Envelopes & Voronoi Diagrams

Outline:

I. Review of duality

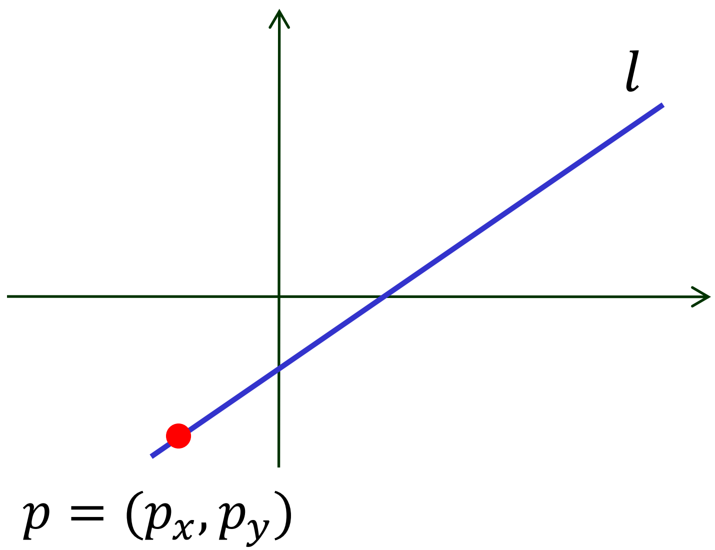
II. Hull-envelope correspondence

III. Voronoi diagram as a 3D convex hull problem

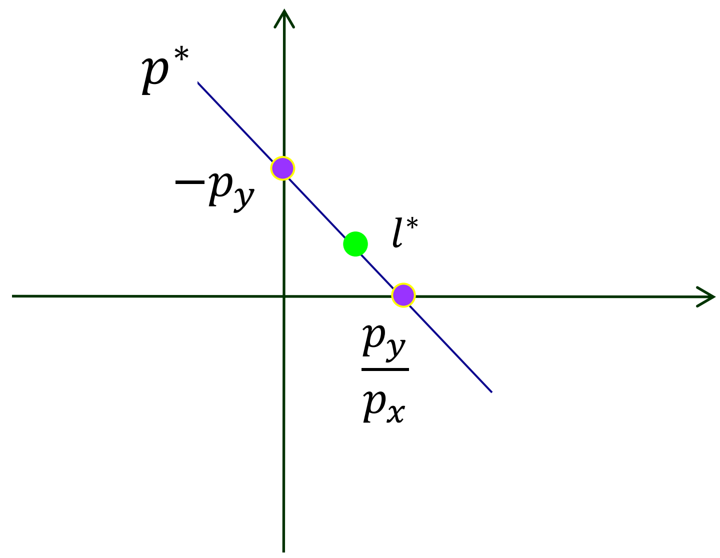
I. Duality: Points ⟷ (Non-vertical) Lines

Point $p = (p_x, p_y)$ $\Longrightarrow$ Line $p^*: y = p_x x - p_y$

Line $l: y = mx + b$ $\Longrightarrow$ Point $l^* = (m, -b)$



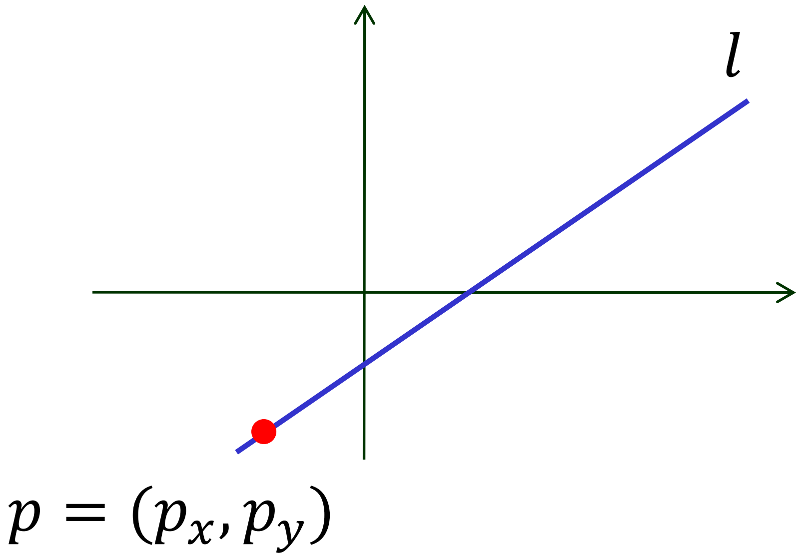
primal plane

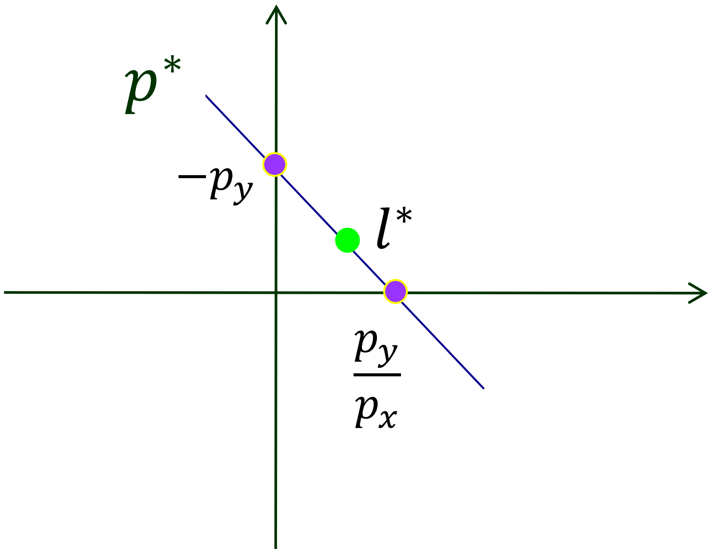


dual plane

Incidence

$$p \in l \Leftrightarrow l^* \in p^*$$





primal plane

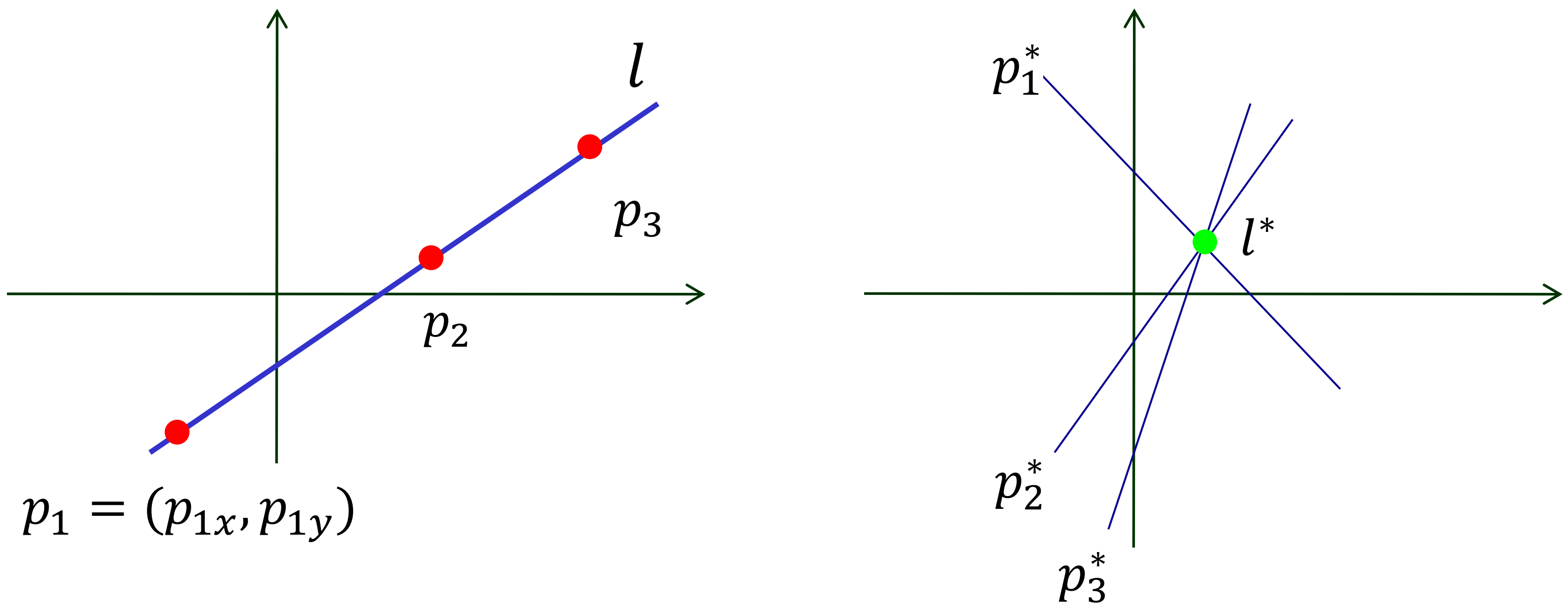
dual plane

Collinearity $\leftrightarrow$ Concurrency

p_1, p_2, p_3 *collinear* on the line l

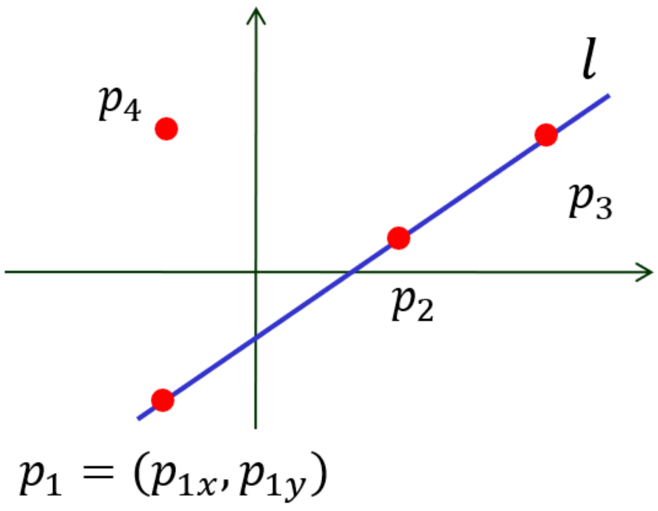
$\Updownarrow$

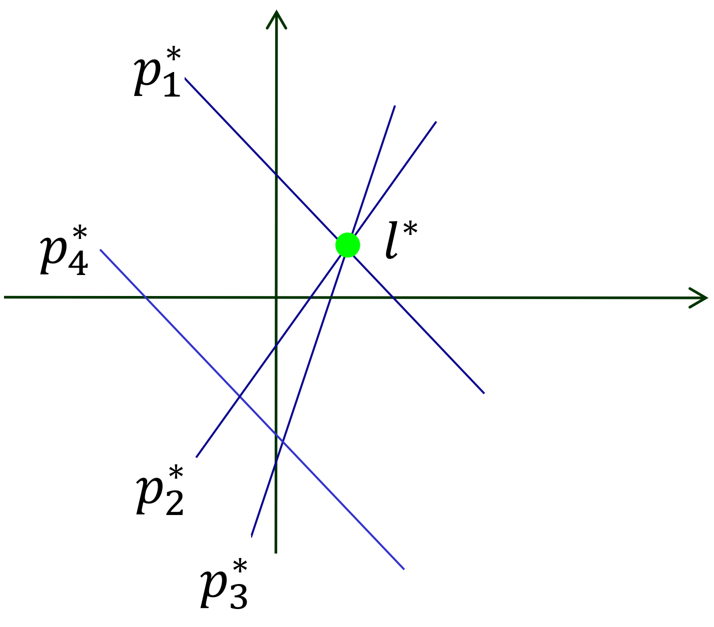
Dual lines p_1^*, p_2^*, p_3^* *concurrent* at the dual point l^*



Point-Line Order Preserving

p lies above l iff l^* lies above p^*.

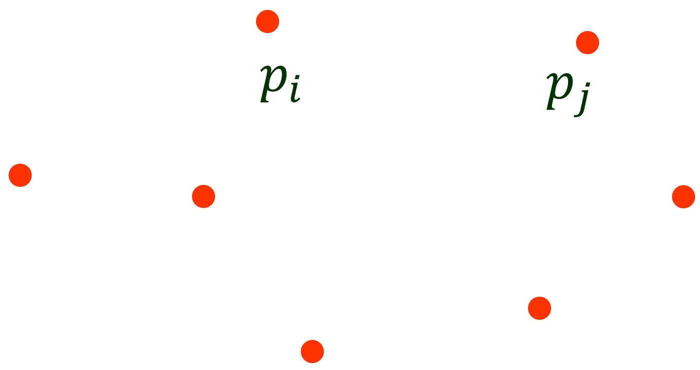




Point Set ↦ Line Arrangement

P: a set of points in the plane.

$P^* = \{p^* | \ p \in P\}$: a line arrangement



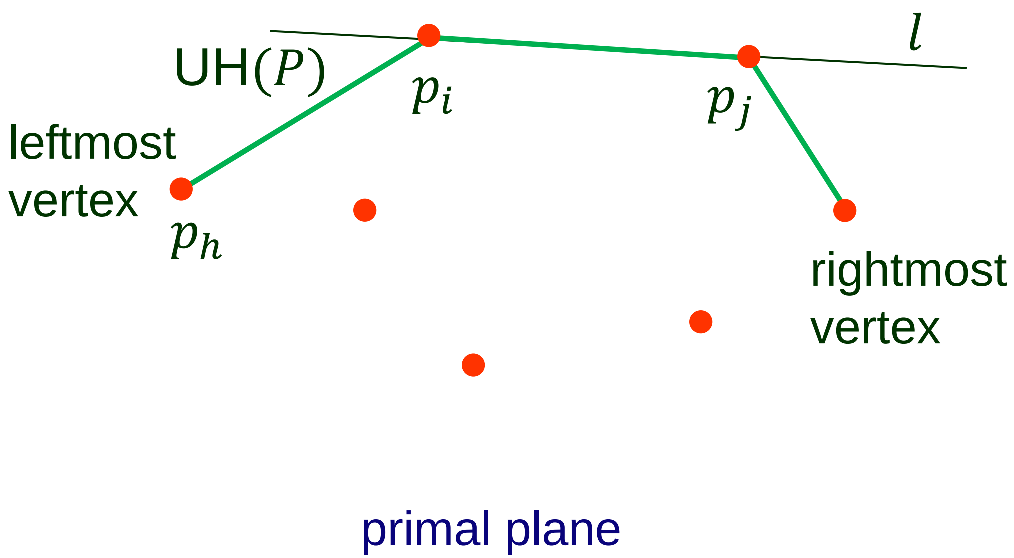
primal plane

dual plane

II. Upper Convex Hull & Lower Envelope

$\mathrm{UH}(P)$: *upper convex hull* of P (part of the boundary from the leftmost vertex to the rightmost one).

l above all points ⇒ l^* below their dual lines



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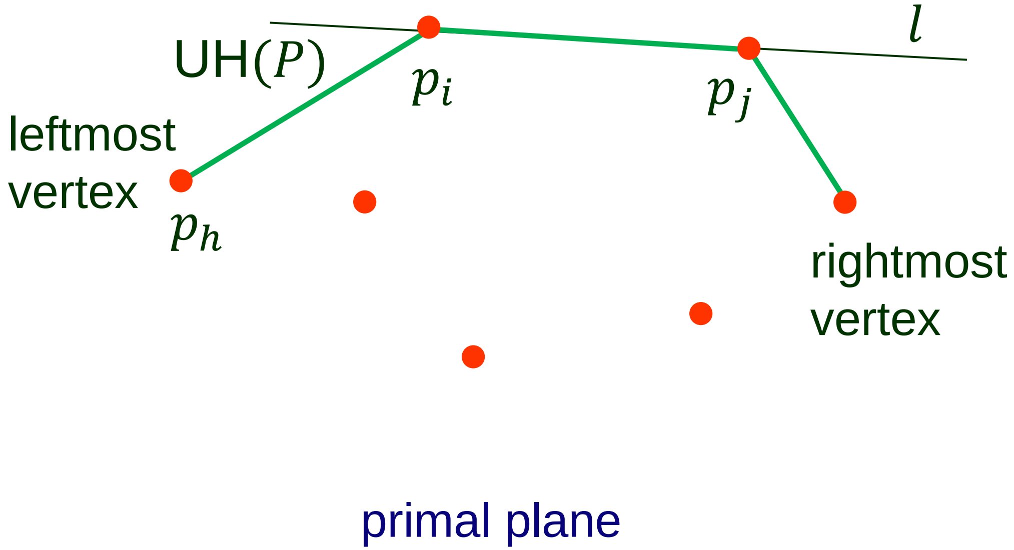

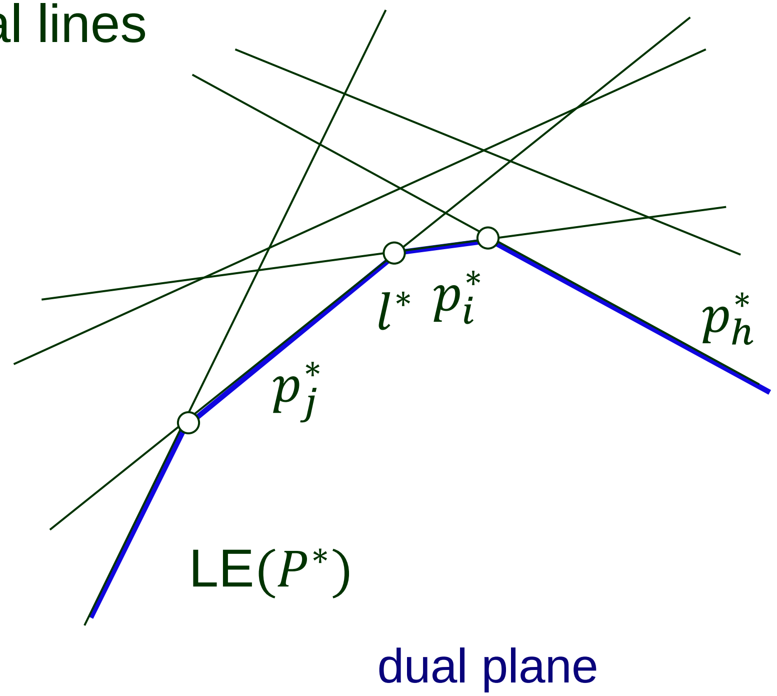

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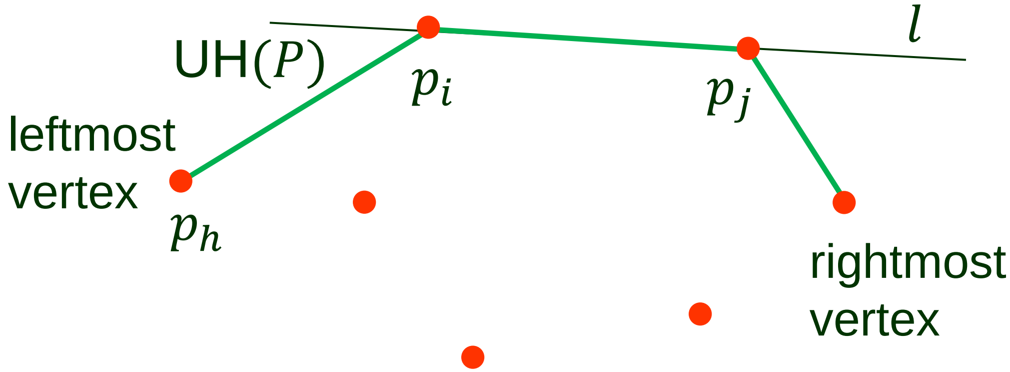
l above all points $\Rightarrow$ l^* below their dual lines



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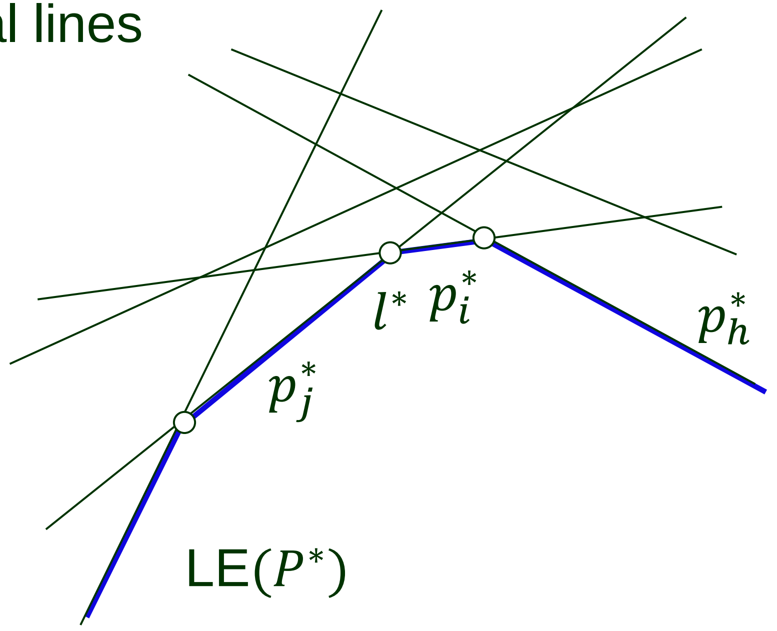

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$\mathrm{LE}(P^*)$: *lower envelope* of P^* is the unique bottom cell of the arrangement.

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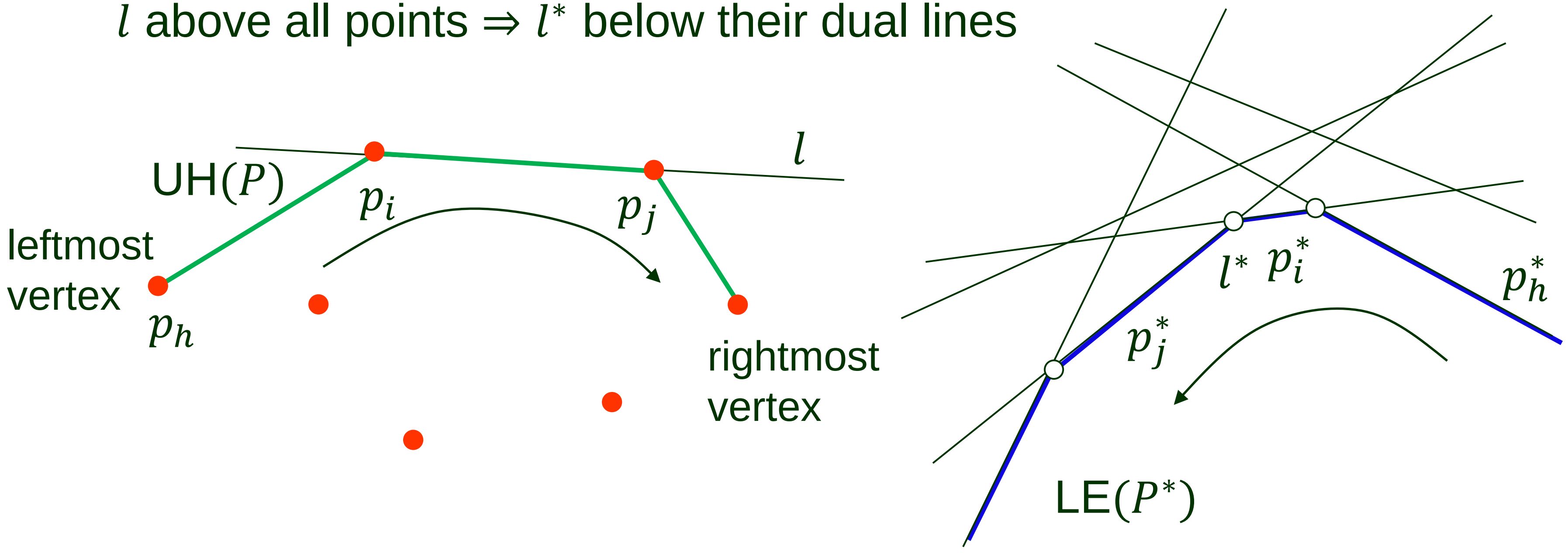

primal plane

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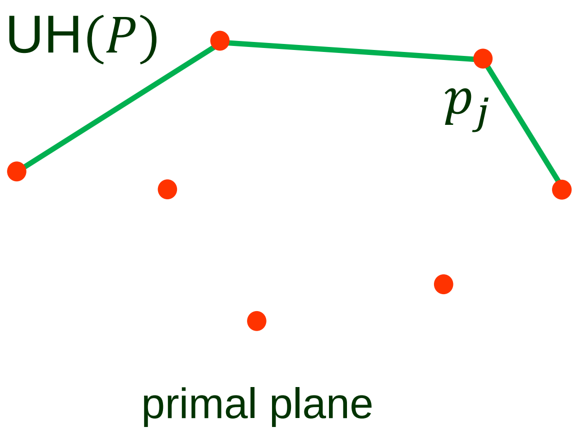
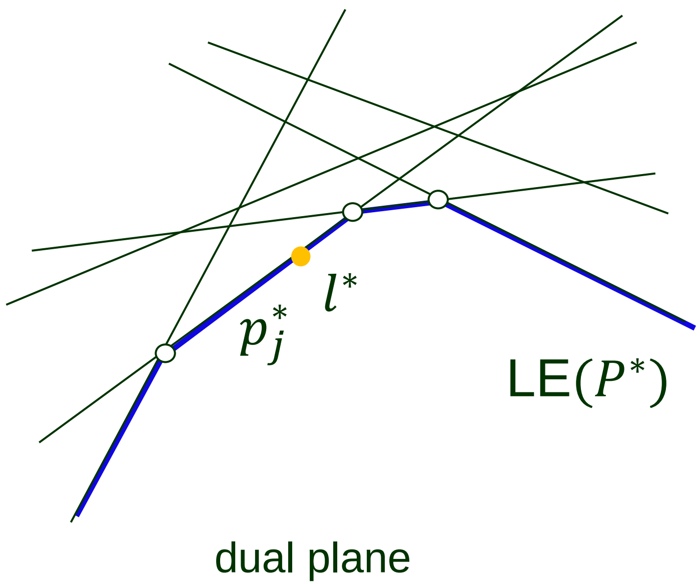
$\mathrm{LE}(P^*)$: *lower envelope* of P^* is the unique bottom cell of the arrangement.

Vertex → Edge



p_j is a vertex of UH(P).



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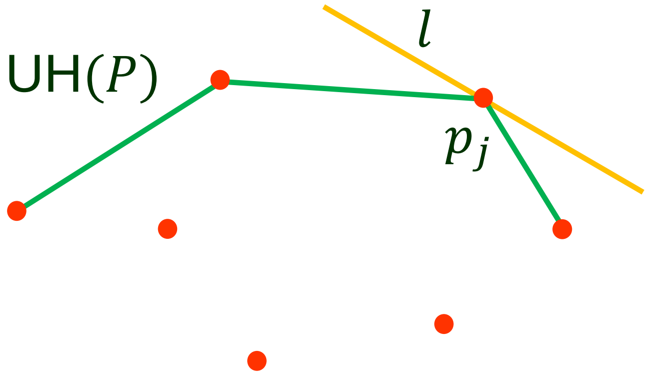
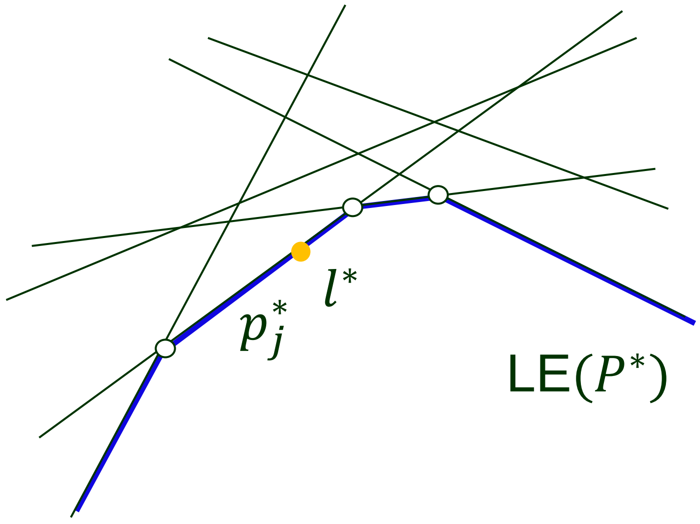


primal plane

p_j is a vertex of UH(P).

⇕

There is a non-vertical line l through p_j such that all other points are below l.



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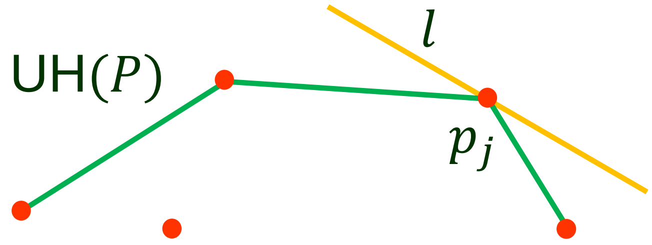

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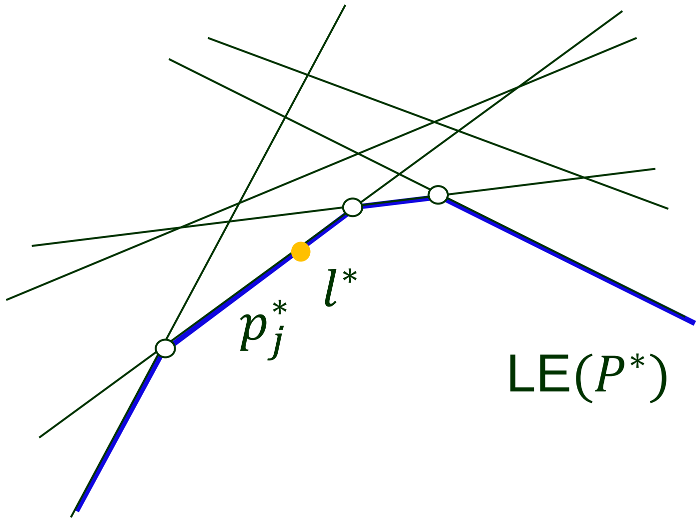

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Its dual point l^* on the line

$$p_j^* \in P^* = \{p^* | \, p \in P\}$$

lies below all other lines of P^*.

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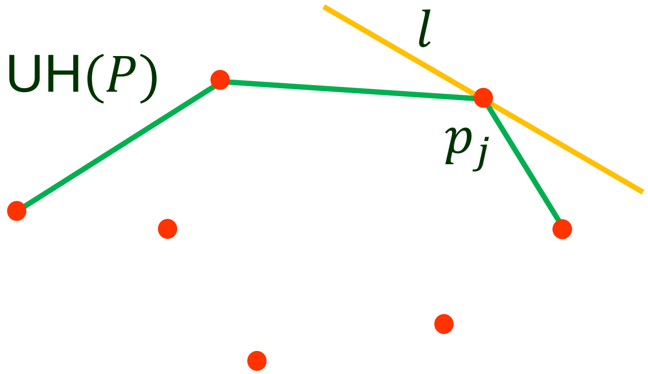

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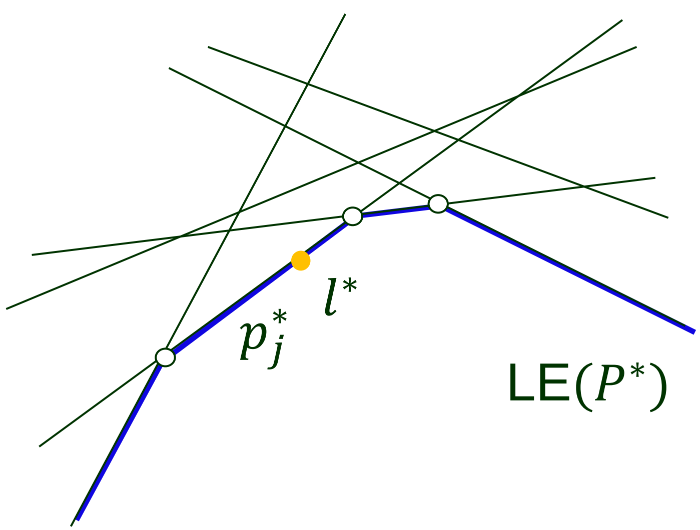

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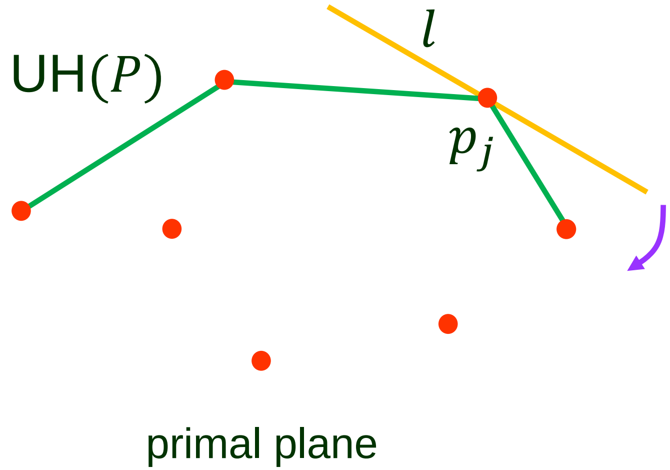

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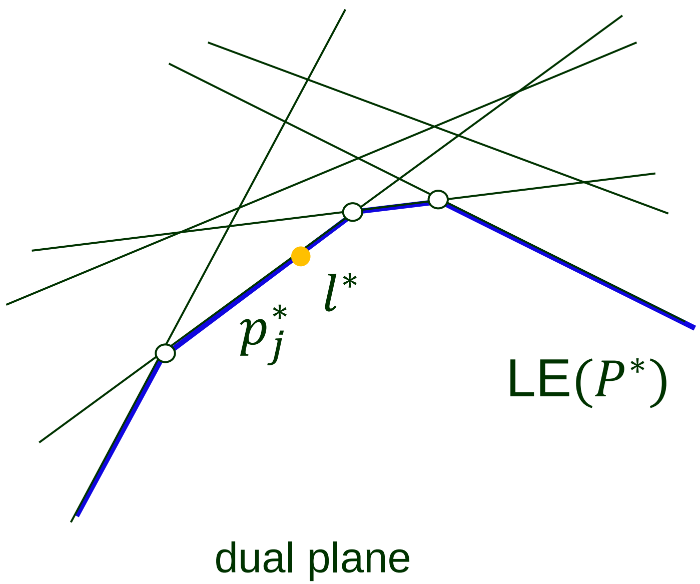

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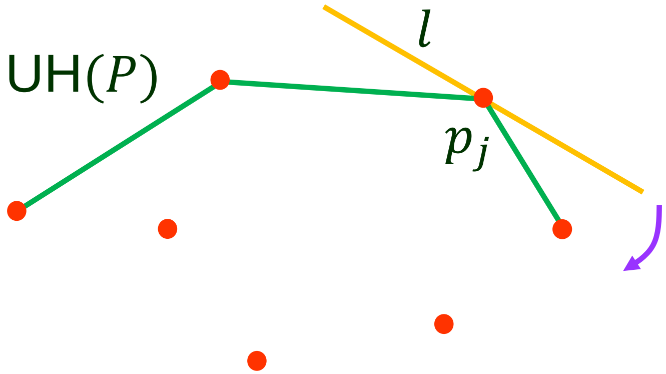

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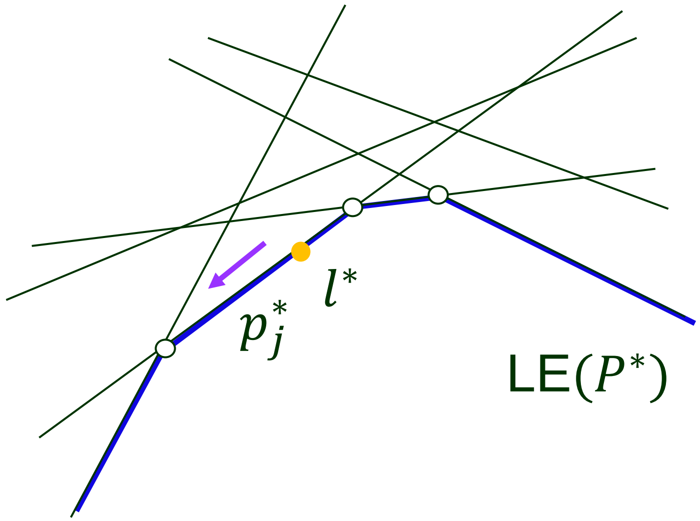

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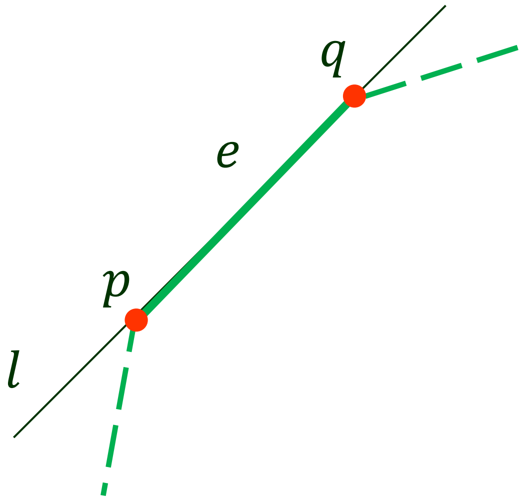

$p, q \in P$ define an edge e in $\mathrm{UH}(P)$.

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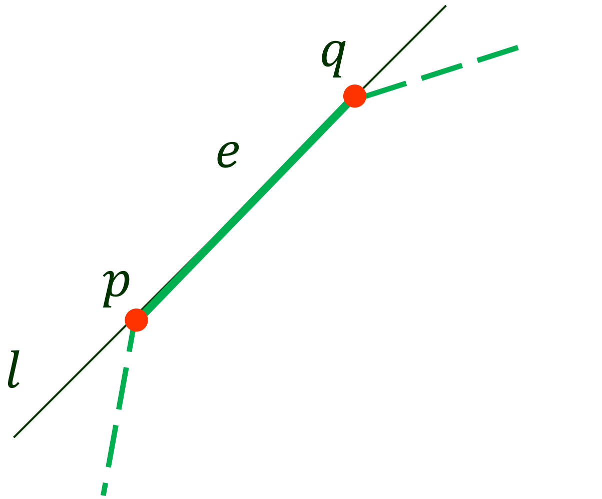

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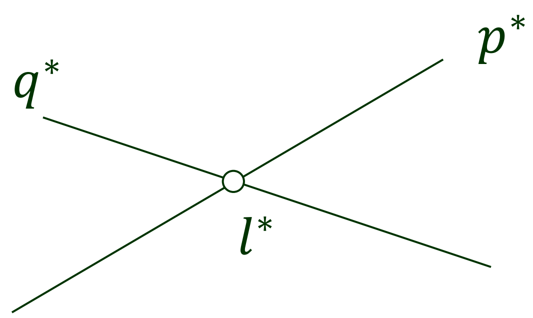

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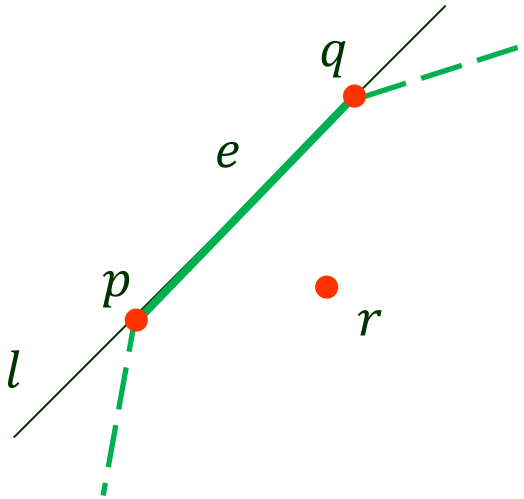

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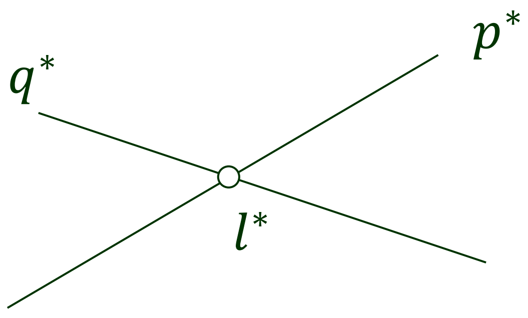

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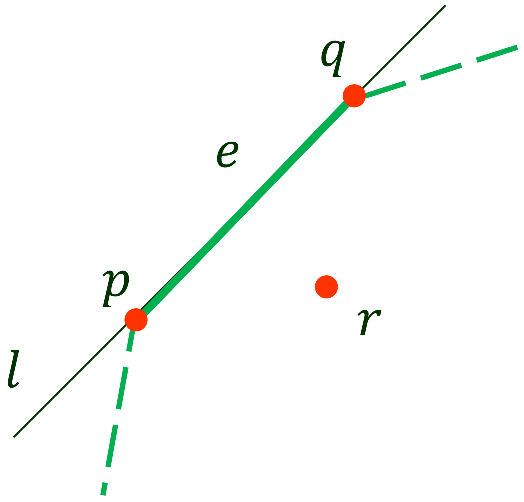

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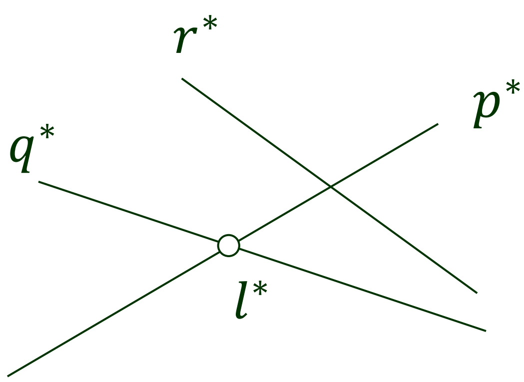

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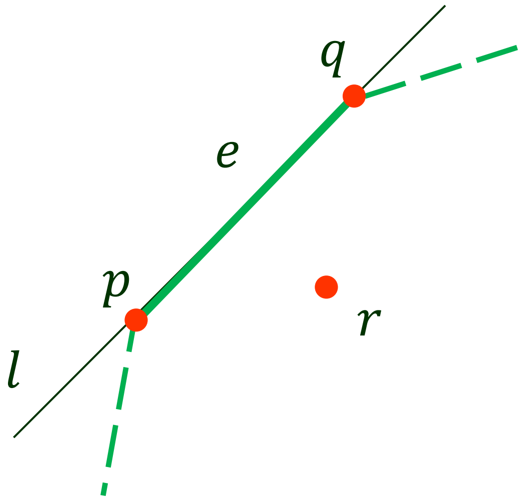

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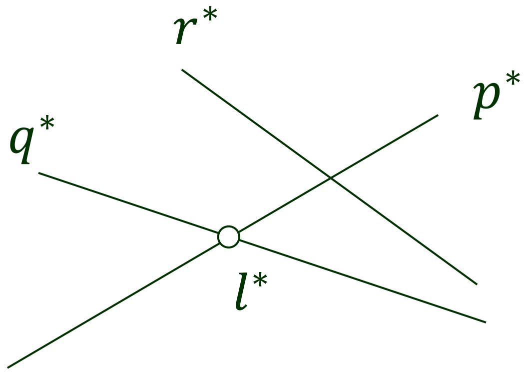

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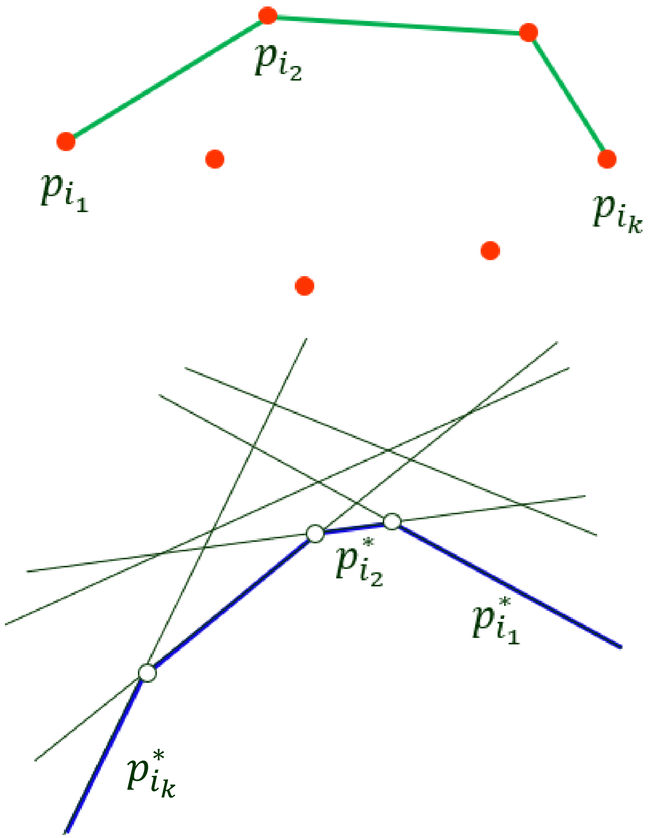
l^* is a vertex of LE(P^*).

Order Reversal

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on UH(P).



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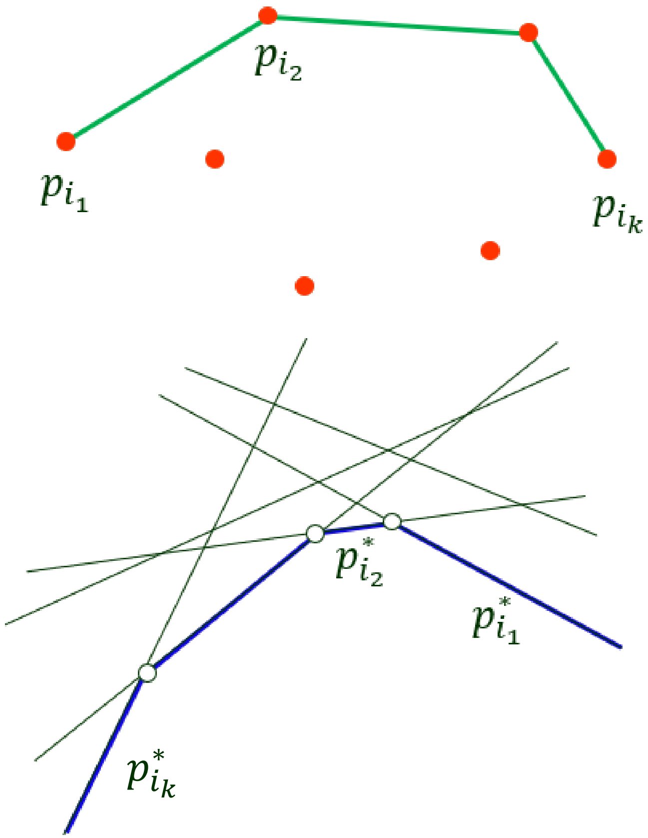

$p_{i_s} = (x_{i_s}, y_{i_s})$ precedes $p_{i_t} = (x_{i_t}, y_{i_t})$

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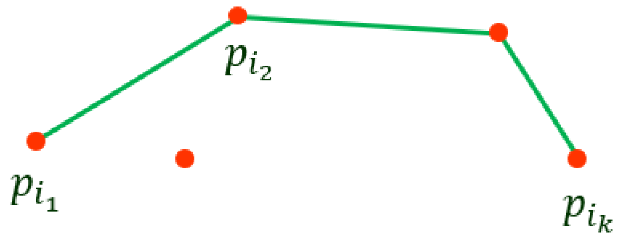

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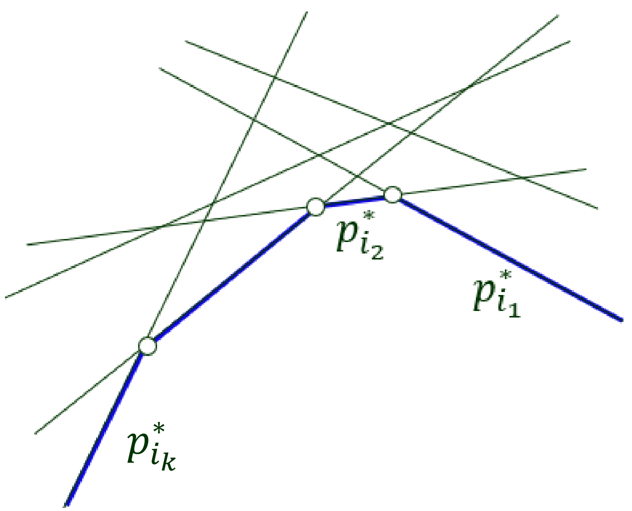
$\Downarrow$ left-to-right

$x_{i_s} < x_{i_t}$



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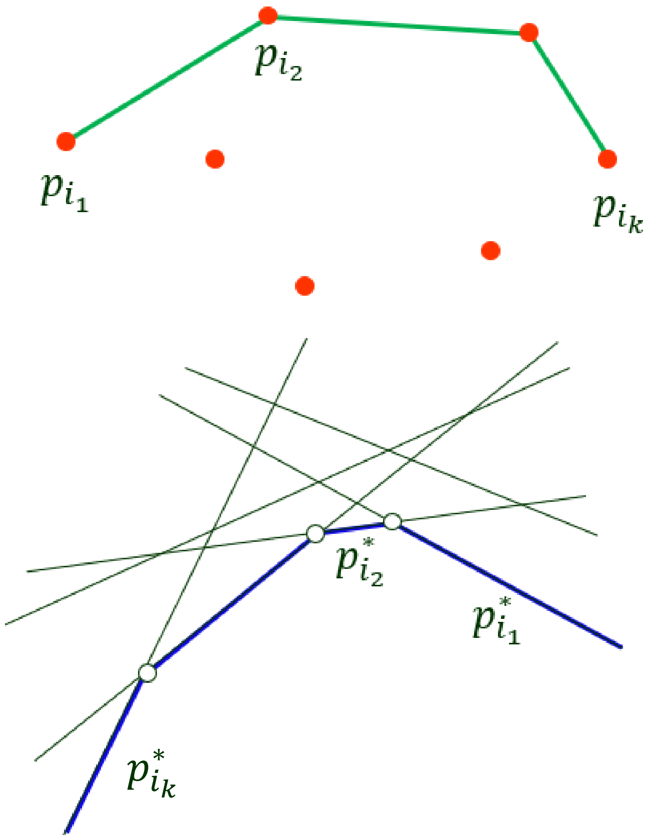

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The dual line $p^*_{i_s}: y = x_{i_s}x - y_{i_s}$ has a smaller slope than the dual line $p^*_{i_t}: y = x_{i_t}x - y_{i_t}$

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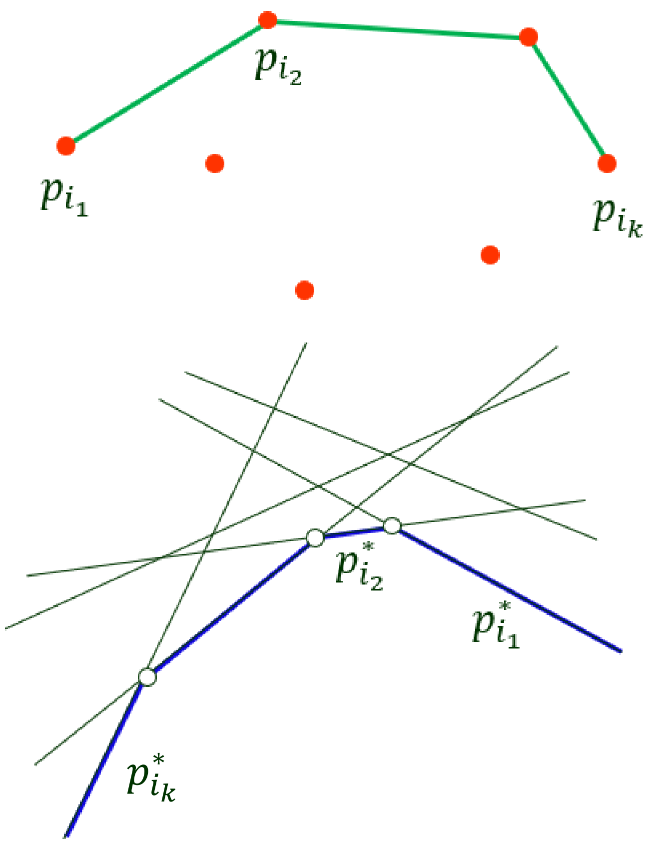

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On the lower envelope, the segment of $p^*_{i_s}$ is before that of $p^*_{i_t}$ in the right-to-left order.

Order Reversal

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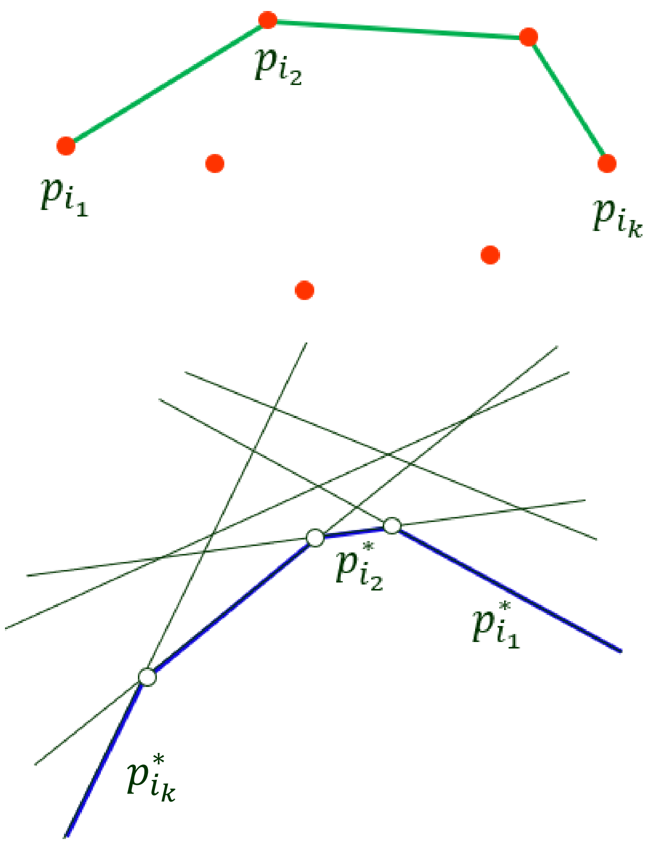

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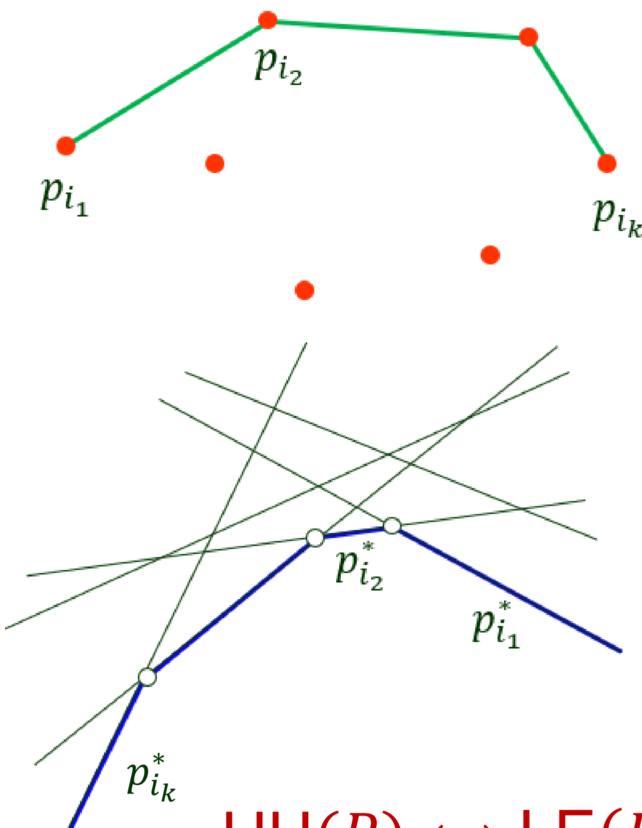

UH$(P) \leftrightarrow$ LE(P^*)

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Lower Convex Hull & Upper Envelope

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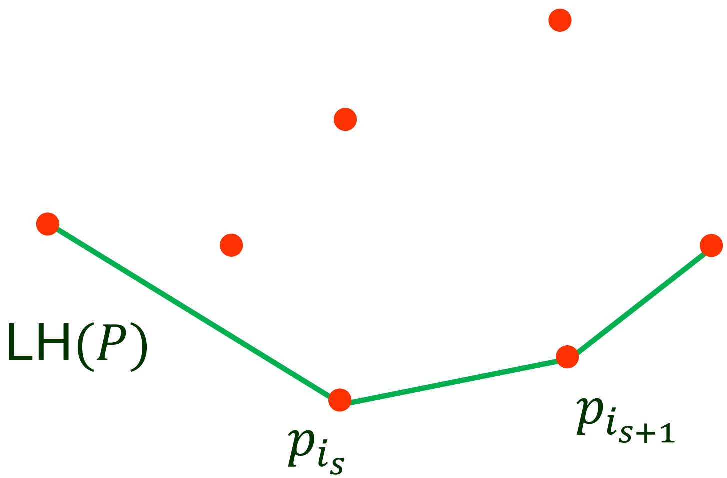
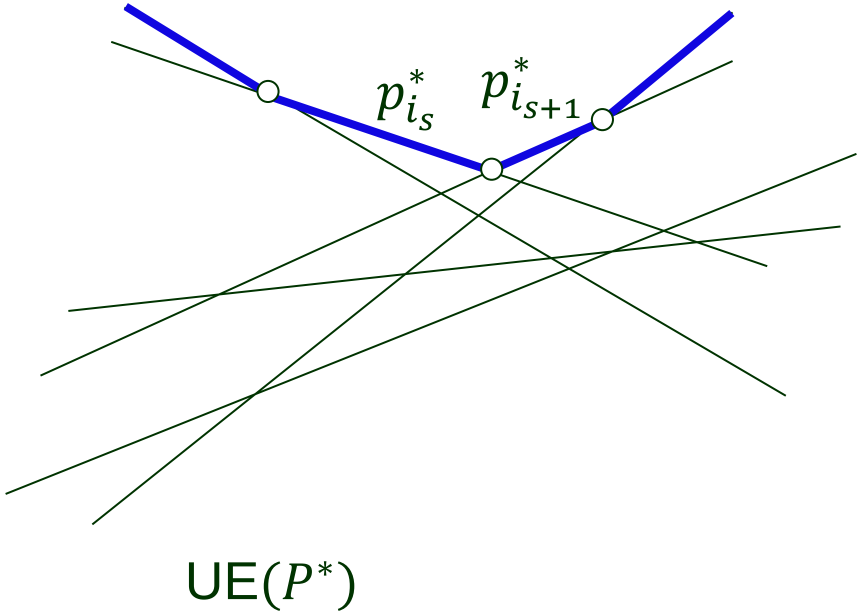
UE(P^*): *upper envelope* of P^*



primal plane



dual plane

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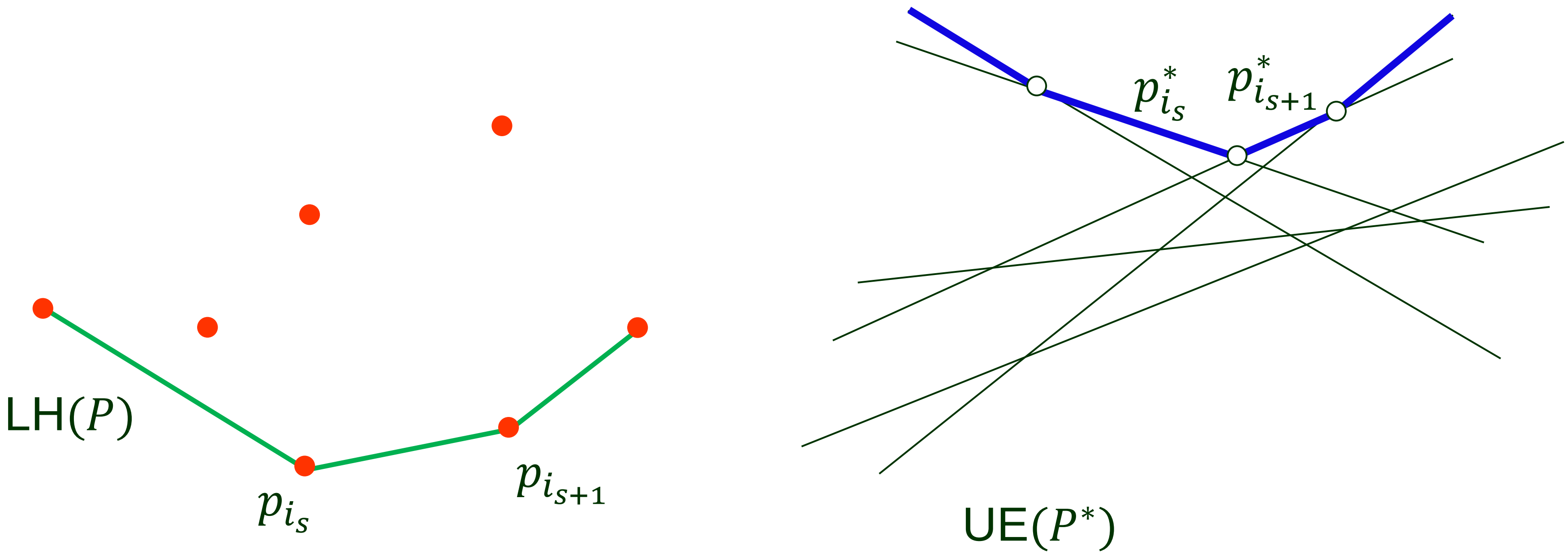

primal plane dual plane

$p_{i_1}, p_{i_2}, \ldots, p_{i_k}$: left-to-right order of vertices on LH(P).

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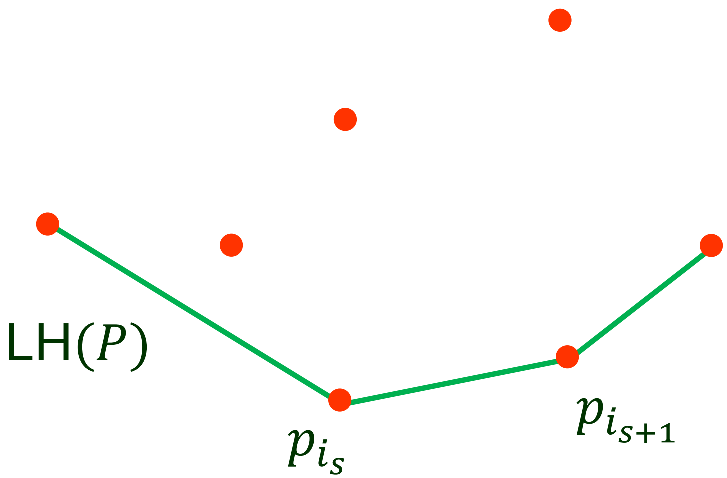

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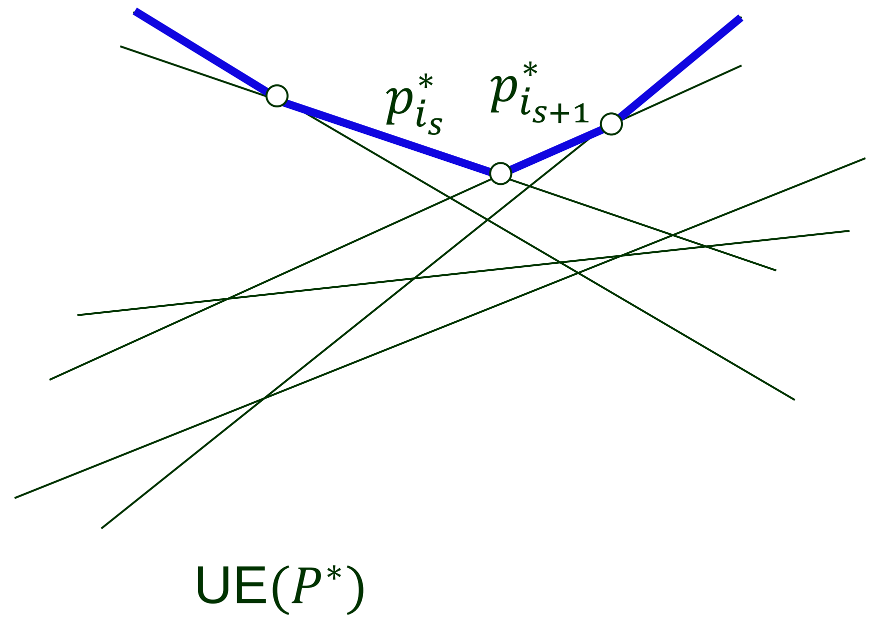

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$$\mathrm{UH}(P) \longleftrightarrow \mathrm{LE}(P^*)$$

By symmetry,

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- The above relationships generalize to convex hulls and intersection of half-spaces in 3D.
- Convex hulls and intersections of half-planes (or half-spaces) are essentially dual concepts.

Hull-Envelope Correspondences

$$\text{UH}(P) \longleftrightarrow \text{LE}(P^*)$$

By symmetry,

$$\text{LH}(P) \longleftrightarrow \text{UE}(P^*)$$

- The above relationships generalize to convex hulls and intersection of half-spaces in 3D.
- Convex hulls and intersections of half-planes (or half-spaces) are essentially dual concepts.

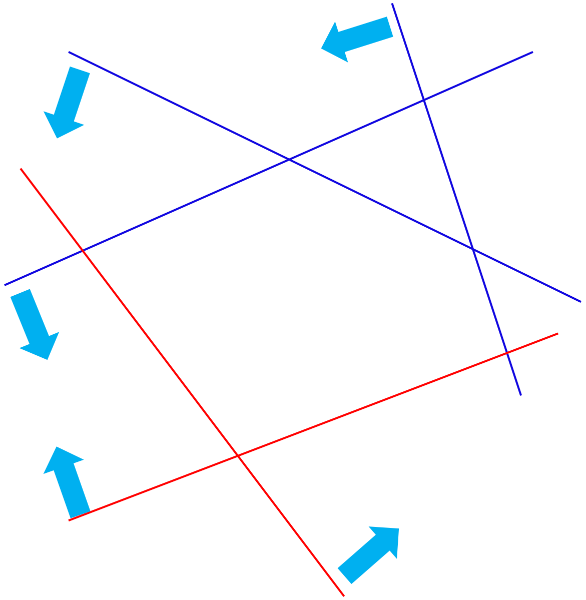
Computing an upper (lower) convex hull

$$\Updownarrow$$

Intersecting lower (upper) half-planes

Algorithm for Half-Plane Intersection

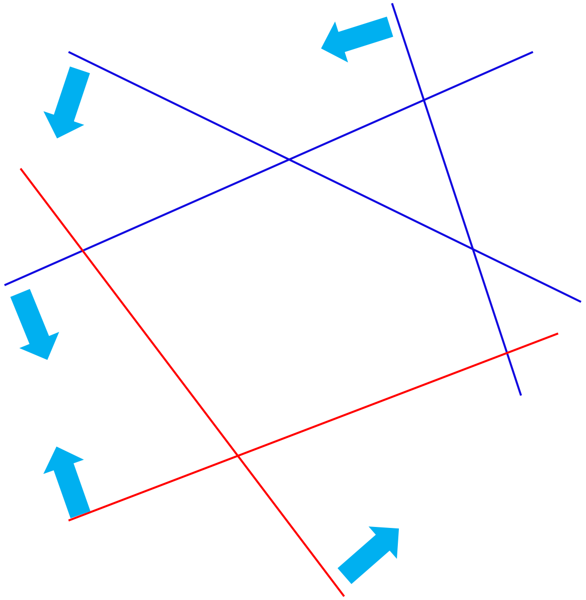
H: a set of half-planes



Idea: Dualize a convex hull algorithm.

Algorithm for Half-Plane Intersection

H: a set of half-planes

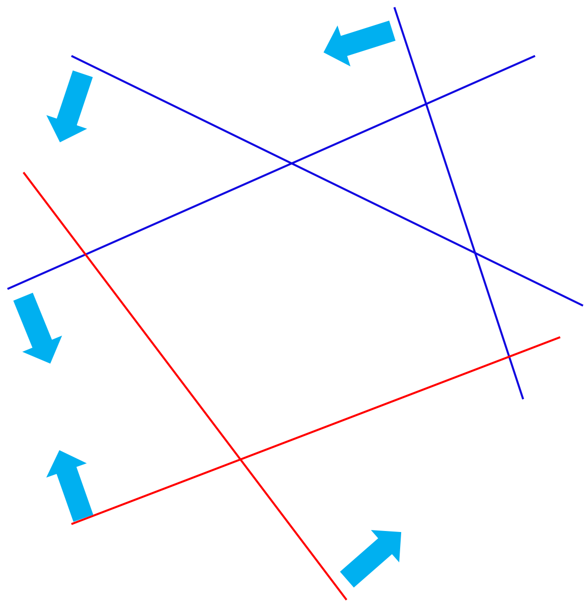


Idea: Dualize a convex hull algorithm.

a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes.

Algorithm for Half-Plane Intersection

H: a set of half-planes

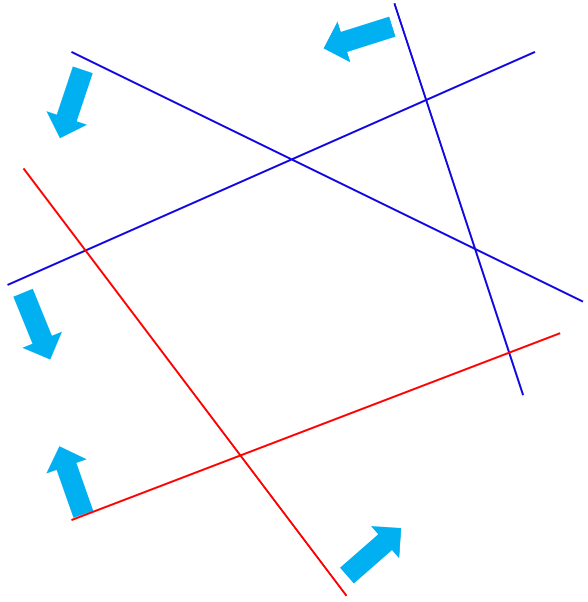


Idea: Dualize a convex hull algorithm.

a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes. $O(n)$

Algorithm for Half-Plane Intersection

H: a set of half-planes



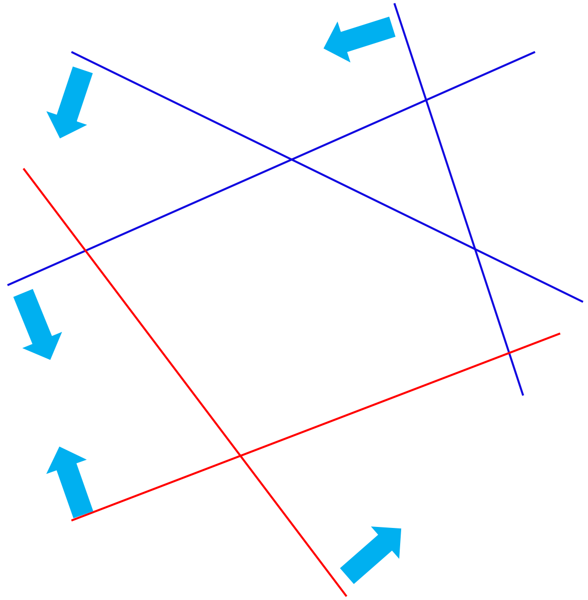
Idea: Dualize a convex hull algorithm.

a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes. $O(n)$

b) Compute $\cap H_+$ by constructing the lower convex hull of H_+^*.

Algorithm for Half-Plane Intersection

H: a set of half-planes



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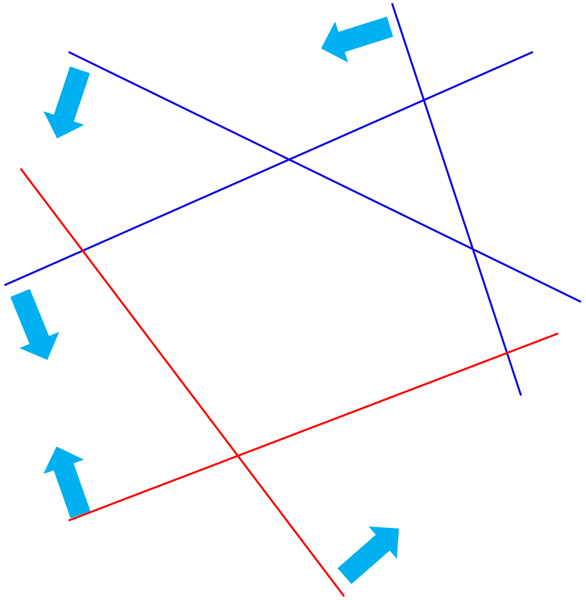
a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes. $O(n)$

b) Compute $\cap\, H_+$ by constructing the lower convex hull of H_+^*. $O(n \log n)$

Algorithm for Half-Plane Intersection

H: a set of half-planes



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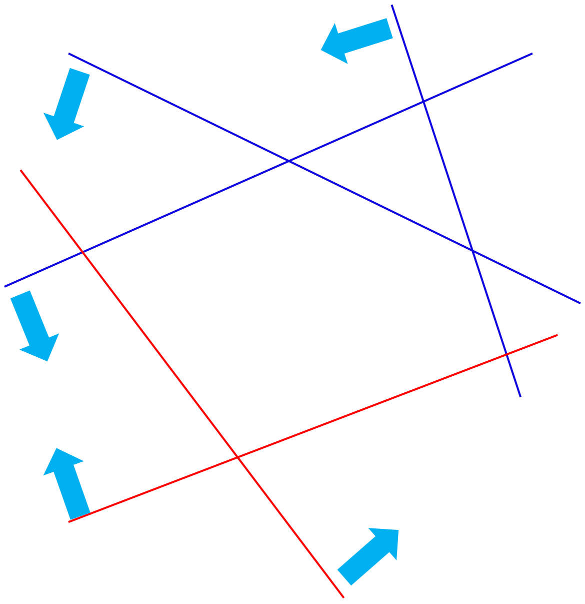
Algorithm for Half-Plane Intersection

H: a set of half-planes



Idea: Dualize a convex hull algorithm.

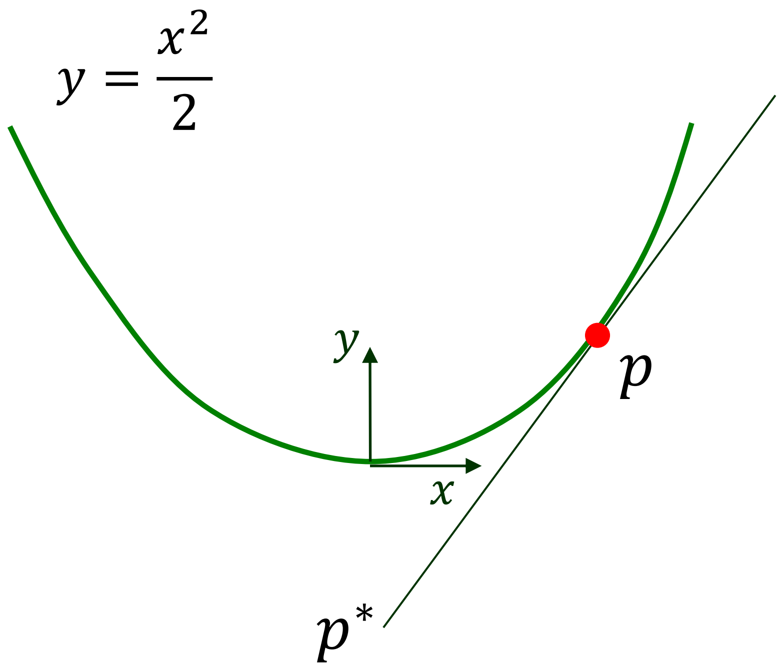
a) Split H into a set H_+ of upper half-planes and a set H_- of lower half-planes. $O(n)$

b) Compute $\cap\, H_+$ by constructing the lower convex hull of H_+^*. $O(n \log n)$

c) Compute $\cap\, H_-$ by constructing the upper convex hull of H_-^*. $O(n \log n)$

d) Intersect H_+ and H_-. $O(n)$

III. Review: Duality with a Parabola

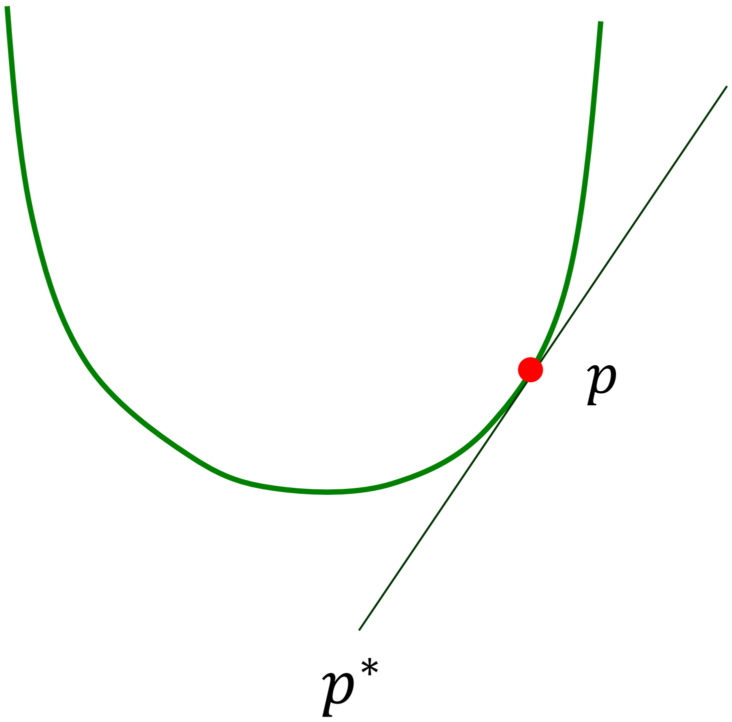


- Dual p^* of p on the parabola is the tangent line at p.

Point Not on a Parabola

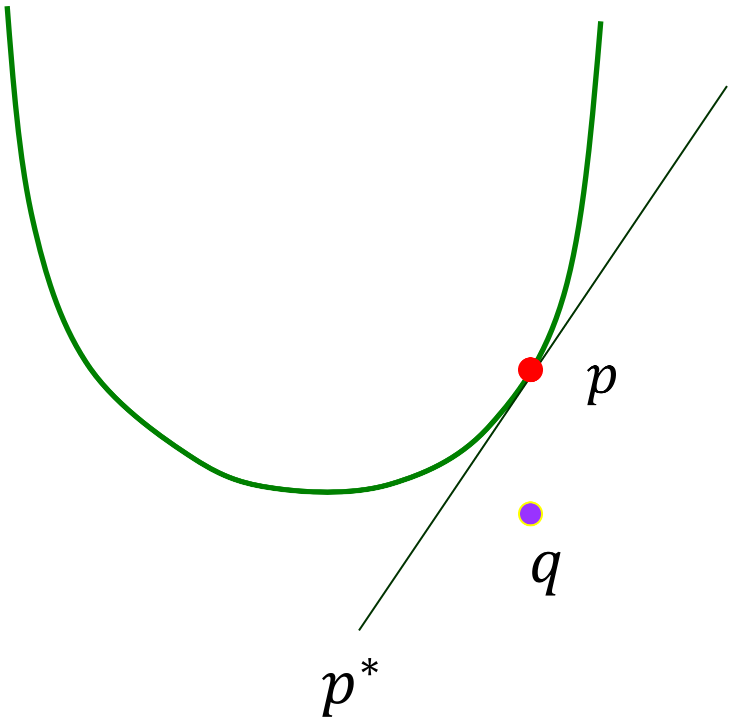
$$y = \frac{x^2}{2}$$

$$p = (p_x, p_y)$$



Point Not on a Parabola

$$y = \frac{x^2}{2}$$



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Point Not on a Parabola

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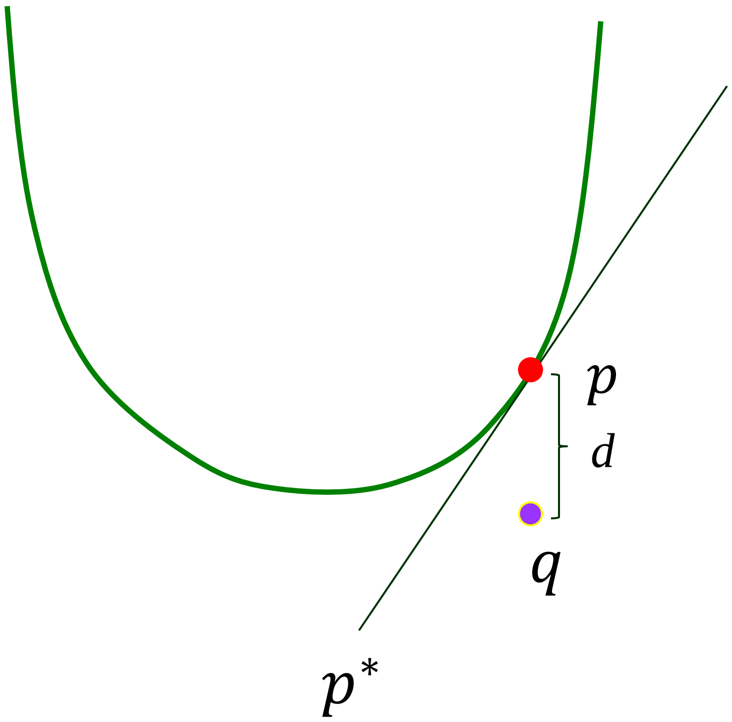
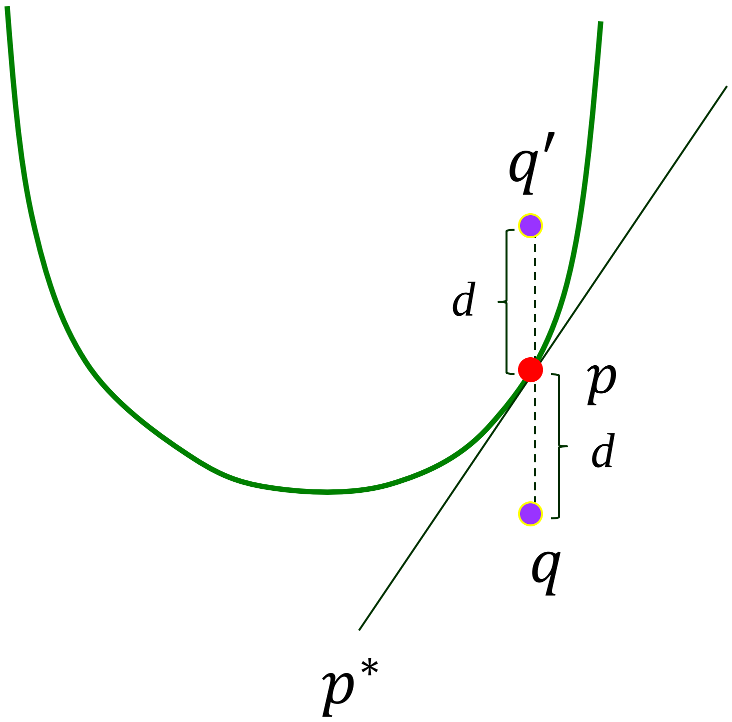


Point Not on a Parabola

$$y = \frac{x^2}{2}$$

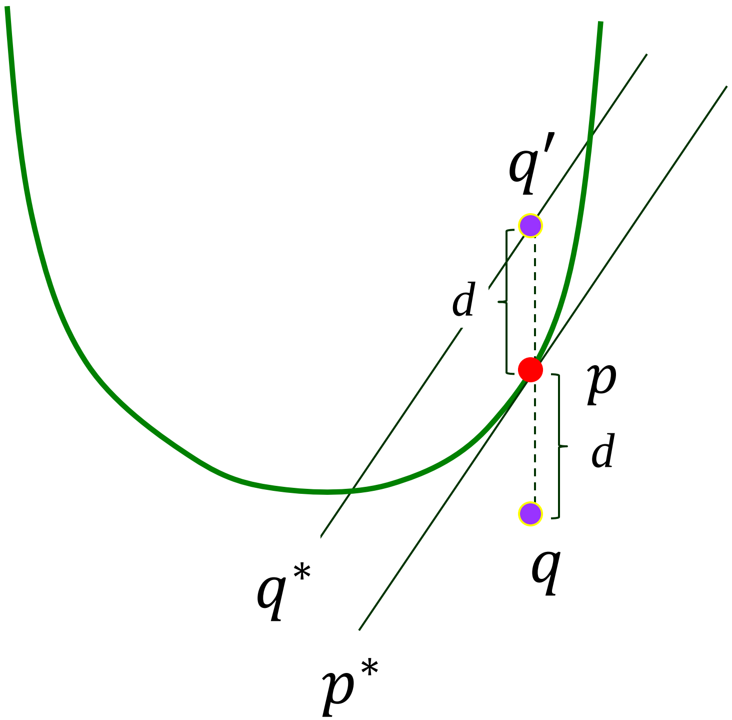


$$p = (p_x, p_y)$$

$$q = (p_x, p_y - d)$$

Point Not on a Parabola

$$y = \frac{x^2}{2}$$

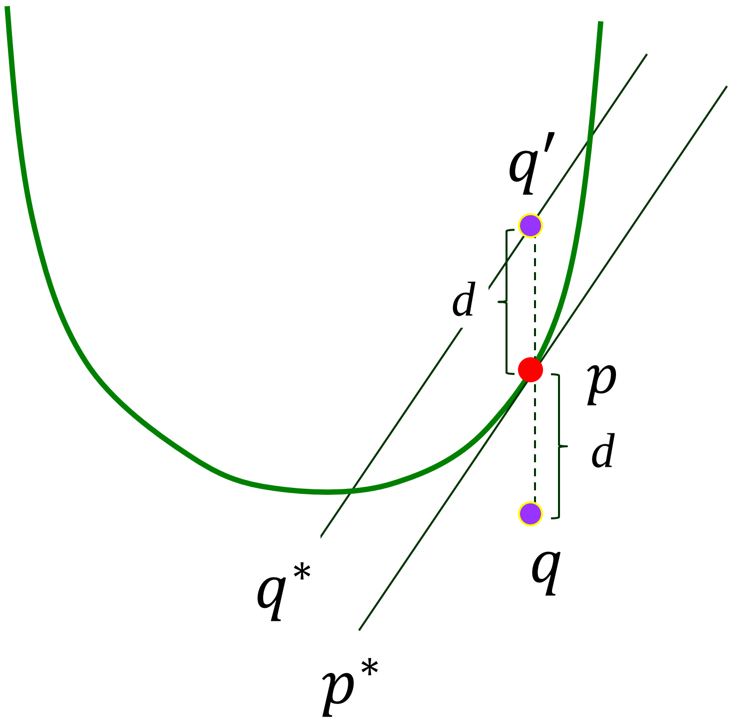


$$p = (p_x, p_y)$$

$$q = (p_x, p_y - d)$$

Point Not on a Parabola

$$y = \frac{x^2}{2}$$



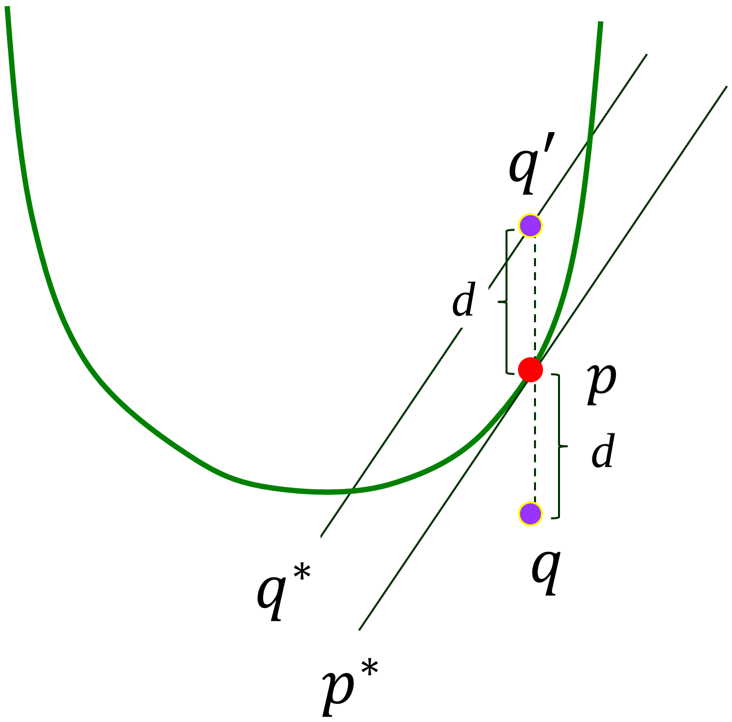
$$p = (p_x, p_y)$$

$$q = (p_x, p_y - d)$$

$$q' = (p_x, p_y + d)$$

Point Not on a Parabola

$$y = \frac{x^2}{2}$$



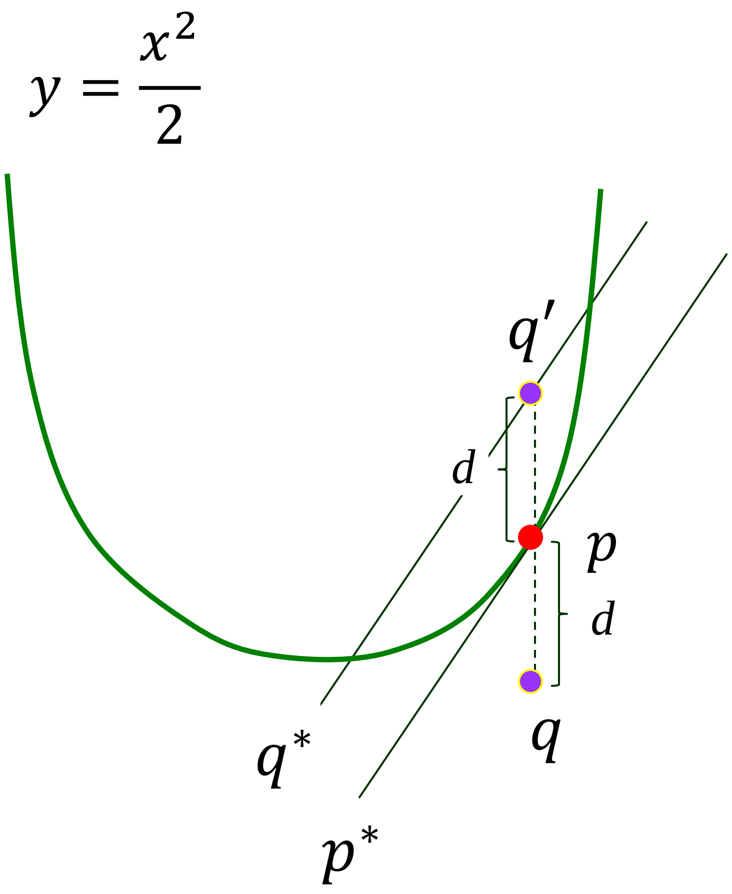
$$p = (p_x, p_y)$$

$$q = (p_x, p_y - d)$$

$$q' = (p_x, p_y + d)$$

$$q' - p = p - q$$

Point Not on a Parabola



$p = (p_x, p_y)$

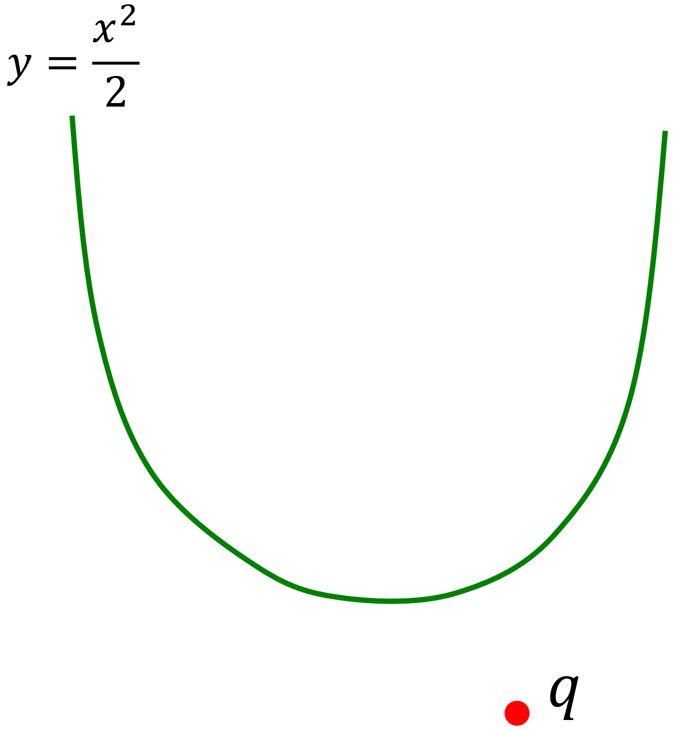
$q = (p_x, p_y - d)$

$q' = (p_x, p_y + d)$

$q' - p = p - q$

- The dual line $q^* \parallel p^*$ and it passes through q'.

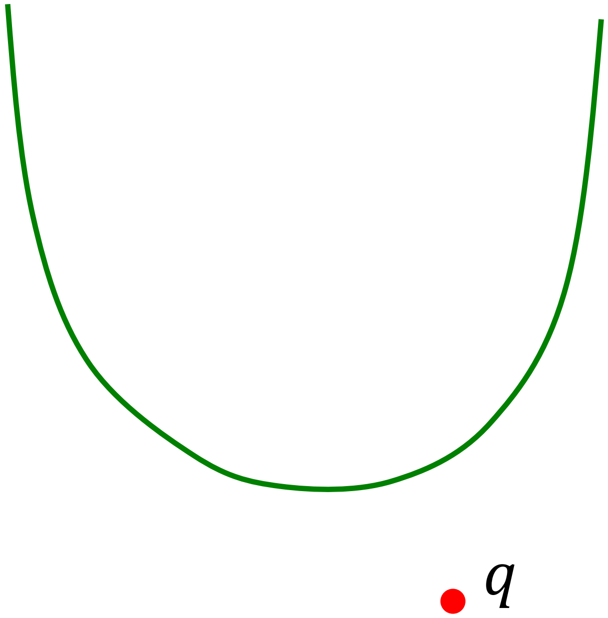
More on Duality



Construct the dual line q^* of q without measuring distances:

More on Duality

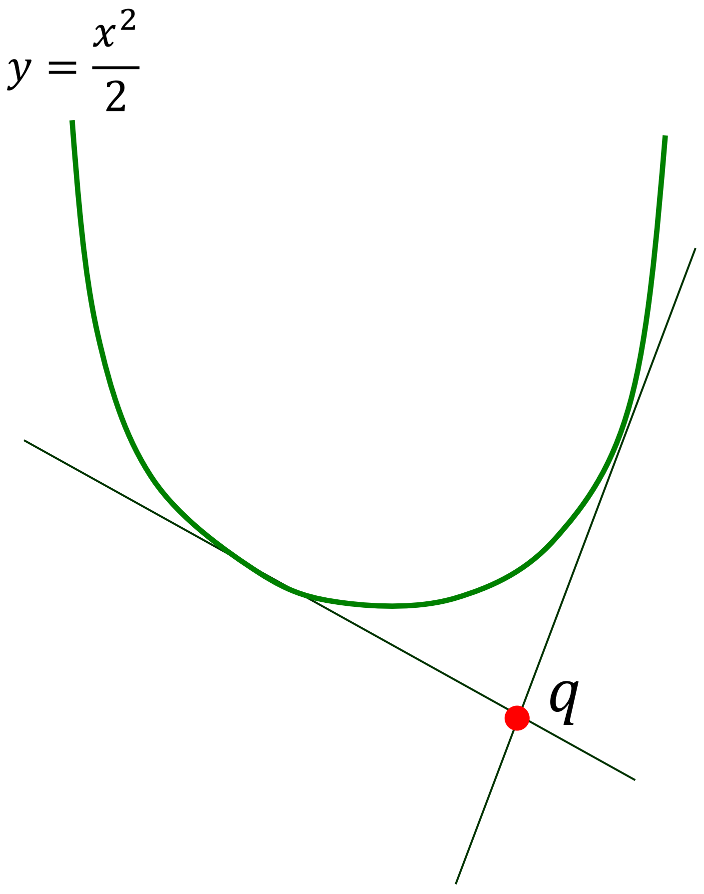
$y = \frac{x^2}{2}$



Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

More on Duality



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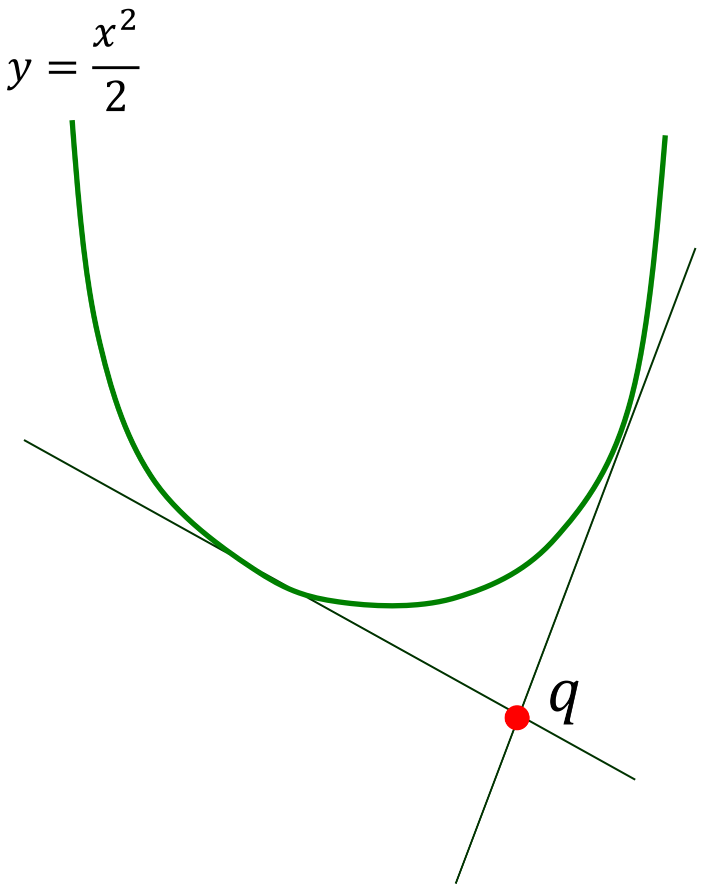
More on Duality

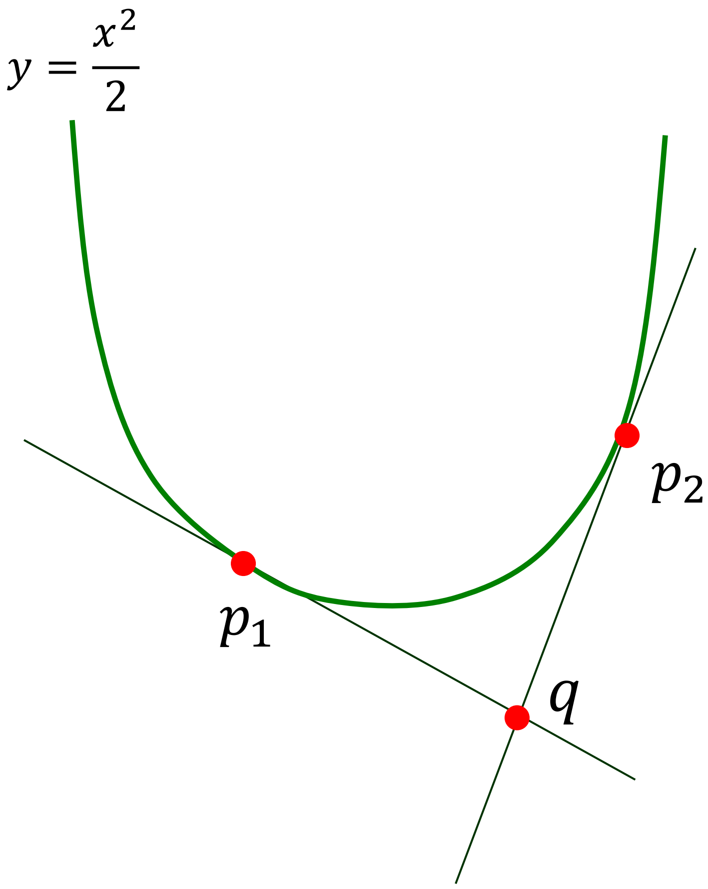


Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

More on Duality



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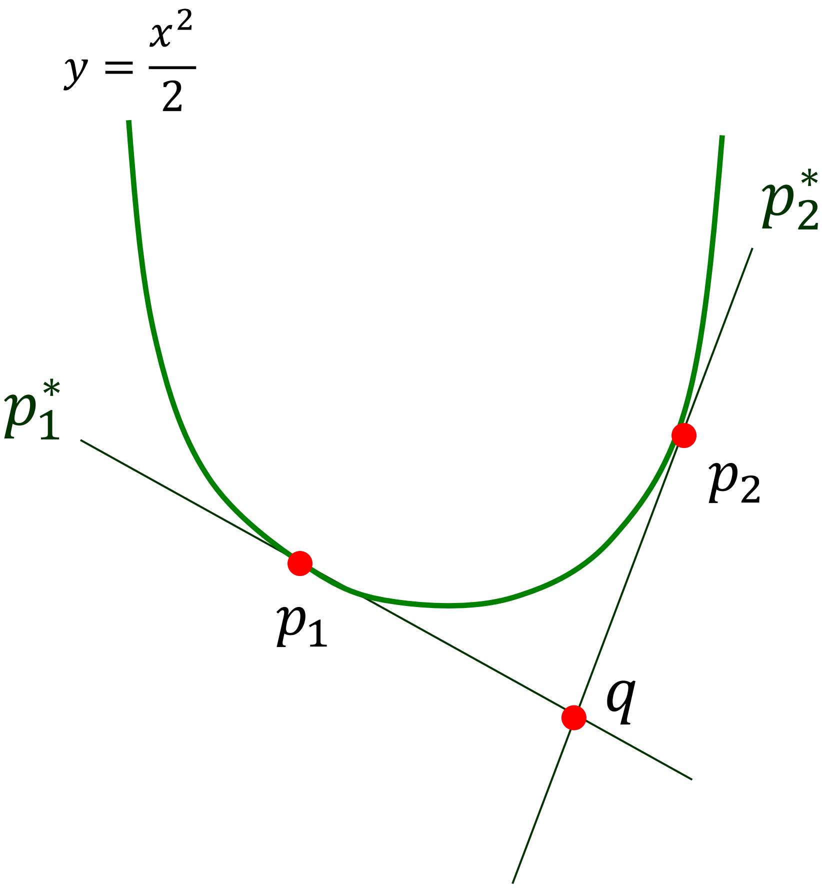
More on Duality



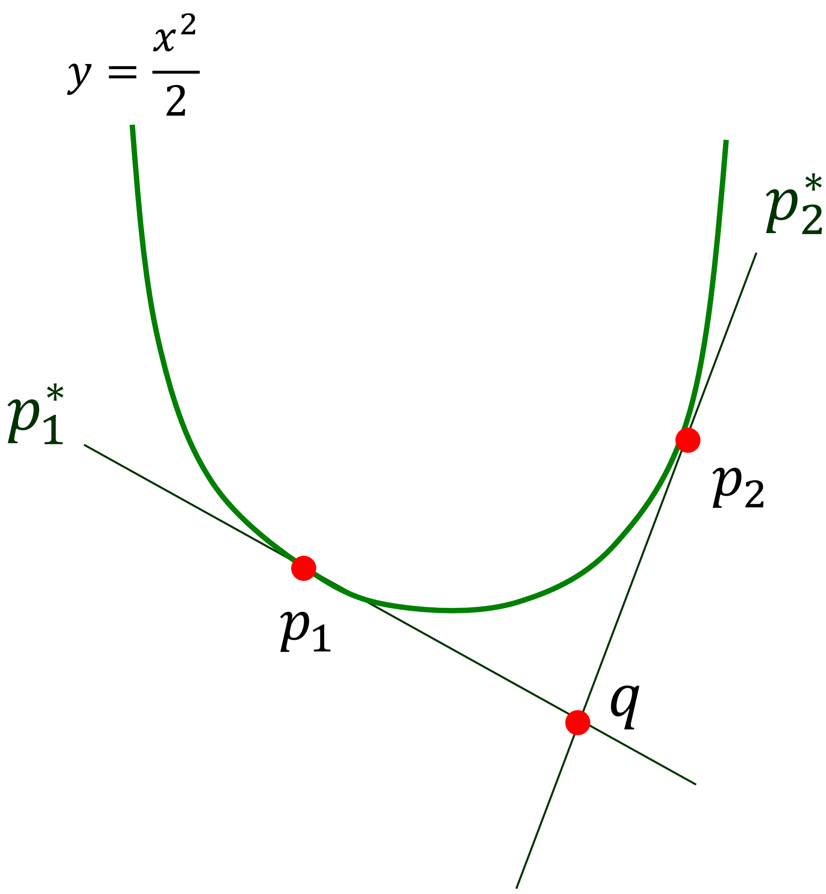
Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

The two tangent lines are p_1^* and p_2^*.

More on Duality



Construct the dual line q^* of q without measuring distances:

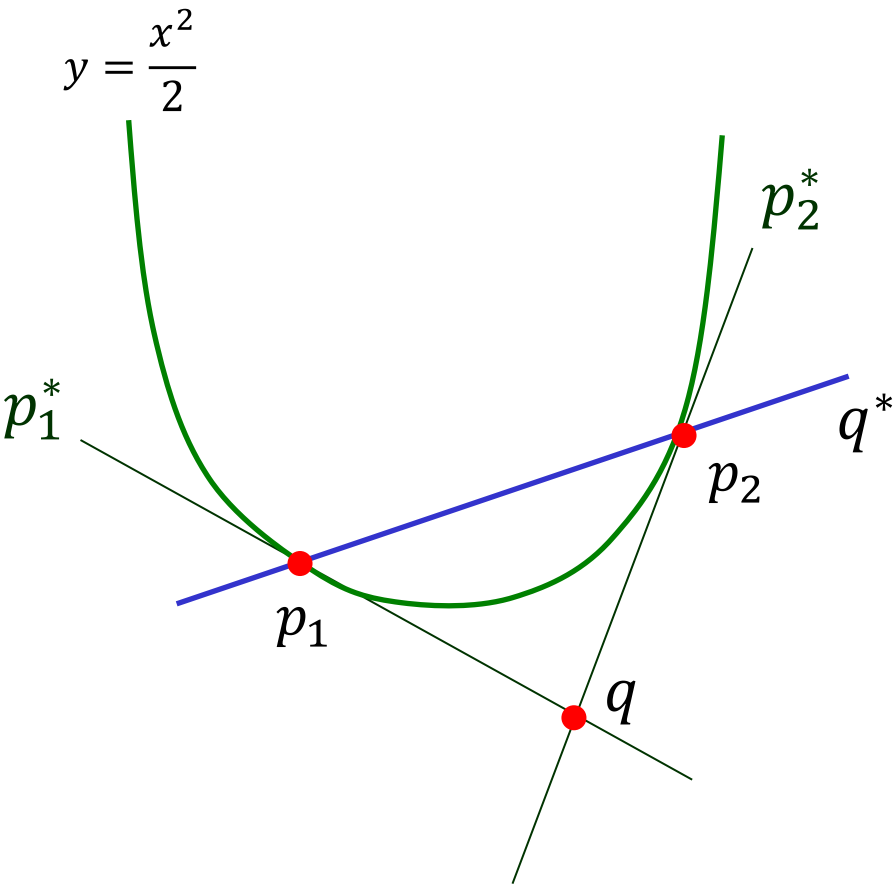
1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

3) q^* is the line through p_1 and p_2.

The two tangent lines are p_1^* and p_2^*.

More on Duality



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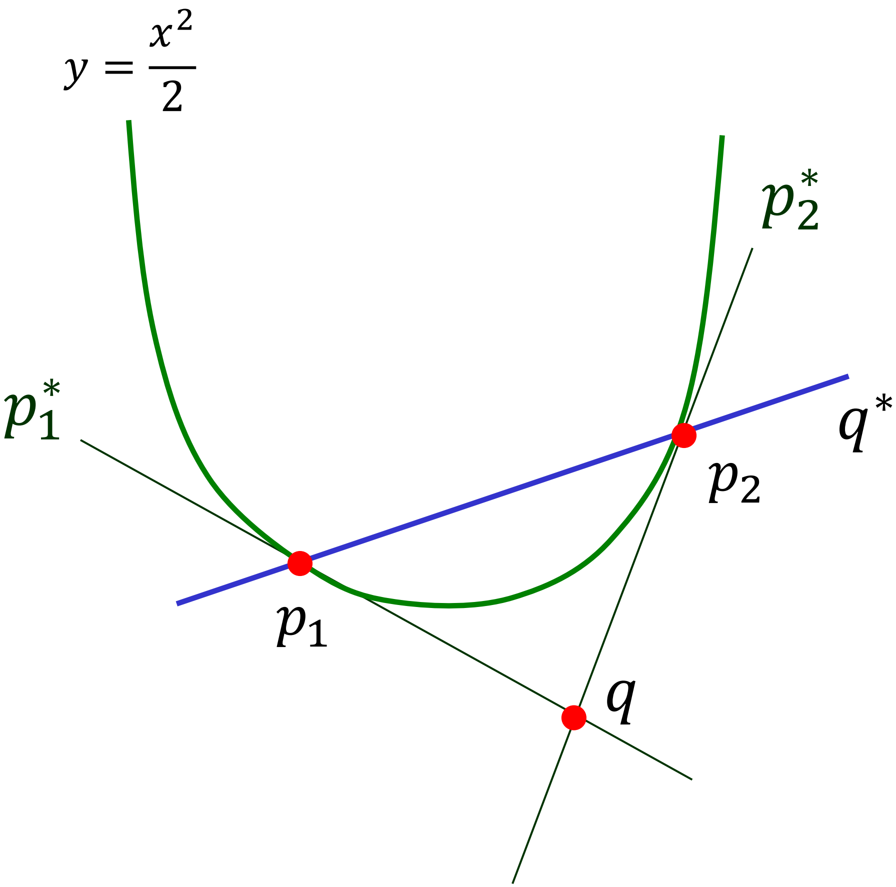
More on Duality



Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

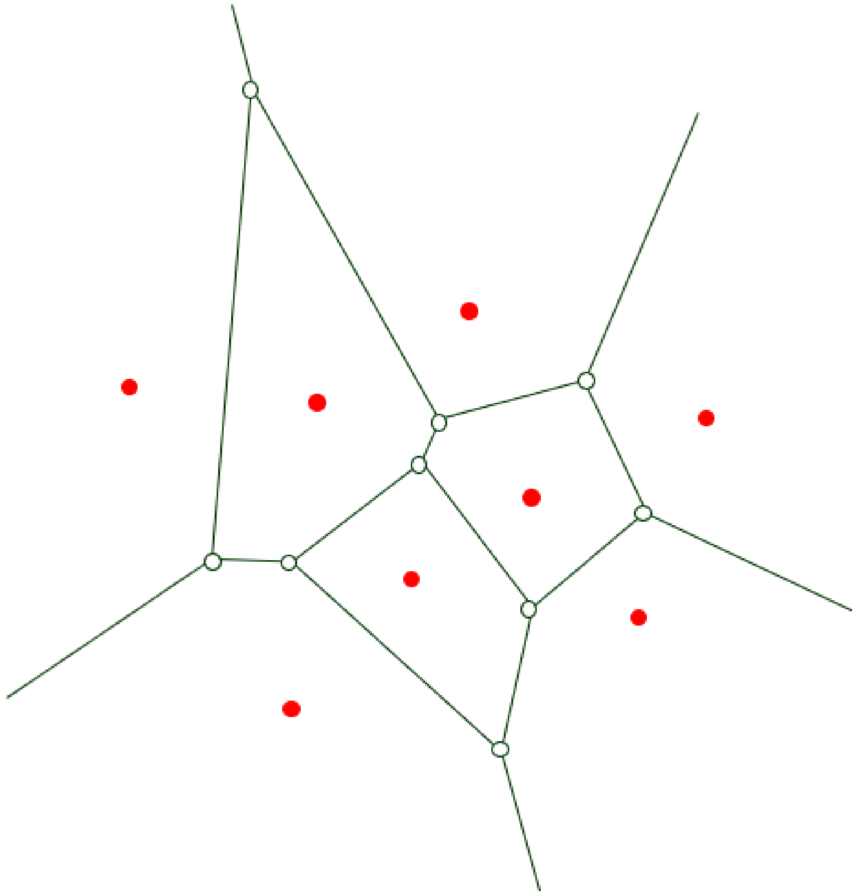
3) q^* is the line through p_1 and p_2.

The two tangent lines are p_1^* and p_2^*.

p_1^* and p_2^* intersect at $q \Leftrightarrow q^*$ passes through p_1 and p_2.

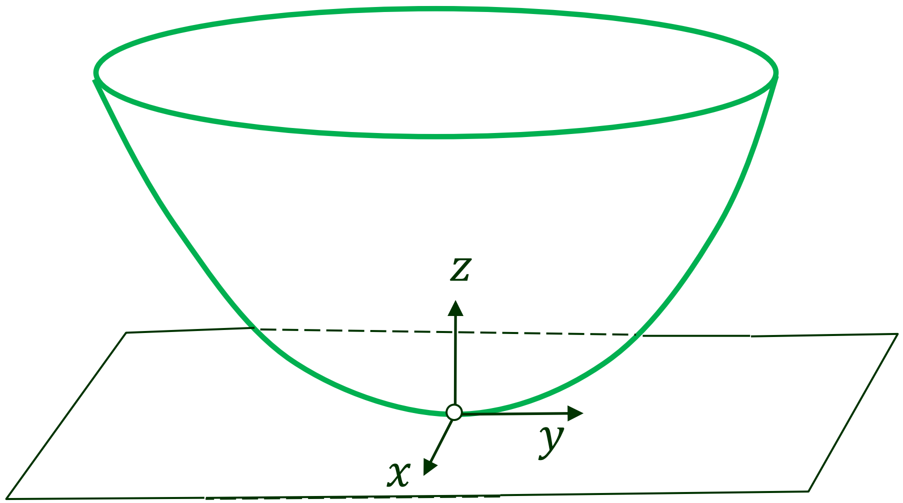
Voronoi Diagram Revisited

P: a set of n sites.



Unit Paraboloid

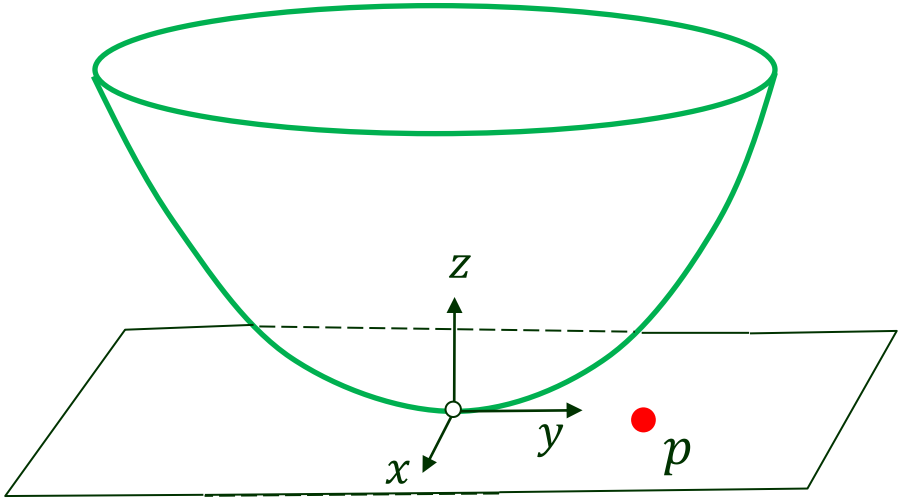
$$U: z = x^2 + y^2$$



Unit Paraboloid

$p = (p_x, p_y, 0)$

$U: z = \ x^2 + y^2$



Unit Paraboloid

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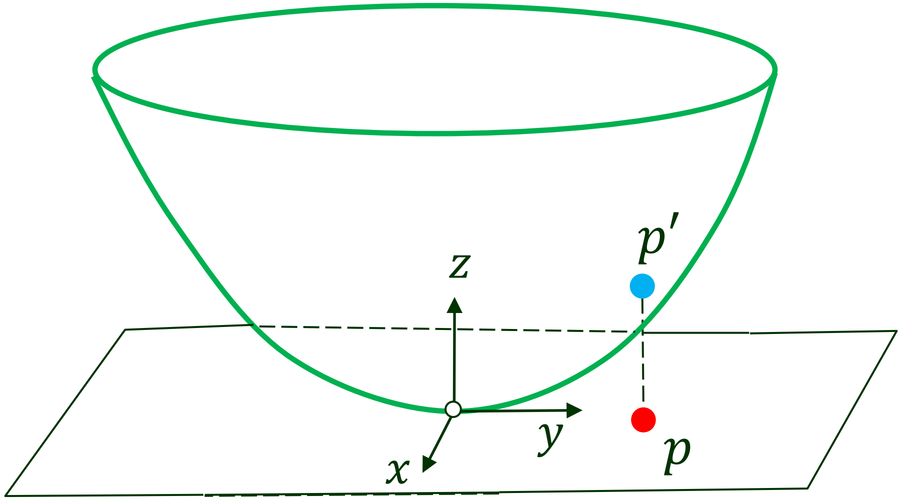


$p = (p_x, p_y, 0)$

Projection of p onto U:

$p' = (p_x, p_y, p_x^2 + p_y^2)$:

Unit Paraboloid

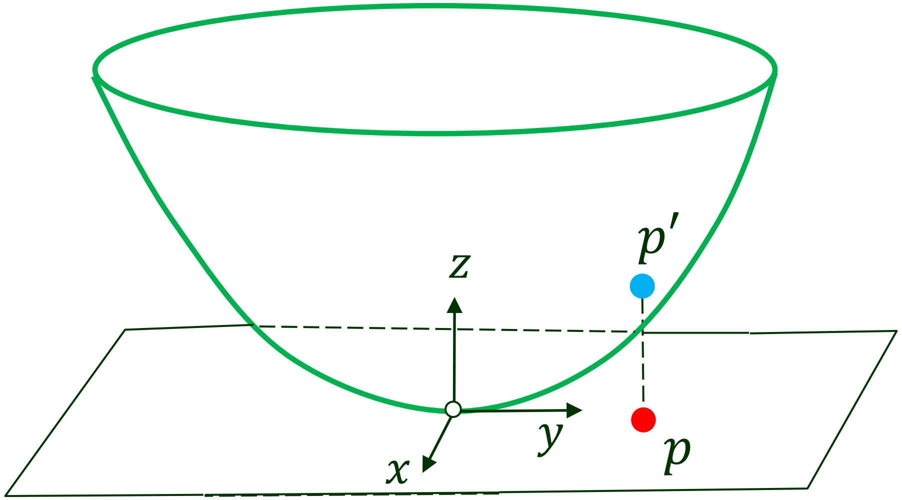
$p = (p_x, p_y, 0)$

$U: z = x^2 + y^2$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$

Projection of p onto U:

$p' = (p_x, p_y, p_x^2 + p_y^2)$:



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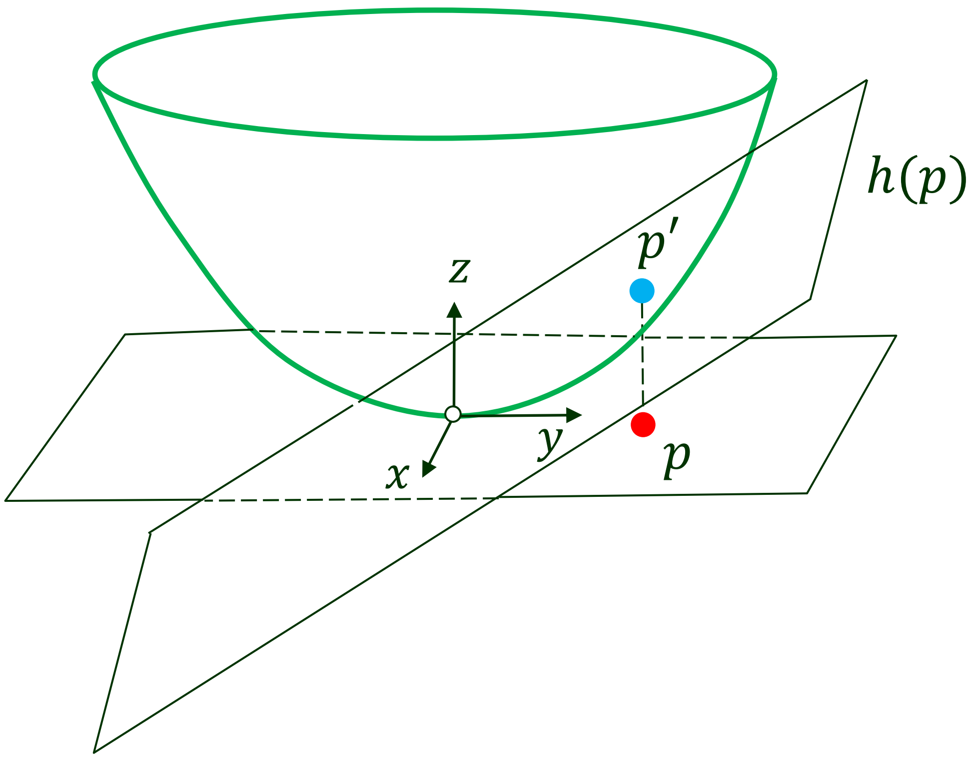


$p = (p_x, p_y, 0)$

Projection of p onto U:

$p' = (p_x, p_y, p_x^2 + p_y^2)$:

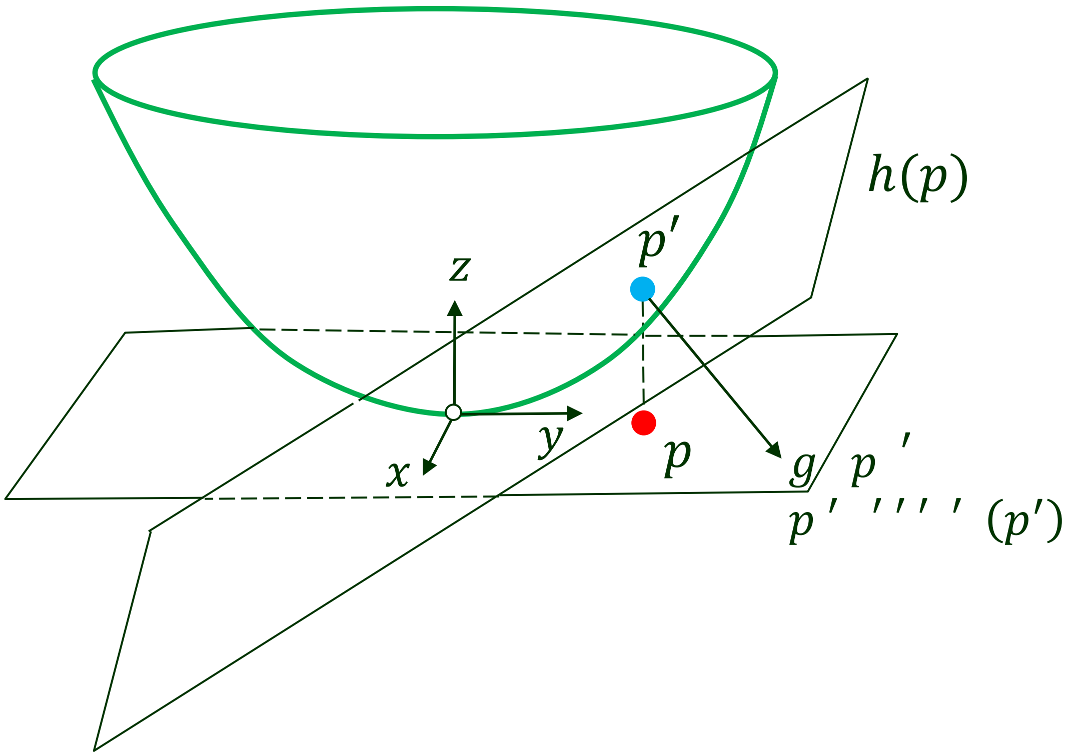
Tangent plane $h(p)$ to U through p' has normal:

$\nabla g(p') = (2p_x, 2p_y, -1)$

Unit Paraboloid

$U: z = \ x^2 + y^2$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



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Projection of p onto U:

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Unit Paraboloid

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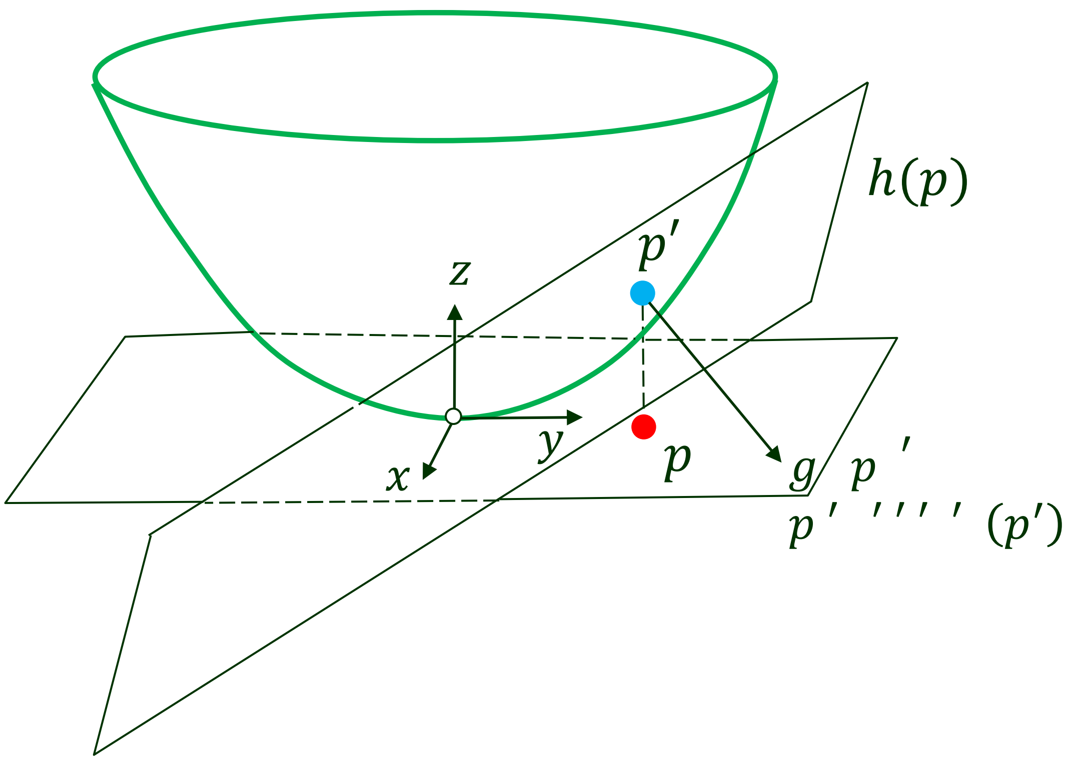
i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



$p = (p_x, p_y, 0)$

Projection of p onto U:

$p' = (p_x, p_y, p_x^2 + p_y^2)$:

Tangent plane $h(p)$ to U through p' has normal:

$\nabla g(p') = (2p_x, 2p_y, -1)$

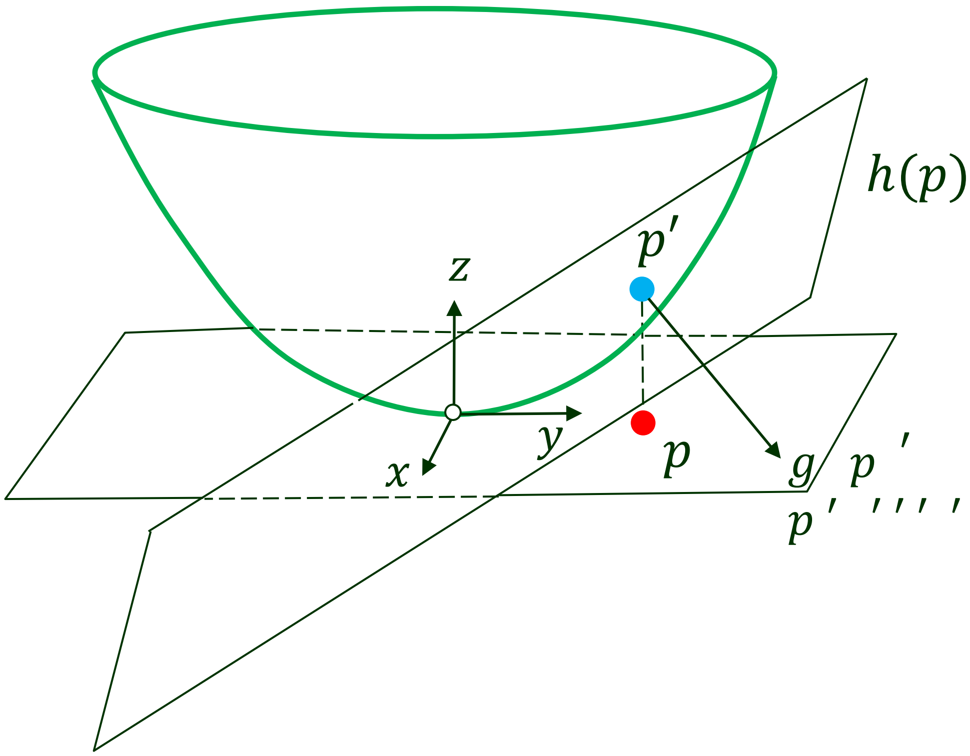
$\Downarrow$

$\big((x, y, z) - p'\big) \cdot \nabla g(p') = 0$

Unit Paraboloid

$$U: z = x^2 + y^2$$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



$$p = (p_x, p_y, 0)$$

Projection of p onto U:

$$p' = (p_x, p_y, p_x^2 + p_y^2):$$

Tangent plane $h(p)$ to U through p' has normal:

$$\nabla g(p') = (2p_x, 2p_y, -1)$$

$$\Downarrow$$

$$\big((x, y, z) - p'\big) \cdot \nabla g(p') = 0$$

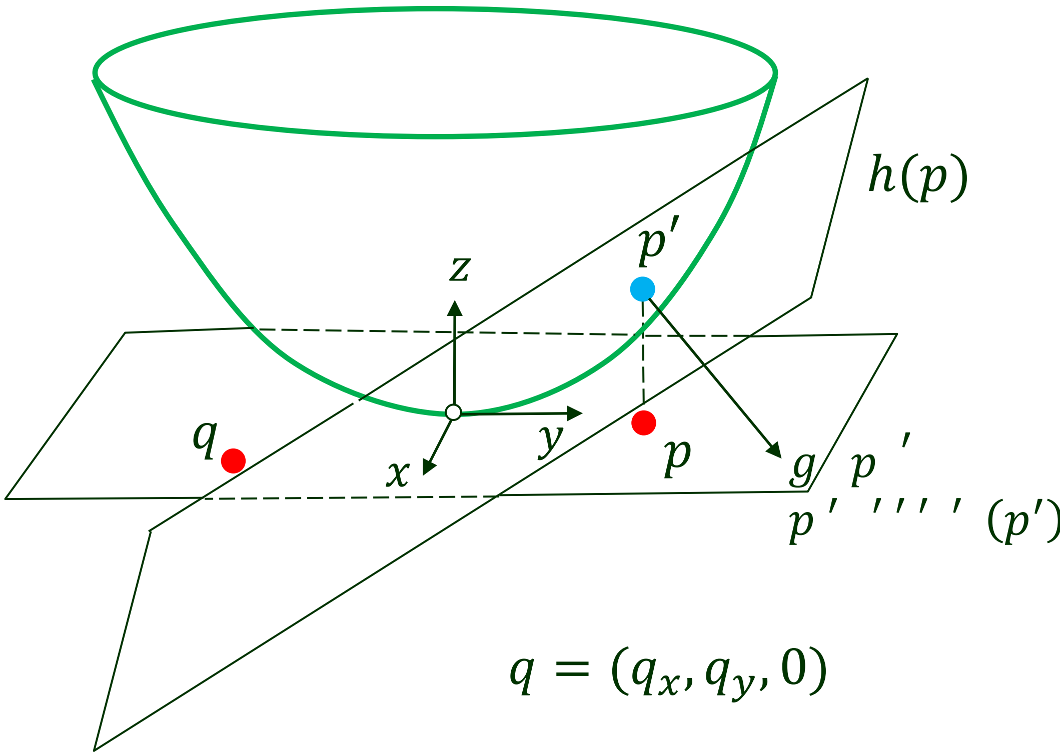
$$\Downarrow$$

$$h(p): z = 2p_x x + 2p_y y - (p_x^2 + p_y^2)$$

Unit Paraboloid

$U: z = x^2 + y^2$

i.e. $g(x, y, z) \equiv x^2 + y^2 - z = 0$



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Projection of p onto U:

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$$\Downarrow$$

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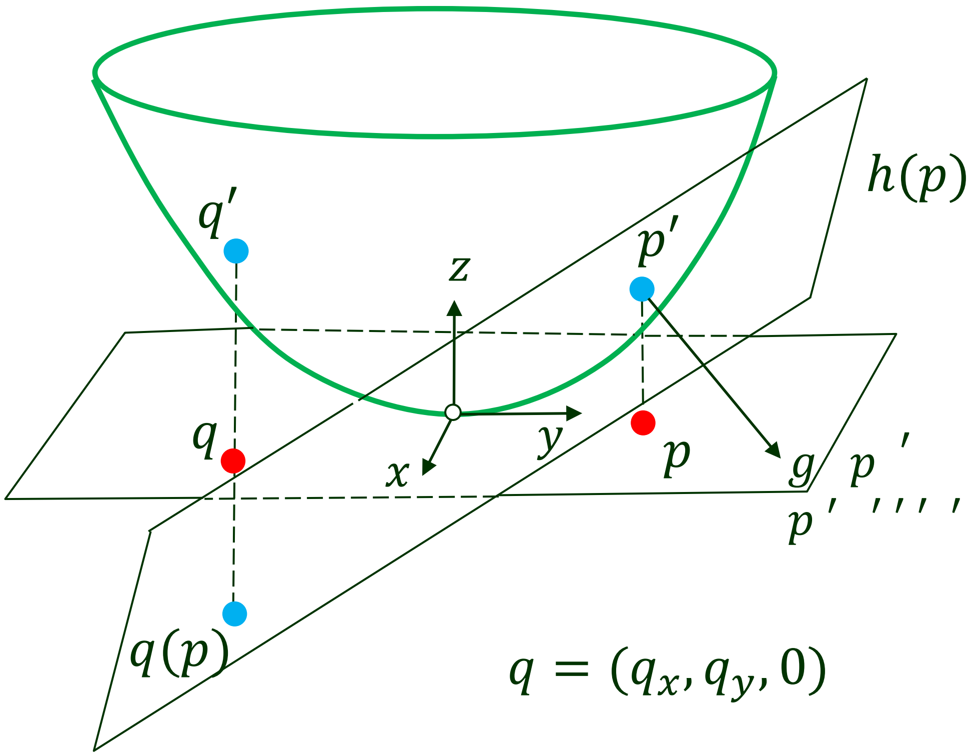
$\Downarrow$

$h(p): z = 2p_x x + 2p_y y - (p_x^2 + p_y^2)$



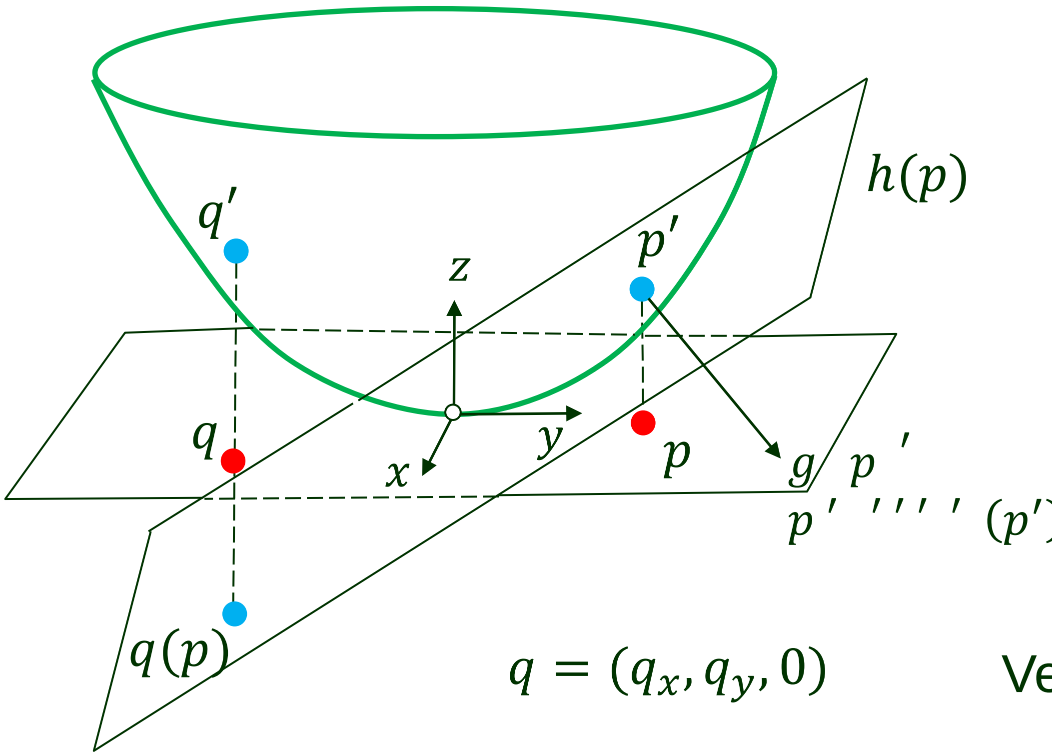
$q = (q_x, q_y, 0)$

Unit Paraboloid

$$p = (p_x, p_y, 0)$$

$$U: z = x^2 + y^2$$

i.e. $g(x,y,z) \equiv x^2 + y^2 - z = 0$



$$q = (q_x, q_y, 0)$$

Projection of p onto U:

$$p' = (p_x, p_y, p_x^2 + p_y^2):$$

Tangent plane $h(p)$ to U through p' has normal:

$$\nabla g(p') = (2p_x, 2p_y, -1)$$

$$\Downarrow$$

$$\big((x,y,z) - p'\big) \cdot \nabla g(p') = 0$$

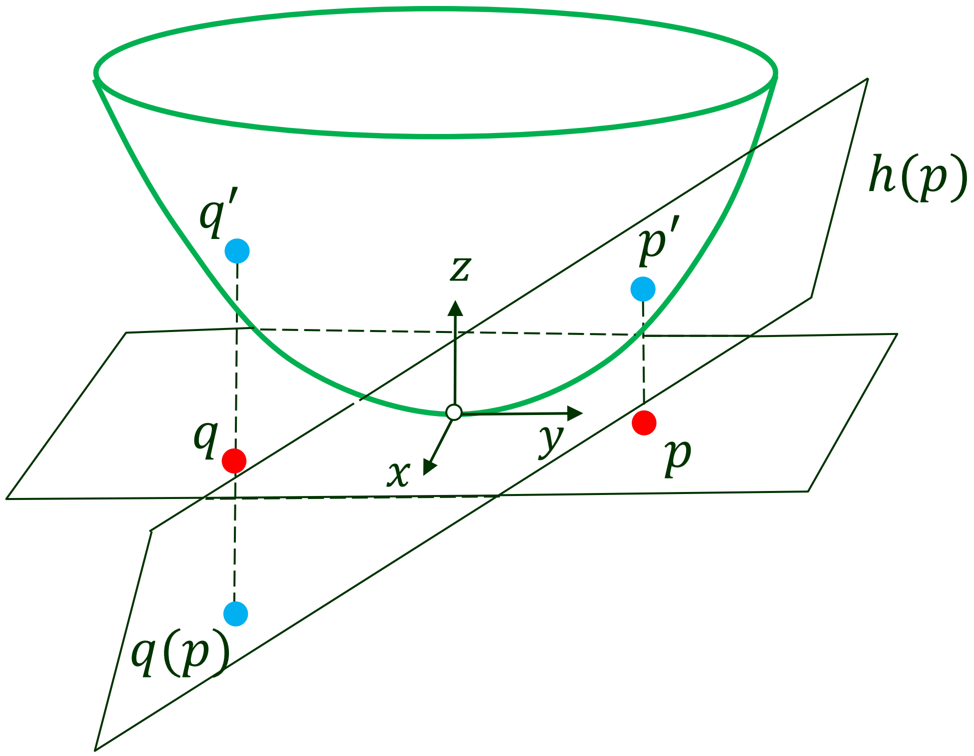
$$\Downarrow$$

$$h(p): z = 2p_x x + 2p_y y - (p_x^2 + p_y^2)$$

Vertical line through q intersects

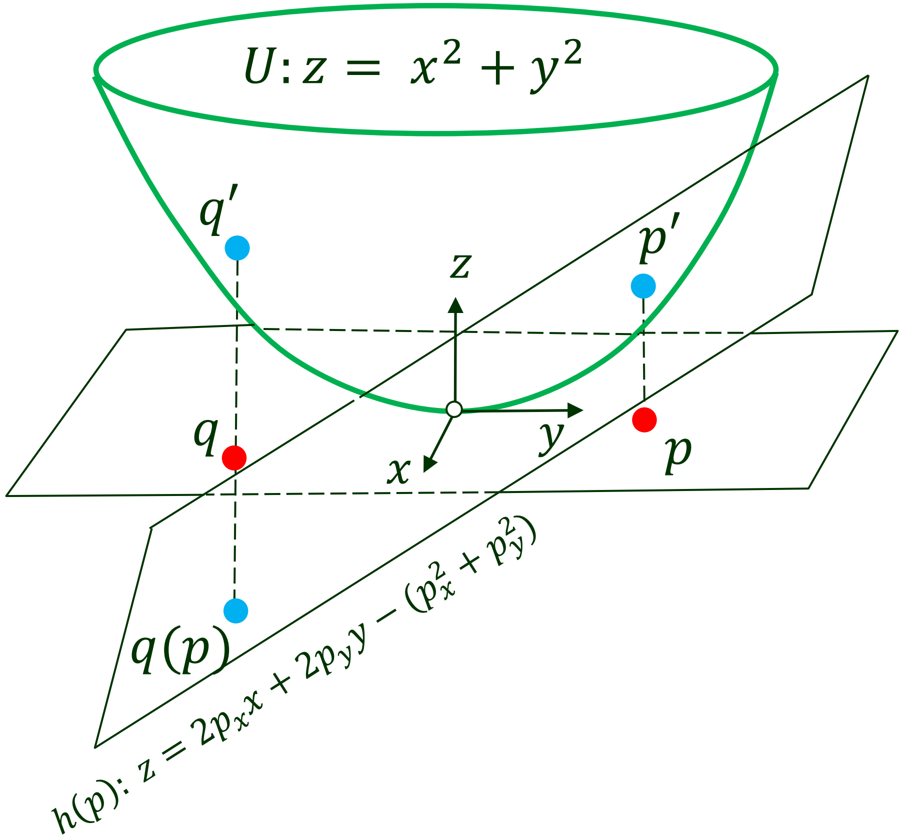
- U at $q' = (q_x, q_y, q_x^2 + q_y^2)$
- $h(p)$ at $q(p)$.

Unit Paraboloid



Distance Encoded in Tangent Plane

$d(p, q)$: distance between two points p and q



Distance Encoded in Tangent Plane

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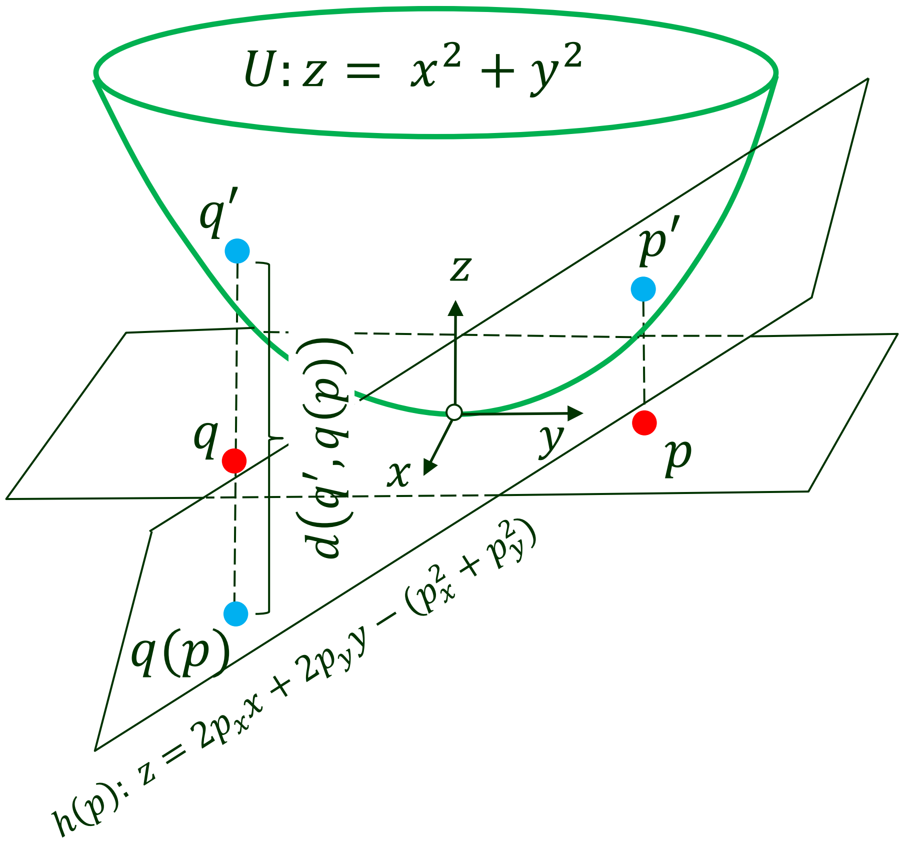
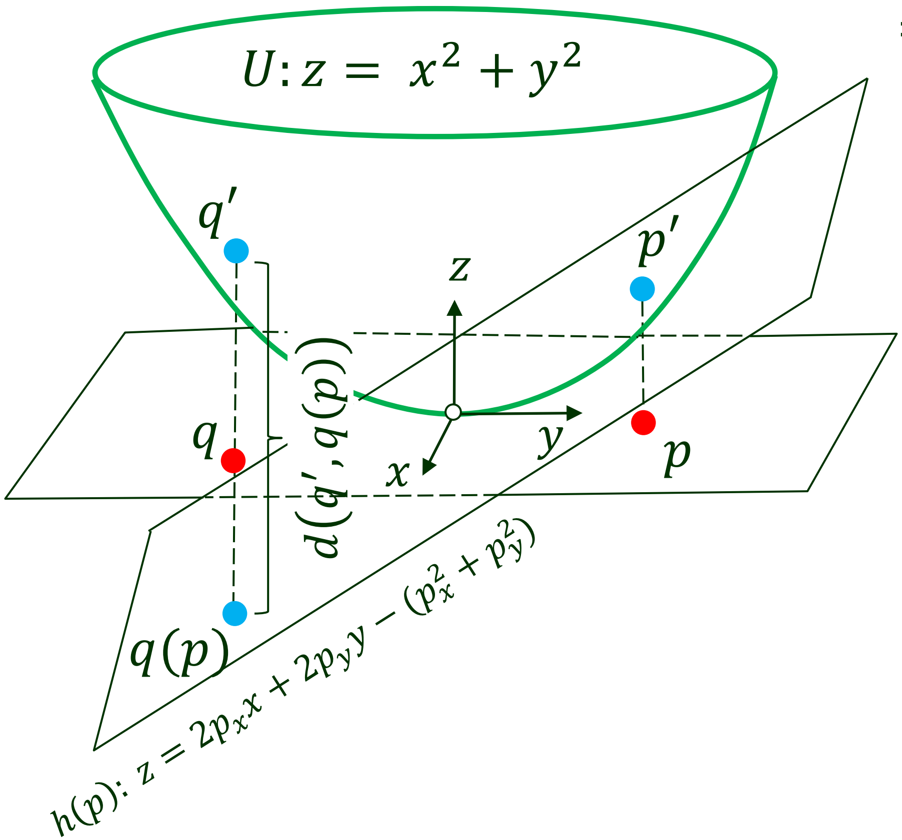


Distance Encoded in Tangent Plane

$d(p, q)$: distance between two points p and q

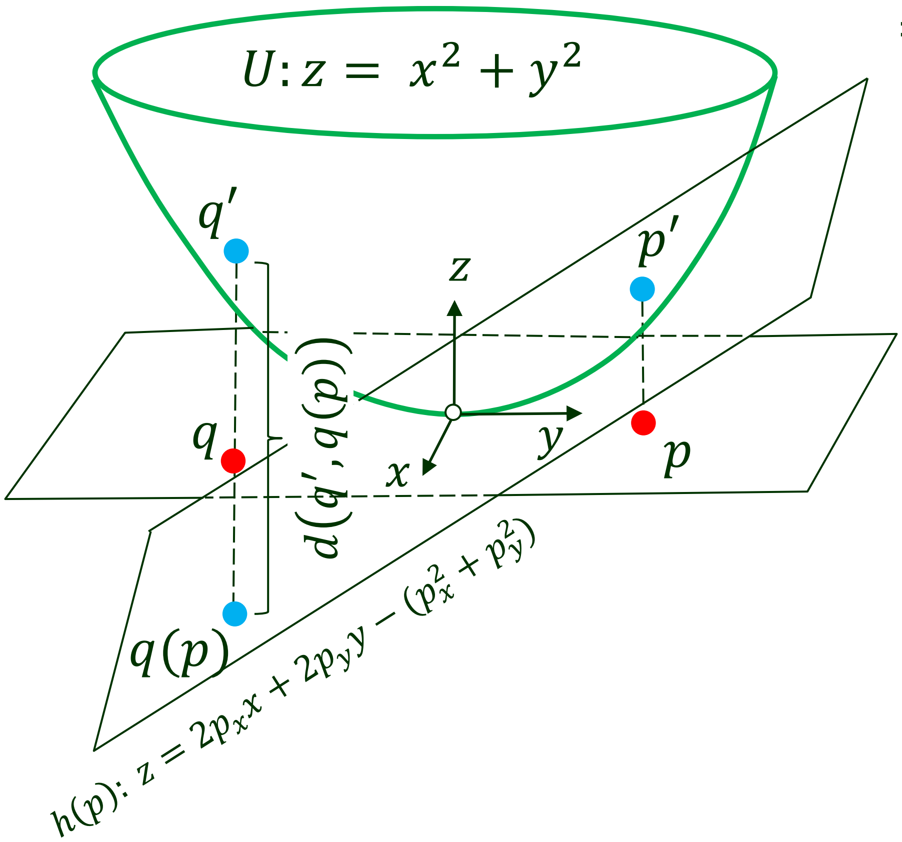
$$d\left(q', q(p)\right)$$

$$= \left(q_x^2 + q_y^2\right) - (2p_x q_x + 2p_y q_y - (p_x^2 + p_y^2))$$



Distance Encoded in Tangent Plane

$d(p,q)$: distance between two points p and q



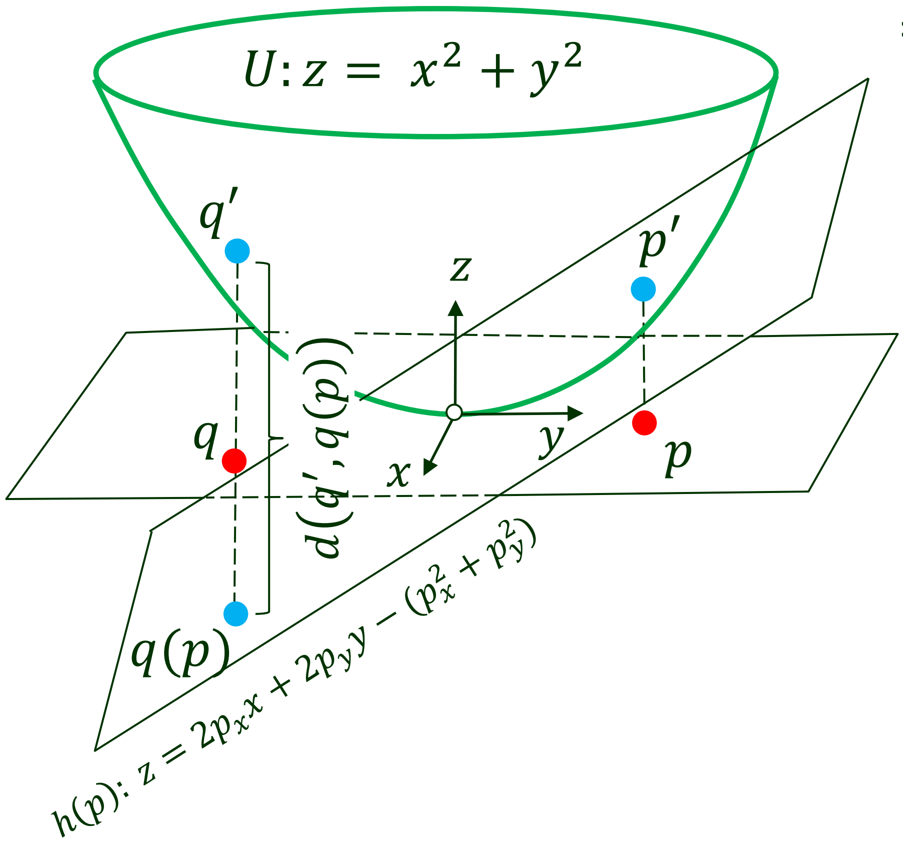
$$d\big(q', q(p)\big)$$

$$= \left(q_x^2 + q_y^2\right) - (2p_x q_x + 2p_y q_y - (p_x^2 + p_y^2))$$

$$= (q_x - p_x)^2 + (q_y - p_y)^2$$

Distance Encoded in Tangent Plane

$d(p,q)$: distance between two points p and q



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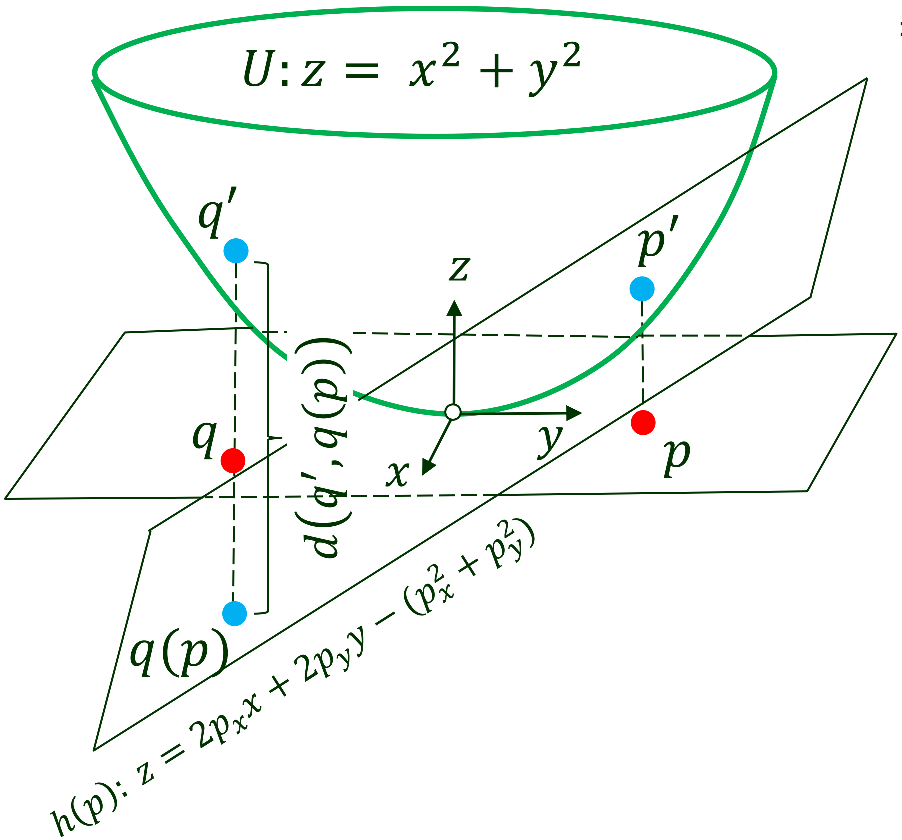

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Thus,

$$q(p) = (q_x, q_y, q_x^2 + q_y^2 - d(p, q)^2)$$

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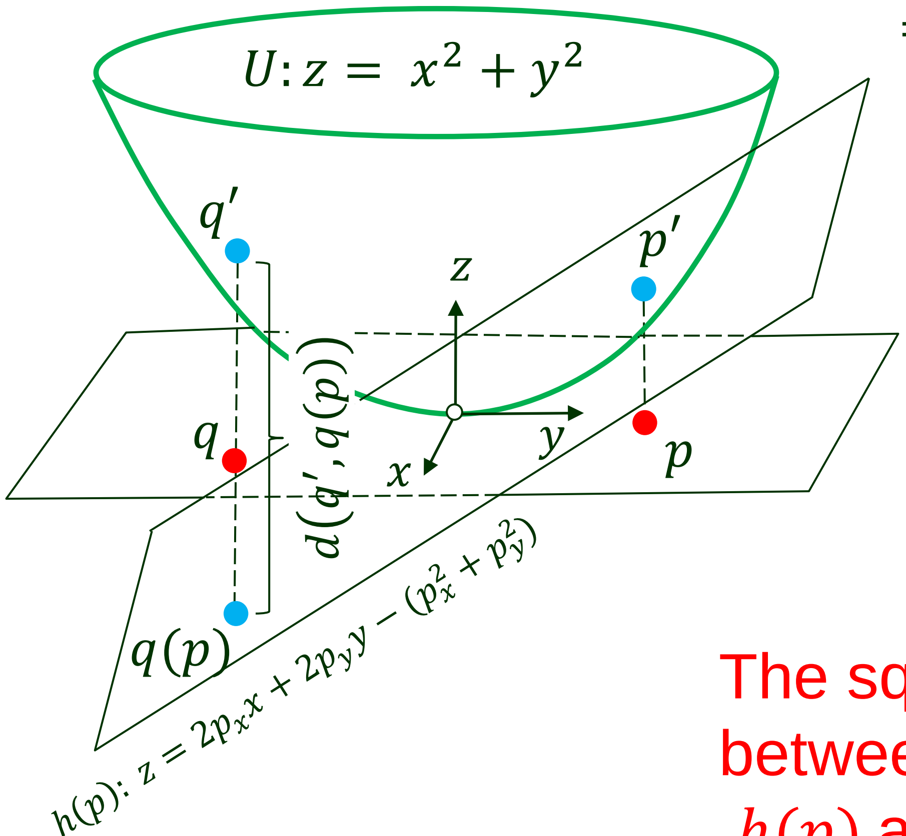

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The square of $d(p,q)$ equals the distance between the two projection points (onto $h(p)$ and U) from q.

Upper Envelope of Planes

H ={ tangent plane $h(p)$ | $p \in P$}

UE(H): upper envelope of the planes in H.

Theorem 1 The projection of UE(H) onto the plane $z = 0$ is the Voronoi diagram of P.

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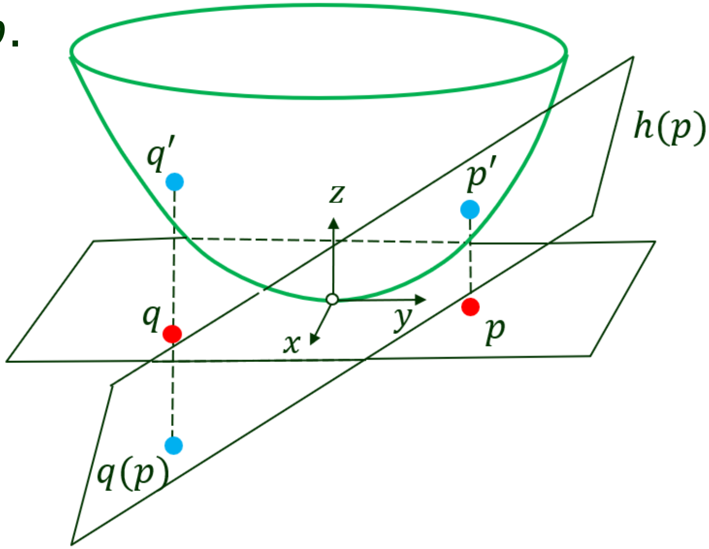

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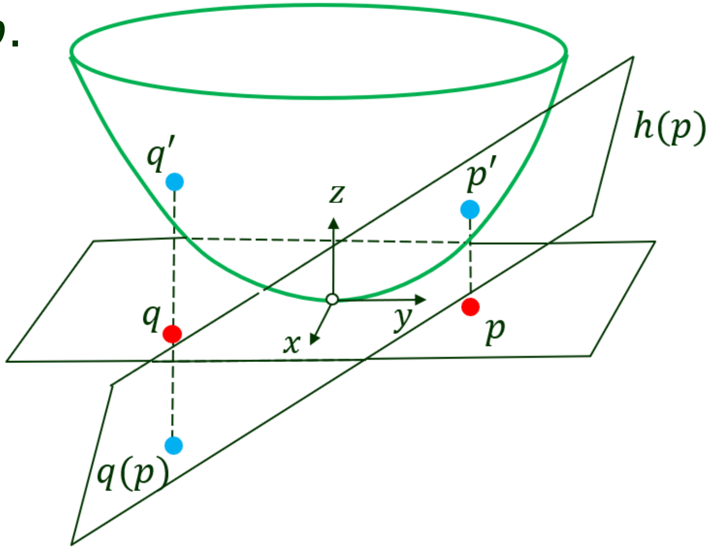

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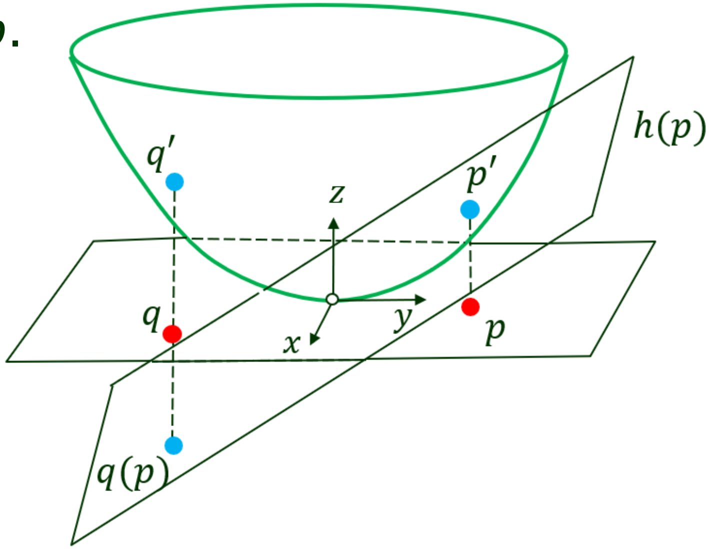

Proof (cont'd)

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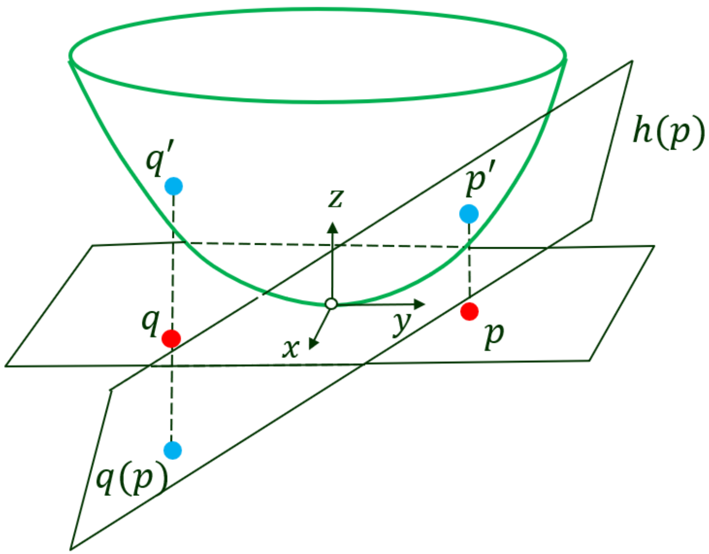

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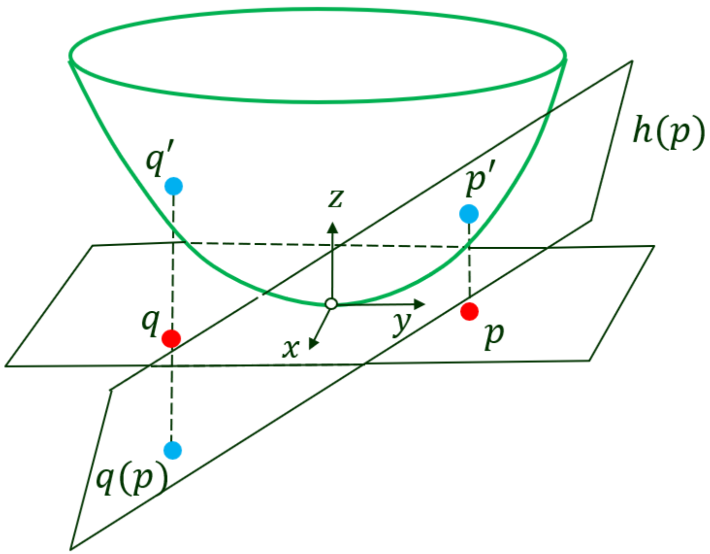

$$\Updownarrow \quad q(p) = (q_x, q_y, q_x^2 + q_y^2 - d(p,q)^2)$$

$$q(p) \cdot (0,0,1) > q(r) \cdot (0,0,1)$$

Proof (cont'd)

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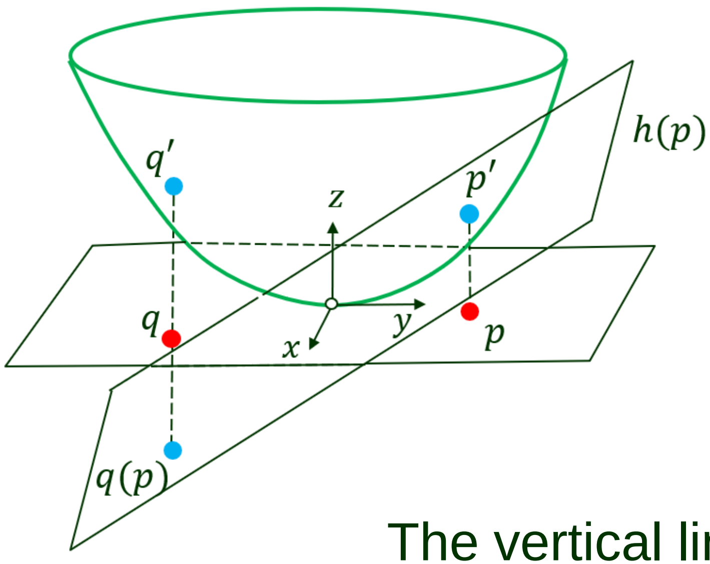

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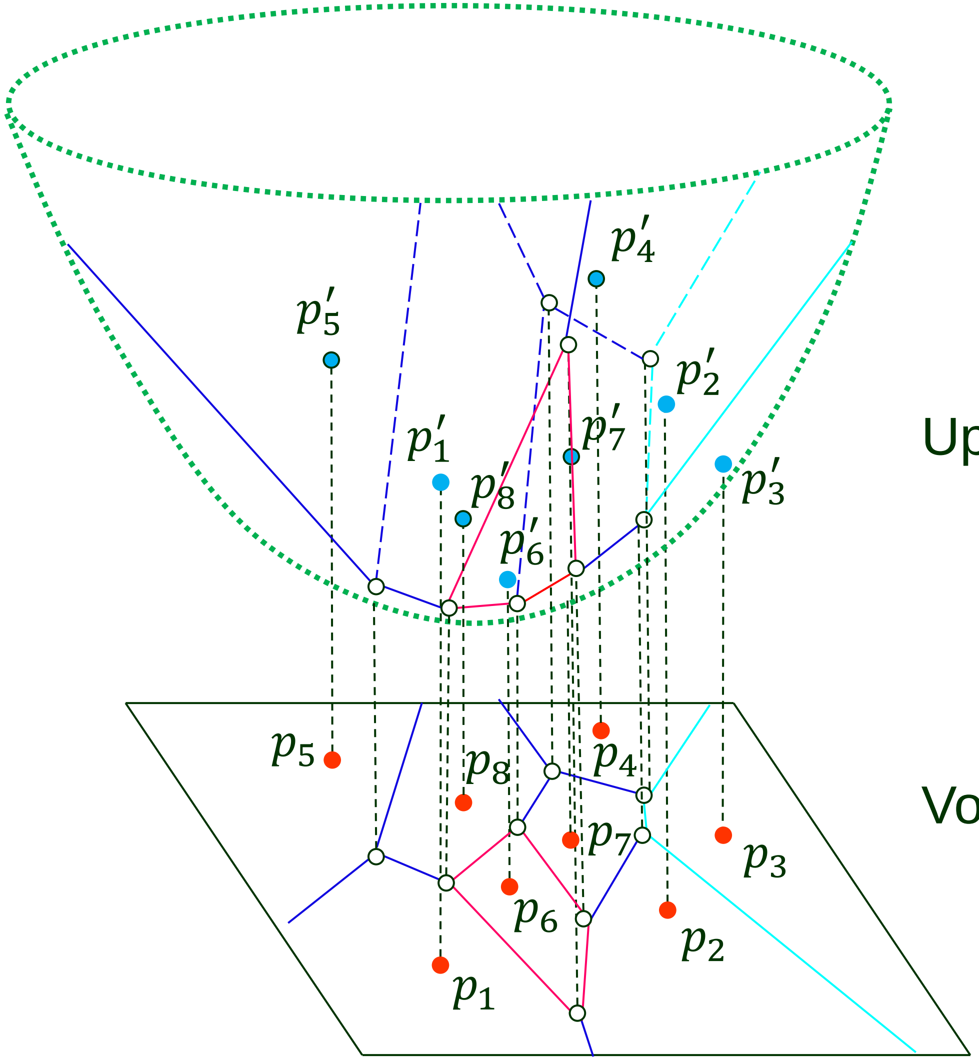
$$q(p) \cdot (0,0,1) > q(r) \cdot (0,0,1)$$

$$\Updownarrow$$

The vertical line through q intersects UE(H) at a point on $h(p)$, i.e., inside the facet contributed by $h(p)$.

Projection of Upper Envelope



Upper envelope UE(H)

Voronoi diagram Vor(P)

Construction of Voronoi Diagram

Constructing the Voronoi diagram of P in 2D

Construction of Voronoi Diagram

Constructing the Voronoi diagram of P in 2D

$\Downarrow$

Computing an upper envelope of the set of planes $H = \{h(p) \mid p \in P\}$ in 3D

Construction of Voronoi Diagram

Constructing the Voronoi diagram of P in 2D

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Computing an upper envelope of the set of planes
$H = \{h(p) \mid p \in P\}$ in 3D

$\Downarrow$

Computing the lower convex hull of the set of dual points
$H^* = \{h(p)^* \mid p \in P\}$ in 3D

Point $p = (p_x, p_y, p_z)$ $\mapsto$ plane $z = p_x x + p_y y - p_z$

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Theorem 2 The projection of the lower convex hull of H^* onto the plane $z = 0$ is the Delaunay graph of P.