Bayesian Networks (Bayes Nets)

Outline

I. Semantics

II. Network construction

III. Conditional independence relations

* Figures are either from the textbook site or by the instructor.
The full joint probability distribution can answer any question, but it also has several drawbacks:

- exponential in the number $n$ of variables and intractable as $n$ grows very large
- unnatural and tedious to specify probabilities of outcomes one by one
- inadequate for representing human reasoning (good at conditional probabilities but poor at joint probabilities)
I. Knowledge in an Uncertain Domain

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♦ The number of probabilities can be greatly reduced by exploring the absolute and conditional independence relationships among the variables.
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  - exponential in the number $n$ of variables and intractable as $n$ grows very large
  - unnatural and tedious to specify probabilities of outcomes one by one
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- The number of probabilities can be greatly reduced by exploring the absolute and conditional independence relationships among the variables.

- These dependencies can be *concisely represented* by a Bayesian network, which can represent any full joint probability distribution.
A *Bayesian network* (aka a *Bayes net*) is a directed acyclic graph (DAG) such that

a) every node corresponds to a random variable, either discrete or continuous;

b) every edge \((X, Y)\) specifies \(X\) (a cause) as a parent of \(Y\) (an effect);

c) every node \(X\) has associated probability information \(\theta(X \mid \text{parent}(X))\) that quantifies the effect of the parents on \(X\).

The network topology specifies the conditional independence relationships that hold in the domain.
The parents of a node $X$ are those judged to be direct causes of $X$ or have direct influence on $X$. 

- Weather
- Cavity
- Toothache
- Catch
BN as a Modeling Tool

- The parents of a node $X$ are those judged to be direct causes of $X$ or have direct influence on $X$.
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The parameters required for model construction are *conditional probabilities* that quantify cause-effect relations, which are

- psychologically meaningful
- often measurable
Burglar Alarm Problem

• A newly installed burglar alarm is fairly reliable at detecting a burglary.
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- But it can also be occasionally set off by earthquakes.
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**Conditional Probability Tables (CPTs)**
A newly installed burglar alarm is fairly reliable at detecting a burglary.
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<table>
<thead>
<tr>
<th>Burglary</th>
<th>$P(B=\text{true})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Earthquake</th>
<th>$P(E=\text{true})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.002</td>
</tr>
</tbody>
</table>

| $A$ | $P(J=\text{true}|A)$ |
|-----|----------------------|
| $t$ | 0.90                 |
| $f$ | 0.05                 |

| $A$ | $P(M=\text{true}|A)$ |
|-----|----------------------|
| $t$ | 0.70                 |
| $f$ | 0.01                 |

conditional probability tables (CPTs)
Burglar Alarm Problem

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- But it can also be occasionally set off by earthquakes.
- Neighbors John and Mary have promised a call when they hear the alarm.
  - John nearly always calls but sometimes confuses the alarm with the telephone ringing.
  - Mary often misses the alarm because she likes playing loud music.

Problem: Estimate the probability of a burglary given the evidence of who has or has not called.

conditional probability tables (CPTs)
Semantics of a Bayes Net

How does the syntax correspond to a joint distribution over the variables?

- $n$ variables $X_1, \ldots, X_n$ in the network
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- an entry in the joint distribution is defined as

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P(x_1, \ldots, x_n) \equiv P(X_1 = x_1 \land \cdots \land X_n = x_n)
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\[
P(x_1, \ldots, x_n) \equiv P(X_1 = x_1 \land \cdots \land X_n = x_n) = \prod_{i=1}^{n} \theta(x_i \mid \text{parents}(X_i))
\]

where

\[
\text{parents}(X_i) = \{x_j \mid x_j \in \text{Parents}(X_i)\},
\]

// the values of \( \text{Parents}(X_i) \) that appear in \( x_1, \ldots, x_n \)
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$$\theta(x_i \mid \text{parents}(X_i))$$

// probability of $X_i = x_i$ given the values of the parents of $X_i$
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Every entry in the joint distribution is the product of the appropriate elements of the local conditional distribution.
BN as a Knowledge Base

\[ P(x_1, \ldots, x_n) = \prod_{i=1}^{n} \theta(x_i | \text{parents}(X_i)) \]

Calculate the probability that the alarm has sounded, but neither a burglary nor an earthquake has occurred, and both John and Mary call.
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\[ P(j, m, a, \neg b, \neg e) \]
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P(j, m, a, \neg b, \neg e) = P(j \mid a) \ P(m \mid a) \ P(a \mid \neg b \land \neg e) P(\neg b) P(\neg e)
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Alarm is the sole parent of JohnCalls.
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Alarm is the sole parent of JohnCalls. Burglary and Earthquake are the only two parents of Alarm.
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Calculate the probability that the alarm has sounded, but neither a burglary nor an earthquake has occurred, and both John and Mary call.

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Alarm is the sole parent of JohnCalls.

Burglary and Earthquake are the only two parents of Alarm.

\[ = 0.90 \times 0.70 \times 0.01 \times 0.999 \times 0.998 \]
BN as a Knowledge Base

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Calculate the probability that the alarm has sounded, but neither a burglary nor an earthquake has occurred, and both John and Mary call.

\[ P(j, m, a, \neg b, \neg e) = P(j \mid a) \cdot P(m \mid a) \cdot P(a \mid \neg b \land \neg e) \cdot P(\neg b) \cdot P(\neg e) \]

Alarm is the sole parent of JohnCalls. Burglary and Earthquake are the only two parents of Alarm.

\[ = 0.90 \times 0.70 \times 0.01 \times 0.999 \times 0.998 \]

\[ = 0.00628 \]
Conditional Probabilities

\[ X_i \]

\[ \text{Parents}(X_i) \]

\[ Y: \text{all variables other than } X_i \text{ and } \text{Parents}(X_i) \]

\[ y: \text{values of } Y \]
Conditional Probabilities

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Conditional Probabilities

\[ P(x_i \mid \text{parents}(X_i)) = \frac{P(x_i, \text{parents}(X_i))}{P(\text{parents}(X_i))} \]

**Parents**($X_i$)

\[ Y: \text{all variables other than } X_i \text{ and } \text{Parents}(X_i) \]

\[ y: \text{values of } Y \]
Conditional Probabilities

- $\mathbf{Y}$: all variables other than $X_i$ and $\text{Parents}(X_i)$
- $\mathbf{y}$: values of $\mathbf{Y}$

$$P(x_i \mid \text{parents}(X_i)) = \frac{P(x_i, \text{parents}(X_i))}{P(\text{parents}(X_i))} = \frac{\sum_{\mathbf{y}} P(x_i, \text{parents}(X_i), \mathbf{y})}{\sum_{x_i', \mathbf{y}} P(x_i', \text{parents}(X_i), \mathbf{y})}$$
Conditional Probabilities

\[ P(x_i | \text{parents}(X_i)) = \frac{P(x_i, \text{parents}(X_i))}{P(\text{parents}(X_i))} = \frac{\sum_y P(x_i, \text{parents}(X_i), y)}{\sum_{x_i', y} P(x_i', \text{parents}(X_i), y)} = \theta(x_i | \text{parents}(X_i)) \]

\( Y \): all variables other than \( X_i \) and \( \text{Parents}(X_i) \)

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Conditional Probabilities

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\[ \therefore \text{proof can be derived} \]

\[ P(x_1, \ldots, x_n) = \prod_{i=1}^{n} \theta(x_i \mid \text{parents}(X_i)) \]

(by definition of the Bayes net)

Full joint distribution:

\[ P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid \text{parents}(X_i)) \]
Correct Domain Representation

Chain rule:

\[ P(x_1, \ldots, x_n) = P(x_n \mid x_{n-1}, \ldots, x_1)P(x_{n-1}, \ldots, x_1) \]
Correct Domain Representation

**Chain rule:**

\[
P(x_1, \ldots, x_n) = P(x_n \mid x_{n-1}, \ldots, x_1)P(x_{n-1}, \ldots, x_1)
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\[
\vdots
\]

\[
= P(x_n \mid x_{n-1}, \ldots, x_1)P(x_{n-1} \mid x_{n-2}, \ldots, x_1) \cdots P(x_2 \mid x_1)P(x_1)
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Meanwhile,

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P(x_1, ..., x_n) = \prod_{i=1}^{n} P(x_i \mid \text{parents}(X_i))
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\[
P(X_i \mid X_{i-1}, \ldots, X_1) = P(X_i \mid \text{Parents}(X_i)) \quad \text{for } i = 2, \ldots, n
\]

if \(\text{Parents}(X_i) \subseteq \{X_1, \ldots, X_{i-1}\}\) for \(i = 2, \ldots, n\)
Topological Order

\[ \text{Parents}(X_i) \subseteq \{X_1, \ldots X_{i-1}\} \text{ for } i = 2, \ldots, n \]

The above is guaranteed if we number the nodes in \textit{topological order} (which exists since the Bayesian network is a DAG).
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Four topological orders:

- \( B, E, A, J, M \)
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- \( E, B, A, J, M \)
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Any one of the four suffices.
II. Construction of the Bayesian Network

\[ P(X_i | X_{i-1}, \ldots, X_1) = P(X_i | \text{Parents}(X_i)) \quad \text{for } i = 2, \ldots, n \]

The Bayesian network is correct only if \( X_i \) is conditionally independent of any \( X_j \), \( 1 \leq j \leq i - 1 \), such that \( X_j \notin \text{Parents}(X_i) \).
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Construction algorithm

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Construction algorithm

1. Determine the set of variables that are required to model the domain.

2. Order them as $X_1, X_2, ..., X_n$.

   Any order works, although network compactness depends on how much the order – whether causes precede effects – is respected.
II. Construction of the Bayesian Network

\[ P(X_i | X_{i-1}, ..., X_1) = P(X_i | \text{Parents}(X_i)) \quad \text{for } i = 2, ..., n \]

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   Any order works, although network compactness depends on how much the order – whether causes precede effects – is respected.

3. For \( i = 1 \) to \( n \) do
   
   a) Choose a minimal set of parents for \( X_i \) from \( X_1, X_2, ..., X_{i-1} \) such that
   
   \[ P(X_i | X_{i-1}, ..., X_1) = P(X_i | \text{Parents}(X_i)) \]
II. Construction of the Bayesian Network

\[ P(X_i \mid X_{i-1}, ..., X_1) = P(X_i \mid \text{Parents}(X_i)) \]

for \( i = 2, ..., n \)

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   b) Add a directed edge from every parent to \( X_i \).
   
   c) Write down the conditional probability table (CPT), \( P(X_i \mid \text{Parents}(X_i)) \).
Chosen order: MaryCalls, JohnCalls, Alarm, Burglary, Earthquake.
Construction (cont’d)

Chosen order:   *MaryCalls*, *JohnCalls*, *Alarm*, *Burglary*, *Earthquake*.  

MaryCalls
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Construction (cont’d)

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\[ P(j \mid m) \quad P(j) \]

// If Mary calls, that probably means
// the alarm has gone off, which
// makes John more likely to call.
Construction (cont’d)

Chosen order: MaryCalls, JohnCalls, Alarm, Burglary, Earthquake.

\[ P(j \mid m) > P(j) \]
// If May calls, that probably means
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Construction (cont’d)

Chosen order: MaryCalls, JohnCalls, Alarm, Burglary, Earthquake.

\[ P(j | m) > P(j) \]

// If May calls, that probably means
// the alarm has gone off, which
// makes John more likely to call.

\[ P(a | m, j) \quad P(a | j), P(a | m), P(a) \]
Construction (cont’d)

Chosen order: MaryCalls, JohnCalls, Alarm, Burglary, Earthquake.

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// makes John more likely to call.

\[ P(a | m, j), P(a | j), P(a | m), P(a) \]
// If both Mary and John call, the alarm
// is more likely to go off than if just
// one calls.
Construction (cont’d)

Chosen order:  *MaryCalls, JohnCalls, Alarm, Burglary, Earthquake.*

\[ P(j \mid m) > P(j) \]

// If May calls, that probably means
// the alarm has gone off, which
// makes John more likely to call.

\[ P(a \mid m, j) > P(a \mid j), P(a \mid m), P(a) \]

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Construction (cont’d)

Chosen order:  *MaryCalls, JohnCalls, Alarm, Burglary, Earthquake.*

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P(a | m, j) > P(a | j), P(a | m), P(a)
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// If both Mary and John call, the alarm // is more likely to go off than if just // one calls.
Construction for the Burglary Example

MaryCalls, JohnCalls, Alarm, Burglary, Earthquake.
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Construction for the Burglary Example

MaryCalls, JohnCalls, Alarm, Burglary, Earthquake.

\[ P(B \mid A, J, M) \quad P(B \mid A) \]

// If the value of A (either a or \( \neg a \)) is known, then the call from John or Mary does not add any information about burglary.
Construction for the Burglary Example

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// If the alarm is on, it is more likely that there has been an earthquake. If there has been a burglary, it is slightly more likely that it happened after an earthquake. In the occurrences of both events, the chance of an earthquake occurrence is even higher.
Construction for the Burglary Example

MaryCalls, JohnCalls, Alarm, Burglary, Earthquake.

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Node Ordering Matters

1 (one probability / parameter)

MaryCalls

2

JohnCalls

(four probabilities)

4

Alarm

2

Burglary

4

Earthquake

1 + 2 + 4 + 2 + 4 = 13 conditional probabilities
Node Ordering Matters

- MaryCalls
- JohnCalls
- Alarm
- Burglary
- Earthquake

1 (one probability / parameter)

2

4

2

4

1 + 2 + 4 + 2 + 4 = 13
conditional probabilities

- More conditional probabilities than needed.
Node Ordering Matters

- More conditional probabilities than needed.

1 + 2 + 4 + 2 + 4 = 13 conditional probabilities

Graph:

- MaryCalls
- JohnCalls
- Alarm
- Burglary
- Earthquake

Conditional probabilities:

<table>
<thead>
<tr>
<th>Burglary</th>
<th>P(B=true)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Earthquake</th>
<th>P(E=true)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.002</td>
</tr>
</tbody>
</table>

| B | E | P(A=true|B,E) |
|---|---|------------|
| t | f | .01        |
| f | t | .70        |
| f | f | .01        |

| A | P(J=true|A) |
|---|-----------|
| t | .90       |
| f | .05       |

| A | P(M=true|A) |
|---|-----------|
| t | .70       |
| f | .01       |
Node Ordering Matters

MaryCalls ➔ JohnCalls ➔ Alarm ➔ Burglary ➔ Earthquake

1 (one probability / parameter)

2

4

1 + 2 + 4 + 2 + 4 = 13 conditional probabilities

More conditional probabilities than needed.

Assessment of unnatural probabilities, e.g., \( P(\text{Earthquake} | \text{Burglary, Alarm}) \).

10 conditional probabilities
Node Ordering Matters

1 (one probability / parameter)

MaryCalls

2

JohnCalls

4

Alarm

(four probabilities) 4

Burglary

2

Earthquake

More conditional probabilities than needed.

Assessment of unnatural probabilities, e.g., $P(\text{Earthquake} | \text{Burglary, Alarm})$.

Sticking to a causal model results in fewer probabilities that are also easier to come up with.

1 + 2 + 4 + 2 + 4 = 13 conditional probabilities

10 conditional probabilities
Bad Node Ordering

MaryCalls, JohnCalls, Earthquake, Burglary, Alarm.

$1 + 2 + 4 + 8 + 16 = 31$ distinct probabilities (exactly the same as the full joint distribution)!
Roles of Casualty

- Deciding conditional independence is hard in noncausal directions. (Causal models and conditional independence seem hardwired for humans!)

- Assessing conditional probabilities is hard in noncausal directions.

- The interpretation of directed acyclic graphs as carriers of independence assumptions does not necessarily imply causation.

- The ubiquity of DAG models in statistical and AI applications stems (often unwittingly) primarily from their causal interpretation.

- In practice, DAG models are rarely used in any variable ordering other than those which respect the direction of time and causation.
Compactness of Bayes Nets

- The full joint distribution contains $2^n$ numbers.
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- With $n$ Boolean variables, the network has $\leq n \cdot 2^k$ numbers.

- To avoid a fully connected network, leave out links that represent slight dependencies.
Every variable is conditionally independent of its non-descendants (including ancestors), given the values of its parents.

Given the value of Alarm, JohnCalls is independent of Burglary, Earthquake, and Marycalls.
III. Non-Descendants Property

Every variable is conditionally independent of its non-descendants (including ancestors), given the values of its parents.

Given the value of Alarm, JohnCalls is independent of Burglary, Earthquake, and Marycalls.

\[ P(x_i \mid \text{parents}(X_i)) = \theta(x_i \mid \text{parents}(X_i)) \]

// network parameter interpretation
\[ P(x_i \mid \text{parents}(X_i)) = \theta(x_i \mid \text{parents}(X_i)) \]
The full joint distribution

\[ P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid \text{parents}(X_i)) \]

**Illustration**
The **Markov blanket** $MB(X)$ of a node $X$ consists of its parents, children, and children’s parents (but *not* the node itself).

$$MB(X) = \text{Children}(X) \cup \text{Parents}(X) \cup \{Y \mid Y \neq X \land (\exists Z Z \in \text{Children}(X) \land Y \in \text{Parents}(Z))\}$$

The Markov blanket is represented by the gray area.
Conditional Independence

It can be shown from the non-descendants property that

A node is conditionally independent of all other nodes given its Markov blanket.
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A node is conditionally independent of all other nodes given its Markov blanket.

For any random variable $Y$ such that $Y \neq X$ and $Y \notin MB(X)$

$$P(X \mid MB(X), Y) = P(X \mid MB(X))$$
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The Markov blanket of Burglary is \{Alarm, Earthquake\}. 
Conditional Independence

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A node is conditionally independent of all other nodes given its Markov blanket.

For any random variable $Y$ such that $Y \neq X$ and $Y \notin MB(X)$

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The Markov blanket of $Burglary$ is

$\{Alarm, Earthquake\}$.

Given $Alarm$ and $Earthquake$, $Burglary$ is independent of $JohnCalls$ and $Marycalls$. 
D-Separation

Q: Is a set of nodes $X$ conditionally independent of another set $Y$, given a third set $Z$?
**D-Separation**

**Q:** Is a set of nodes $X$ conditionally independent of another set $Y$, given a third set $Z$?

This question can be answered as follows:

1. Start with the *ancestral subgraph* consisting of $X$, $Y$, $Z$, and their ancestors (and edges between them).
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   b. Otherwise, $X$ and $Y$ are *not necessarily* conditionally independent, given $Z$.
Examples

1. \( X = \{ \text{Burglary} \} \)
   \( Y = \{ \text{Earthquake} \} \)
   \( Z = \{ \} \)

**Q:** \( X \) conditionally independent of \( Y \) given \( Z \)?
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Moral graph:

\[
\begin{array}{c|c|c}
B & E & P(A=\text{true}|B,E) \\
\hline
\text{t} & \text{t} & 0.70 \\
\text{t} & \text{f} & 0.01 \\
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Q: \( X \) conditionally independent of \( Y \) given \( Z \)?

Moral graph:

\( X \) and \( Y \) are separated, thus d-separated by \( Z \). They (\emph{Burglary} and \emph{Earthquake}) are independent given the empty set.
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Burglary and Earthquake are not necessarily independent given Alarm.
Examples

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   \( d \)-separated by \( Z \).

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   \( Z = \{ \text{Alarm} \} \)

   \( X \) and \( Y \) are separated, thus

   \( \text{Q: } X \) conditionally independent of \( Y \) given \( Z \)?

2. \( X = \{ \text{JohnCalls} \} \)
   \( Y = \{ \text{MaryCalls} \} \)

   \( Z = \{ \text{Alarm} \} \)

   \( X \) and \( Y \) are separated, thus

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   \text{JohnCalls} and \text{MaryCalls} are conditionally independent given \text{Alarm}.

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