Envelopes & Voronoi Diagrams

Outline:

I. Algorithm for 3D Convex Hulls

II. Review of duality

III. Hull-envelope correspondence

IV. Voronoi diagram as a 3D convex hull problem
I. Algorithm for 3D Convex Hulls

ConvexHull($P$)

1. find $p_1, p_2, p_3, p_4 \in P$ that form a tetrahedron
2. $C \leftarrow CH(\{p_1, p_2, p_3, p_4\})$
3. compute a random permutation $p_5, p_6, ..., p_n$
4. initialize the conflict graph $G$ over all facets of $C$ and $p_5, p_6, ..., p_n$
5. for $r \leftarrow 5$ to $n$
   6. do // insert $p_r$ to $C$
   7. if $F_{conflict}(p_r) \neq \emptyset$ // $p_r$ lies outside $C$
      8. then
      9. delete all facets in $F_{conflict}(p_r)$ from $C$
10. find the horizon by walking along the boundary of the visible region of $p_r$
10. for each edge $e$ on the horizon
11. do connect $e$ to $p_r$ to form a triangle $f$
Execution Example – Initialization

\[ G: \]

\[ p_1 \]

\[ p_2 \]

\[ p_3 \]

\[ p_4 \]

\[ p_5 \]

\[ p_6 \]

\[ f_1 \]

\[ f_2 \]

\[ f_3 \]

\[ f_4 \]

\[ \triangle p_1 p_2 p_4 \]

\[ \triangle p_2 p_3 p_4 \]

\[ \triangle p_1 p_3 p_4 \]

\[ \triangle p_1 p_2 p_3 \]
Iteration of Adding $p_5$

$r = 5$

$F_{conflict}(p_5) = \{f_1, f_3, f_4\}$

$G:$

$p_5$

$p_6$

$p_1$

$p_2$

$p_3$

$p_4$
Iteration of Adding $p_5$

$r = 5$

$F_{\text{conflict}}(p_5) = \{f_1, f_3, f_4\}$

$G:$

$p_5$
Iteration of Adding $p_5$

$r = 5$

$F_{conflict}(p_5) = \{f_1, f_3, f_4\}$

$G:$

$p_5 \rightarrow f_2$

$p_6 \rightarrow f_3$

$p_5 \rightarrow f_4$
Iteration of Adding $p_5$

$r = 5$

$F_{\text{conflict}}(p_5) = \{f_1, f_3, f_4\}$

$G:$

$p_5$\[\begin{array}{c}
\circ f_1 \\
\circ f_2 \\
\circ f_3 \\
\circ f_4 \\
\end{array}\]
Iteration of Adding $p_5$

$r = 5$

$F_{conflict}(p_5) = \{f_1, f_3, f_4\}$

$G:$

$\bigcirc f_1$

$\bigcirc f_2$

$\bigcirc f_3$

$\bigcirc f_4$
12. if $f$ is coplanar with its neighbor facet $f'$ along $e$
13. then merge $f$ with $f'$ and the merged facet inherits the latter’s conflict set
14. else // determine conflicts for $f$
15. create a node for $f$ in $G$
16. $f_1, f_2$: facets incident to $e$
17. for all $p \in P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2)$
18. do
19. if $f$ is visible from $p$
20. add $\langle p, f \rangle$ to $G$ // update $P_{\text{conflict}}(f)$ and $F_{\text{conflict}}(p)$
21. delete the node corresponding to $p_r$ and the nodes corresponding to the facets in $F_{\text{conflict}}(p_r)$ from $G$, along with incident arcs
22. return $C$
Iteration (cont’d)

\[ r = 5 \]

\[ G: \]

\[ \begin{align*}
   &p_1 & & & & & & & & & & & f_1 \\
   &p_2 & & & & & & & & & & & f_2 \\
   &p_3 & & & & & & & & & & & f_3 \\
   &p_4 & & & & & & & & & & & f_4 \\
   &p_5 & & & & & & & & & & & f_5 \\
   &p_6 & & & & & & & & & & & f_6
\end{align*} \]
Iteration (cont’d)

\[ r = 5 \]

\[ G: \]

\[ \begin{array}{cccc}
\circ f_1 \\
\circ f_2 \\
\circ f_3 \\
\circ f_4 \\
\circ f_5: \triangle p_2 p_4 p_5 
\end{array} \]
Iteration (cont’d)

\[ r = 5 \]

\[ G: \]

\[ f_1, f_2, f_3, f_4, f_5 \]

\[ p_1, p_2, p_3, p_4, p_5, p_6 \]

\[ f_5 \text{ is not visible from } p_6, \text{ the point that sees } e = \overline{p_2p_4} \text{ and its (old) bordering facets } f_1, f_2. \]
Iteration (cont’d)

$r = 5$

$G$: $\triangle p_2p_4p_5$

$f_5$: $\triangle p_3p_4p_5$

$f_5$ is not visible from $p_6$, the point that sees $e = \overline{p_2p_4}$ and its (old) bordering facets $f_1, f_2$. 

$p_5$ $p_4$ $p_6$ $p_1$ $p_2$ $p_3$ $p_6$ $p_5$

horizon
Iteration (cont’d)

\[ r = 5 \]

\[ f_5 \text{ is not visible from } p_6, \text{ the point that sees } e = \overrightarrow{p_2 p_4} \text{ and its (old) bordering facets } f_1, f_2. \]

\[ f_6 \text{ is not visible from } p_6. \]
Iteration (cont’d)

\[r = 5\]

\(G:\)

\[f_5: \triangle \overline{p_2p_4p_5}\]
\[f_6: \triangle \overline{p_3p_4p_5}\]
\[f_7: \triangle \overline{p_2p_3p_5}\]

\(p_5\) is not visible from \(p_6\), the point that sees \(e = \overline{p_2p_4}\) and its (old) bordering facets \(f_1, f_2\).

\(f_6\) is not visible from \(p_6\).
Iteration (cont’d)

\[ r = 5 \]

\[ G: \]

- \( f_1 \)
- \( f_2 \)
- \( f_3 \)
- \( f_4 \)
- \( f_5: \triangle p_2p_4p_5 \)
- \( f_6: \triangle p_3p_4p_5 \)
- \( f_7: \triangle p_2p_3p_5 \)

- \( f_5 \) is not visible from \( p_6 \), the point that sees \( e = \overline{p_2p_4} \) and its (old) bordering facets \( f_1, f_2 \).
- \( f_6 \) is not visible from \( p_6 \).
- \( f_7 \) is not visible from \( p_6 \).
Iteration (cont’d)

$r = 5$

$f_5$ is not visible from $p_6$, the point that sees $e = \overline{p_2p_4}$ and its (old) bordering facets $f_1, f_2$.

$f_6$ is not visible from $p_6$.

$f_7$ is not visible from $p_6$. 

$G$: 

- $\bigcirc f_1$
- $\bigcirc f_2$
- $\bigcirc f_3$
- $\bigcirc f_4$
- $\bigcirc f_5 \colon \triangle p_2p_4p_5$
- $\bigcirc f_6 \colon \triangle p_3p_4p_5$
- $\bigcirc f_7 \colon \triangle p_2p_3p_5$
Iteration (cont’d)

\[ r = 5 \]

\( G \):

\( p_1 \)

\( p_2 \)

\( p_3 \)

\( p_4 \)

\( p_5 \)

\( p_6 \)

\( f_5 \) is not visible from \( p_6 \), the point that sees \( e = \overline{p_2p_4} \) and its (old) bordering facets \( f_1, f_2 \).

\( f_6 \) is not visible from \( p_6 \).

\( f_7 \) is not visible from \( p_6 \).
Iteration for $p_6$

$r = 6$

$G:$

$p_6 \xrightarrow{f_2} f_5 \xrightarrow{f_6} p_6 \xrightarrow{f_7} f_6 \xrightarrow{f_5} f_2$
Iteration for $p_6$

$r = 6$

$G$:  

$p_6 \quad o \quad f_2 \quad o \quad f_5 \quad o \quad f_6 \quad o \quad f_7$
Iteration for $p_6$

$r = 6$

$G:$

$p_6 \bigcirc$

$igcirc f_2$

$igcirc f_5$

$igcirc f_6$

$igcirc f_7$
Iteration for $p_6$

$r = 6$

$G: \quad \bigcirc f_2 \quad \circ f_5 \quad \circ f_6 \quad \circ f_7 \quad \circ f_8 : \triangle p_2 p_4 p_6$
Iteration for $p_6$

$r = 6$

$G:$

- $f_2$
- $f_5$
- $f_6$
- $f_7$
- $f_8: \triangle p_2 p_4 p_6$
- $f_9: \triangle p_3 p_4 p_6$
Iteration for $p_6$

$r = 6$

$G$: $f_2$

$p_6 \bigcirc

f_5

f_6

f_7

f_8

f_9

f_{10}$

$p_6 \triangle p_2 p_4 p_6$

$p_6 \triangle p_3 p_4 p_6$

$p_6 \triangle p_2 p_3 p_6$
Iteration for $p_6$

$r = 6$

$G:$

$p_6 \circ$

$\circ f_5$

$\circ f_6$

$\circ f_7$

$\circ f_8 \triangle p_2p_4p_6$

$\circ f_9 \triangle p_3p_4p_6$

$\circ f_{10} \triangle p_2p_3p_6$
Analysis

**Theorem**  The randomized incremental algorithm computes the convex hull of $n$ points in 3D in $O(n \log n)$ expected time.

**Proof**  (omitted)
II. Duality: Points ↔ (Non-vertical) Lines

Point $p = (p_x, p_y) \quad \rightarrow \quad \text{Line } p^*: y = p_x x - p_y$

Line $l: y = mx + b \quad \rightarrow \quad \text{Point } l^* = (m, -b)$
Incidence

\[ p \in l \Leftrightarrow l^* \in p^* \]

\[ p = (p_x, p_y) \]
Collinearity ↔ Concurrency

$p_1, p_2, p_3$ collinear on the line $l$

Dual lines $p_1^*, p_2^*, p_3^*$ concurrent at the dual point $l^*$
Point-Line Order Preserving

$p$ lies above $l$ iff $l^*$ lies above $p^*$.
Point Set $\mapsto$ Line Arrangement

$P$: a set of points in the plane.

$P^* = \{p^* | p \in P\}$: a line arrangement
III. Upper Convex Hull & Lower Envelope

\[ \text{UH}(P) : \text{upper convex hull of } P \text{ (part of the boundary from the leftmost vertex to the rightmost one).} \]

\[ l \text{ above all points } \Rightarrow l^* \text{ below their dual lines} \]
III. Upper Convex Hull & Lower Envelope

UH(P): *upper convex hull* of $P$ (part of the boundary from the leftmost vertex to the rightmost one).

$l$ above all points $\Rightarrow l^*$ below their dual lines
III. Upper Convex Hull & Lower Envelope

UH\( (P) \): *upper convex hull* of \( P \) (part of the boundary from the leftmost vertex to the rightmost one).

\( l \) above all points \( \Rightarrow l^* \) below their dual lines

LE\( (P^*) \): *lower envelope* of \( P^* \) is the unique bottom cell of the arrangement.
III. Upper Convex Hull & Lower Envelope

UH(P): *upper convex hull* of P (part of the boundary from the leftmost vertex to the rightmost one).

\( l \) above all points \( \Rightarrow \) \( l^* \) below their dual lines

\( p_i \), \( p_j \) leftmost vertex, rightmost vertex

\( l \), \( p_h \) above all points \( \Rightarrow l^* \) below their dual lines

LE(P\(^*\)): *lower envelope* of \( P^* \) is the unique bottom cell of the arrangement.

\( p_{h*} \), \( p_i^* \), \( p_j^* \) in the dual plane

\( p_h \) above all points \( \Rightarrow p_{h*} \) below their dual lines

primal plane, dual plane
III. Upper Convex Hull & Lower Envelope

**UH(P):** upper convex hull of \( P \) (part of the boundary from the leftmost vertex to the rightmost one).

\( l \) above all points \( \Rightarrow \ l^* \) below their dual lines

**LE(P*)**: lower envelope of \( P^* \) is the unique bottom cell of the arrangement.

Slope of \( p^*_h < \) Slope of \( p^*_i < \) Slope of \( p^*_j \)
III. Upper Convex Hull & Lower Envelope

**UH**($P$): *upper convex hull* of $P$ (part of the boundary from the leftmost vertex to the rightmost one).

$l$ above all points $\Rightarrow$ $l^*$ below their dual lines

**LE**($P^*$): *lower envelope* of $P^*$ is the unique bottom cell of the arrangement.

Slope of $p_h^* <$ Slope of $p_i^* <$ Slope of $p_j^*$
Vertex $\rightarrow$ Edge

$p_j$ is a vertex of $UH(P)$.

$UH(P)$

primal plane

$p_j$

$dual plane$

$p_j^*$

$LE(P^*)$
Vertex $\rightarrow$ Edge

$p_j$ is a vertex of $\text{UH}(P)$. There is a non-vertical line $l$ through $p_j$ such that all other points are below $l$. 

primal plane

$\text{UH}(P)$

$p_j$ is a vertex of $\text{UH}(P)$. There is a non-vertical line $l$ through $p_j$ such that all other points are below $l$. 

dual plane

$\text{LE}(P^*)$
Vertex $\rightarrow$ Edge

$p_j$ is a vertex of $\text{UH}(P)$.

There is a non-vertical line $l$ through $p_j$ such that all other points are below $l$.

Its dual point $l^*$ on the line $p_j^* \in P^* = \{p^* | p \in P\}$ lies below all other lines of $P^*$.
$p_j$ is a vertex of $\text{UH}(P)$.  

There is a non-vertical line $l$ through $p_j$ such that all other points are below $l$.  

Its dual point $l^*$ on the line $p_j^* \in P^* = \{p^* | p \in P\}$ lies below all other lines of $P^*$.  

$l^* \in p_j^*$ is on the boundary of the bottom cell; i.e., $p_j^*$ contributes an edge to $\text{LE}(P^*)$.  

$\bullet$ $p_j \in \text{UH}(P)$  

$\bullet$ $l$  

$\bullet$ $l^* \in p_j^*$  

$\bullet$ $\text{LE}(P^*)$
Vertex → Edge

$p_j$ is a vertex of $\text{UH}(P)$.

There is a non-vertical line $l$ through $p_j$ such that all other points are below $l$.

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Vertex → Edge

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$l^* \in p_j^*$ is on the boundary of the bottom cell; i.e., $p_j^*$ contributes an edge to $\text{LE}(P^*)$. 
$p, q \in P$ define an edge $e$ in $\text{UH}(P)$. 

\[ l \rightarrow \text{Vertex} \]
$p, q \in P$ define an edge $e$ in $UH(P)$. 

**Edge → Vertex**
$p, q \in P$ define an edge $e$ in $\text{UH}(P)$.

All the points $r \in P \setminus e$ lie below the line $l$ through $p$ and $q$. 
Edge $\rightarrow$ Vertex

$p, q \in P$ define an edge $e$ in $\text{UH}(P)$.

All the points $r \in P \setminus e$ lie below the line $l$ through $p$ and $q$.

All the lines $r^*, r \in P \setminus e$ lie above $l^*$. 

Diagram: 
- Points $p$, $q$, $l$, $r$, $r^*$, $p^*$, $q^*$, $l^*$
- Line $e$ connecting $p$ and $q$
$p, q \in P$ define an edge $e$ in $\text{UH}(P)$.

All the points $r \in P \setminus e$ lie below the line $l$ through $p$ and $q$.

All the lines $r^*, r \in P \setminus e$ lie above $l^*$.

$l^*$ is a vertex of $\text{LE}(P^*)$. 
Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on $\text{UH}(P)$. 
Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on $\text{UH}(P)$.

$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$
Order Reversal

\[ p_{i_1}, p_{i_2}, \ldots, p_{i_k} : \text{left-to-right order of vertices on UH}(P). \]

\[ p_{i_s} = (x_{i_s}, y_{i_s}) \text{ precedes } p_{i_t} = (x_{i_t}, y_{i_t}) \]

\[ x_{i_s} < x_{i_t} \]
Order Reversal

$p_i_1, p_i_2, \ldots, p_i_k$: left-to-right order of vertices on UH($P$).

$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$

$\downarrow$ left-to-right

$x_{i_s} < x_{i_t}$

The dual line $p_{i_s}^*: y = x_{i_s}x - y_{i_s}$ has a smaller slope than the dual line $p_{i_t}^*: y = x_{i_t}x - y_{i_t}$
Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH($P$).

$p_s = (x_{i_s}, y_{i_s})$ precedes $p_t = (x_{i_t}, y_{i_t})$

\[ x_{i_s} < x_{i_t} \]

The dual line $p_s^* : y = x_{i_s}x - y_{i_s}$ has a smaller slope than the dual line $p_t^* : y = x_{i_t}x - y_{i_t}$

On the lower envelope, the segment of $p_s^*$ is before that of $p_t^*$ in the right-to-left order.
Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH($P$).

$p_{i_1}^*, p_{i_2}^*, \ldots, p_{i_k}^*$: right-to-left order of edges on LE($P^*$).

$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$

\[ x_{i_s} < x_{i_t} \]

The dual line $p_{i_s}^*: y = x_{i_s}x - y_{i_s}$ has a smaller slope than the dual line $p_{i_t}^*: y = x_{i_t}x - y_{i_t}$

On the lower envelope, the segment of $p_{i_s}^*$ is before that of $p_{i_t}^*$ in the right-to-left order.
Order Reversal

\( p_{i_1}, p_{i_2}, \ldots, p_{i_k} \): left-to-right order of vertices on \( \text{UH}(P) \).

\( p_{i_1}^*, p_{i_2}^*, \ldots, p_{i_k}^* \): right-to-left order of edges on \( \text{LE}(P^*) \).

\[ p_{i_s} = (x_{i_s}, y_{i_s}) \] precedes \( p_{i_t} = (x_{i_t}, y_{i_t}) \)

\[ x_{i_s} < x_{i_t} \]

The dual line \( p_{i_s}^*: y = x_{i_s} x - y_{i_s} \) has a smaller slope than the dual line \( p_{i_t}^*: y = x_{i_t} x - y_{i_t} \)

On the lower envelope, the segment of \( p_{i_s}^* \) is before that of \( p_{i_t}^* \) in the right-to-left order.
Lower Convex Hull & Upper Envelope

\( \text{LH}(P) \): \textit{lower convex hull} of \( P \)

\( \text{UE}(P^*) \): \textit{upper envelope} of \( P^* \)

\[ \text{primal plane} \]

\[ \text{dual plane} \]
Lower Convex Hull & Upper Envelope

LH(P): *lower convex hull* of P

UE(P*): *upper envelope* of P*

\[ p_{i_1}, p_{i_2}, \ldots, p_{i_k} : \text{left-to-right order of vertices on LH(P)}. \]
Lower Convex Hull & Upper Envelope

\textbf{LH}(P): lower convex hull of } P

\textbf{UE}(P^*): upper envelope of } P^*

\(p_{i_1}, p_{i_2}, \ldots, p_{i_k}\): \underline{left-to-right} order of vertices on LH(P).

\(p_{i_1^*}, p_{i_2^*}, \ldots, p_{i_k^*}\): \underline{left-to-right} order of edges on UE(P*).
Hull-Envelope Correspondences

\[ \text{UH}(P) \leftrightarrow \text{LE}(P^*) \]

By symmetry,

\[ \text{LH}(P) \leftrightarrow \text{UE}(P^*) \]
Hull-Envelope Correspondences

\[ \text{UH}(P) \leftrightarrow \text{LE}(P^*) \]

By symmetry,

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- The above relationships generalize to convex hulls and intersection of half-spaces in 3D.
Hull-Envelope Correspondences

\[ \text{UH}(P) \leftrightarrow \text{LE}(P^*) \]

By symmetry,

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- The above relationships generalize to convex hulls and intersection of half-spaces in 3D.
- Convex hulls and intersections of half-planes (or half-spaces) are essentially dual concepts.
Hull-Envelope Correspondences

\[ \text{UH}(P) \leftrightarrow \text{LE}(P^*) \]

By symmetry,

\[ \text{LH}(P) \leftrightarrow \text{UE}(P^*) \]

- The above relationships generalize to convex hulls and intersection of half-spaces in 3D.
- Convex hulls and intersections of half-planes (or half-spaces) are essentially dual concepts.

Computing an upper (lower) convex hull

\[ \uparrow \downarrow \]

Intersecting lower (upper) half-planes
Algorithm for Half-Plane Intersection

$H$: a set of half-planes

Idea: Dualize a convex hull algorithm.
Algorithm for Half-Plane Intersection

\( H \): a set of half-planes

**Idea**: Dualize a convex hull algorithm.

a) Split \( H \) into a set \( H_+ \) of upper half-planes and a set \( H_- \) of lower half-planes.
Algorithm for Half-Plane Intersection

\( H \): a set of half-planes

**Idea:** Dualize a convex hull algorithm.

a) Split \( H \) into a set \( H_+ \) of upper half-planes and a set \( H_- \) of lower half-planes. \( O(n) \)
Algorithm for Half-Plane Intersection

$H$: a set of half-planes

Idea: Dualize a convex hull algorithm.

a) Split $H$ into a set $H_+$ of upper half-planes and a set $H_-$ of lower half-planes. $O(n)$

b) Compute $\cap H_+$ by constructing the lower convex hull of $H_+^*$. 
Algorithm for Half-Plane Intersection

$H$: a set of half-planes

**Idea**: Dualize a convex hull algorithm.

a) Split $H$ into a set $H_+$ of upper half-planes and a set $H_-$ of lower half-planes. $O(n)$

b) Compute $\cap H_+$ by constructing the lower convex hull of $H^*_+$. $O(n \log n)$
Algorithm for Half-Plane Intersection

$H$: a set of half-planes

Idea: Dualize a convex hull algorithm.

a) Split $H$ into a set $H_+$ of upper half-planes and a set $H_-$ of lower half-planes. $O(n)$

b) Compute $\cap H_+$ by constructing the lower convex hull of $H^*_+$. $O(n \log n)$

c) Compute $\cap H_-$ by constructing the upper convex hull of $H^*_-$.$O(n \log n)$
Algorithm for Half-Plane Intersection

$H$: a set of half-planes

**Idea**: Dualize a convex hull algorithm.

1. **Split $H$ into** a set $H_+$ of upper half-planes and a set $H_-$ of lower half-planes. \( O(n) \)
2. **Compute** $\cap H_+$ by constructing the lower convex hull of $H_+^*$. \( O(n \log n) \)
3. **Compute** $\cap H_-$ by constructing the upper convex hull of $H_-^*$. \( O(n \log n) \)
4. **Intersect** $H_+$ and $H_-$. \( O(n) \)
IV. Review: Duality with a Parabola

- Dual $p^*$ of $p$ on the parabola is the tangent line at $p$. 

\[
y = \frac{x^2}{2}
\]
Point Not on a Parabola

\[ y = \frac{x^2}{2} \]

\[ p = (p_x, p_y) \]
Point Not on a Parabola

\[ y = \frac{x^2}{2} \]

\[ p = (p_x, p_y) \]
Point Not on a Parabola

\[ y = \frac{x^2}{2} \]

\[ p = (p_x, p_y) \]
Point Not on a Parabola

\[ y = \frac{x^2}{2} \]

\[ p = (p_x, p_y) \]

\[ q = (p_x, p_y - d) \]
Point Not on a Parabola

\[ y = \frac{x^2}{2} \]

\[ p = (p_x, p_y) \]

\[ q = (p_x, p_y - d) \]
Point Not on a Parabola

\[ y = \frac{x^2}{2} \]

\[ p = (p_x, p_y) \]

\[ q = (p_x, p_y - d) \]

\[ q' = (p_x, p_y + d) \]
Point Not on a Parabola

\[ y = \frac{x^2}{2} \]

\[ p = (p_x, p_y) \]

\[ q = (p_x, p_y - d) \]

\[ q' = (p_x, p_y + d) \]

\[ q' - p = p - q \]
Point Not on a Parabola

\[ y = \frac{x^2}{2} \]

\[ p = (p_x, p_y) \]

\[ q = (p_x, p_y - d) \]

\[ q' = (p_x, p_y + d) \]

\[ q' - p = p - q \]

\[ \diamond \text{ The dual line } q^* \parallel p^* \text{ and it passes through } q'. \]
More on Duality

Construct the dual line $q^*$ of $q$ without measuring distances:

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1) Through $q$ draw two tangent lines to the parabola.
More on Duality

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$y = \frac{x^2}{2}$
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2) Let $p_1$ and $p_2$ be the points of tangency, respectively.

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The two tangent lines are $p_1^*$ and $p_2^*$. 

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$p_1^*$ and $p_2^*$ intersect at $q \iff q^*$ passes through $p_1$ and $p_2$. 

\[ y = \frac{x^2}{2} \]
Voronoi Diagram Revisited

$P$: a set of $n$ sites.
Unit Paraboloid

\[ U: z = x^2 + y^2 \]
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\[ p = (p_x, p_y, 0) \]
**Unit Paraboloid**

\[ U: z = x^2 + y^2 \]

Projection of \( p \) onto \( U \):

\[ p' = (p_x, p_y, p_x^2 + p_y^2) \]

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i.e. \( g(x, y, z) \equiv x^2 + y^2 - z = 0 \)

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\[ p' = (p_x, p_y, p_x^2 + p_y^2) : \]
Tangent plane \( h(p) \) to \( U \) through \( p' \) has normal:
\[ \nabla g(p') = \left( \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) \bigg|_{p'} = (2p_x, 2p_y, -1) \]

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\[ ((x, y, z) - p') \cdot \nabla g(p') = 0 \]
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- Vertical line through \( q \) intersects
  - \( U \) at \( q' = (q_x, q_y, q_x^2 + q_y^2) \)
  - \( h(p) \) at \( q(p) \).
Distance Encoded in Tangent Plane

\[ d(p, q) \]: distance between two points \( p \) and \( q \)

\[ q' = (q_x, q_y, q_x^2 + q_y^2) \]
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$d(p, q)$: distance between two points $p$ and $q$

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$d(q', q(p)) = q'_z - (q(p))_z$
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\]

\[
= (q_x - p_x)^2 + (q_y - p_y)^2
\]
Distance Encoded in Tangent Plane

*d(p, q)*: distance between two points *p* and *q*

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\[ d(q', q(p)) = q'_z - (q(p))_z \]

\[ = (q_x^2 + q_y^2) - (2p_xq_x + 2p_yq_y - (p_x^2 + p_y^2)) \]

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\[ = d(p, q)^2 \]
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\[ = (q_x^2 + q_y^2) - (2p_x q_x + 2p_y q_y - (p_x^2 + p_y^2)) \]

\[ = (q_x - p_x)^2 + (q_y - p_y)^2 \]

\[ = d(p, q)^2 \]

The square of \(d(p, q)\) equals the distance between the two projection points (onto \(h(p)\) and \(U\)) from \(q\).
Upper Envelope of Planes

\[ H = \{ \text{tangent plane } h(p) \mid p \in P \} \]

\( \text{UE}(H) \): upper envelope of the planes in \( H \).

**Theorem 1** The projection of \( \text{UE}(H) \) onto the plane \( z = 0 \) is the Voronoi diagram of \( P \).
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**Proof** A point \( q \in \text{Vor}(p) \), the Voronoi cell of \( p \).
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UE(\(H\)): upper envelope of the planes in \(H\).

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**Proof** A point \(q \in \text{Vor}(p)\), the Voronoi cell of \(p\).

\[ d(q, p) < d(q, r) \text{ for } r \in P \text{ and } r \neq p \]

[Diagram showing the upper envelope and Voronoi diagram]
Upper Envelope of Planes

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Proof (cont’d)

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\[ q(p) = (q_x, q_y, q_x^2 + q_y^2 - d(p, q)^2) \]

\[ q(p) \cdot (0,0,1) > q(r) \cdot (0,0,1) \]
Proof (cont’d)

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The vertical line through \( q \) intersects UE(\( H \)) at a point on \( h(p) \), i.e., inside the facet contributed by \( h(p) \).
Proof (cont’d)

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Projection of Upper Envelope

Upper envelope $\text{UE}(H)$

Voronoi diagram $\text{Vor}(P)$
Construction of Voronoi Diagram

Constructing the Voronoi diagram of $P$ in 2D
Construction of Voronoi Diagram

Constructing the Voronoi diagram of \( P \) in 2D

\[ \downarrow \text{reduces to} \]

Computing an upper envelope of the set of planes \( H = \{ h(p) \mid p \in P \} \) in 3D

Point \( p = (p_x, p_y, p_z) \) \( \mapsto \) plane \( h(p): z = p_x x + p_y y - p_z \)
Constructing the Voronoi diagram of $P$ in 2D

\[ \downarrow \text{ reduces to } \]

Computing an upper envelope of the set of planes

\[ H = \{ h(p) \mid p \in P \} \] in 3D

\[ \downarrow \text{ reduces to } \]

Computing the lower convex hull of the set of dual points

\[ H^* = \{ h(p)^* \mid p \in P \} \] in 3D

Point $p = (p_x, p_y, p_z)$ $\leftrightarrow$ plane $h(p)$: $z = p_x x + p_y y - p_z$
Construction of Voronoi Diagram

Constructing the Voronoi diagram of $P$ in 2D

\[ \downarrow \text{reduces to} \]

Computing an upper envelope of the set of planes

$H = \{ h(p) | p \in P \}$ in 3D

\[ \downarrow \text{reduces to} \]

Computing the lower convex hull of the set of dual points

$H^* = \{ h(p)^* | p \in P \}$ in 3D

Point $p = (p_x, p_y, p_z)$ $\mapsto$ plane $h(p)$: $z = p_x x + p_y y - p_z$

**Theorem 2** The projection of the lower convex hull of $H^*$ onto the plane $z = 0$ is the Delaunay graph of $P$. 
Summary

$P$: a set of $n$ sites.

$H = \{ h(p): z = 2p_x x + 2p_y y - (p_x^2 + p_y^2) | p \in P \}$

\[
\begin{align*}
P \rightarrow H \rightarrow \text{UE}(H) \rightarrow \text{LH}(H^*) \\
\downarrow \text{projected onto the } x-y \text{ plane} \quad \downarrow \text{projected onto the } x-y \text{ plane}
\end{align*}
\]

$\text{Vor}(P) \quad \text{DG}(P)$