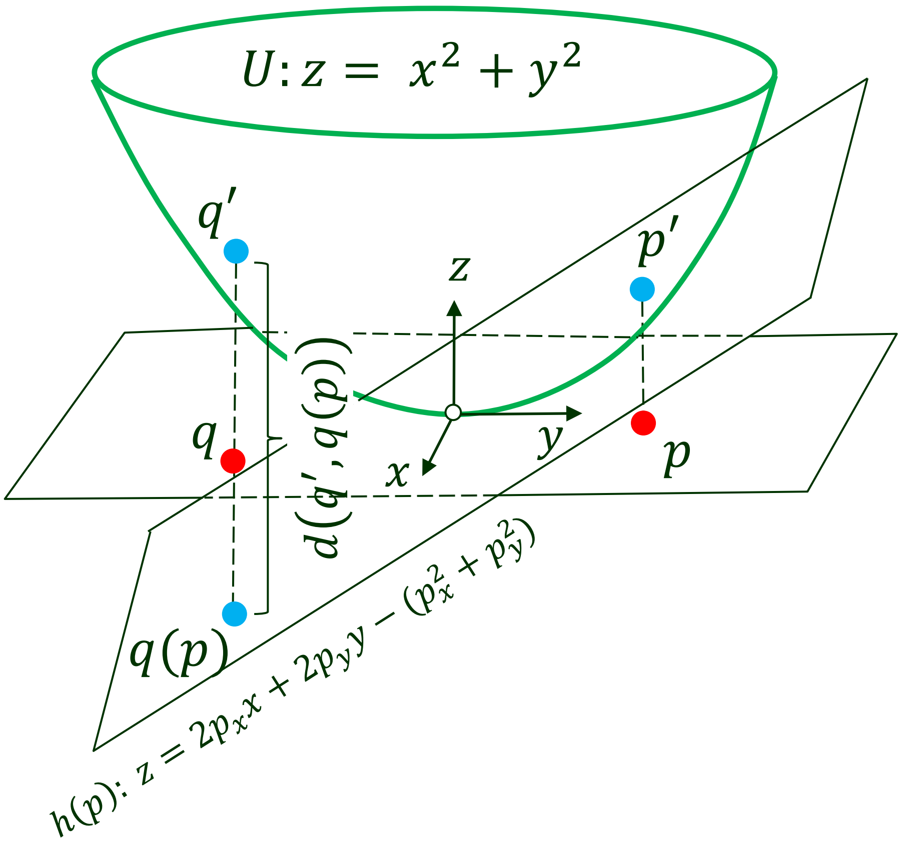
$$q' = (q_x, q_y, q_x^2 + q_y^2)$$



Distance Encoded in Tangent Plane

$d(p,q)$: distance between two points p and q

$$q' = (q_x, q_y, q_x^2 + q_y^2)$$

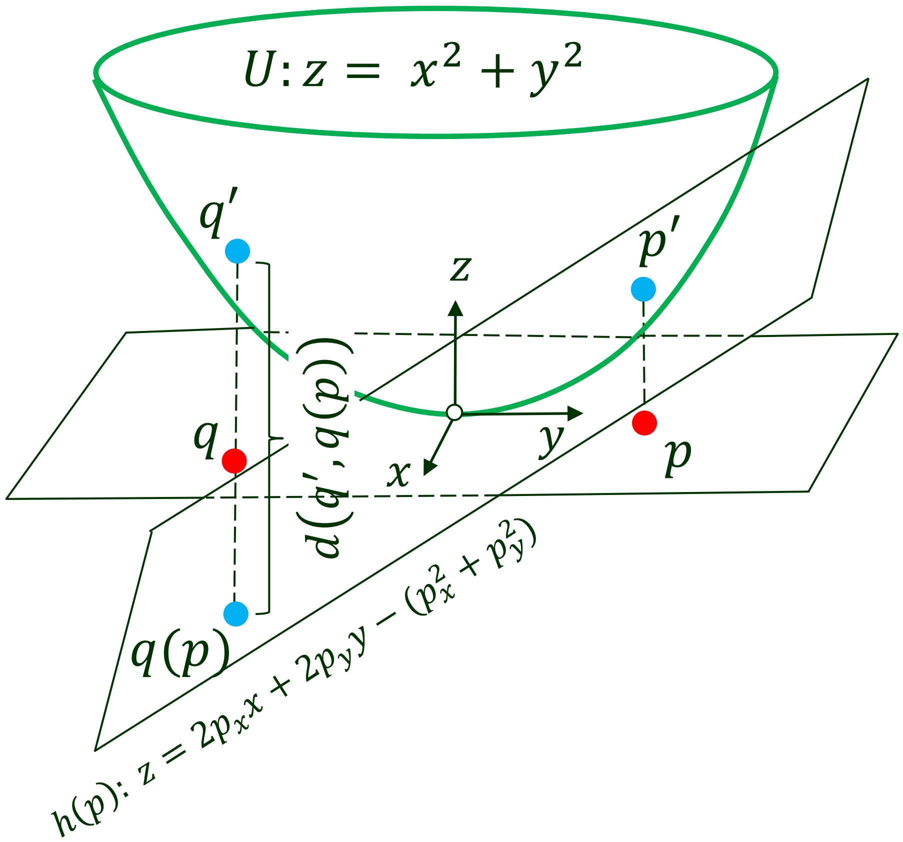


Distance Encoded in Tangent Plane

$d(p,q)$: distance between two points p and q

$$q' = (q_x, q_y, q_x^2 + q_y^2)$$

$$d\big(q', q(p)\big) = q'_z - \big(q(p)\big)_z$$

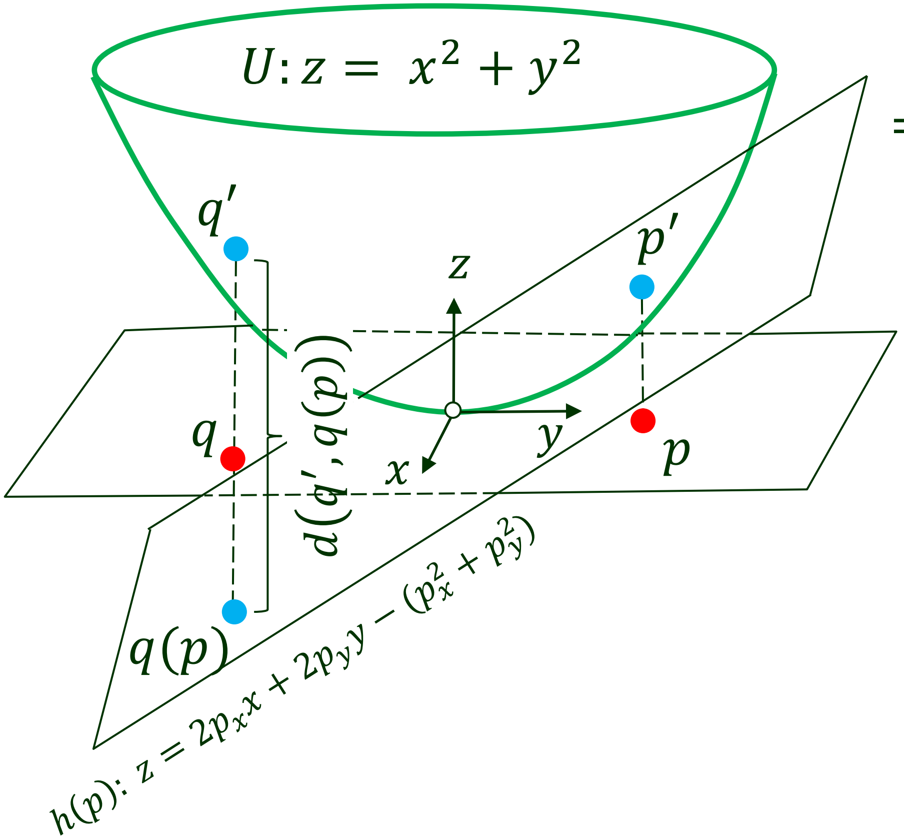


Distance Encoded in Tangent Plane

$d(p, q)$: distance between two points p and q

$$q' = (q_x, q_y, q_x^2 + q_y^2)$$

$$d\big(q', q(p)\big) = q'_z - \big(q(p)\big)_z$$
$$= \big(q_x^2 + q_y^2\big) - \big(2p_x q_x + 2p_y q_y - (p_x^2 + p_y^2)\big)$$



Distance Encoded in Tangent Plane

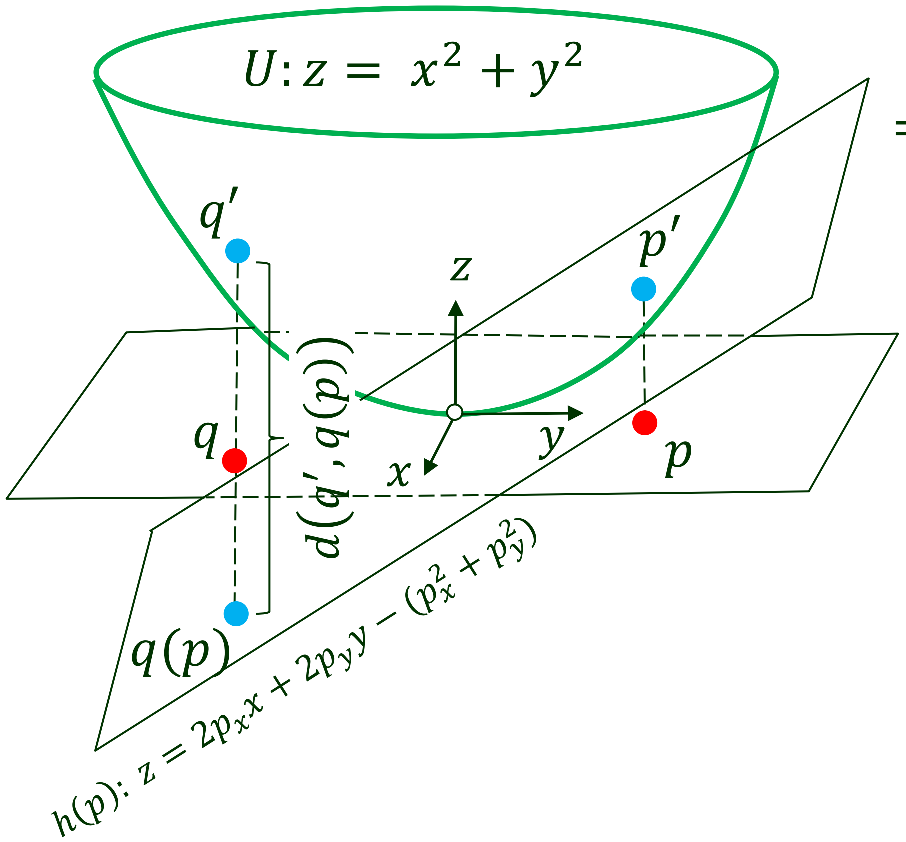
$d(p,q)$: distance between two points p and q

$$q' = (q_x, q_y, q_x^2 + q_y^2)$$

$$d\big(q', q(p)\big) = q'_z - \big(q(p)\big)_z$$

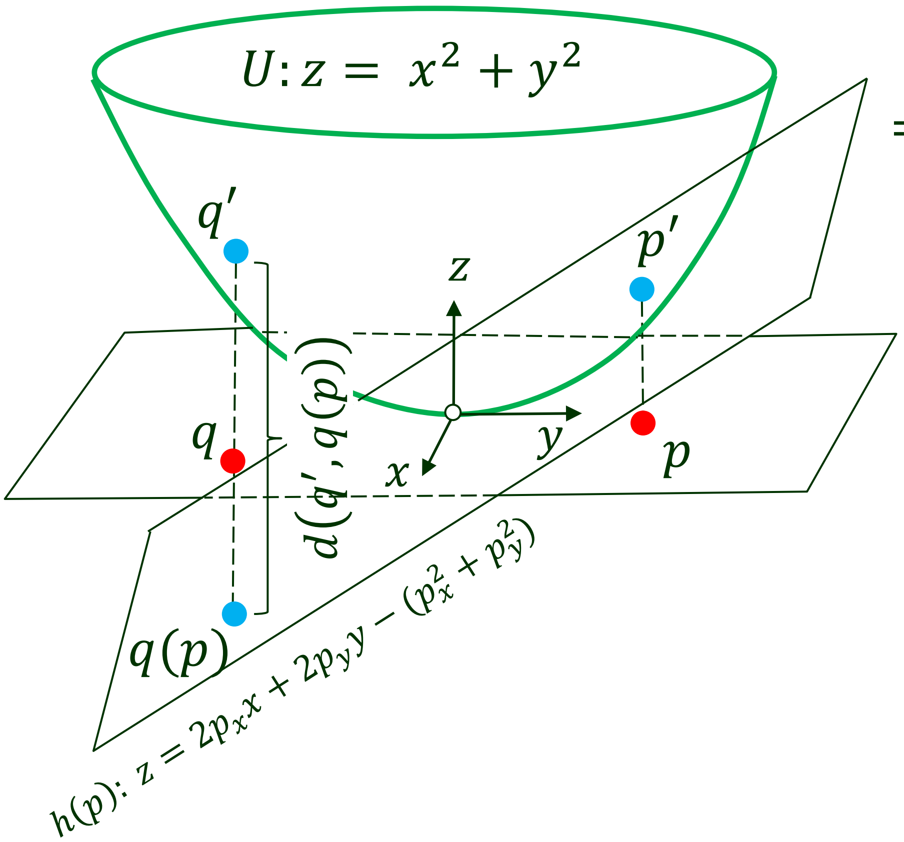
$$= \left(q_x^2 + q_y^2\right) - \left(2p_x q_x + 2p_y q_y - (p_x^2 + p_y^2)\right)$$

$$= (q_x - p_x)^2 + (q_y - p_y)^2$$



Distance Encoded in Tangent Plane

$d(p,q)$: distance between two points p and q



$$q' = (q_x, q_y, q_x^2 + q_y^2)$$

$$d\big(q', q(p)\big) = q'_z - \big(q(p)\big)_z$$

$$= \left(q_x^2 + q_y^2\right) - \left(2p_x q_x + 2p_y q_y - (p_x^2 + p_y^2)\right)$$

$$= (q_x - p_x)^2 + (q_y - p_y)^2$$

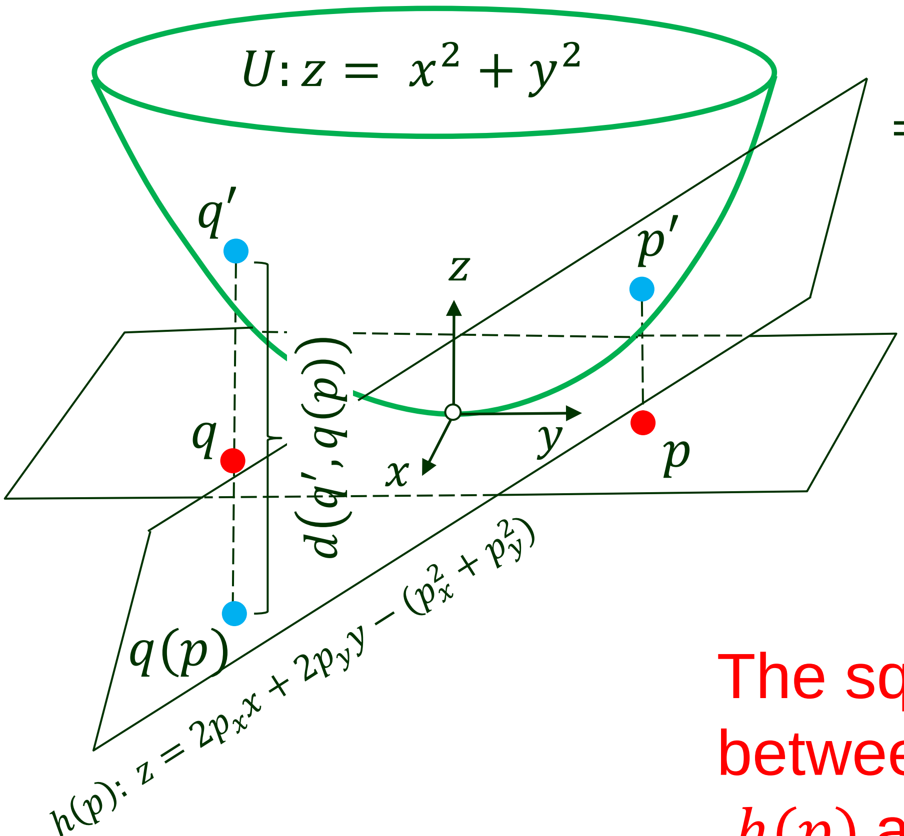
$$= d(p,q)^2$$

Distance Encoded in Tangent Plane

$d(p,q)$: distance between two points p and q

$$q' = (q_x, q_y, q_x^2 + q_y^2)$$

$$d\big(q', q(p)\big) = q'_z - \big(q(p)\big)_z$$
$$= \big(q_x^2 + q_y^2\big) - \big(2p_x q_x + 2p_y q_y - (p_x^2 + p_y^2)\big)$$
$$= (q_x - p_x)^2 + (q_y - p_y)^2$$
$$= d(p,q)^2$$



The square of $d(p,q)$ equals the distance between the two projection points (onto $h(p)$ and U) from q.

Upper Envelope of Planes

$H = \{$ tangent plane $h(p) \mid p \in P\}$

UE(H): upper envelope of the planes in H.

Theorem 1 The projection of UE(H) onto the plane $z = 0$ is the Voronoi diagram of P.

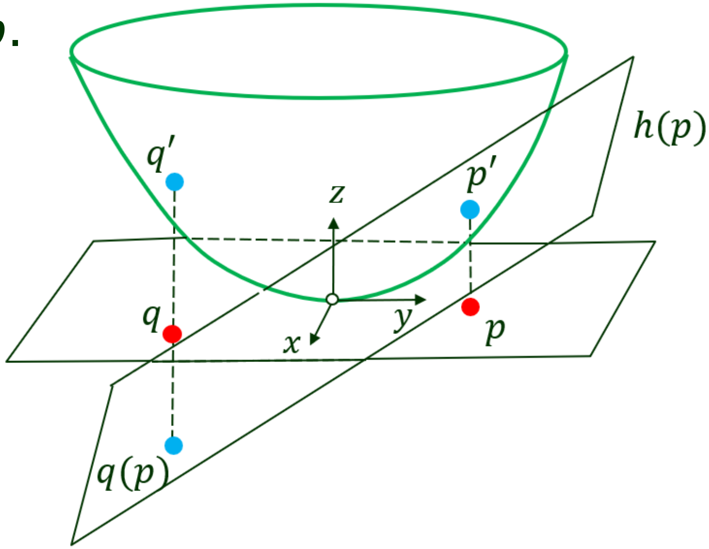
Upper Envelope of Planes

H ={ tangent plane $h(p)$ | $p \in P$}

UE(H): upper envelope of the planes in H.

Theorem 1 The projection of UE(H) onto the plane $z = 0$ is the Voronoi diagram of P.

Proof A point $q \in$ Vor(p), the Voronoi cell of p.



Upper Envelope of Planes

H ={ tangent plane $h(p)$ | $p \in P$}

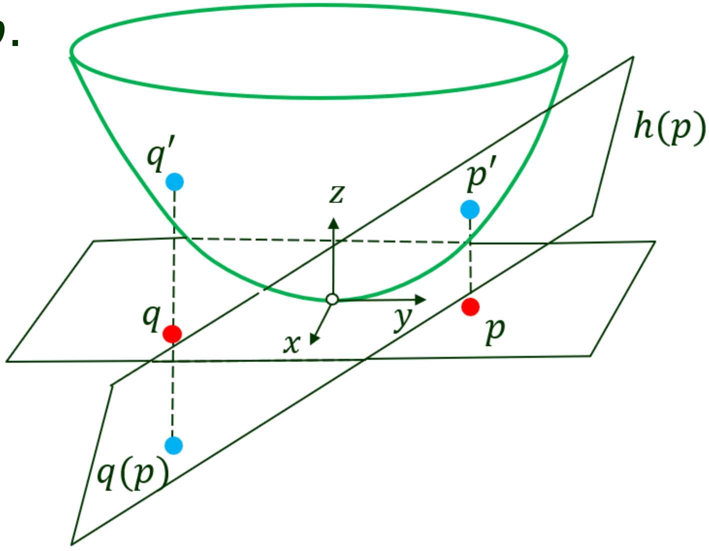
UE(H): upper envelope of the planes in H.

Theorem 1 The projection of UE(H) onto the plane $z = 0$ is the Voronoi diagram of P.

Proof A point $q \in$ Vor(p), the Voronoi cell of p.

$\Updownarrow$

$d(q,p) < d(q,r)$ for $r \in P$ and $r \neq p$



Upper Envelope of Planes

H ={ tangent plane $h(p)$ | $p \in P$}

UE(H): upper envelope of the planes in H.

Theorem 1 The projection of UE(H) onto the plane $z = 0$ is the Voronoi diagram of P.

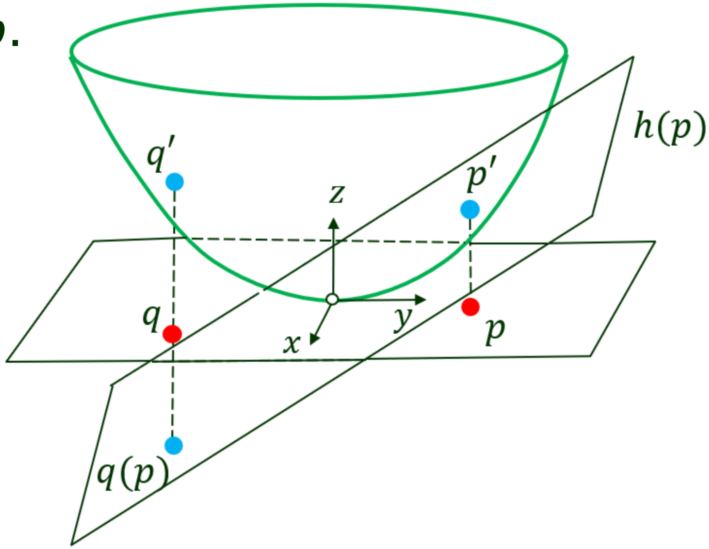
Proof A point $q \in$ Vor(p), the Voronoi cell of p.

$$\Updownarrow$$

$d(q,p) < d(q,r)$ for $r \in P$ and $r \neq p$

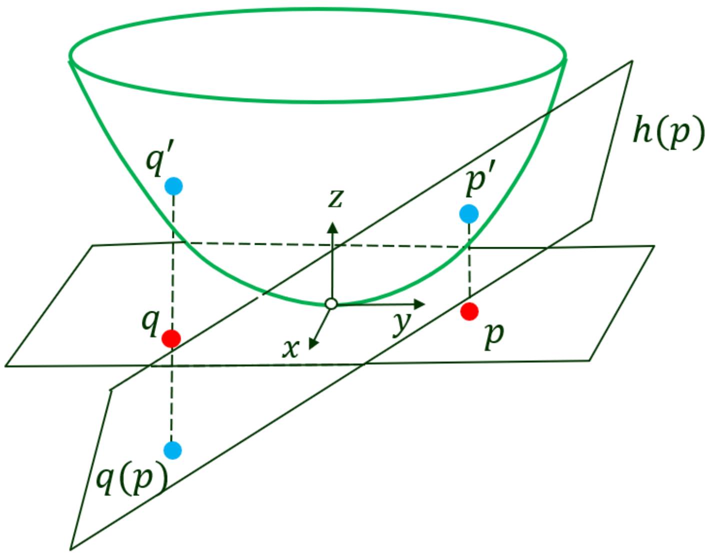
$$\Updownarrow$$

$$q_x^2 + q_y^2 - d(q,p)^2 > q_x^2 + q_y^2 - d(q,r)^2$$



Proof (cont'd)

$$q_x^2 + q_y^2 - d(q,p)^2 > q_x^2 + q_y^2 - d(q,r)^2$$

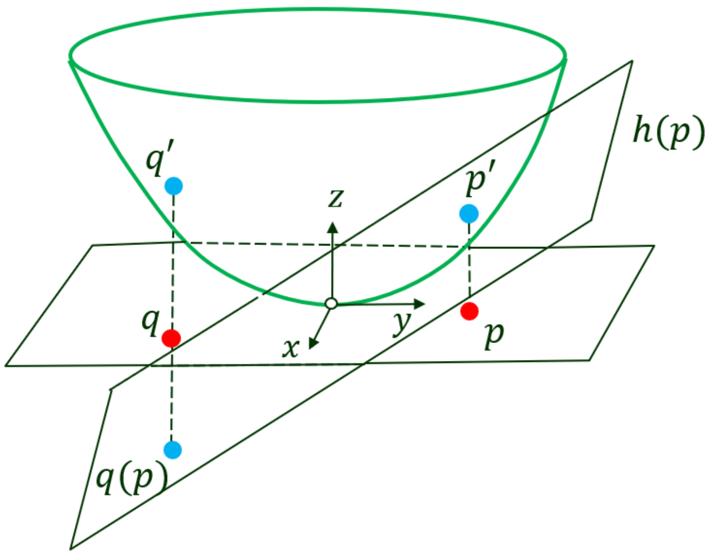


Proof (cont'd)

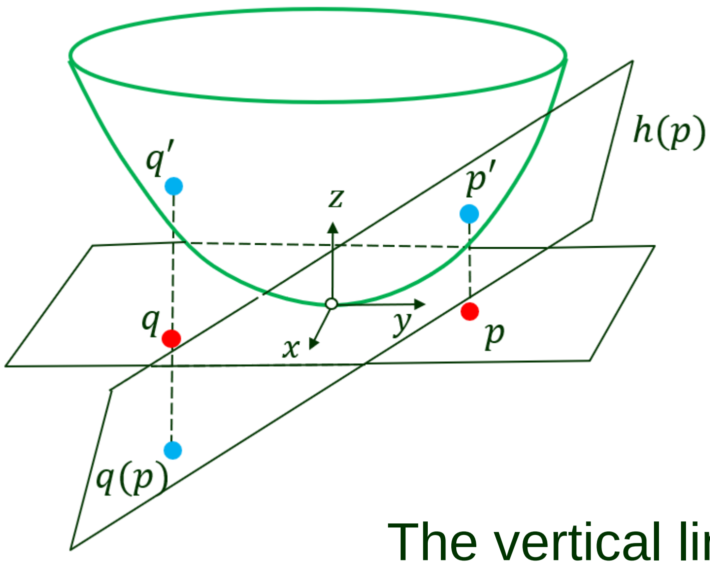
$$q_x^2 + q_y^2 - d(q,p)^2 > q_x^2 + q_y^2 - d(q,r)^2$$

$$\Updownarrow \quad q(p) = (q_x, q_y, q_x^2 + q_y^2 - d(p,q)^2)$$

$$q(p) \cdot (0,0,1) > q(r) \cdot (0,0,1)$$



Proof (cont'd)



$$q_x^2 + q_y^2 - d(q,p)^2 > q_x^2 + q_y^2 - d(q,r)^2$$

$$\Updownarrow \quad q(p) = (q_x, q_y, q_x^2 + q_y^2 - d(p,q)^2)$$

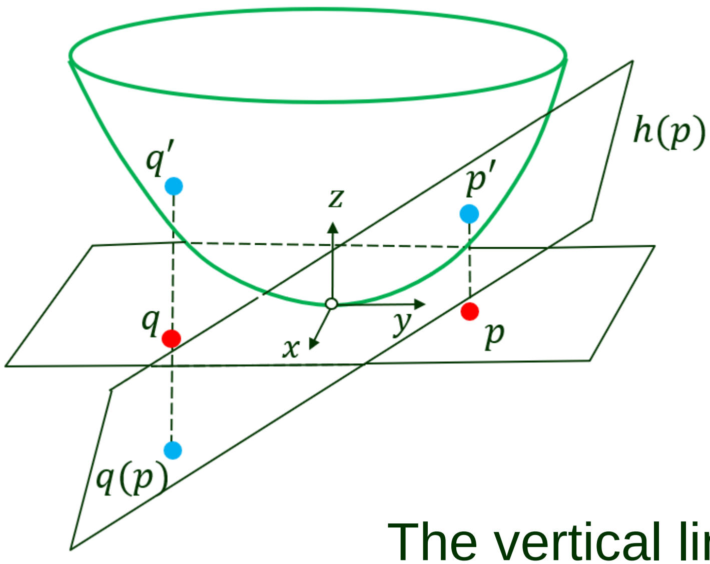
$$q(p) \cdot (0,0,1) > q(r) \cdot (0,0,1)$$

$$\Updownarrow$$

The vertical line through q intersects UE(H) at a point on $h(p)$, i.e., inside the facet contributed by $h(p)$.

Proof (cont'd)

$$q_x^2 + q_y^2 - d(q,p)^2 > q_x^2 + q_y^2 - d(q,r)^2$$



$$\Updownarrow \quad q(p) = (q_x, q_y, q_x^2 + q_y^2 - d(p,q)^2)$$

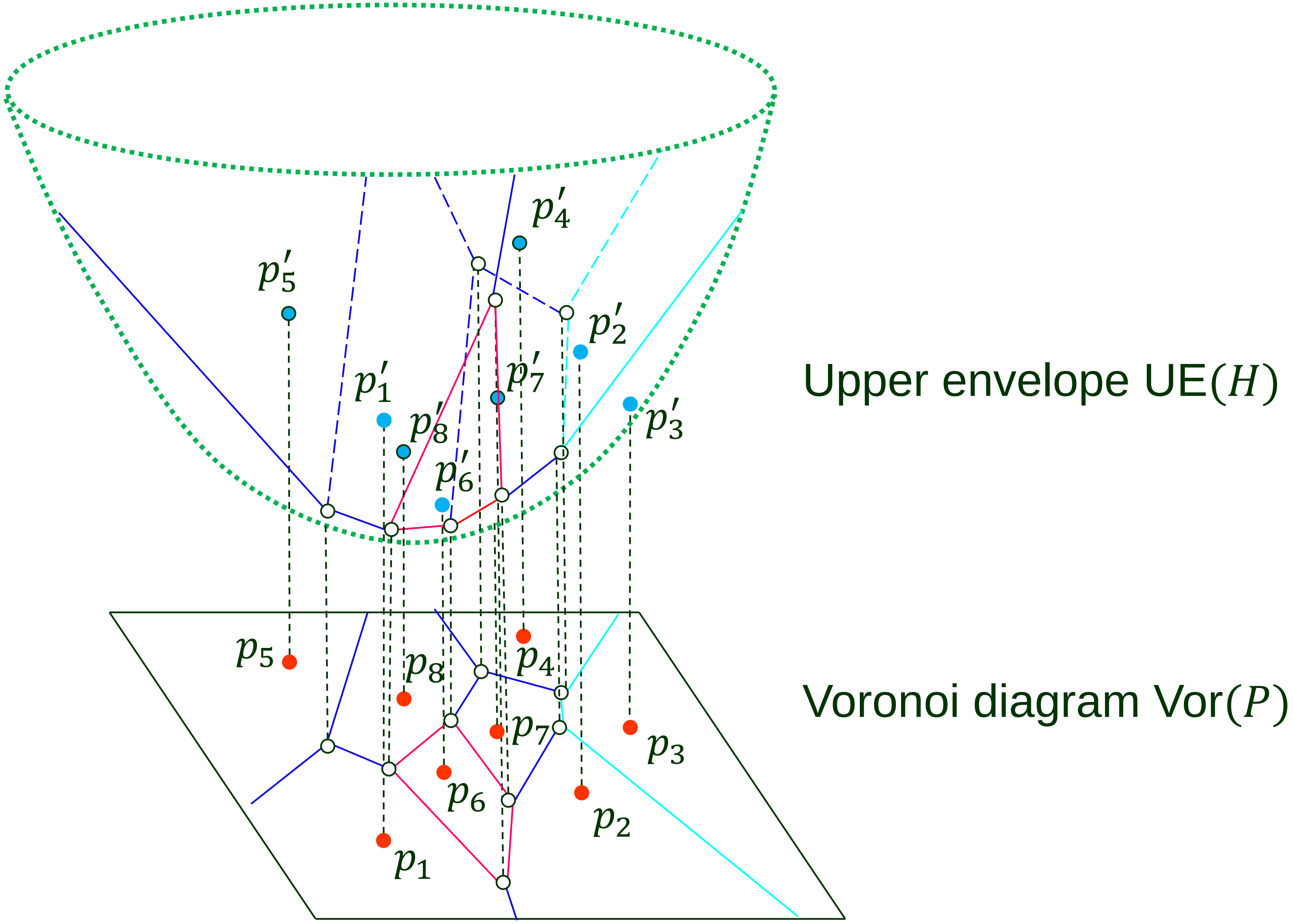
$$q(p) \cdot (0,0,1) > q(r) \cdot (0,0,1)$$

$$\Updownarrow$$

The vertical line through q intersects UE(H) at a point on $h(p)$, i.e., inside the facet contributed by $h(p)$.

$\square$

Projection of Upper Envelope



Construction of Voronoi Diagram

Constructing the Voronoi diagram of P in 2D

Construction of Voronoi Diagram

Constructing the Voronoi diagram of P in 2D

$\Downarrow$ reduces to

Computing an upper envelope of the set of planes
$H = \{h(p) \mid p \in P\}$ in 3D

Point $p = (p_x, p_y, p_z) \quad \mapsto \quad$ plane $h(p)$: $z = p_x x + p_y y - p_z$

Construction of Voronoi Diagram

Constructing the Voronoi diagram of P in 2D

$\Downarrow$ reduces to

Computing an upper envelope of the set of planes
$H = \{h(p) \mid p \in P\}$ in 3D

$\Downarrow$ reduces to

Computing the lower convex hull of the set of dual points
$H^* = \{h(p)^* \mid p \in P\}$ in 3D

Point $p = (p_x, p_y, p_z) \quad \mapsto \quad$ plane $h(p)$: $z = p_x x + p_y y - p_z$

Construction of Voronoi Diagram

Constructing the Voronoi diagram of P in 2D

$\Downarrow$ reduces to

Computing an upper envelope of the set of planes
$H = \{h(p) \mid p \in P\}$ in 3D

$\Downarrow$ reduces to

Computing the lower convex hull of the set of dual points
$H^* = \{h(p)^* \mid p \in P\}$ in 3D

Point $p = (p_x, p_y, p_z) \mapsto$ plane $h(p)$: $z = p_x x + p_y y - p_z$

Theorem 2 The projection of the lower convex hull of H^* onto the plane $z = 0$ is the Delaunay graph of P.

Summary

P: a set of n sites.

$H = \{\, h(p)\colon z = 2p_x x + 2p_y y - (p_x^2 + p_y^2) \mid p \in P\}$

$$
\begin{array}{ccccccc}
P & \rightarrow & H & \rightarrow & \text{UE}(H) & \rightarrow & \text{LH}(H^*) \\
 & & & & \big\downarrow \text{projected onto the } x\text{-}y \text{ plane} & & \big\downarrow \text{projected onto the } x\text{-}y \text{ plane} \\
 & & & & \text{Vor}(P) & & \text{DG}(P)
\end{array}
$$