Bayesian Networks (Bayes Nets)

Outline

I. Semantics

II. Network construction

III. Conditional independence relations

* Figures are either from the textbook site or by the instructor.

I. Knowledge in an Uncertain Domain

- The full joint probability distribution can answer any question, but it also has several drawbacks:
  - exponential in the number n of variables and intractable as n grows very large
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- These dependencies can be *concisely represented* by a Bayesian network, which can represent any full joint probability distribution.

Bayesian Network

A *Bayesian network* (aka a *Bayes net*) is a directed acyclic graph (DAG) such that

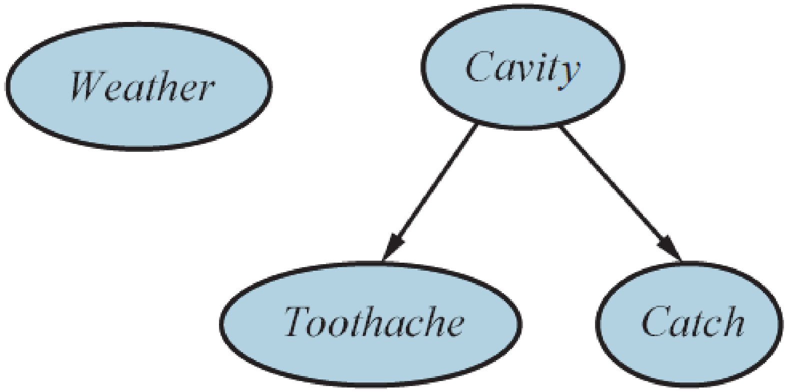
a) every node corresponds to a random variable, either discrete or continuous;

b) every edge (X, Y) specifies X (a cause) as a parent of Y (an effect);

c) every node X has associated probability information $\theta(X \mid \textit{parent}(X))$ that quantifies the effect of the parents on X.

The network topology specifies the conditional independence relationships that hold in the domain.

BN as a Modeling Tool



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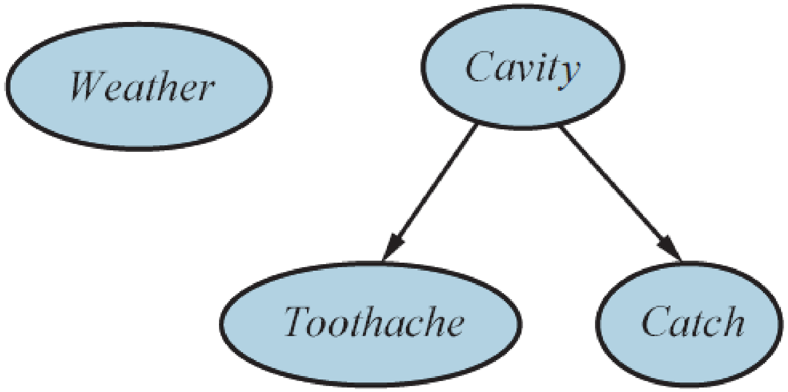

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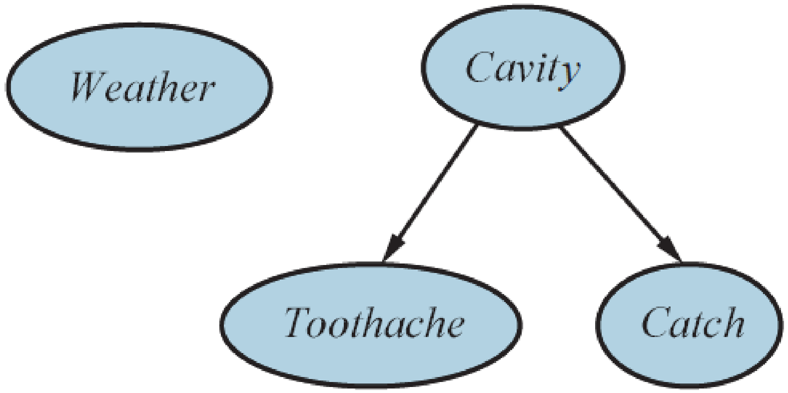

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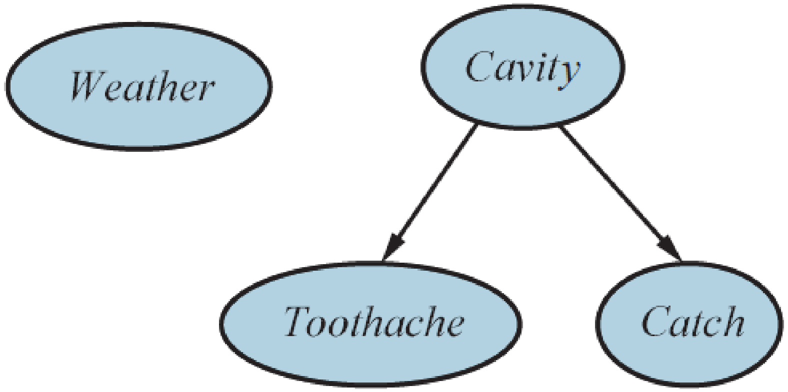

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- The *parameters* required for model construction are *conditional probabilities* that quantify cause-effect relations, which are
  - psychologically meaningful
  - often measurable

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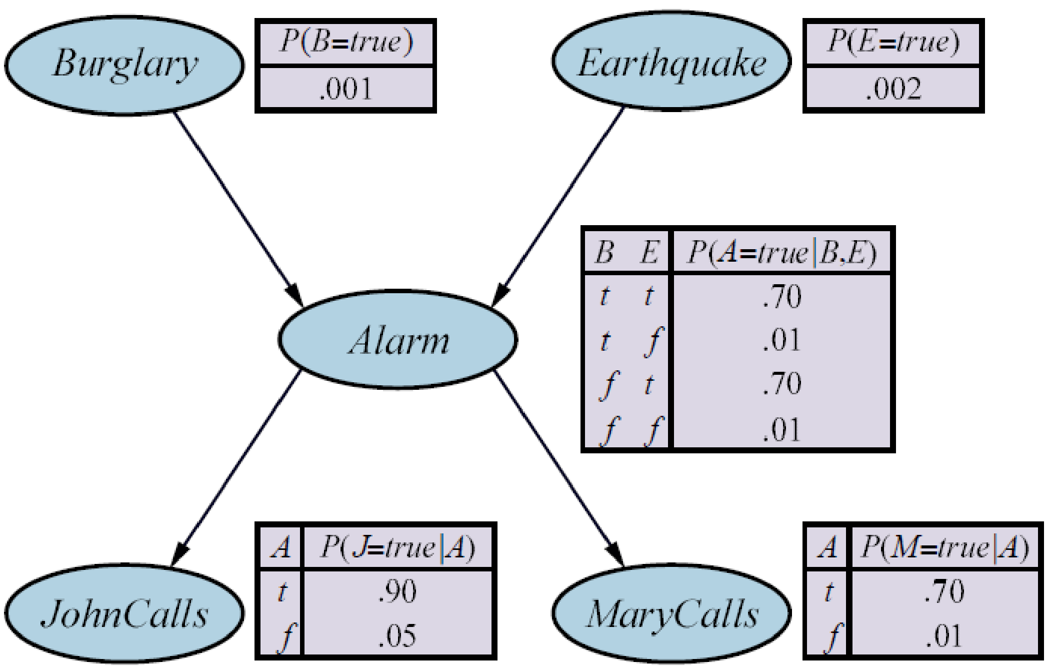

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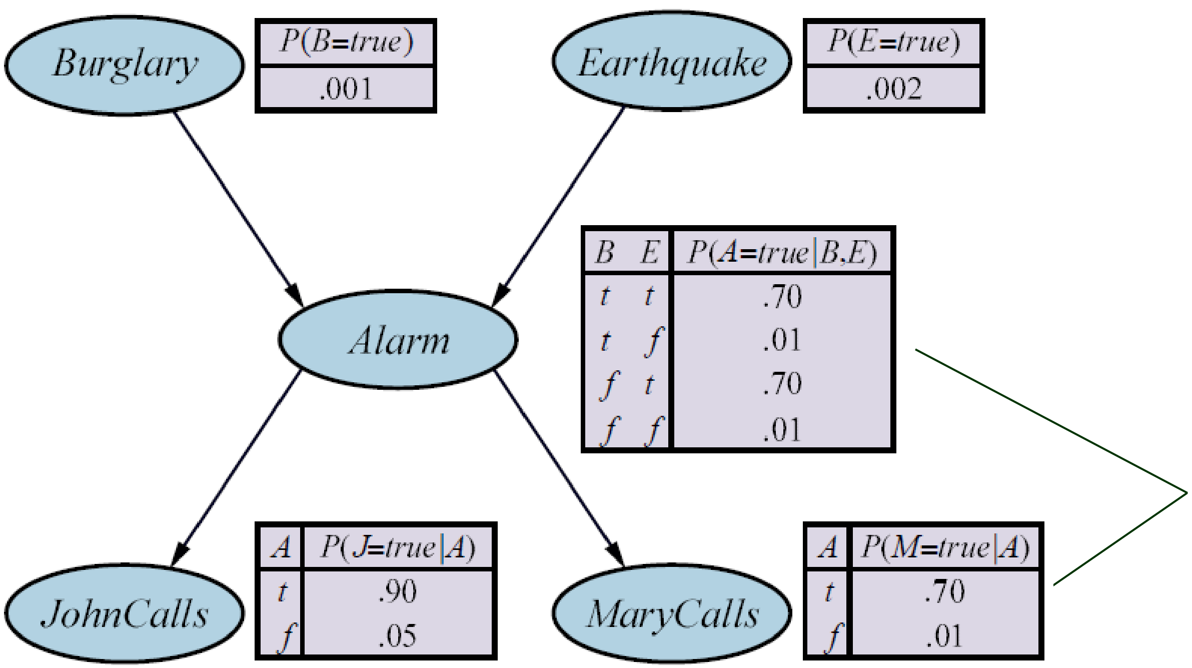

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.001

P(E=true)
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B	E	P(A=true|B,E)
t	t	.70
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conditional probability tables (CPTs)

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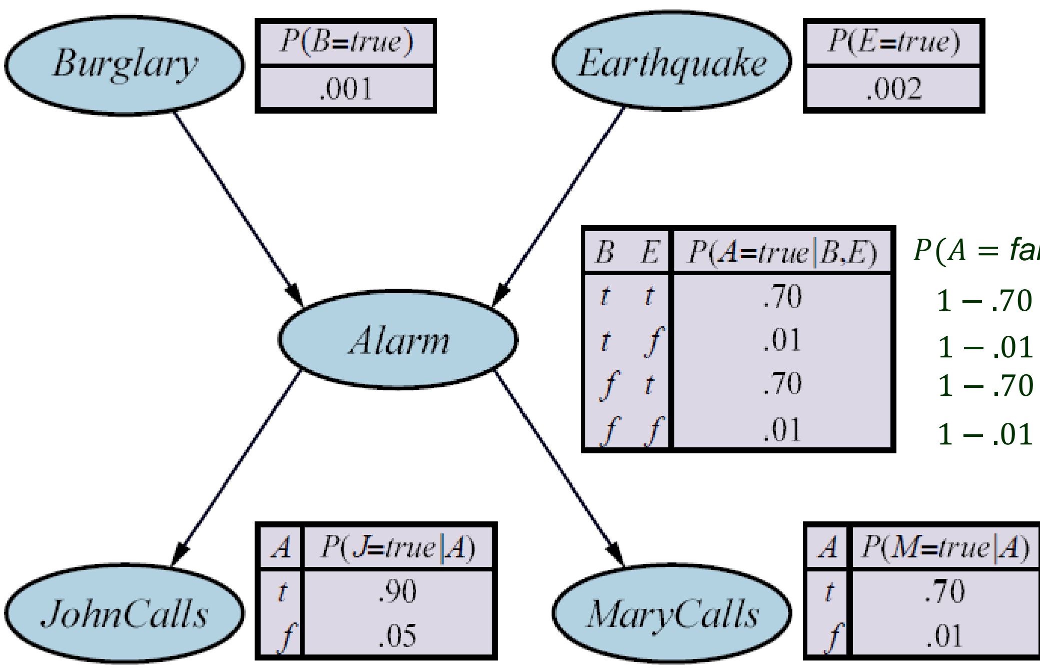

$P(A = \textit{false} \mid B, E)$

$1 - .70 = .30$

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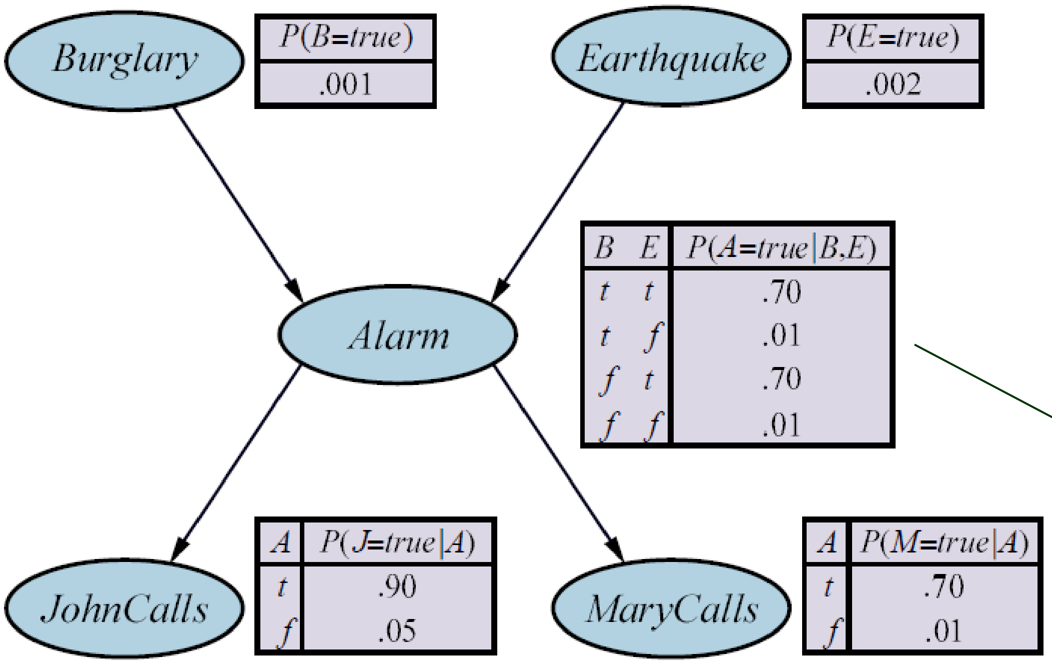

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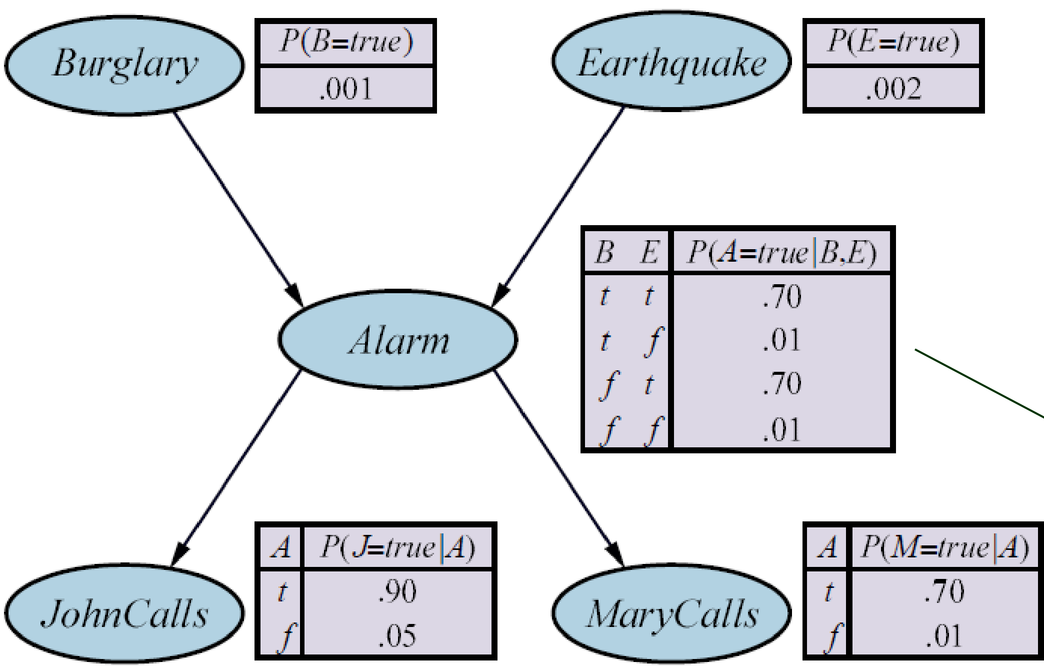

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Problem Estimate the probability of a burglary given the evidence of who has or has not called.

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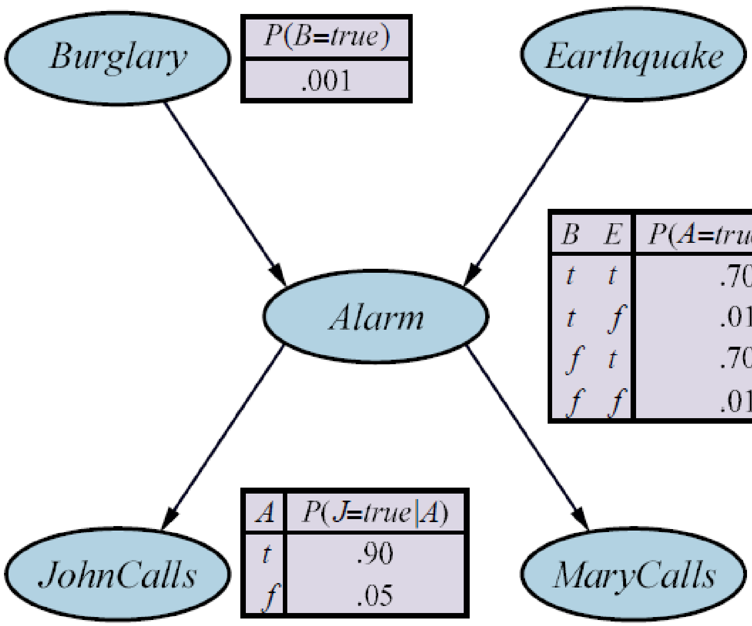
Every entry in the joint distribution is the product of the appropriate elements of the local conditional distribution.

BN as a Knowledge Base

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} \theta(x_i \mid parents(X_i))$$

Calculate the probability that the alarm has sounded, but neither a burglary nor an earthquake has occurred, and both John and Mary call.



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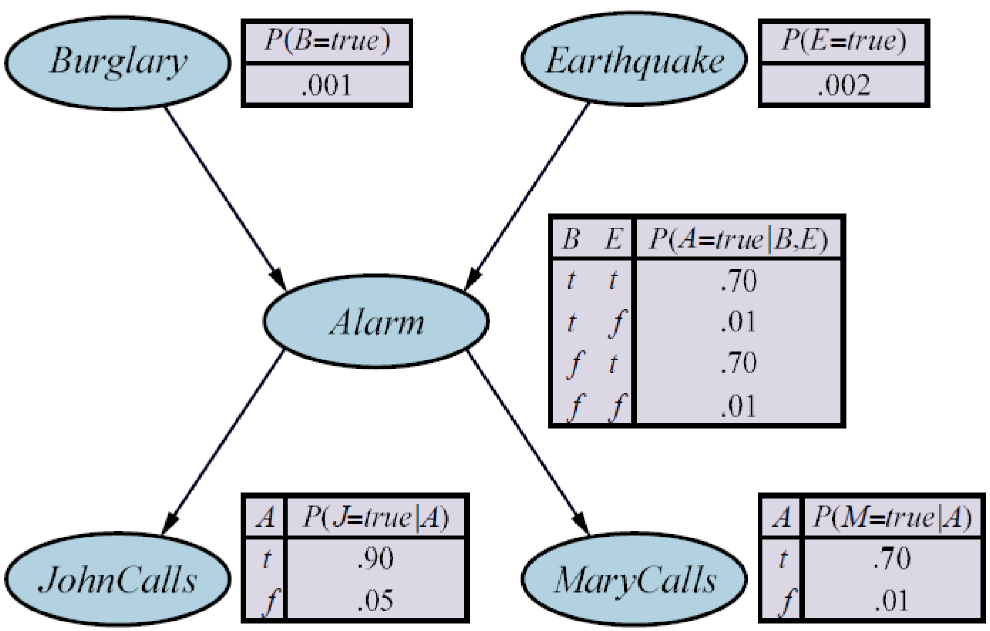

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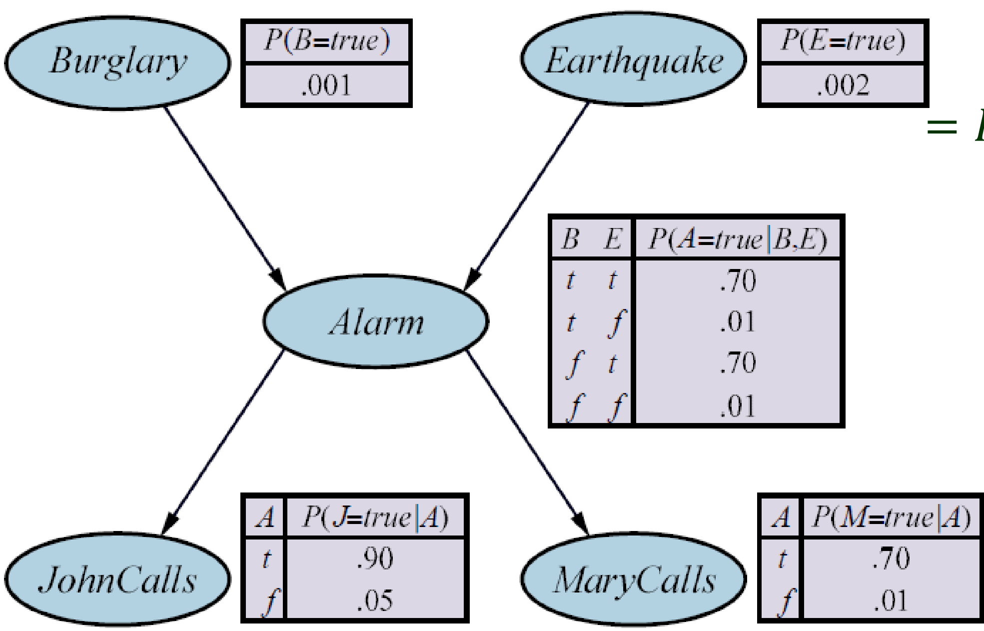

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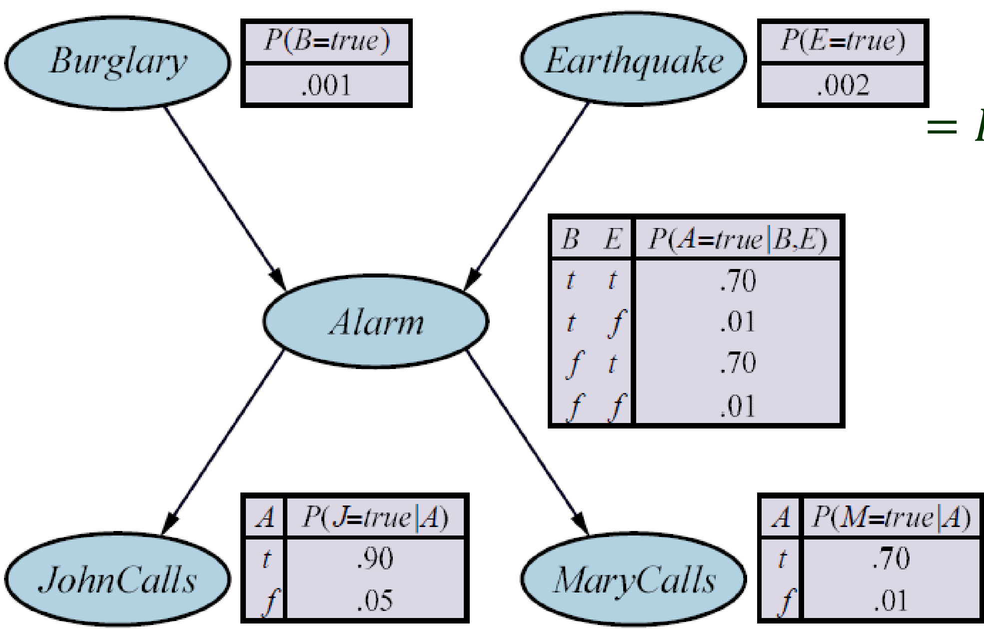

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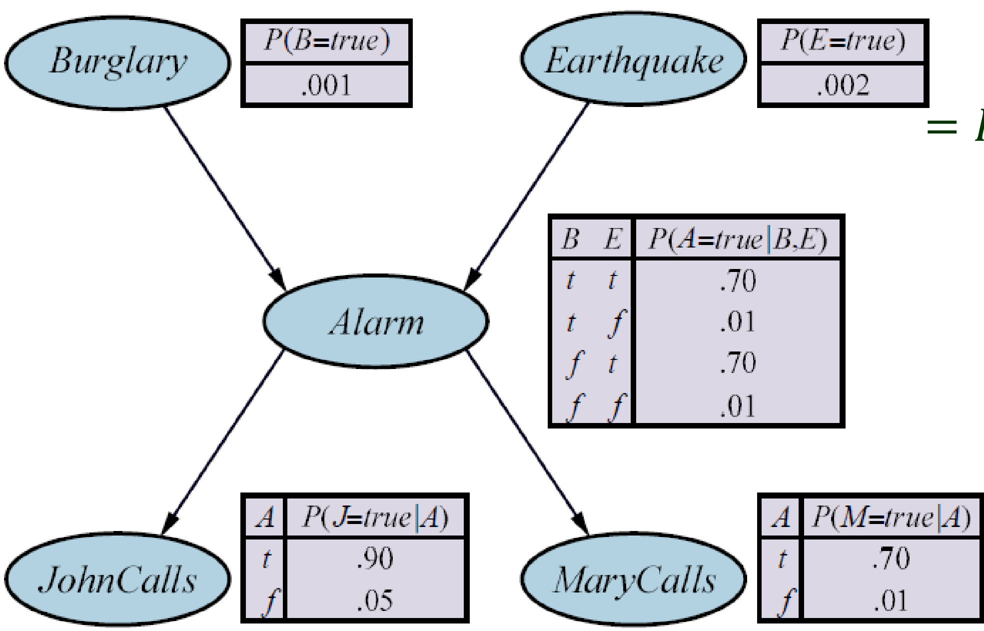

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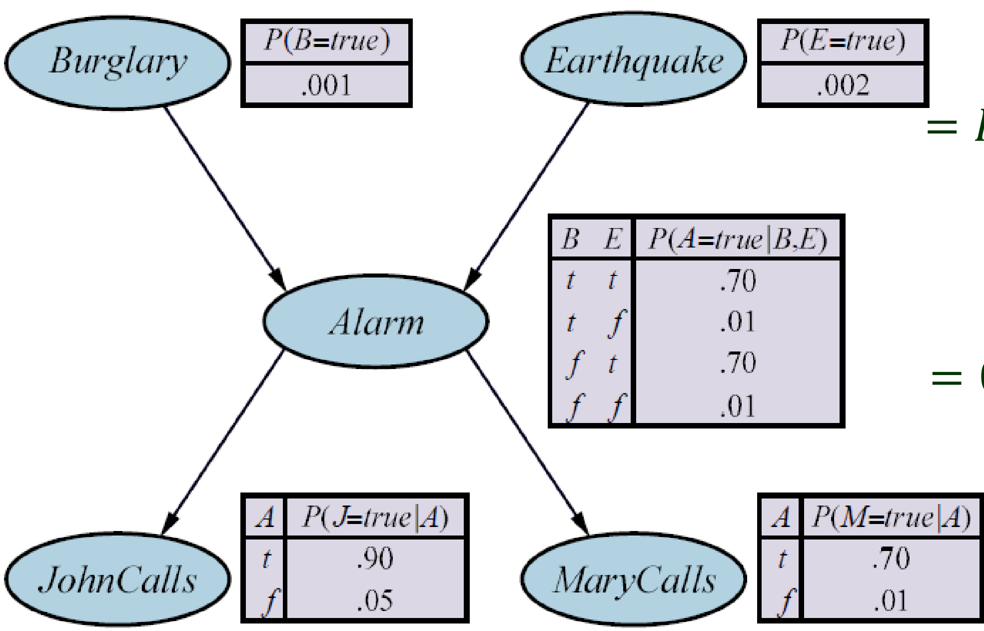

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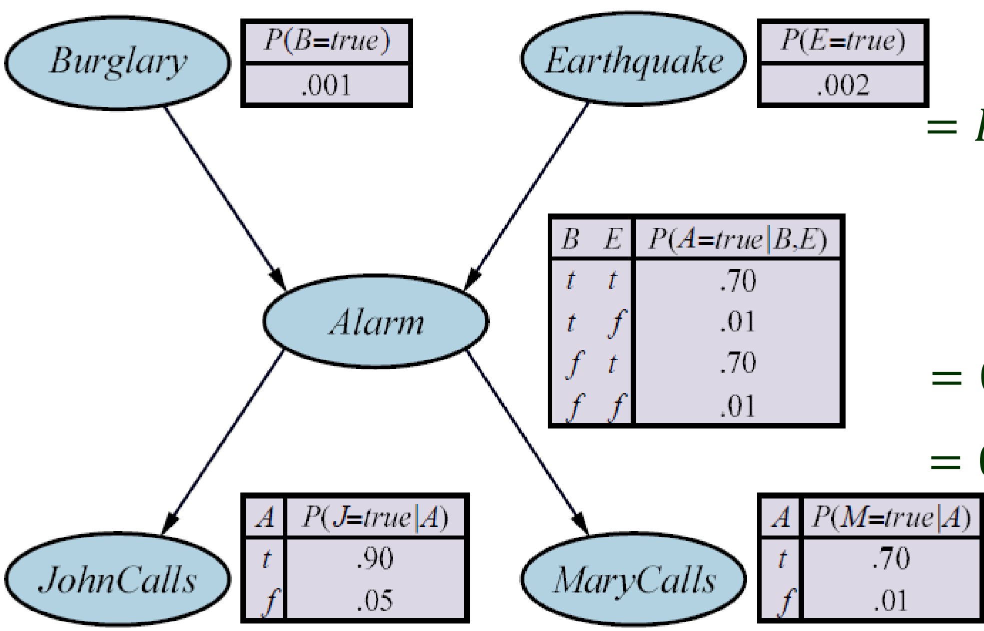

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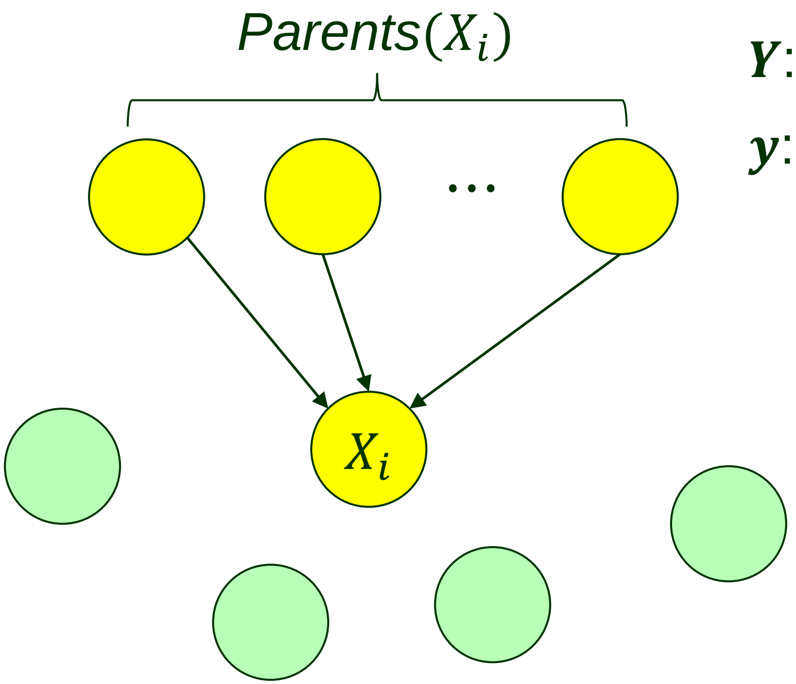

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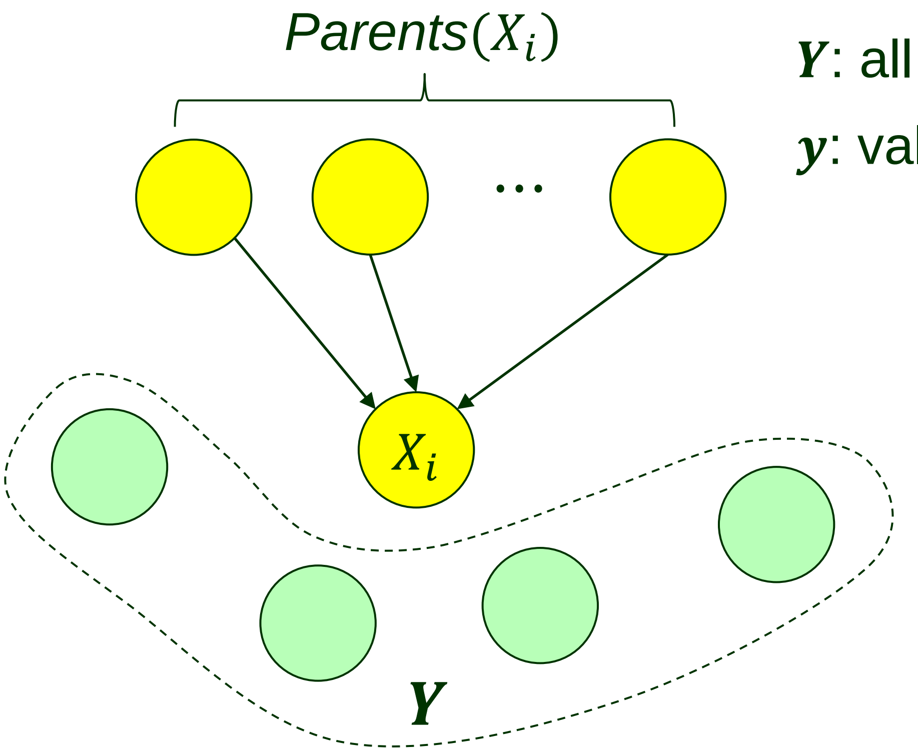

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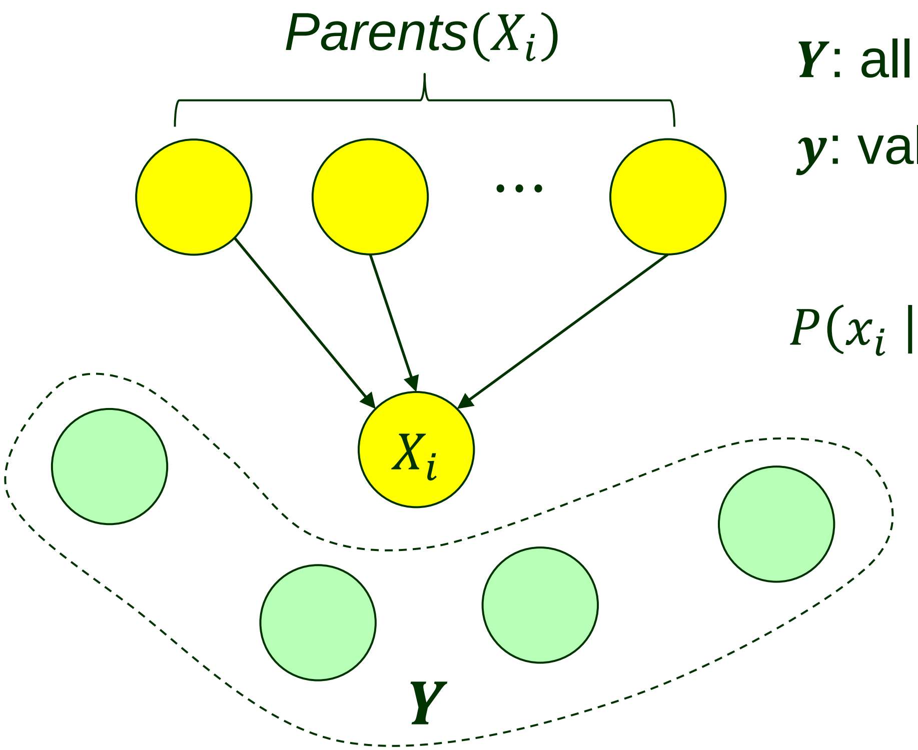

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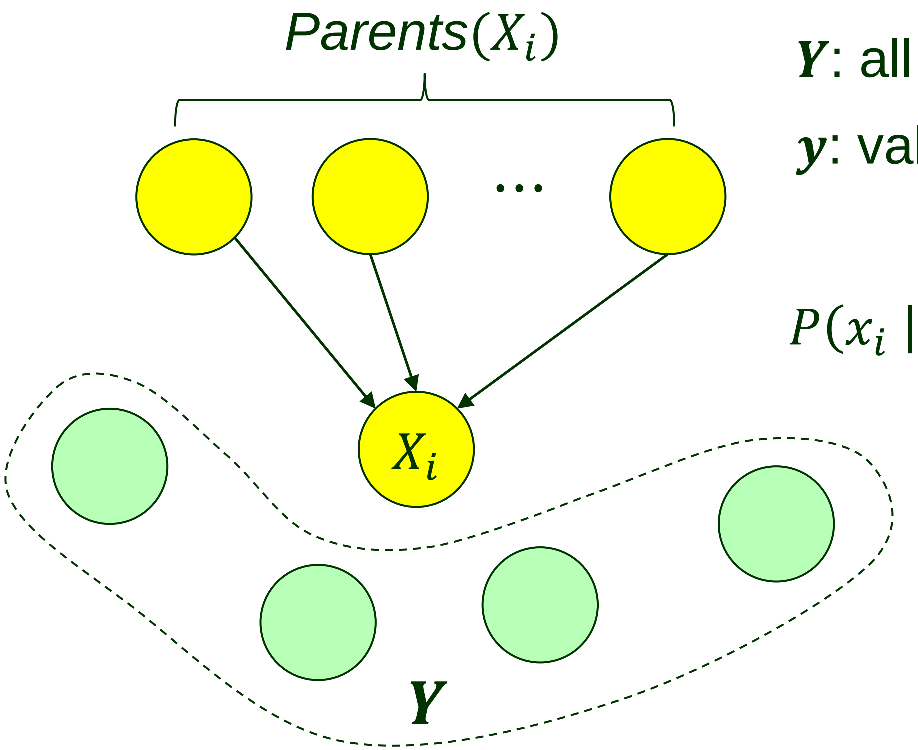

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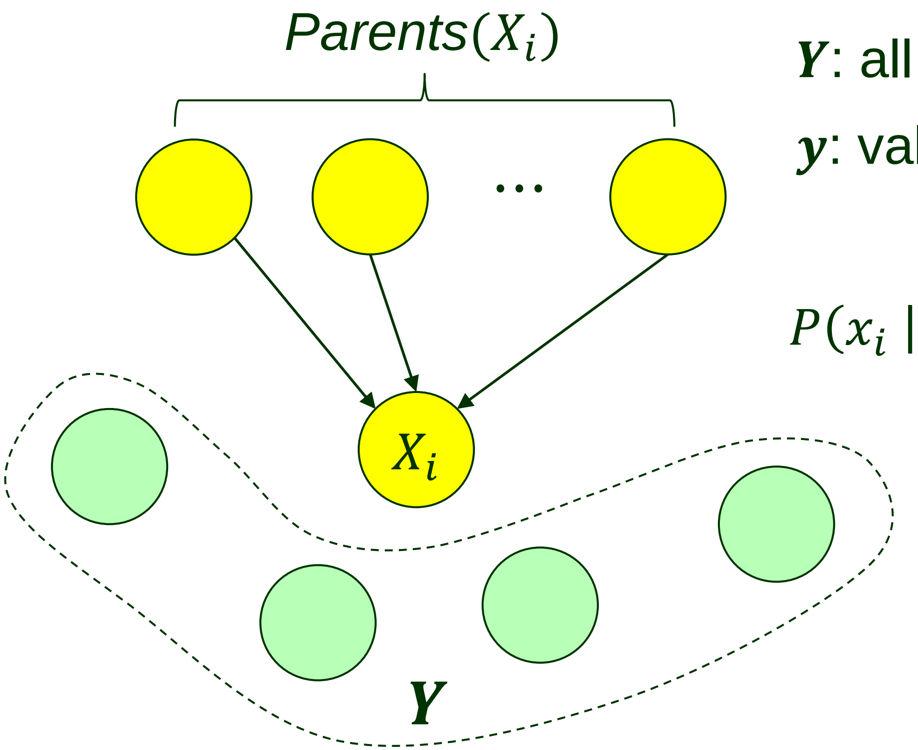

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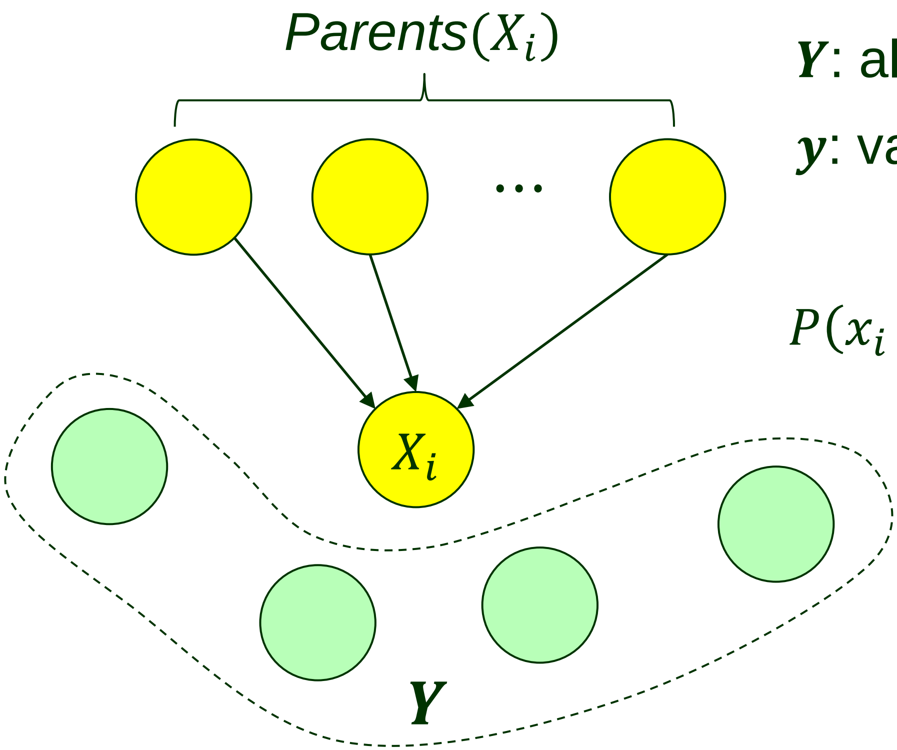

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$$P(x_1, \dots, x_n) = \prod_{i=1}^{n} \theta(x_i \mid parents(X_i))$$

(by definition of the Bayes net)

Full joint distribution: $$P(x_1, \dots, x_n) = \prod_{i=1}^{n} P(x_i \mid parents(X_i))$$

Correct Domain Representation

Chain rule:

$$P(x_1, \dots, x_n) = P(x_n \mid x_{n-1}, \dots, x_1) P(x_{n-1}, \dots, x_1)$$

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&= P(x_n \mid x_{n-1}, \dots, x_1) P(x_{n-1} \mid x_{n-2}, \dots, x_1) \cdots P(x_2 \mid x_1) P(x_1) \\
&= \prod_{i=1}^{n} P(x_i \mid x_{i-1}, \dots, x_1)
\end{aligned}
$$

Meanwhile,

$$P(x_1, \dots, x_n) = \prod_{i=1}^{n} P(x_i \mid \textit{parents}(X_i))$$

$$\prod_{i=1}^{n} P(x_i \mid x_{i-1}, \dots, x_1) = \prod_{i=1}^{n} P(x_i \mid \textit{parents}(X_i))$$

$\Updownarrow$

$\boldsymbol{P}(X_i \mid X_{i-1}, \dots, X_1) = \boldsymbol{P}(X_i \mid \textit{Parents}(X_i))$ for $i = 2, \dots, n$

if $\textit{Parents}(X_i) \subseteq \{X_1, \dots X_{i-1}\}$ for $i = 2, \dots, n$

Topological Order

$Parents(X_i) \subseteq \{X_1, \ldots X_{i-1}\}$ for $i = 2, \ldots, n$

The above is guaranteed if we number the nodes in *topological order* (which exists since the Bayesian network is a DAG).

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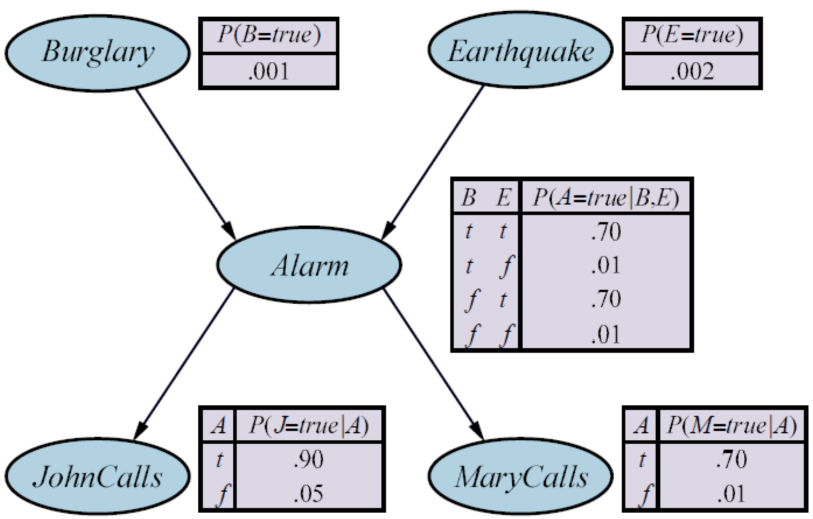

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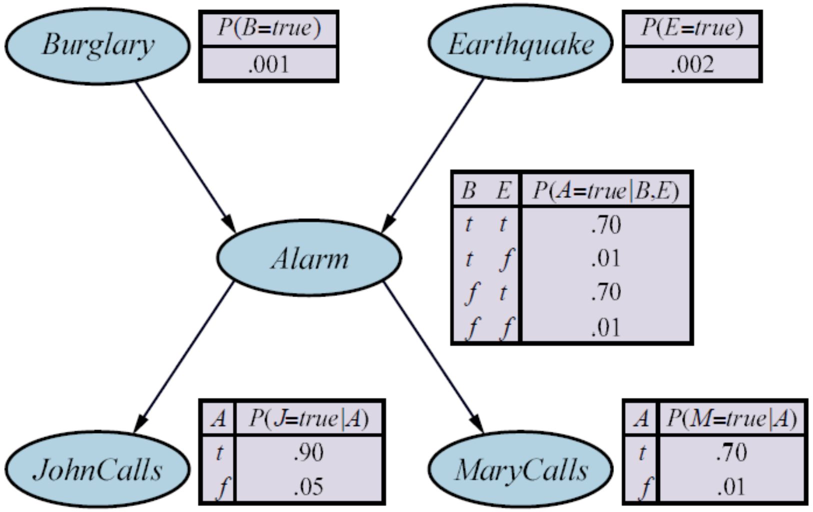

Four topological orders:

B, E, A, J, M

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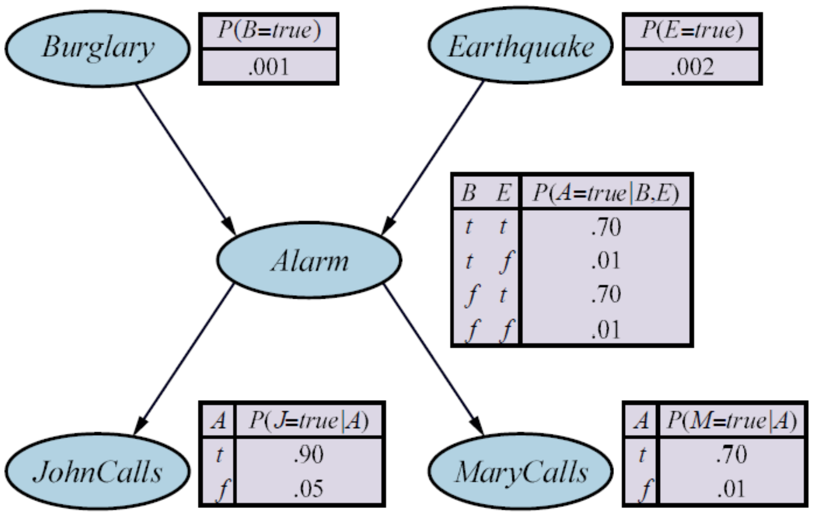

Four topological orders:

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B, E, A, M, J
E, B, A, J, M
E, B, A, M, J

Any one of the four suffices.

II. Construction of the Bayesian Network

$$\boldsymbol{P}(X_i \mid X_{i-1}, \ldots, X_1) = \boldsymbol{P}(X_i \mid \textit{Parents}(X_i)) \quad \text{for } i = 2, \ldots, n$$

The Bayesian network is correct only if X_i is conditionally independent of any X_j, $1 \le j \le i-1$, such that $X_j \notin \textit{Parents}(X_i)$.

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3. For $i = 1$ to n do

   a) Choose a minimal set of parents for X_i from $X_1, X_2, \ldots, X_{i-1}$ such that

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   b) Add a directed edge from every parent to X_i.

   c) Write down the conditional probability table (CPT), $\boldsymbol{P}(X_i \mid \mathit{Parents}(X_i))$.

Construction (cont'd)

Chosen order: *MaryCalls*, *JohnCalls*, *Alarm*, *Burglary*, *Earthquake*.

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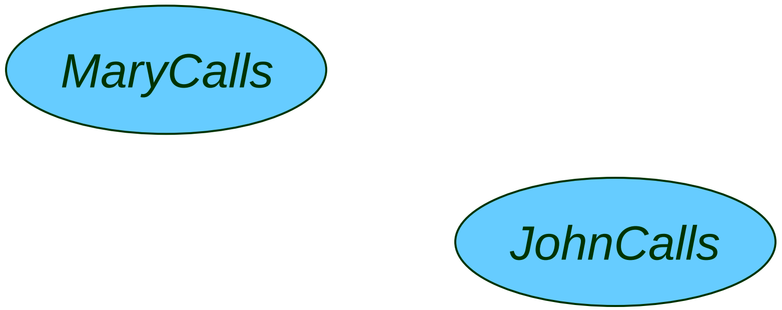

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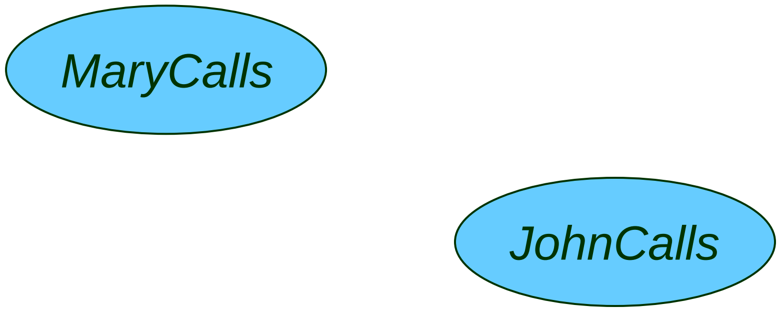

$P(j \mid m) \quad P(j)$

Construction (cont'd)

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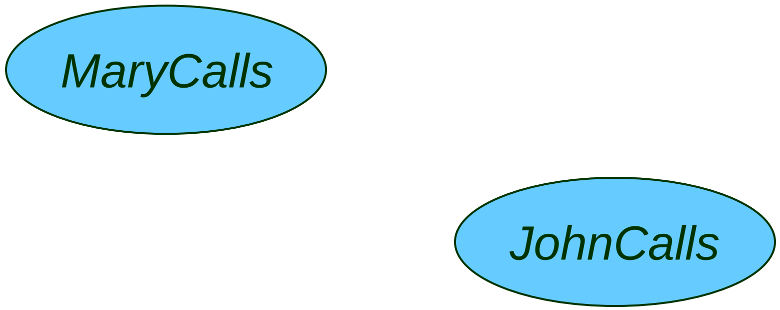

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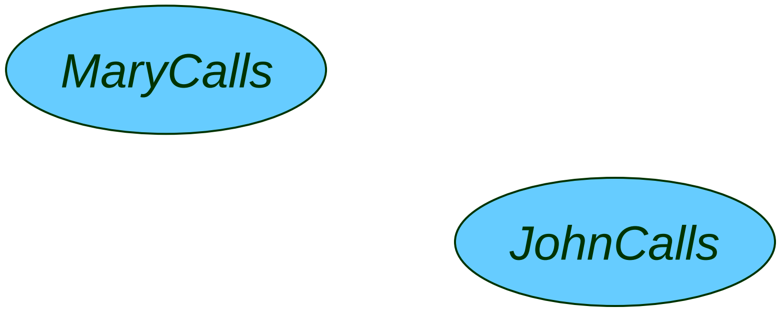

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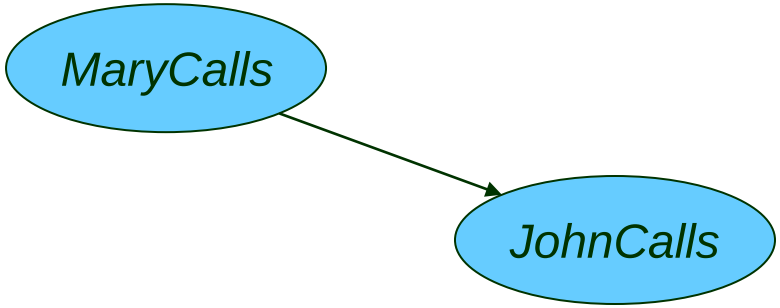

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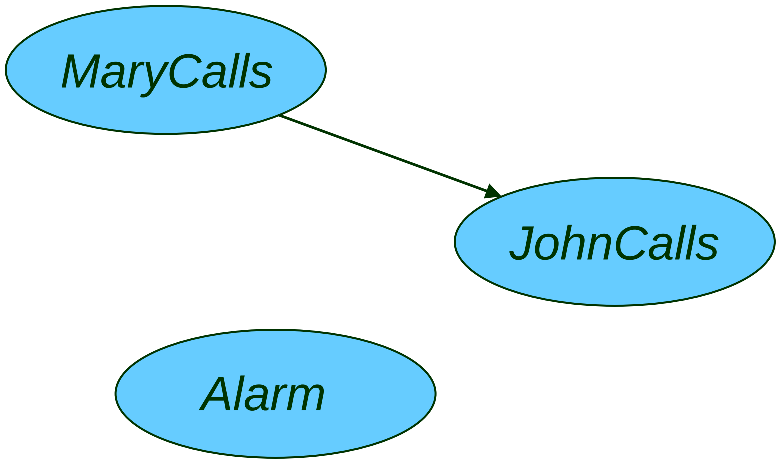

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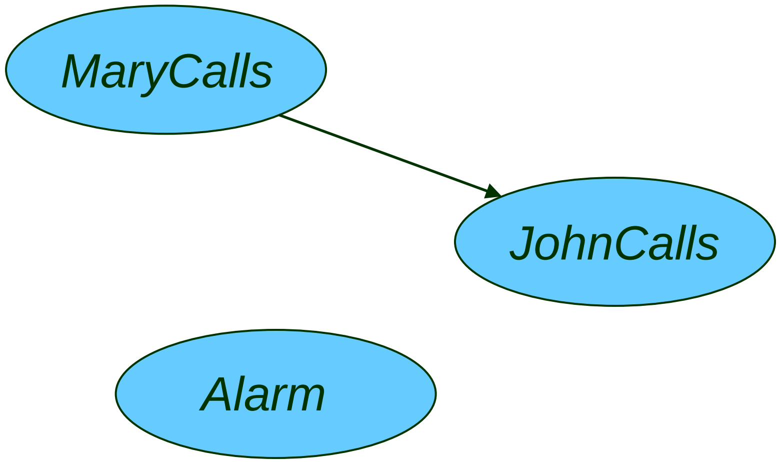

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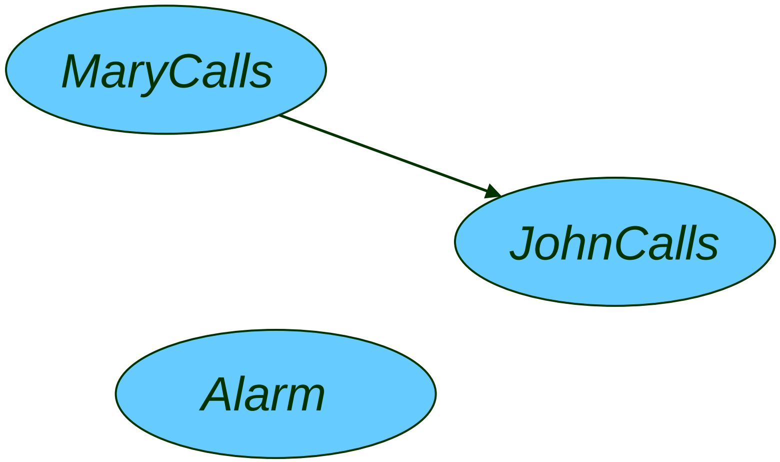

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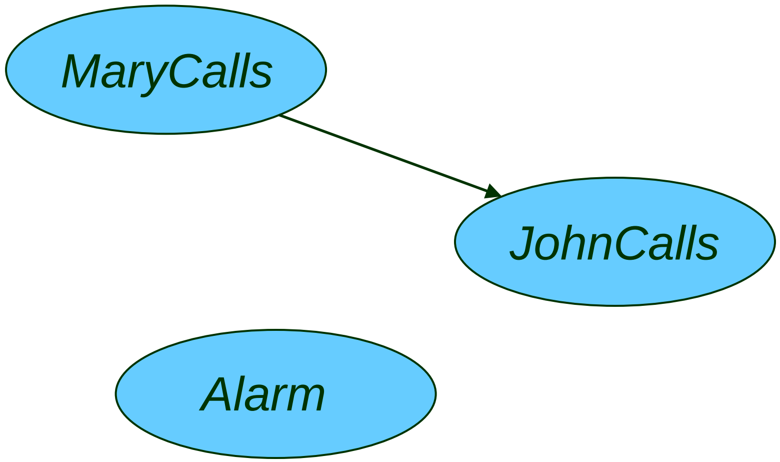

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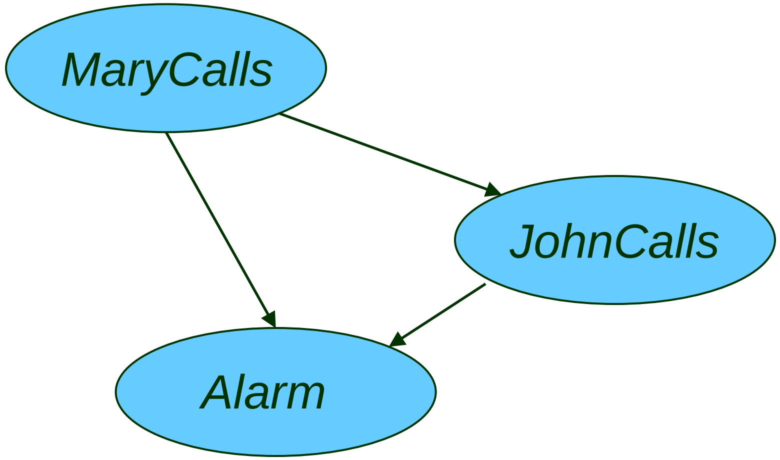

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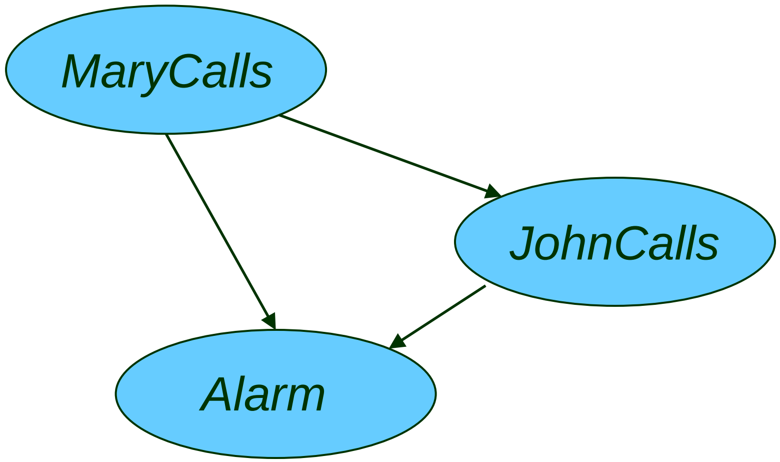

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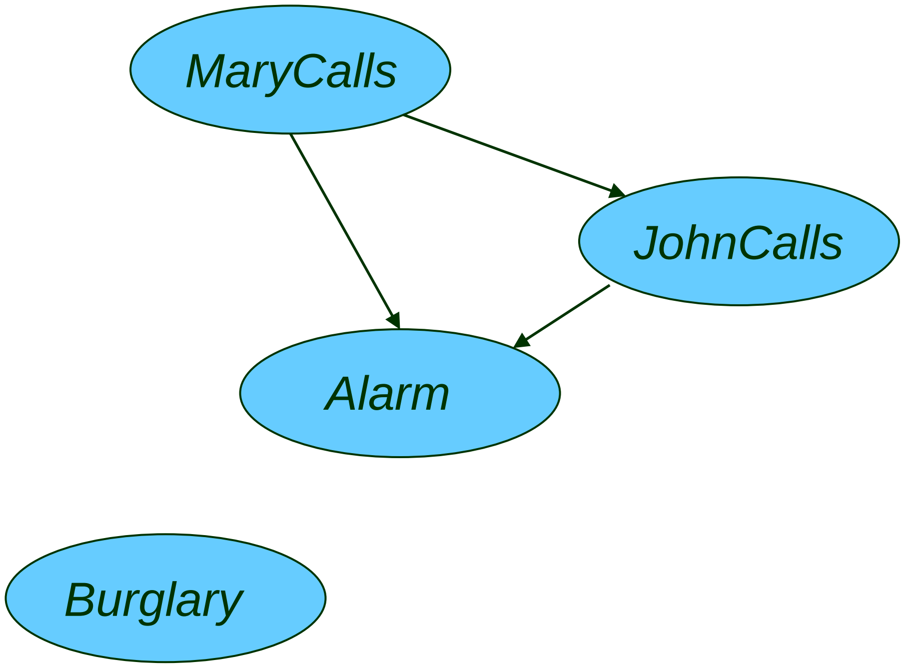

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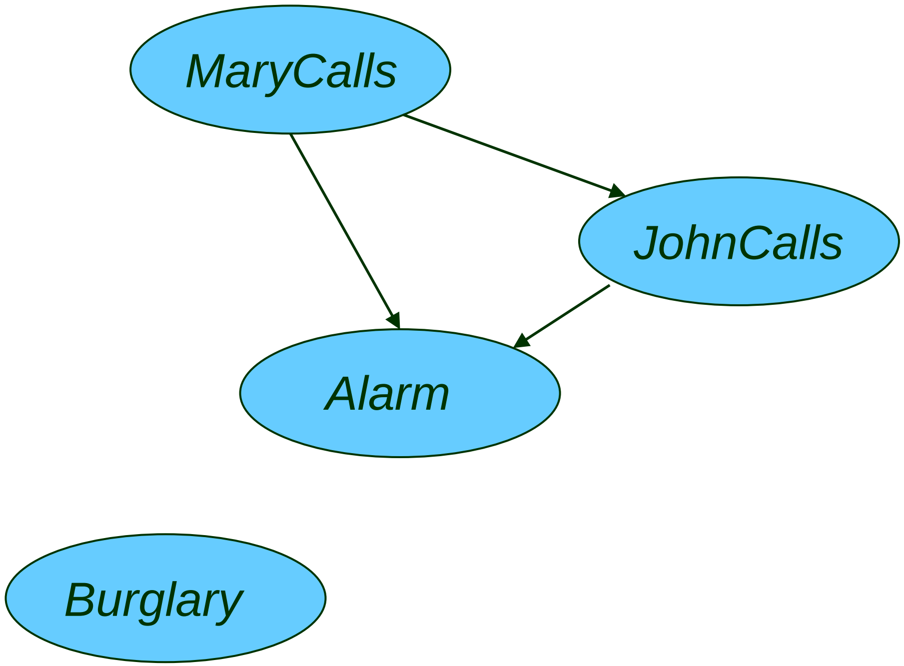

$\boldsymbol{P}(B \mid A, J, M)$ $\boldsymbol{P}(B \mid A)$

// If the value of A (either a or $\neg a$) is
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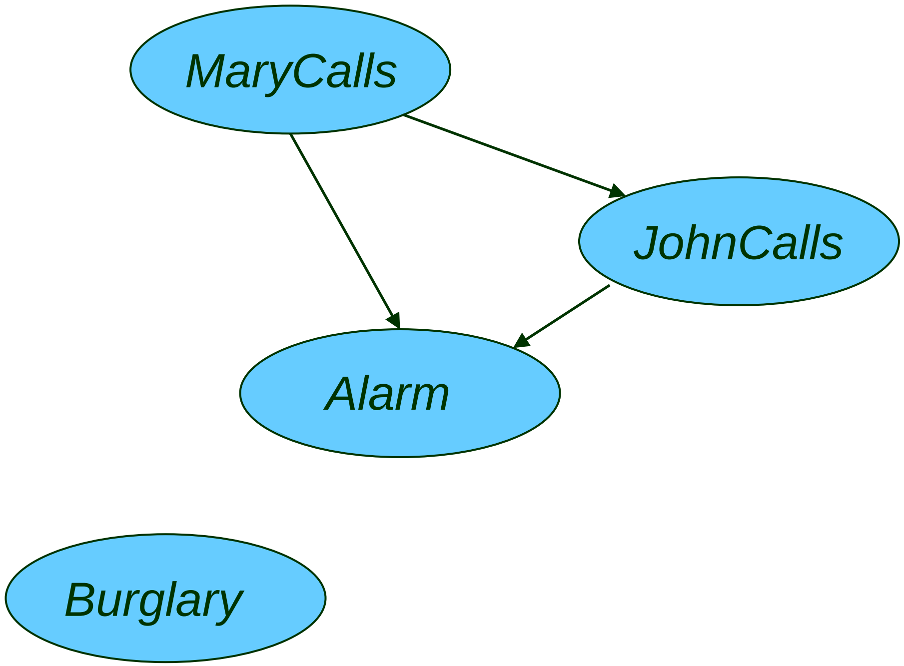

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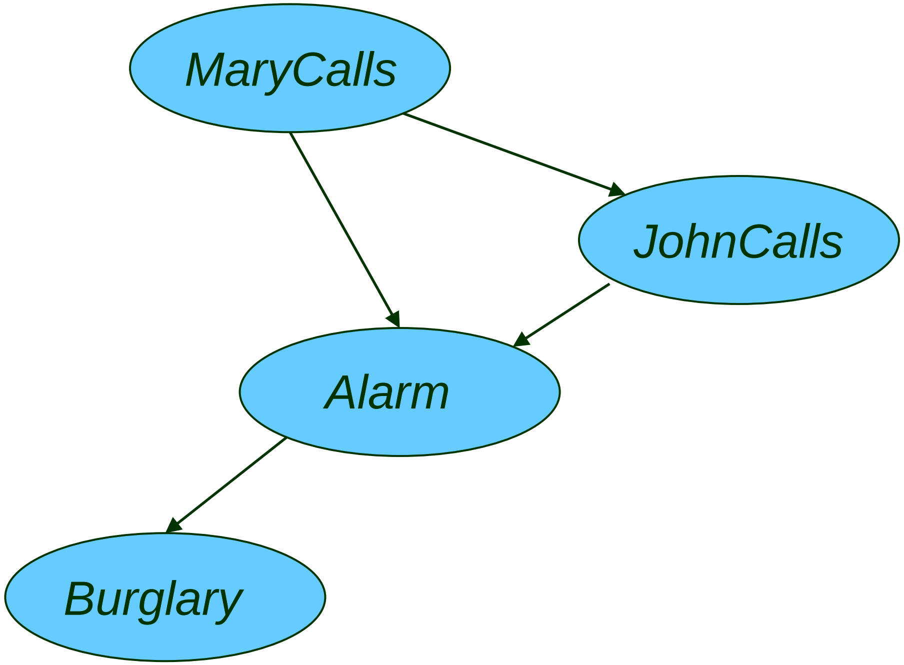

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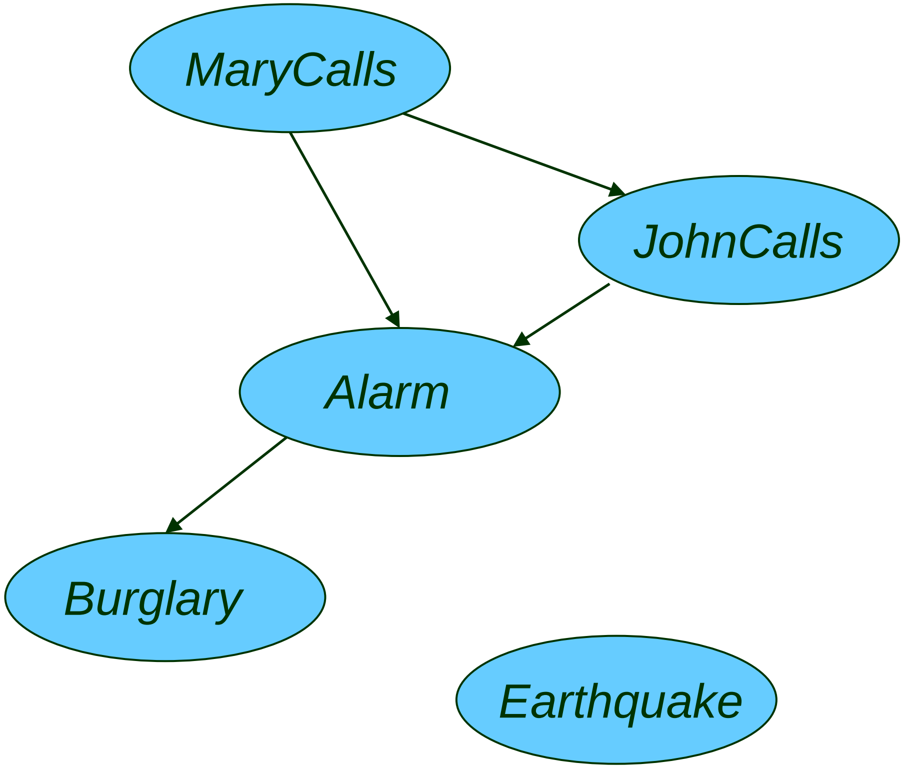

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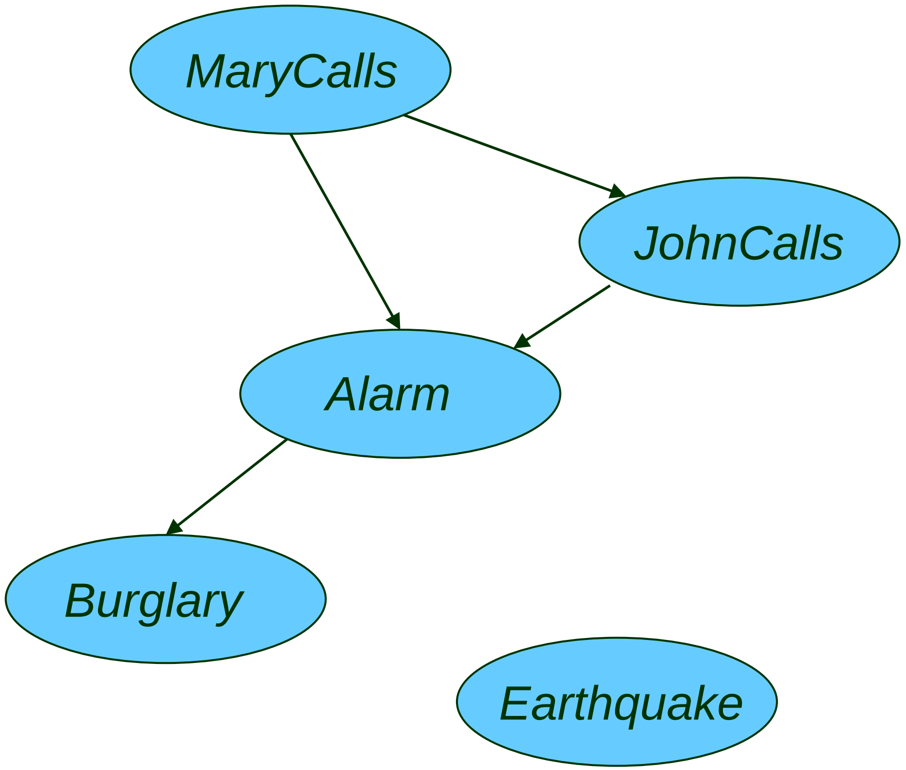

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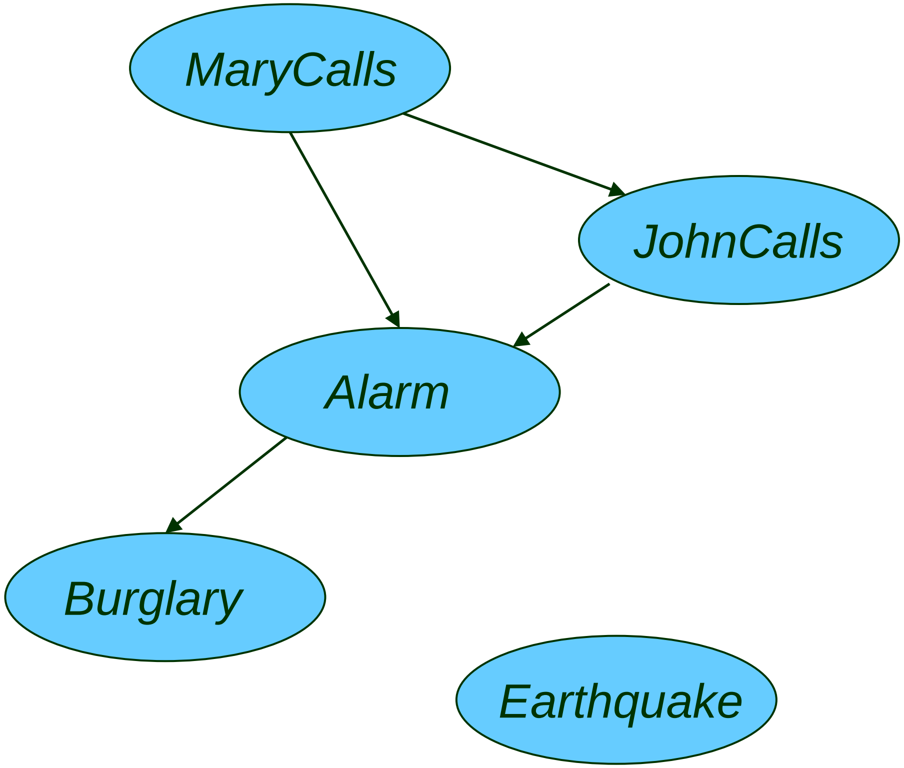

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// If the alarm is on, it is more likely that
// there has been earthquake. If there
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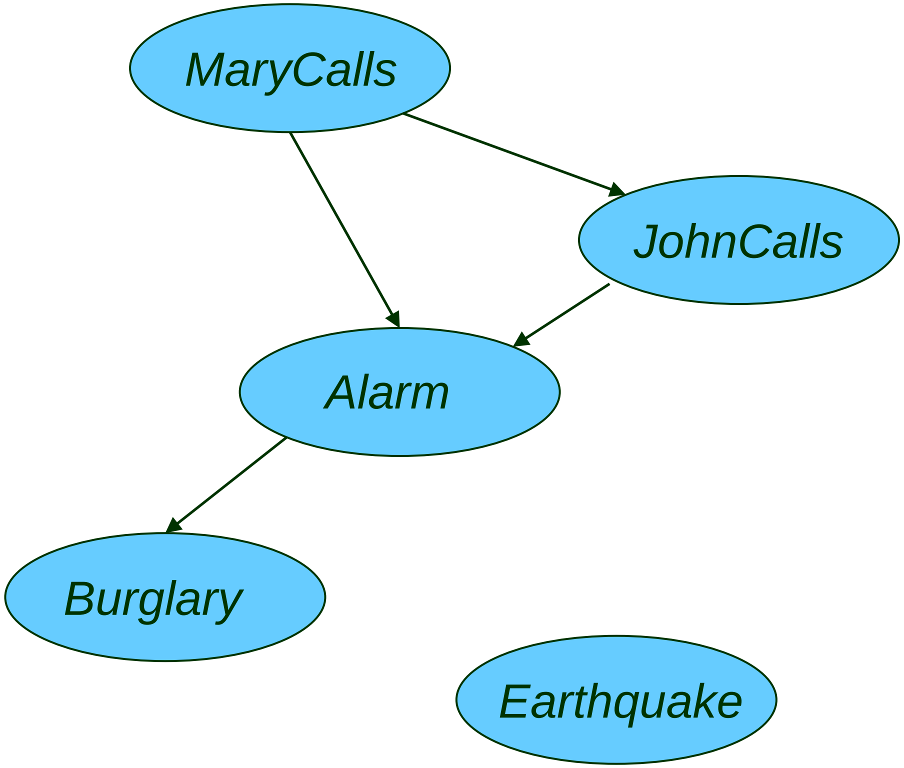
Construction for the Burglary Example

MaryCalls, *JohnCalls*, *Alarm*, *Burglary*, *Earthquake*.



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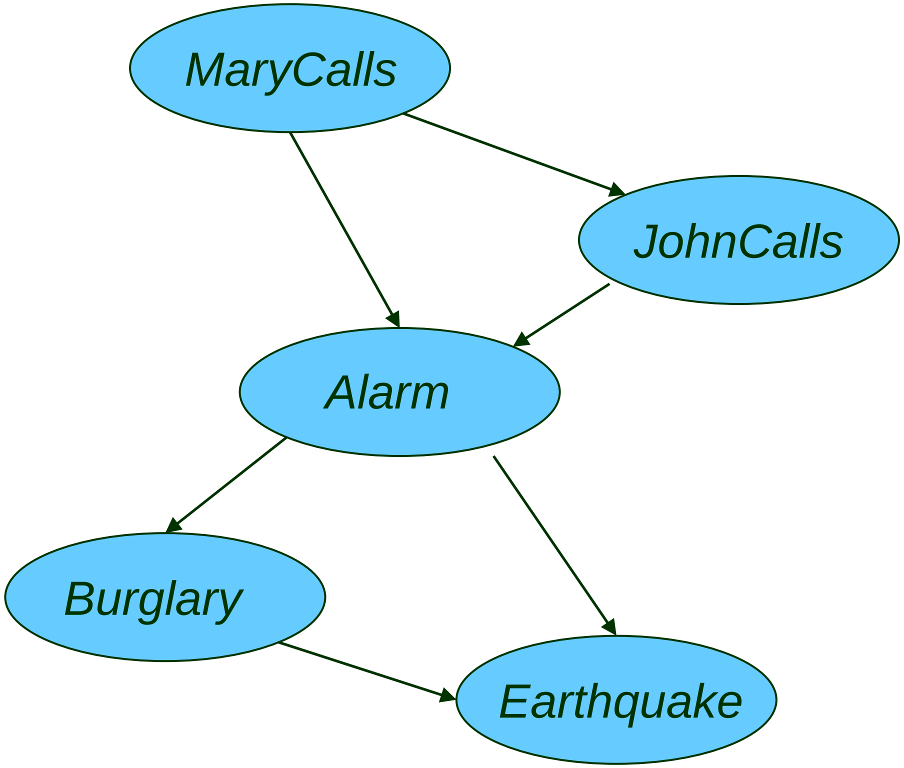

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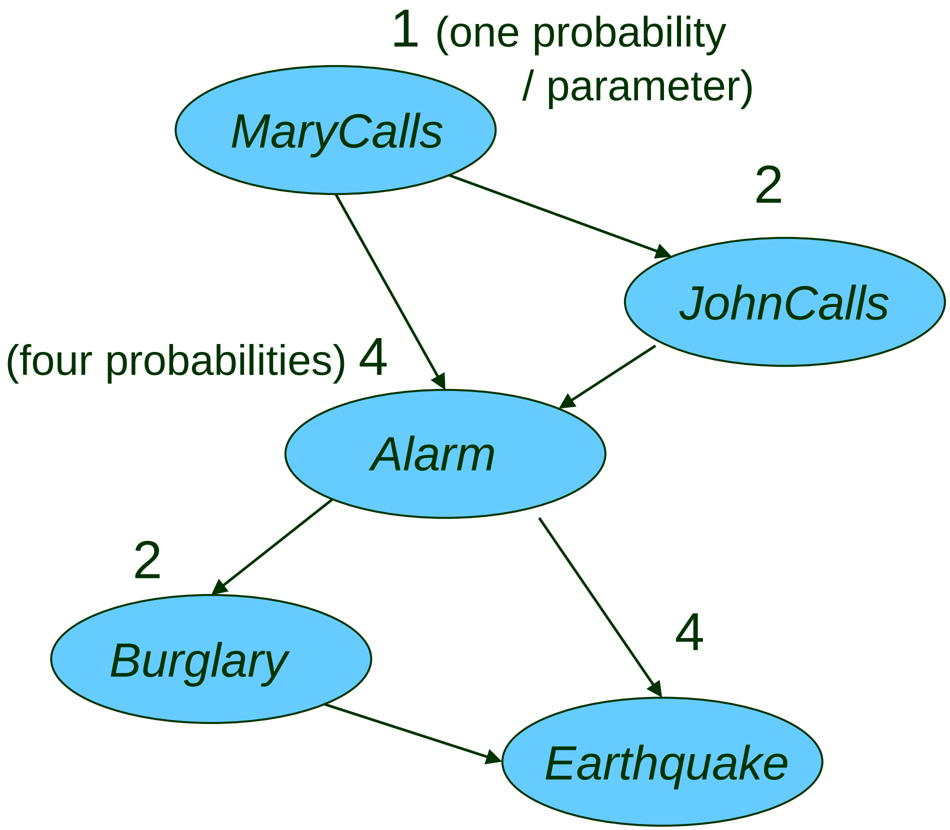
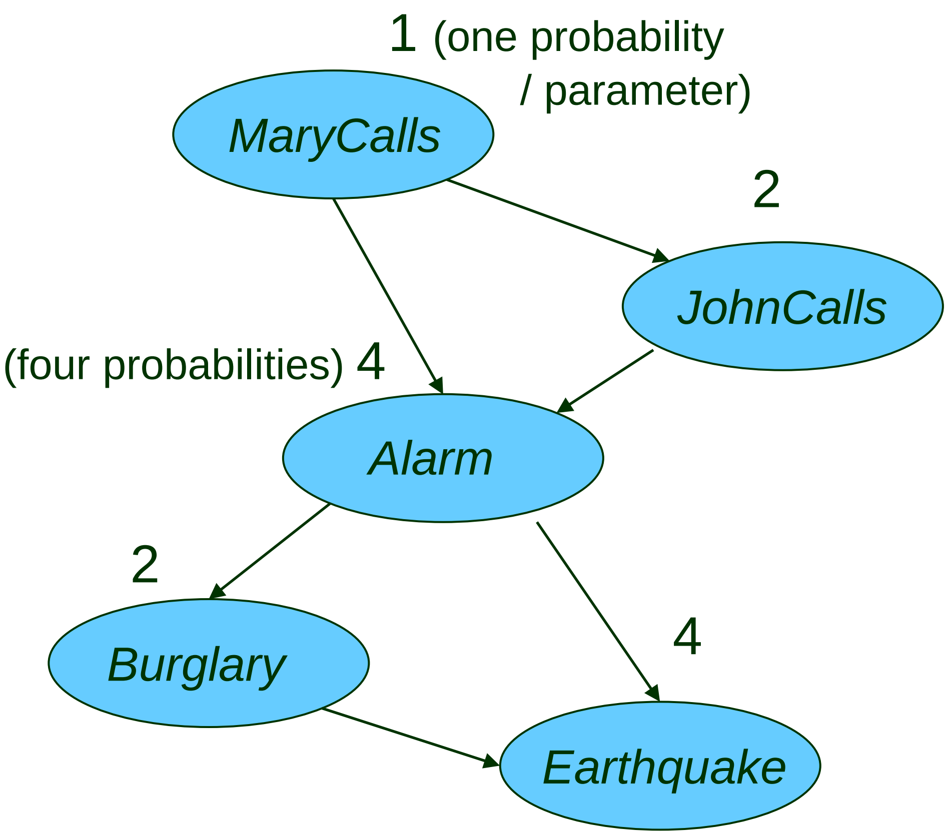
Node Ordering Matters



$1 + 2 + 4 + 2 + 4 = 13$
conditional probabilities

Node Ordering Matters



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- ♣ More conditional probabilities than needed.

Node Ordering Matters

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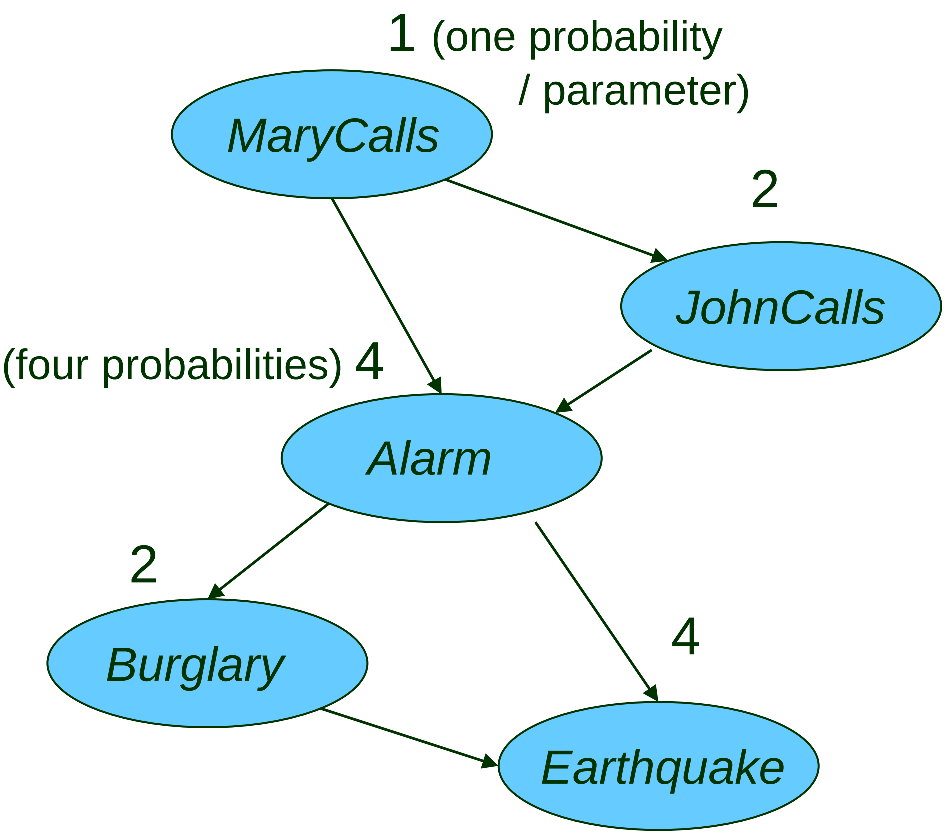
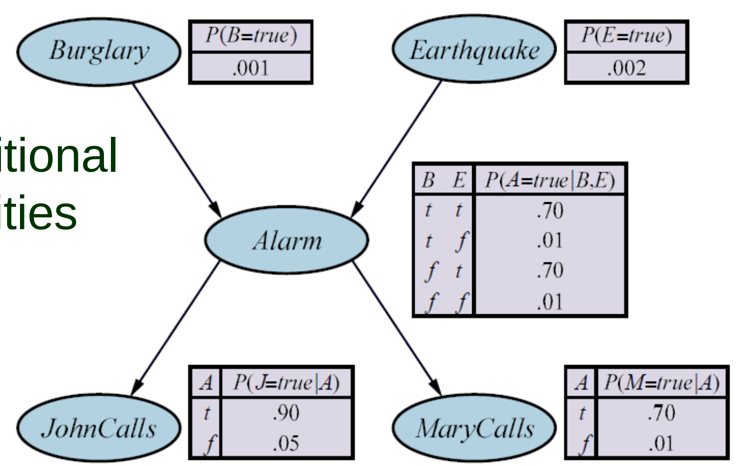


$1 + 2 + 4 + 2 + 4 = 13$
conditional probabilities

10 conditional probabilities



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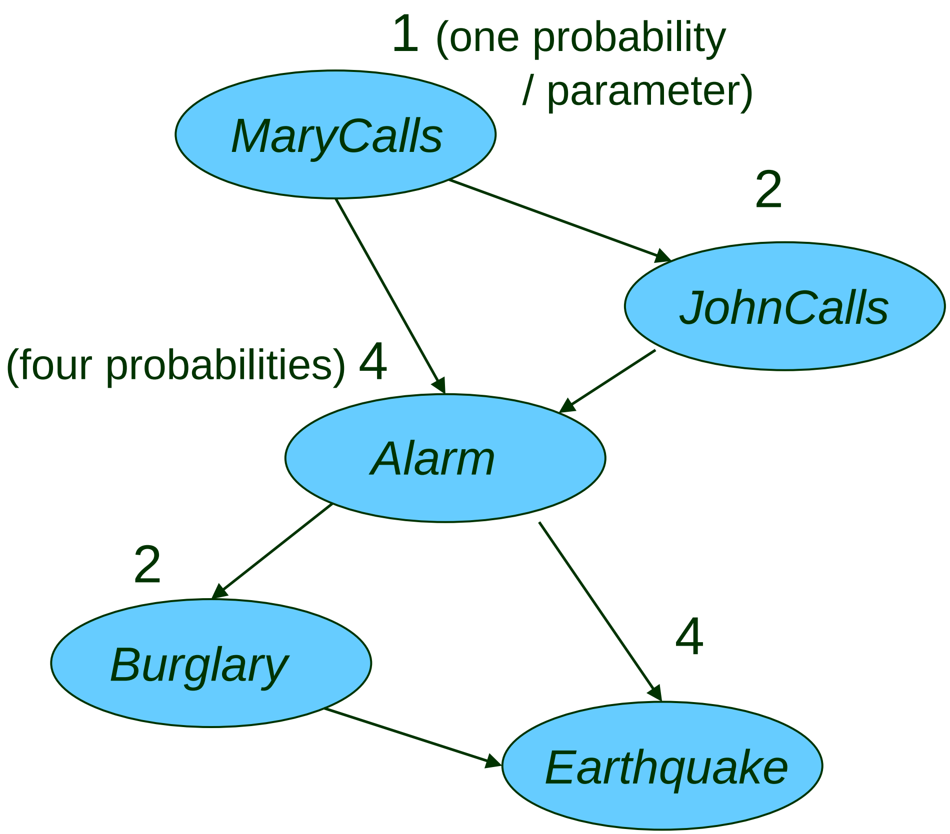
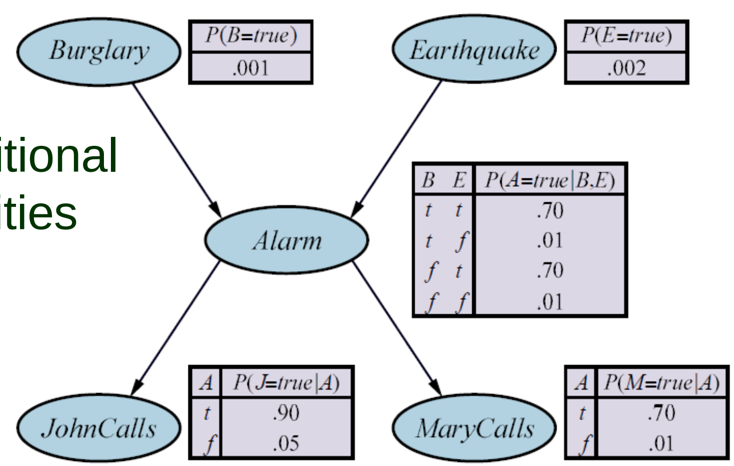


- More conditional probabilities than needed.
- Assessment of unnatural probabilities, e.g., $\boldsymbol{P}$(*Earthquake* | *Burglary, Alarm*}.

$1 + 2 + 4 + 2 + 4 = 13$ conditional probabilities

10 conditional probabilities



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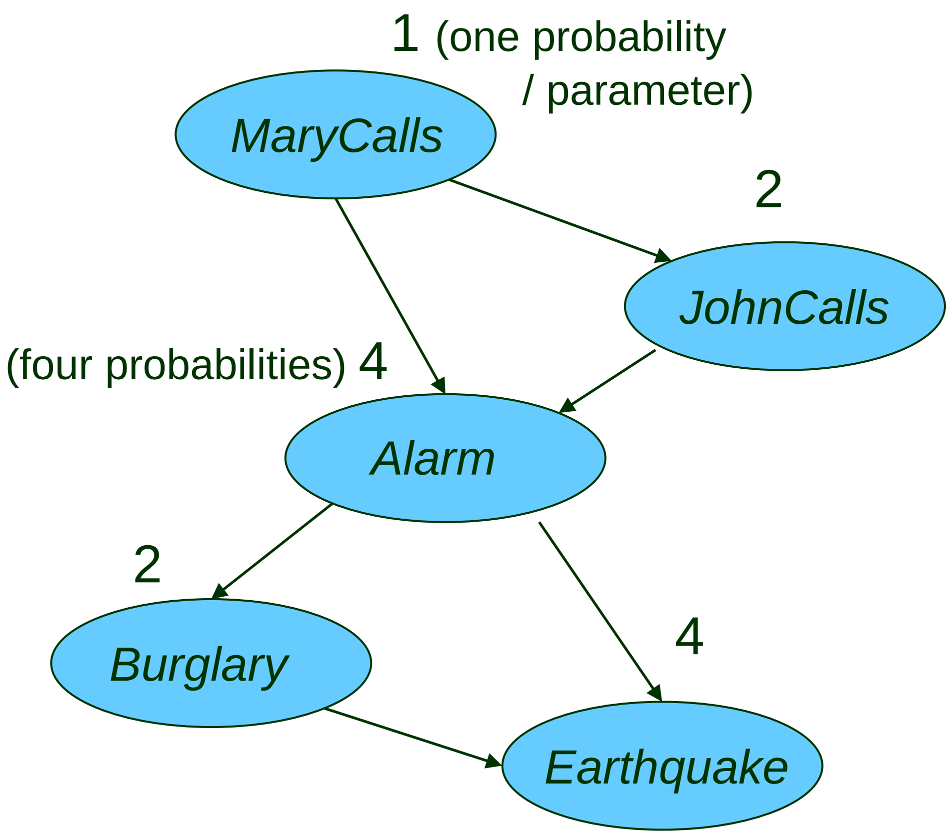


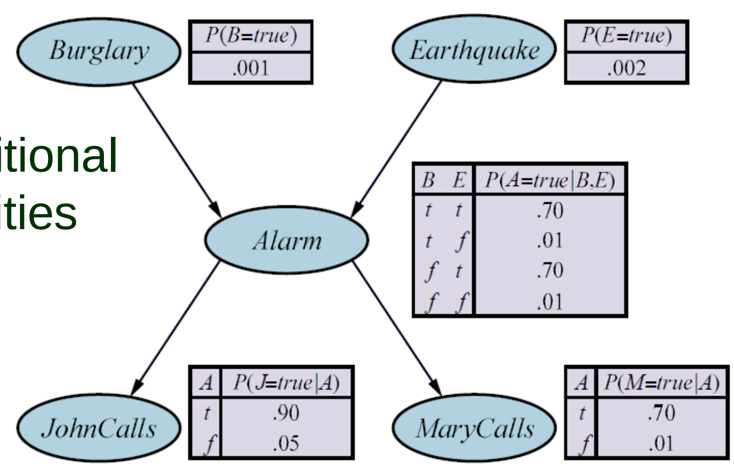
$1 + 2 + 4 + 2 + 4 = 13$
conditional probabilities

♣ More conditional probabilities than needed.

♣ Assessment of unnatural probabilities, e.g., $\boldsymbol{P}$(*Earthquake* | *Burglary, Alarm*}.

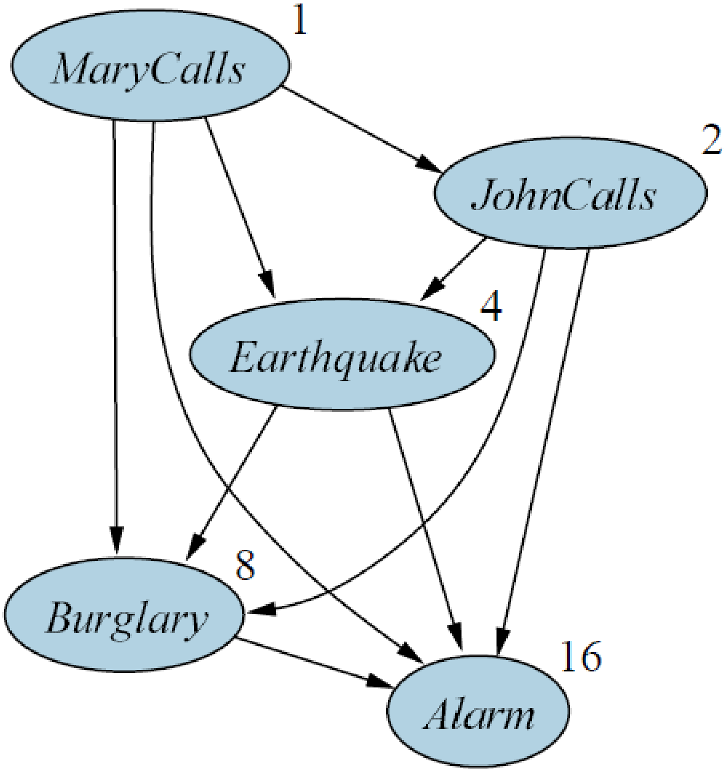
♦ Sticking to a causal model results in fewer probabilities that are also easier to come up with.

10 conditional probabilities



Bad Node Ordering

MaryCalls, *JohnCalls*, *Earthquake, Burglary*, *Alarm*.



$1 + 2 + 4 + 8 + 16 = 31$ distinct probabilities (exactly the same as the full joint distribution)!

Roles of Casualty

- Deciding conditional independence is hard in noncausal directions.
  (Causal models and conditional independence seem hardwired for humans!)

- Assessing conditional probabilities is hard in noncausal directions.

- The interpretation of directed acyclic graphs as carriers of independence assumptions does not necessarily imply causation.

- The ubiquity of DAG models in statistical and AI applications stems (often unwittingly) primarily from their causal interpretation.

- In practice, DAG models are rarely used in any variable ordering other than those which respect the direction of time and causation.

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- ♦ To avoid a fully connected network, leave out links that represent slight dependencies.

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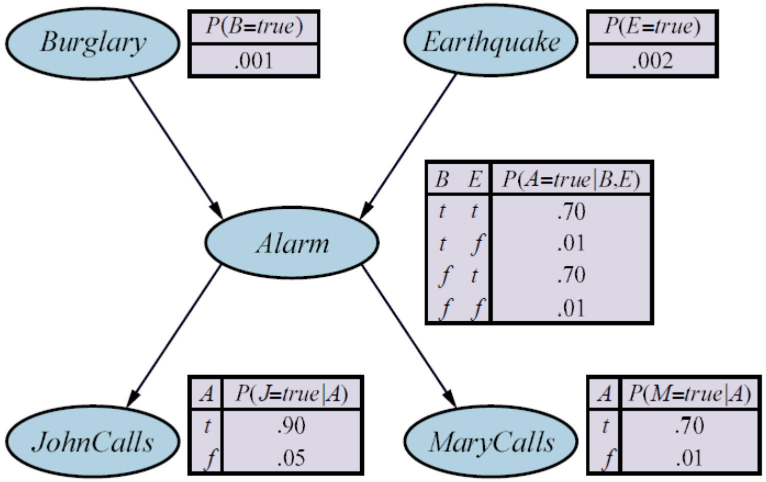
Every variable is conditionally independent of its non-descendants (including ancestors), given the values of its parents.



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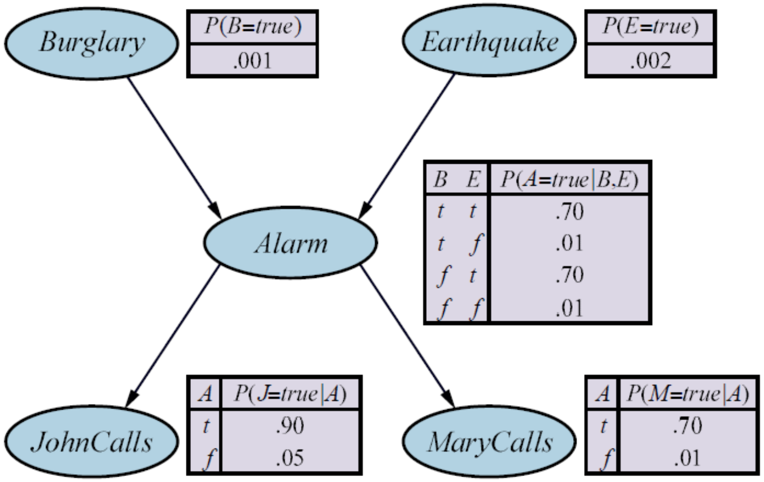

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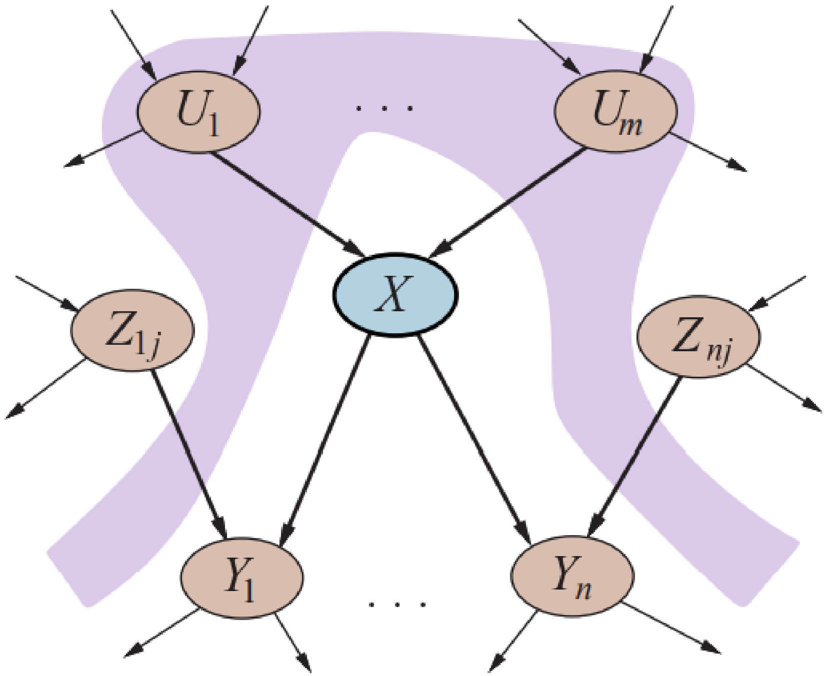
// network parameter interpretation

Illustration



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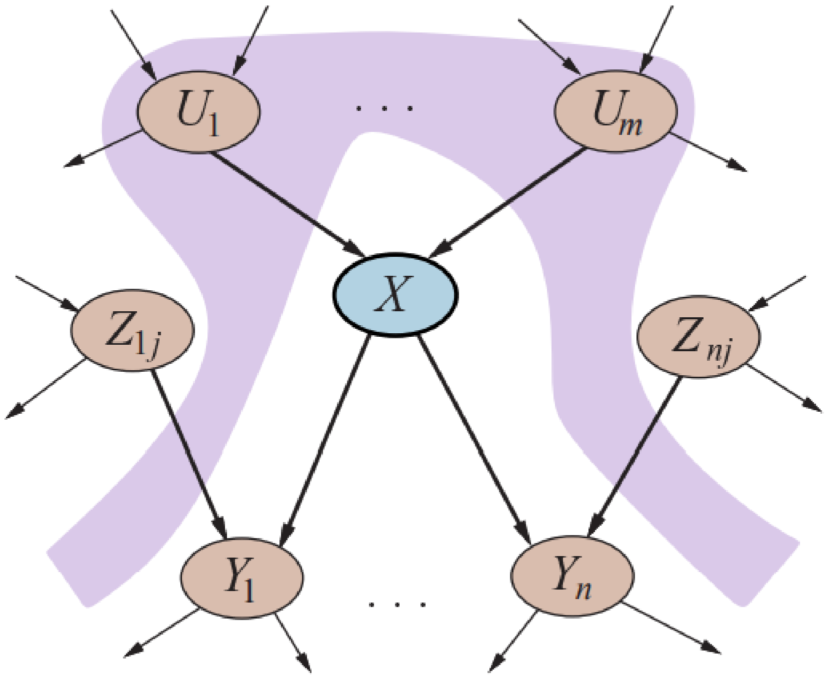

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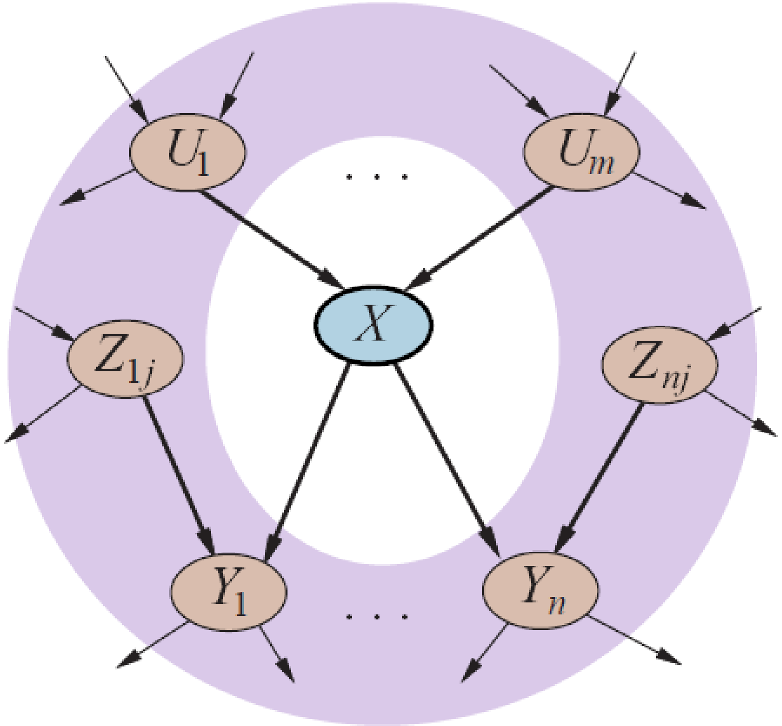
$$\Downarrow$$

The full joint distribution $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid parents(X_i))$

Markov Blanket

The *Markov blanket* $MB(X)$ of a node X consists of its parents, children, and children's parents (but *not* the node itself).

$$MB(X) = Children(X) \cup Parents(X)$$
$$\cup \{Y \mid Y \neq X \wedge (\exists Z \;\; Z \in Children(X) \wedge Y \in Parents(Z))\}$$



The Markov blanket is represented by the gray area.

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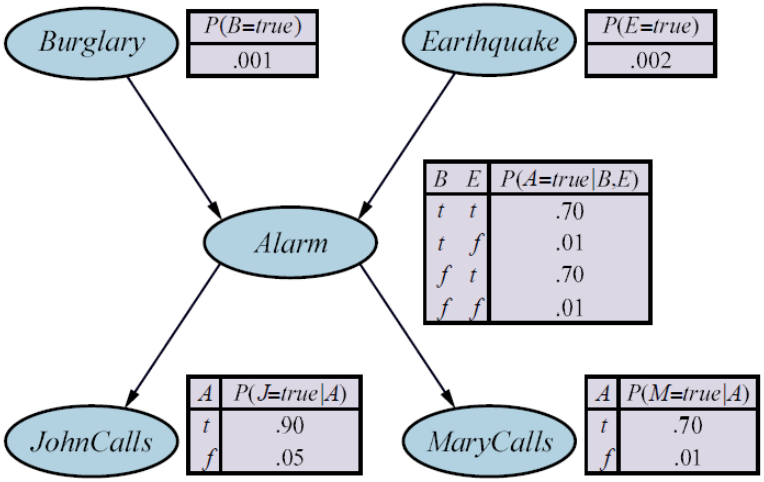

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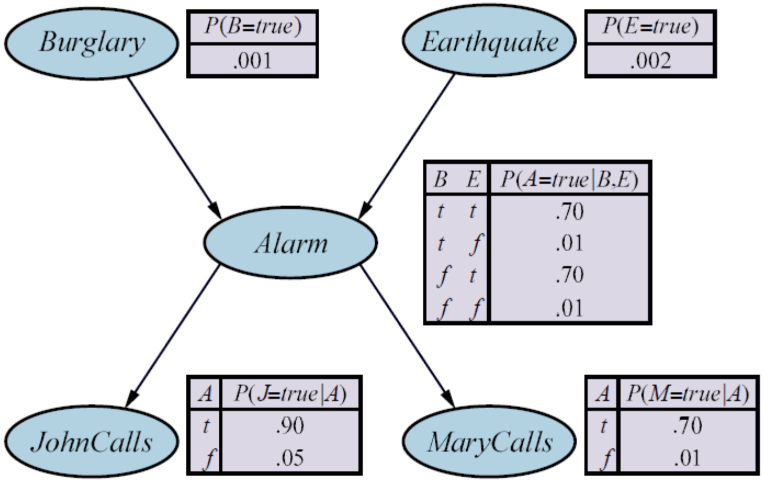

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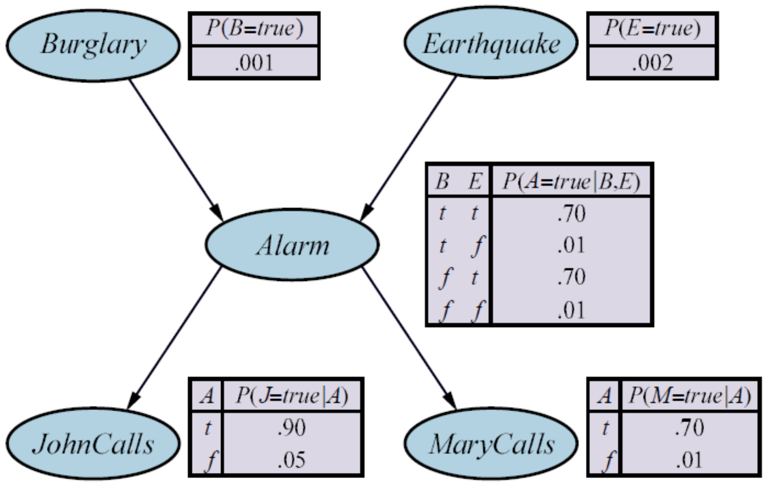

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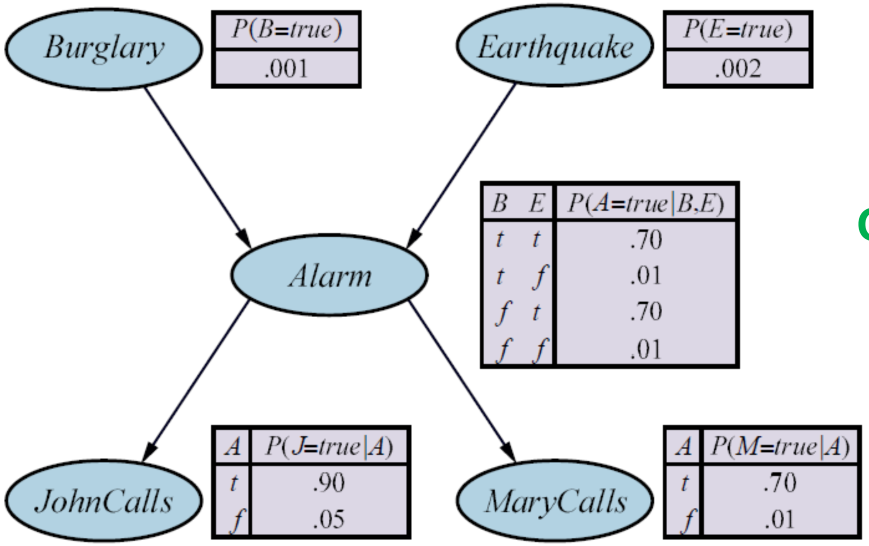
   b. Otherwise, X and Y are *not necessarily* conditionally independent, given Z.

Examples



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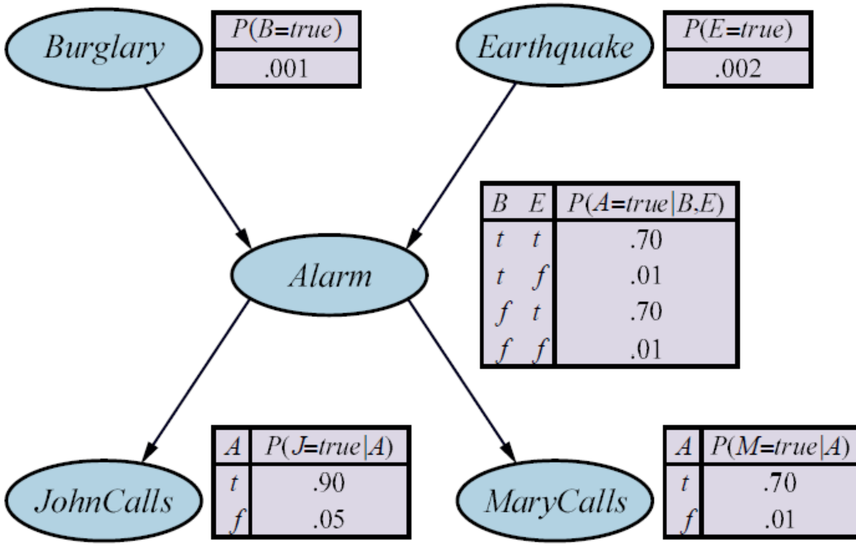

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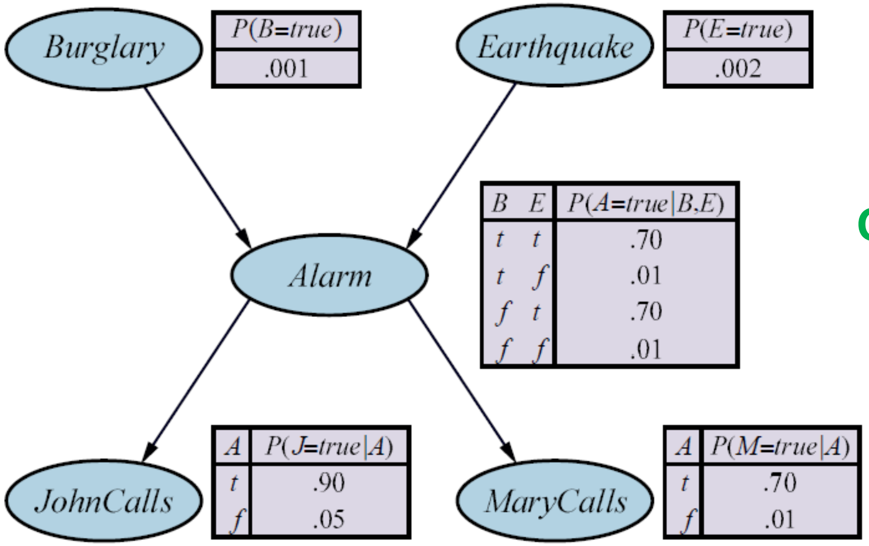

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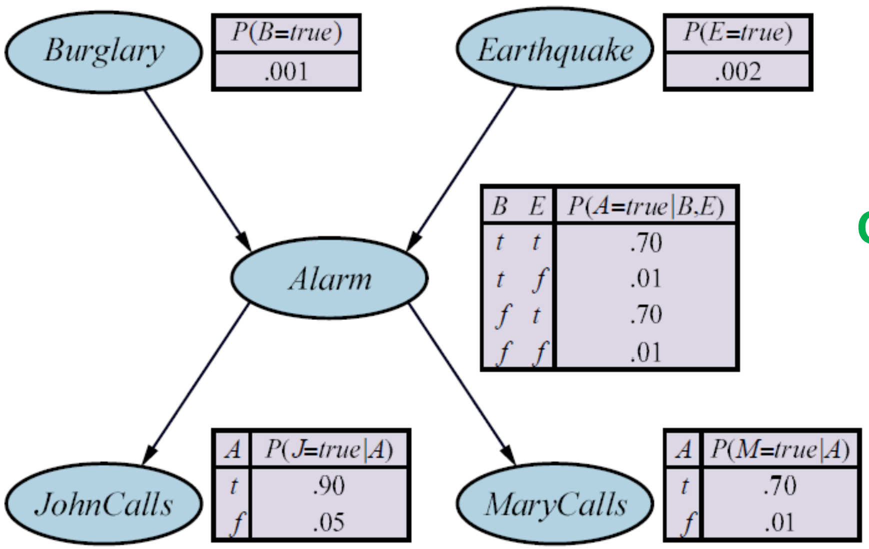

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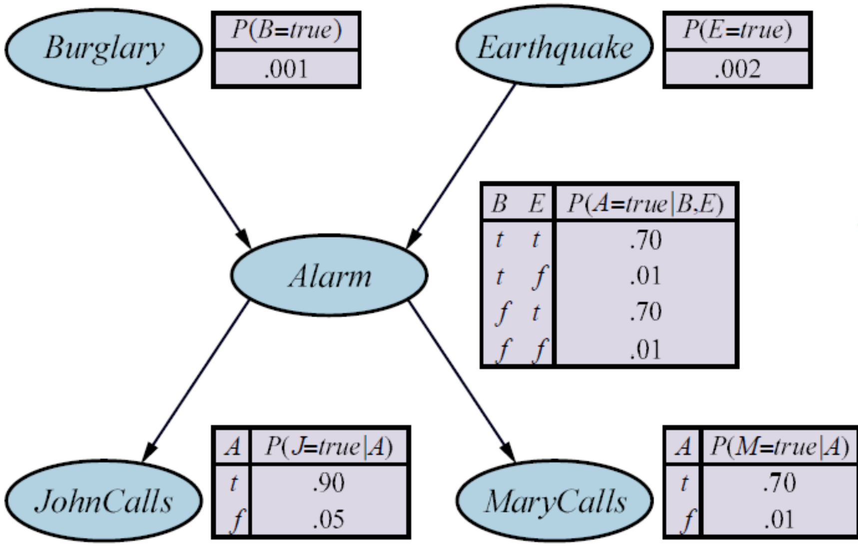

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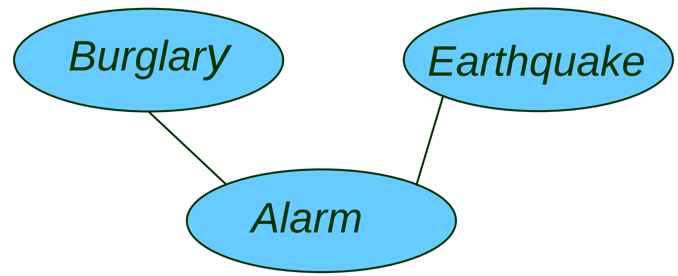

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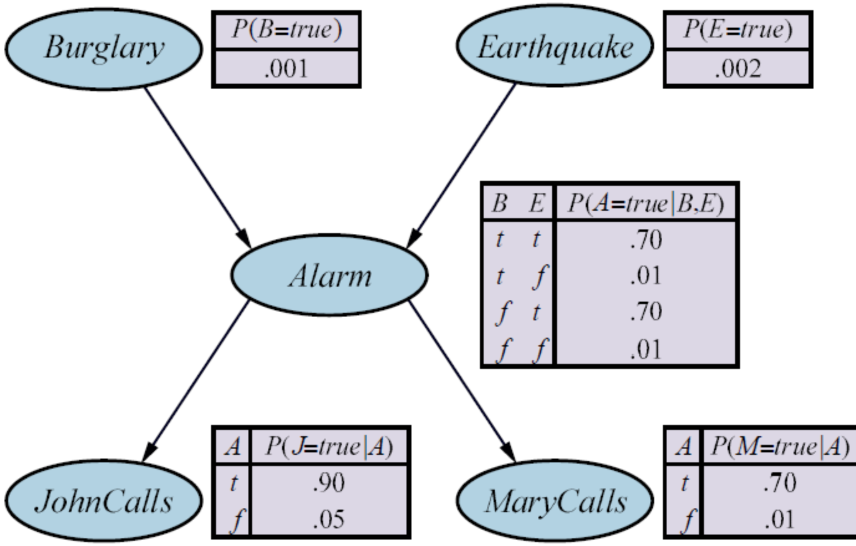

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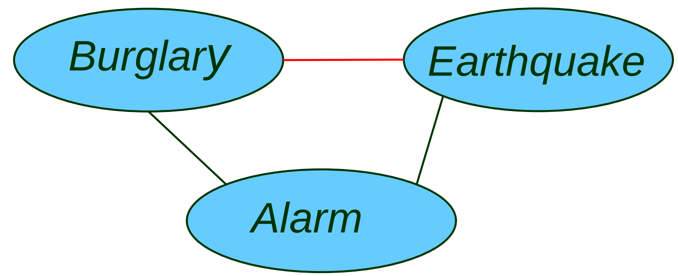

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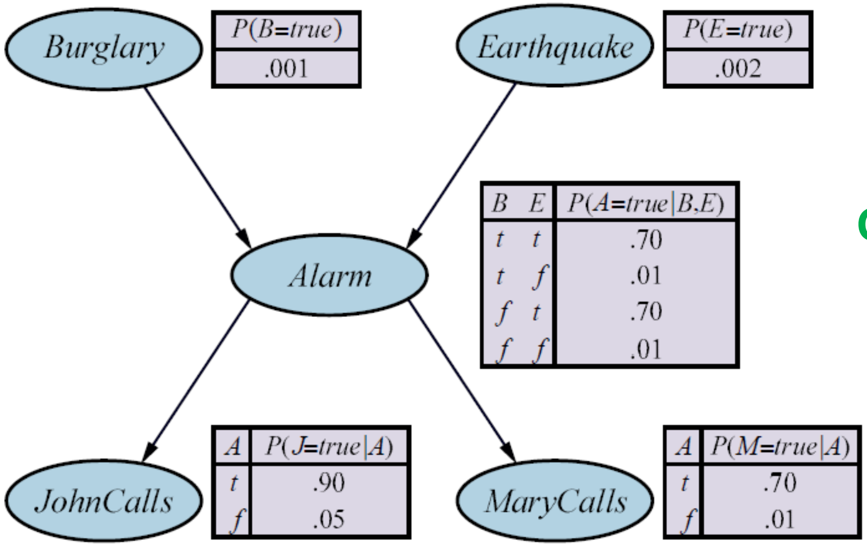

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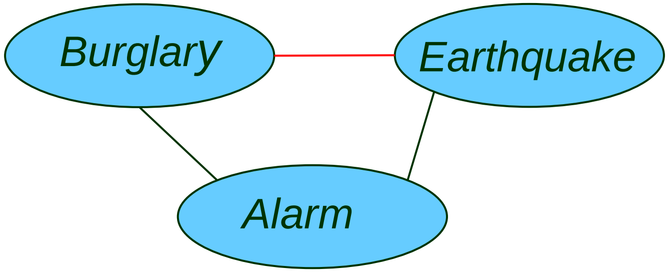

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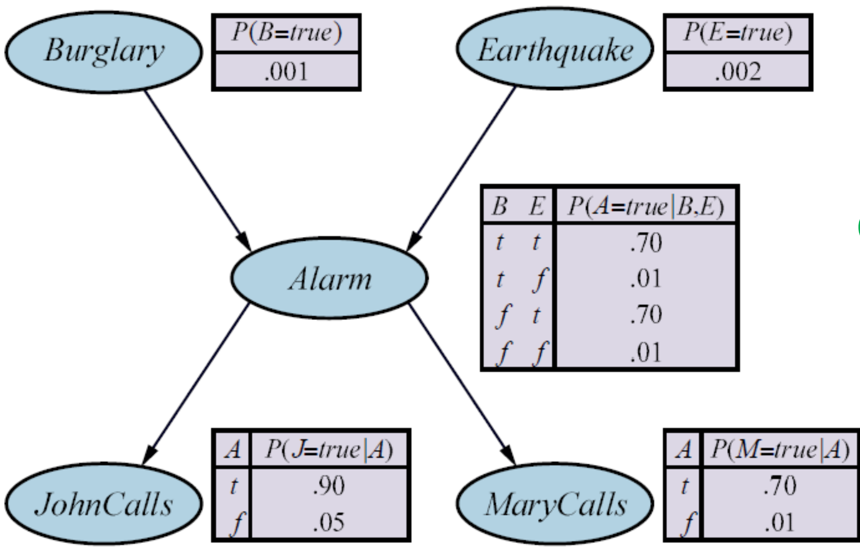

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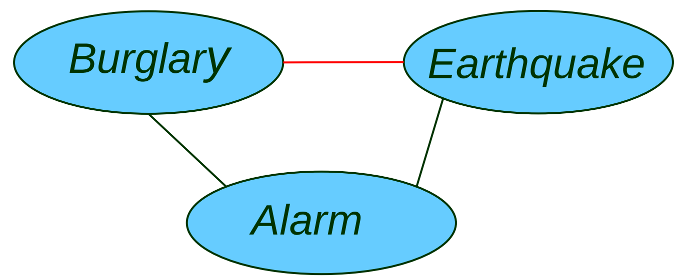

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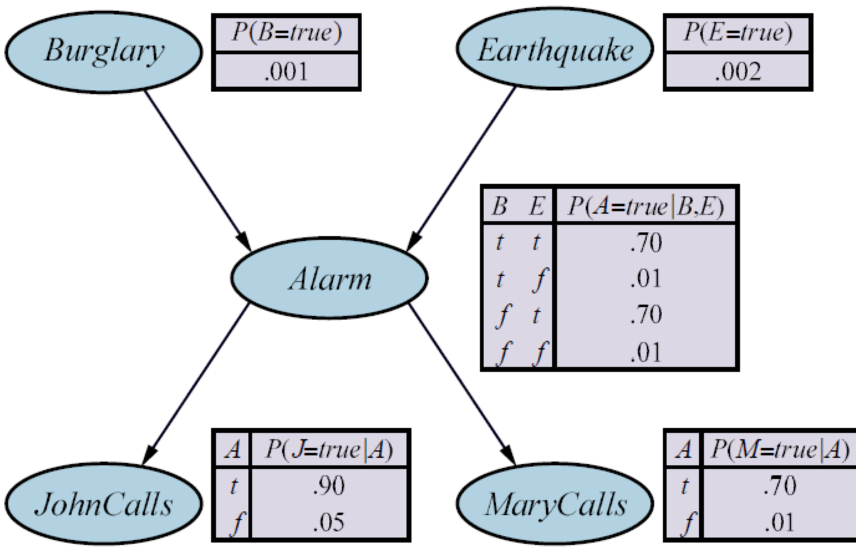

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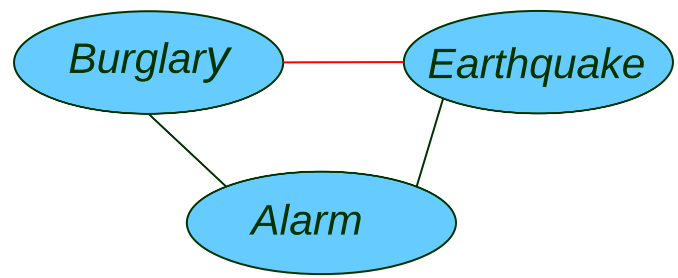

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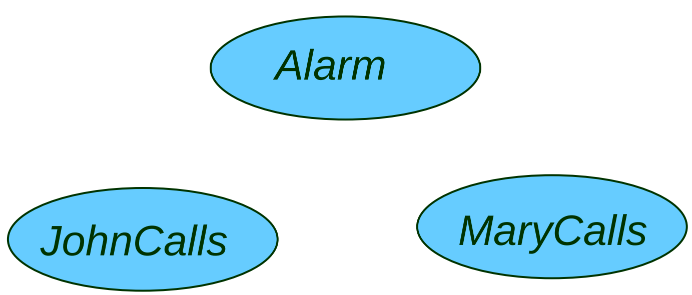

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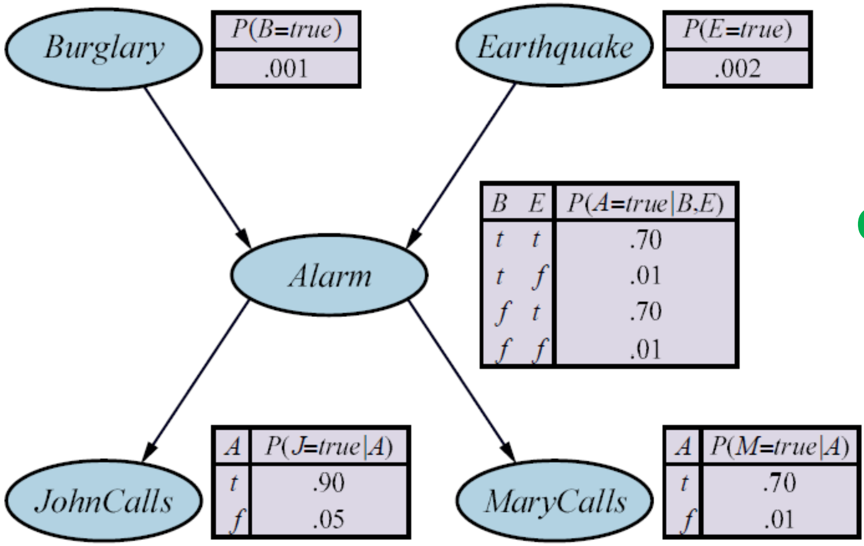

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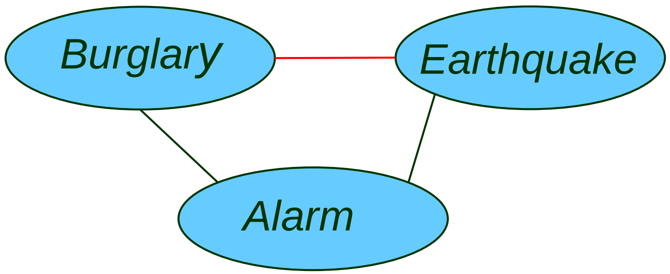

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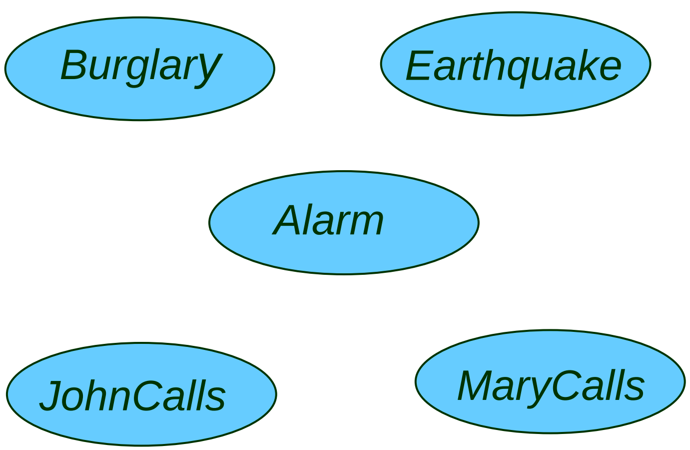

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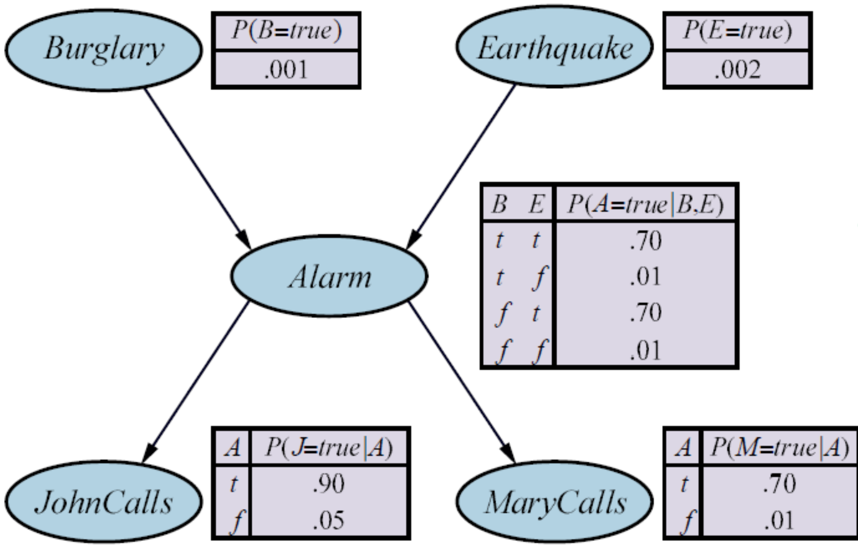

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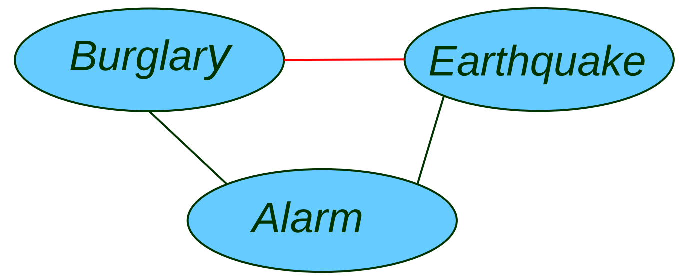

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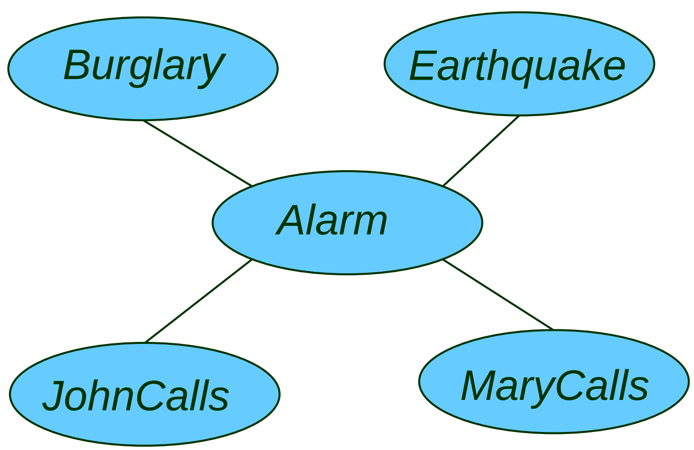

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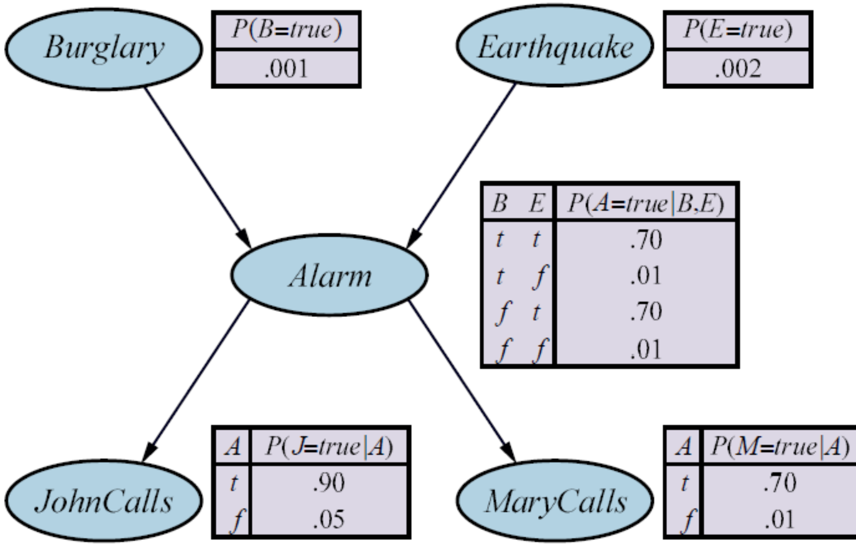

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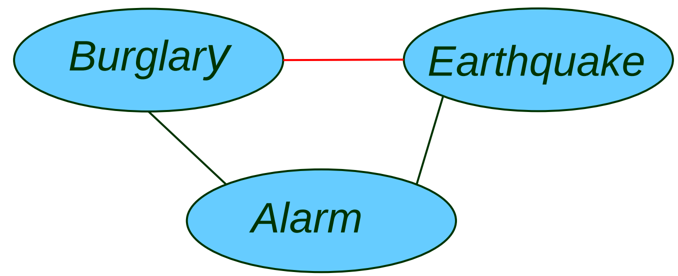

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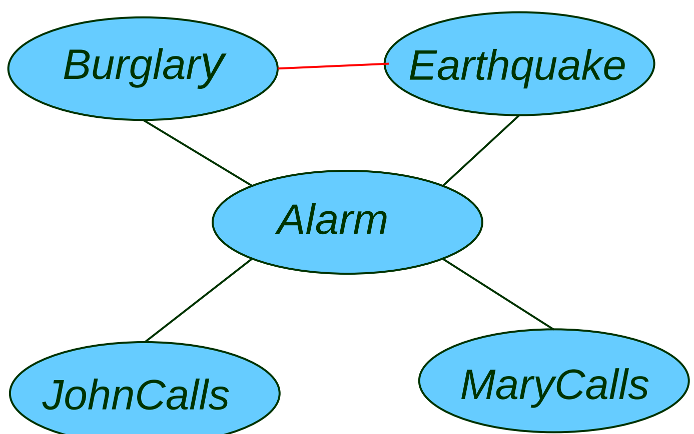

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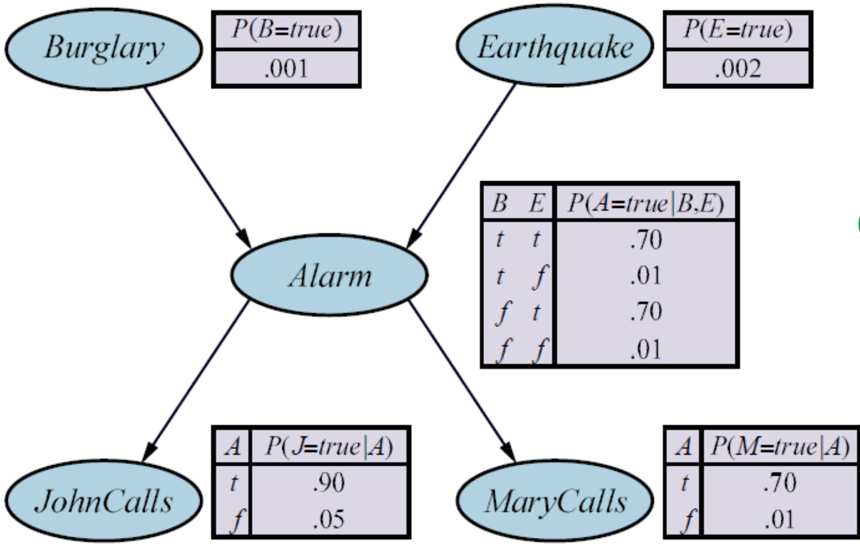

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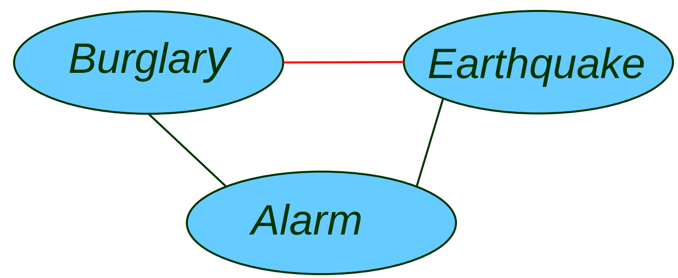

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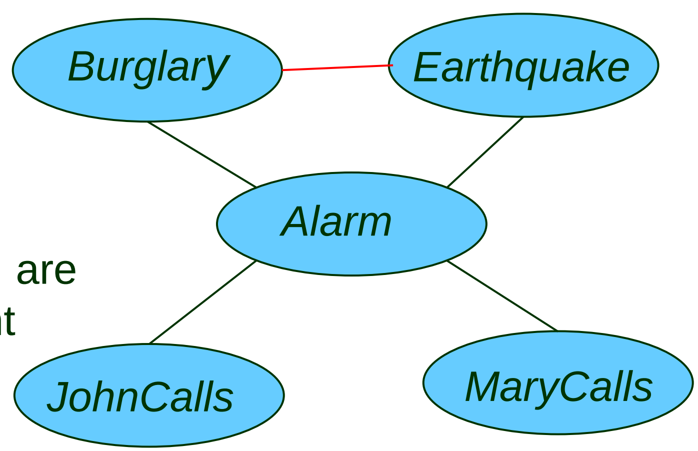

JohnCalls and *MayCalls* are conditionally independent given *Alarm*.