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# Convex Hulls in 3D

## Outline:

I. Algebraic definition

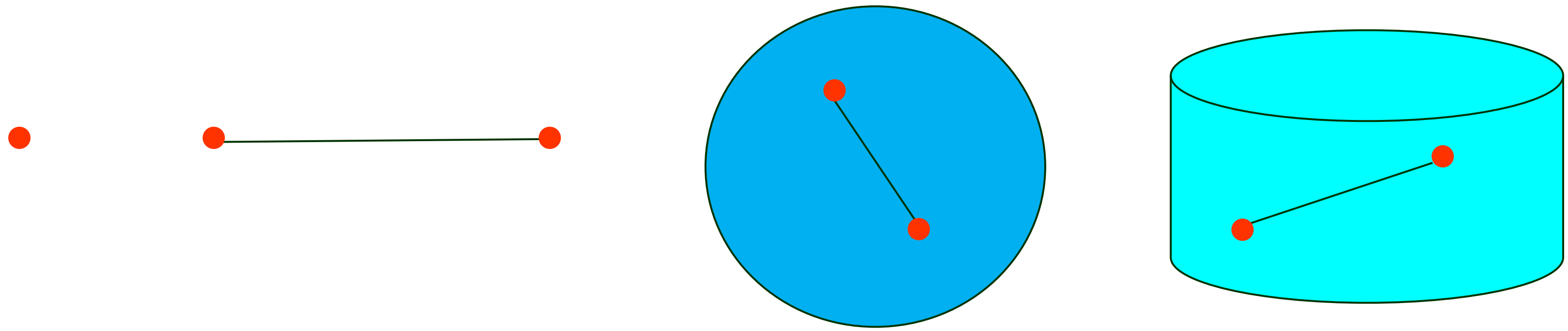
II. Complexity of a convex hull

III. Visible facets

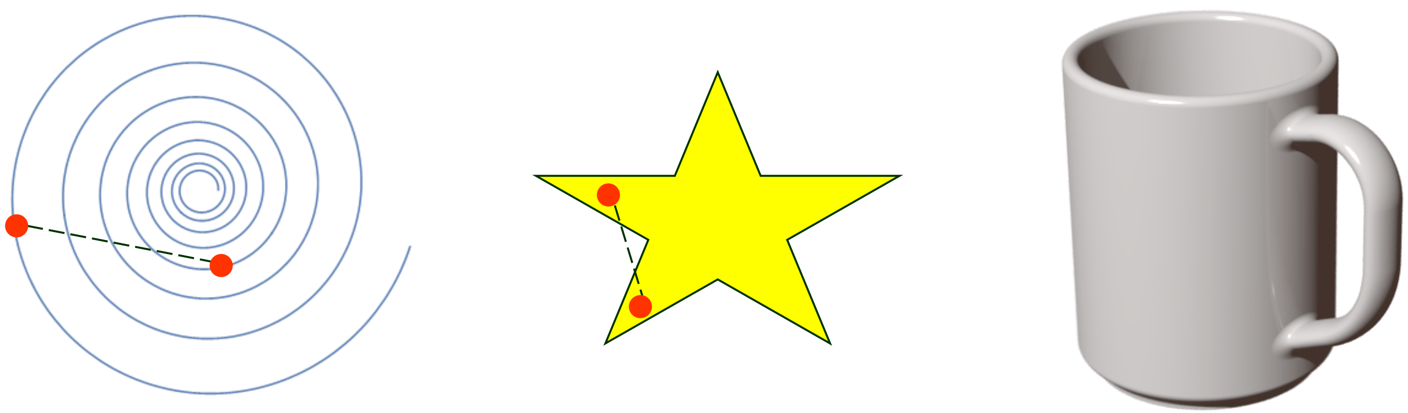
IV. Conflict sets

# I. Convex Sets

A set $S \subseteq \mathbb{R}^n$ is *convex* if the line segment $\overline{pq} \subset S$ for any pair of points $p, q \in S$.



It is *concave* if the set does not contain all the line segments.



# Convex Hulls

The *convex hull* of a set of points $S \subseteq \mathbb{R}^n$ is the *intersection* of all convex sets containing $S$.

Every $x \epsilon [x_1, x_2]$ satisfies

$$x = \lambda_1 x_1 + \lambda_2 x_2$$

where $\lambda_1, \lambda_2 \geq 0$

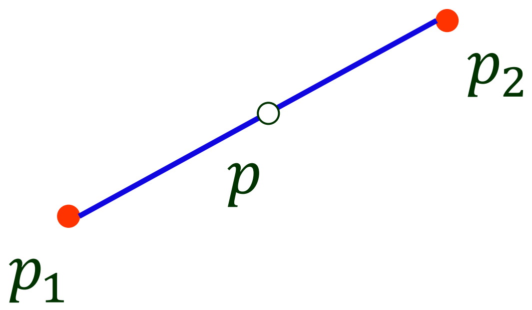
$$\lambda_1 + \lambda_2 = 1$$

$\lambda_1, \lambda_2$: *barycentric coordinates*

$$\lambda_1 = \frac{x_2 - x}{x_2 - x_1} \qquad \lambda_2 = \frac{x - x_1}{x_2 - x_1}$$

# Line Segment

$S = \{p_1, p_2\}$




A point $p$ on the segment $\overline{p_1 p_2}$

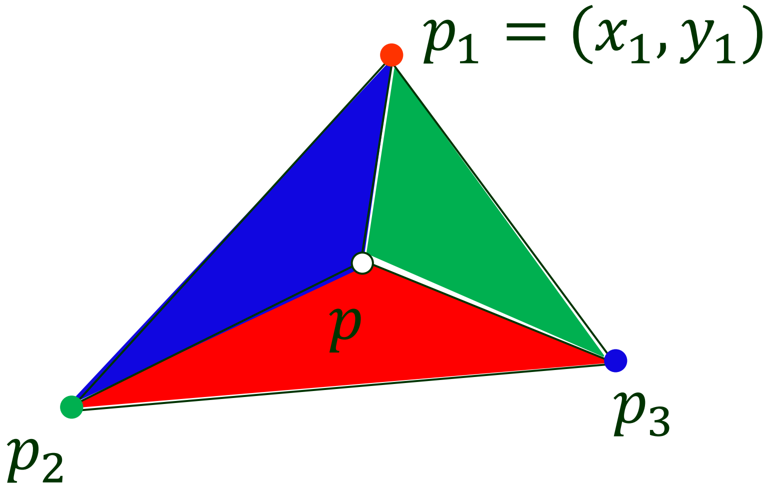
$$p = \lambda_1 p_1 + \lambda_2 p_2$$

where $\lambda_1, \lambda_2 \geq 0$

$\lambda_1 + \lambda_2 = 1$

$$\lambda_1 = \frac{||p - p_2||}{||p_2 - p_1||} \qquad \lambda_2 = \frac{||p - p_1||}{||p_2 - p_1||}$$

# Three Non-Collinear Points in 2D




A point $p$ in the convex hull (bounded by triangle $\Delta p_1 p_2 p_3$):

$$p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3$$

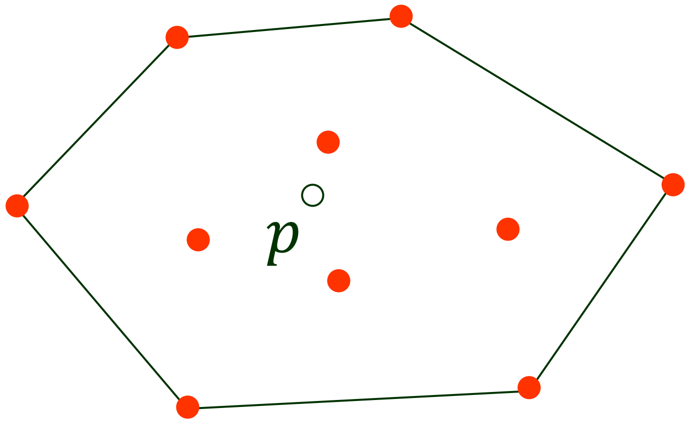
where $\lambda_1, \lambda_2, \lambda_3 \geq 0$

$\lambda_1 + \lambda_2 + \lambda_3 = 1$

In fact, let $A = area(\Delta p_1 p_2 p_3)$

$$\lambda_1 = \frac{area(\Delta p_2 p_3 p)}{A} \qquad \lambda_2 = \frac{area(\Delta p_3 p_1 p)}{A} \qquad \lambda_3 = \frac{area(\Delta p_1 p_2 p)}{A}$$

# $n$ Points in the Plane




$n$ points $p_1, p_2, \ldots, p_n$

A point $p$ in the convex hull has

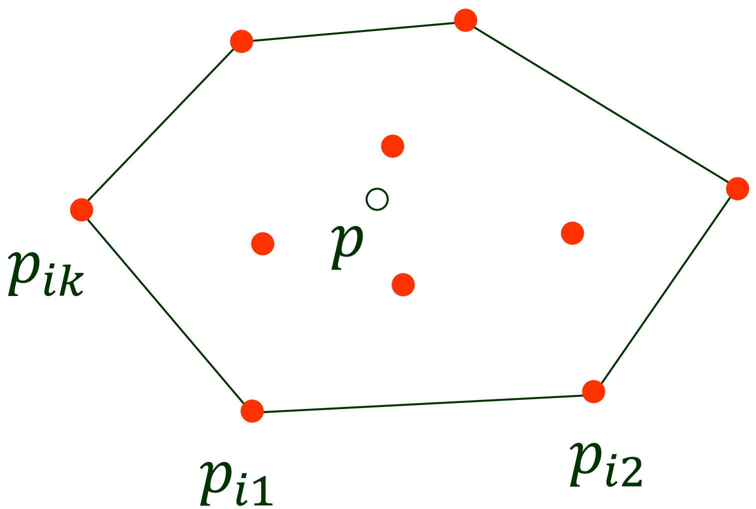
$$p = \sum_{i=1}^{n} \lambda_i p_i$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$

$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are not uniquely determined when $n > 3$.

# Vertices of the Convex Hull




$k$ vertices: $p_{i1}, p_{i2}, \ldots, p_{ik}$

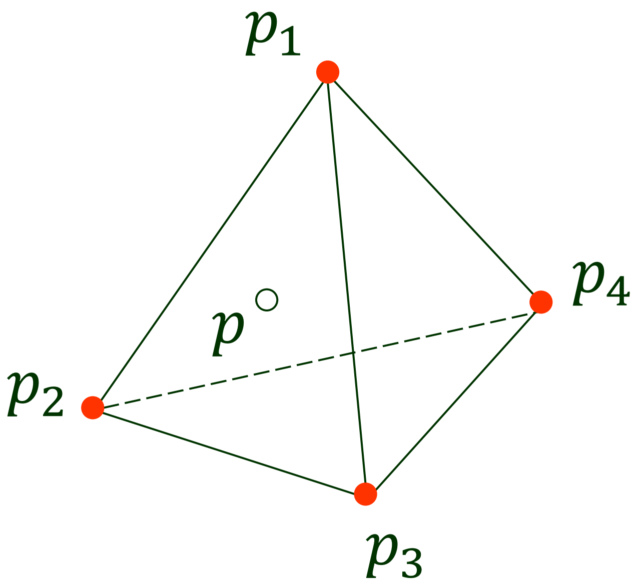
$$p = \sum_{j=1}^{k} \mu_j p_{ij}$$

where $\mu_1, \mu_2, \ldots, \mu_k \geq 0$

$$\mu_1 + \mu_2 + \cdots + \mu_k = 1$$

$\mu_1, \mu_2, \ldots, \mu_k$ are not uniquely determined when $k > 3$.

# Non-Coplanar Points in 3D




Tetrahedron (for 4 points)

$$p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \lambda_4 p_4$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$$

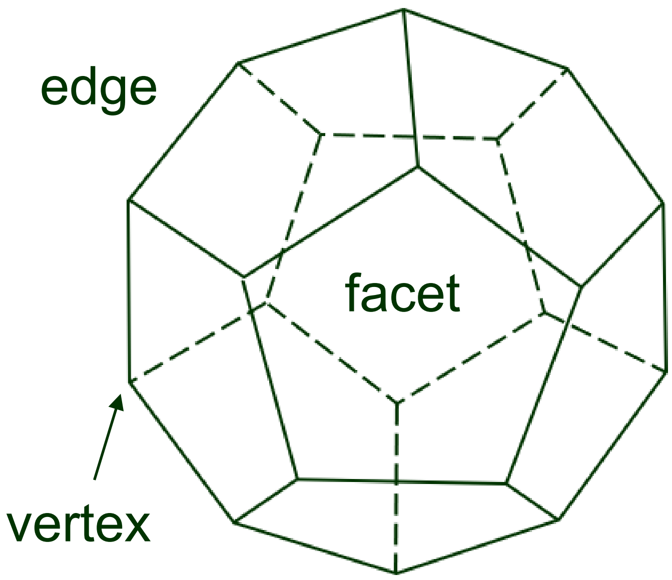
*Convex polyhedron* (for $n$ points)

$$p = \sum_{i=1}^{n} \lambda_i p_i$$

where $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$$

# II. Faces vs Facets




A dodecahedron has

20 vertices
30 edges
12 facets

*Faces* are features of all dimensions on a polyhedron.

0-faces: vertices

1-faces: edges

2-faces (*facets*): polygonal faces

- The generalization of a polyhedron in the $d$-dimensional ($d$-D) space is called a *polytope*.
- A $d$-D polytope $P$ has 0-faces, 1-faces, …, $(d-1)$-faces.
- The facets of $P$ are its $(d-1)$-faces.

# Platonic Solids

- Convex polyhedra with equivalent faces composed of congruent convex regular polygons.
- Also called the *regular solids* or *regular polyhedra*.
- Only five types (proved by Euclid).

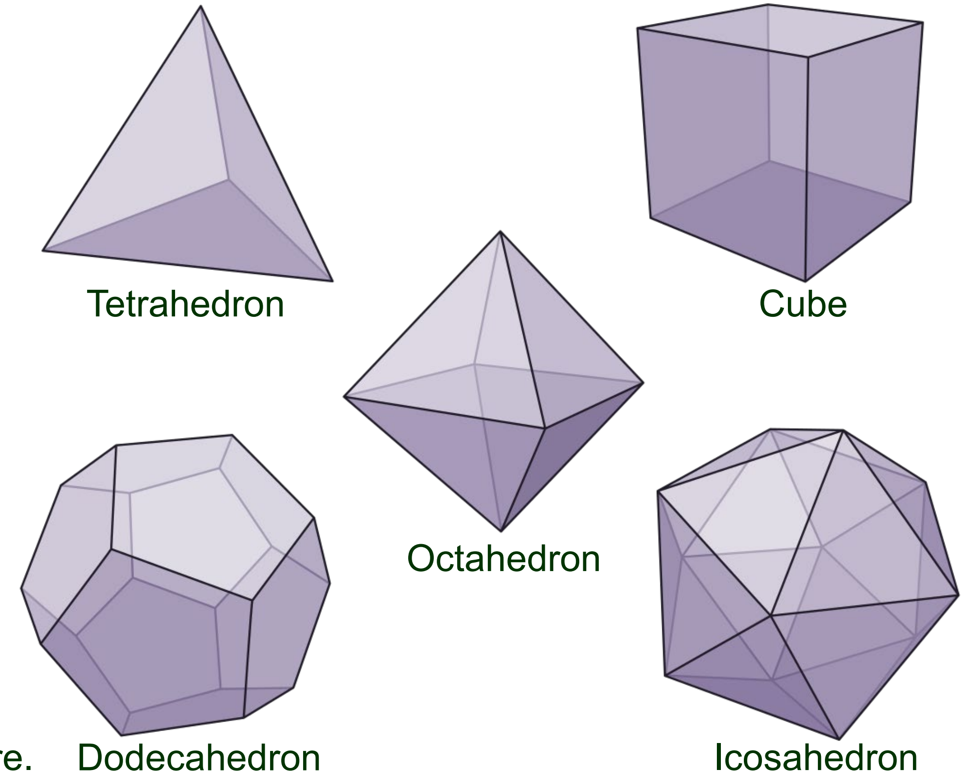



Image from Platonic solid - Wikidata.

See Platonic Solid -- from Wolfram MathWorld for more.

# Complexity of a Convex Hull in 3D

$S$: a set of $n$ points $P$: convex hull of $S$ (a convex polyhedron)

**Theorem** #edges $\leq 3n - 6$ and #facets $\leq 2n - 4$

**Proof** The surface of a convex polyhedron can be seen as a planar graph.

facet $\longmapsto$ face

top facet $\longmapsto$ unbounded face

Apply Euler's formula:

$$n_v - n_e + n_f = 2$$

# Proof (cont'd)

Every facet of the polyhedron has $\geq 3$ edges.

$\Downarrow$

Every face of the planar graph has $\geq 3$ edges.

Every edge is adjacent to two faces.

$\Rightarrow \quad 2n_e \geq 3n_f$

$\Downarrow \quad n_v - n_e + n_f = 2$

$$n_v + n_f - 2 = n_e \geq \frac{3}{2} n_f$$

$\Downarrow \quad n \geq n_v$

$$n + n_f - 2 \geq \frac{3}{2} n_f$$

$\Downarrow$

$$n_f \leq 2n - 4$$

$\Downarrow \quad n_e \leq n + n_f - 2$

$$n_e \leq 3n - 6$$

$\square$

# Simplicial Polytope

**Corollary** The complexity of the convex hull of $n$ points in 3D is $O(n)$.

A *simplicial polytope* has every facet as a triangle.

$$\left.\begin{array}{l} 2n_e = 3n_f \\ n_v = n \end{array}\right\} \xrightarrow[\text{the theorem}]{\text{Proof of}} \left\{\begin{array}{l} n_e = 3n - 6 \\ n_f = 2n - 4 \end{array}\right.$$

# III. Computing a Convex Hull

## Randomized incremental construction

- Choose four points $p_1, p_2, p_3, p_4 \in S$ that are not co-planar. $O(n)$

  Their convex hull is a tetrahedron.

- Compute a random permutation $p_5, p_6, \dots, p_n$.

$$P_r = \{p_1, p_2, , \dots, p_r\} \qquad r \geq 1$$

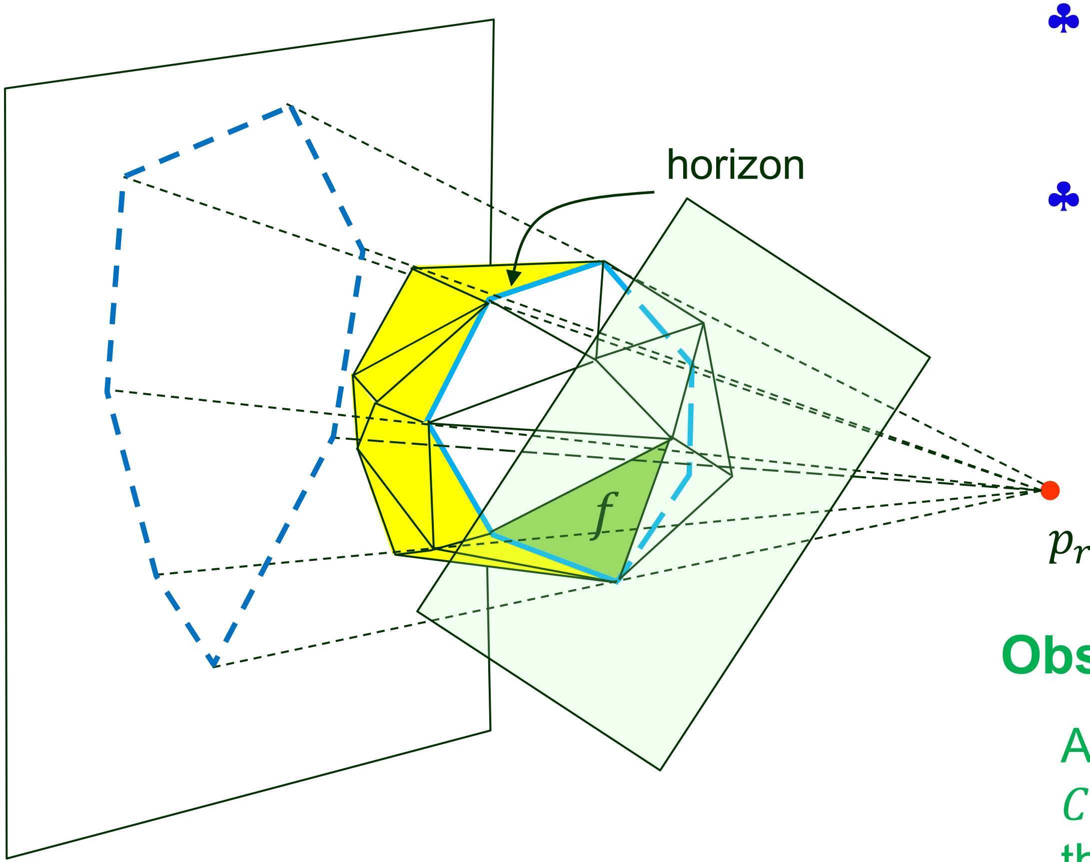
- For $r \geq 5$, add $p_r$ to the convex hull $CH(P_{r-1})$ .

  - $p_r$ inside $CH(P_{r-1})$ or on its boundary.

$$CH(P_r) = CH(P_{r-1})$$

# Visible Facets

- $p_r$ outside $CH(P_{r-1})$.



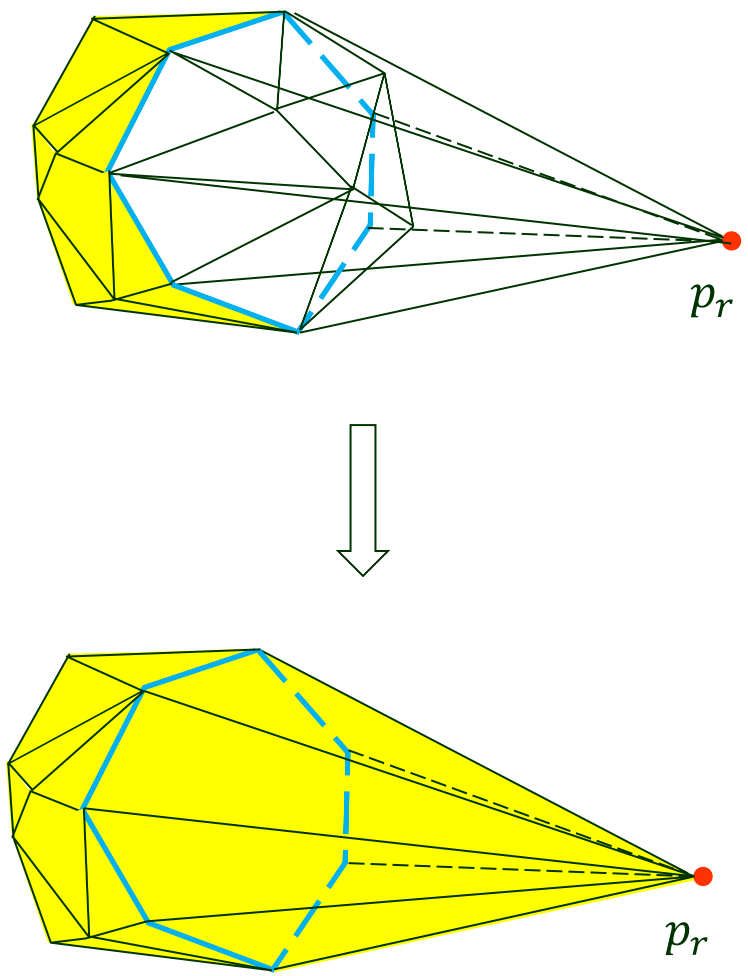

♣ Visible facets form a connected region on the surface of $CH(P_{r-1})$.

♣ Boundary of this visible region is called the *horizon* of $CH(P_{r-1})$.

## Observation

A facet $f$ is visible from $p_r$ if $p_r$ and $CH(P_{r-1})$ lie on opposite sides of the plane containing $f$.

# Hull Update

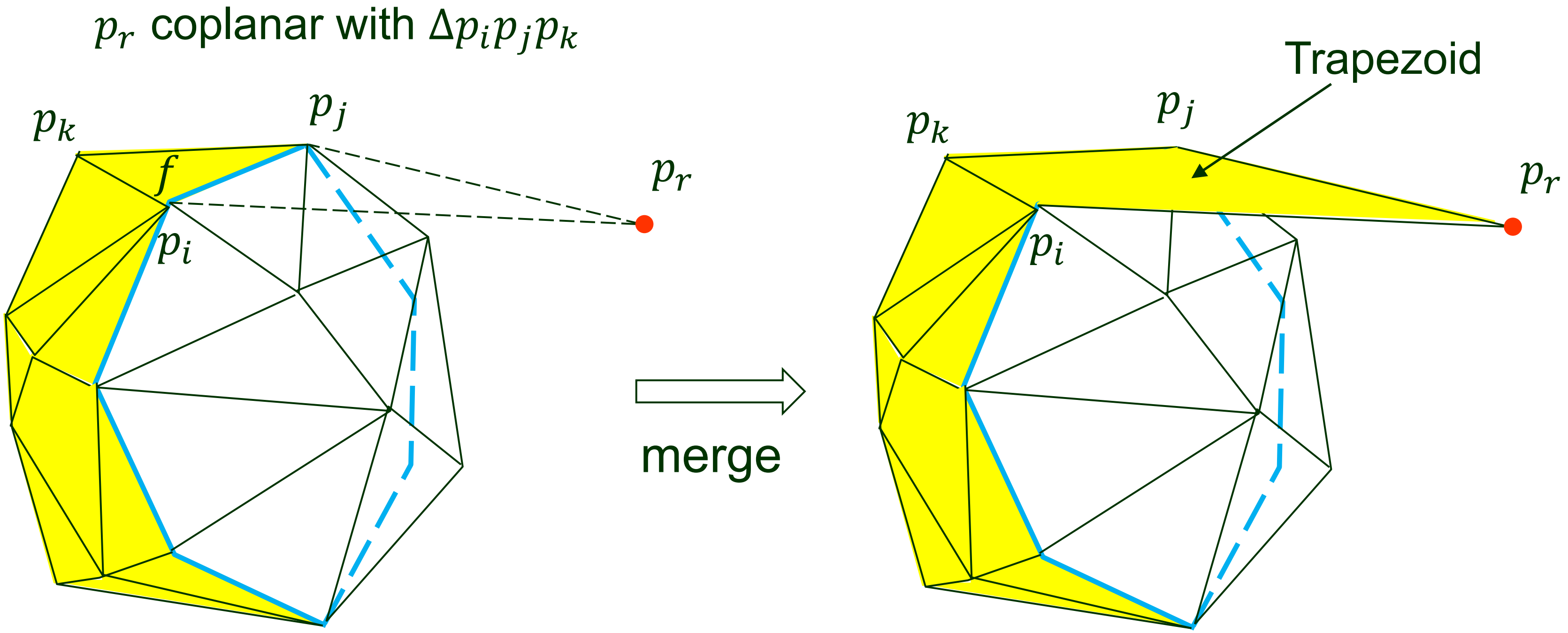



Strategy:

- Keep all invisible facets.
- Replace visible facets with facets connecting $p_r$ to its horizon.

# Degeneracy Handling

- ♣ Check if $p_r$ lies in the plane of a facet of $CH(P_{r-1})$.



# Data Structure

Doubly-connected edge list (DCEL)

because convex hull can be interpreted as a planar graph.

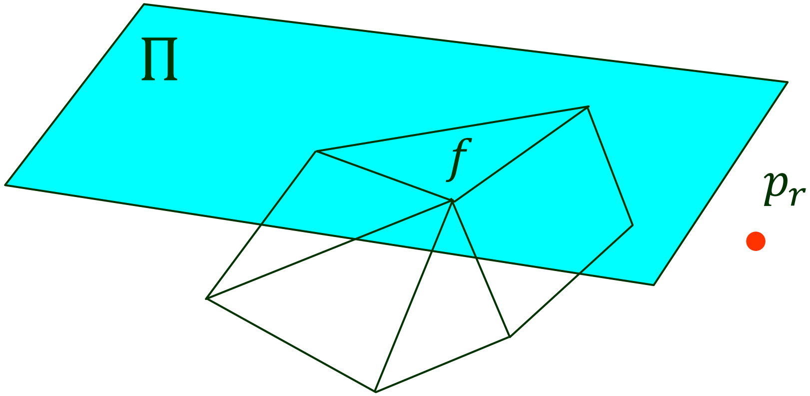
- Every vertex represents a point in space.
- Every edge represents an edge on the convex hull.
- Transforming DCEL$_{r-1}$ for CH$(P_{r-1})$ to DCEL$_r$ for CH$(P_r)$ takes time linear in the total complexity of the visible facets.

# IV. Finding Visible Facets

Which facets of $CH(P_{r-1})$ are visible to $p_r$?

Slow strategy

Test every facet $f$ whether $p_r$ and $CH(P_{r-1})$ are on the opposite sides of the plane $\Pi$ containing $f$.




- $O(1)$ for each facet.
- $O(n)$ for all facets.

Algorithm runs in $O(n^2)$ time.

# Faster Testing – the Conflict Graph

**Heuristic** Maintain additional information related to $CH(P_{r-1})$.

- A *conflict graph* for $CH(P_{r-1})$ is a bipartite graph $G$:

  ♣ Vertices are from two sets:

  ♦ $\{p_r, \dots, p_n\}$ // points yet to be added

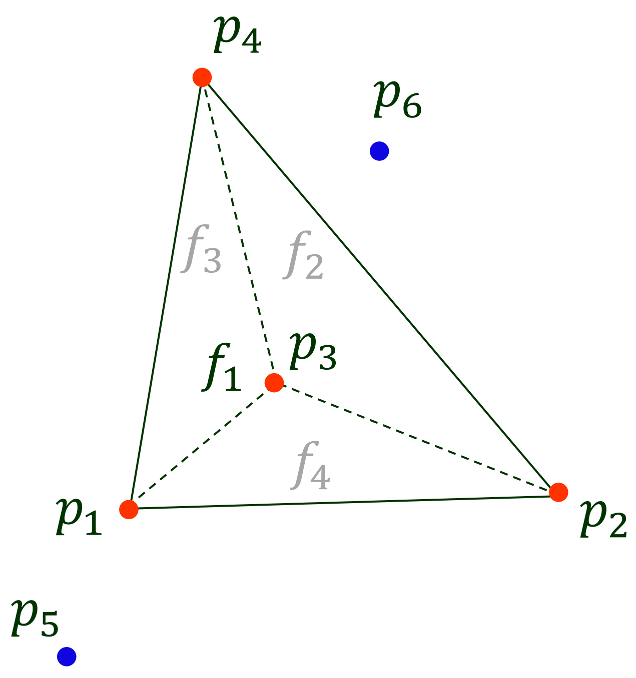
  ♦ facets of $CH(P_{r-1})$

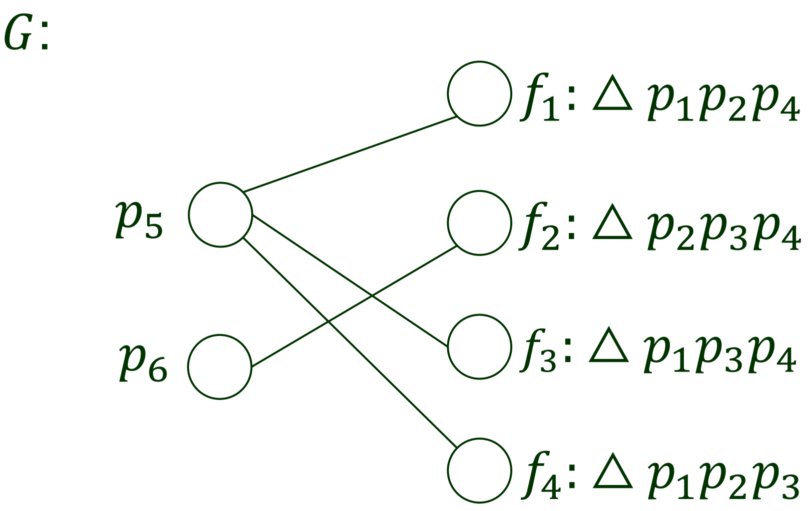
  ♣ Every edge connects a point and a facet.

  An edge $\langle p_t, f \rangle$ exists if $f$ is visible from $p_t$, $r \le t \le n$.

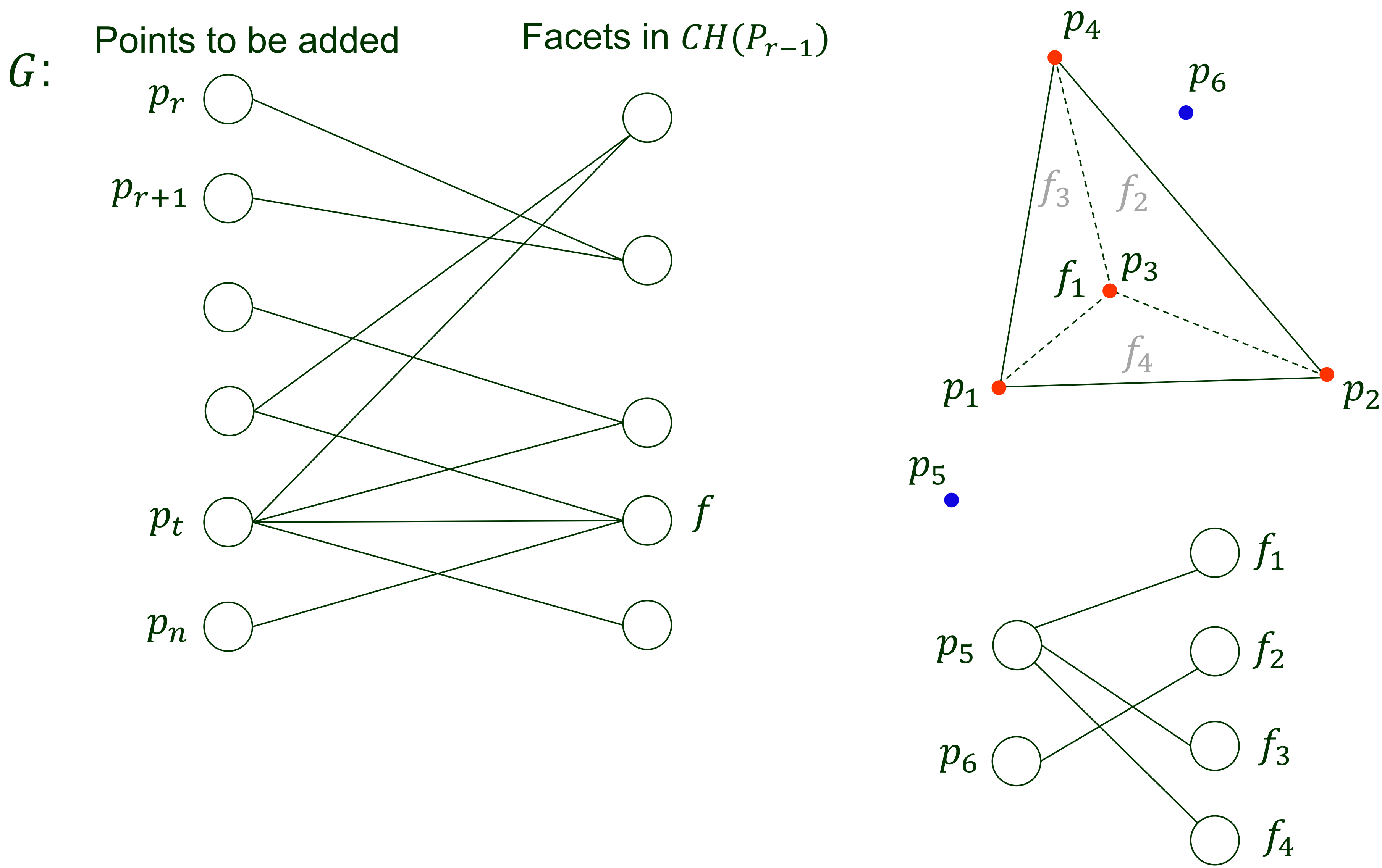
# Example of the Conflict Graph

$r = 5$

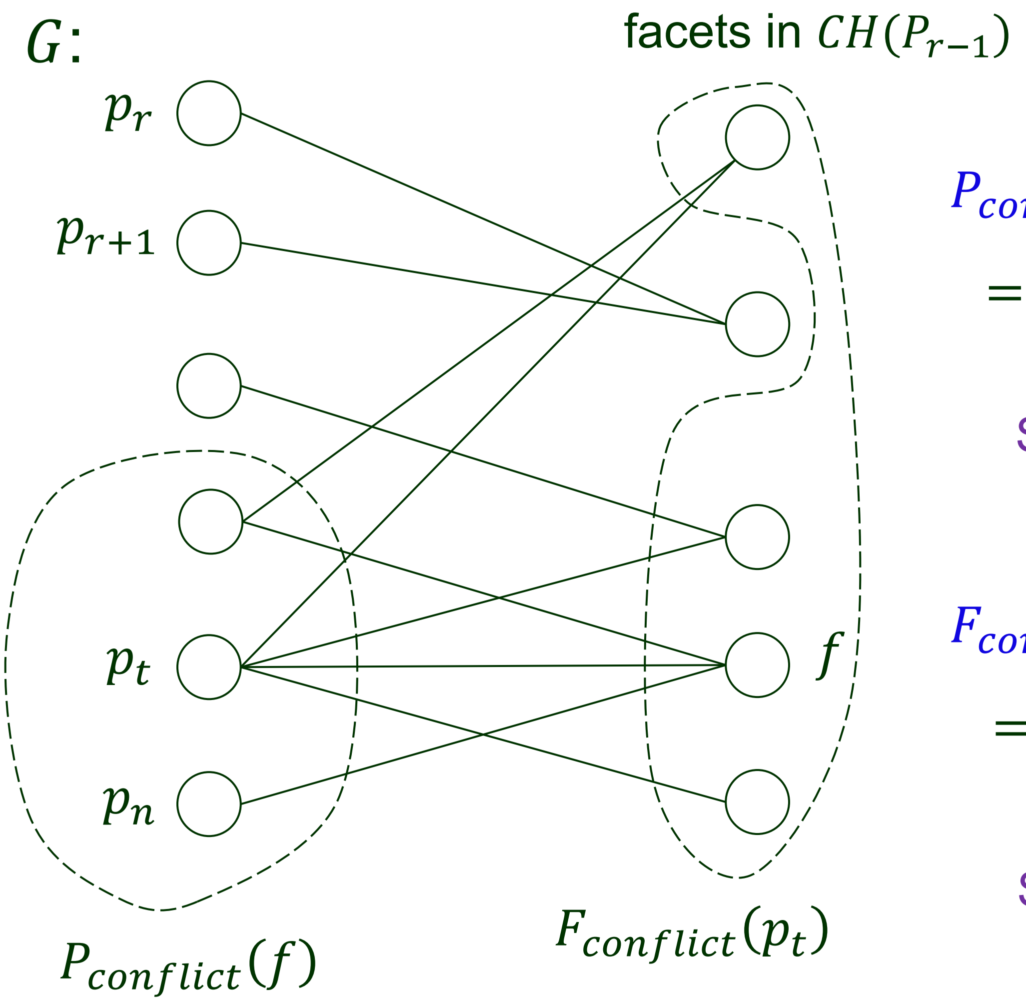


# Conflict Graph for $CH(P_{r-1})$



# Conflict Sets




$P_{conflict}(f)$

$= \{p_t \mid r \le t \le n$ and $f$ visible from $p_t\}$

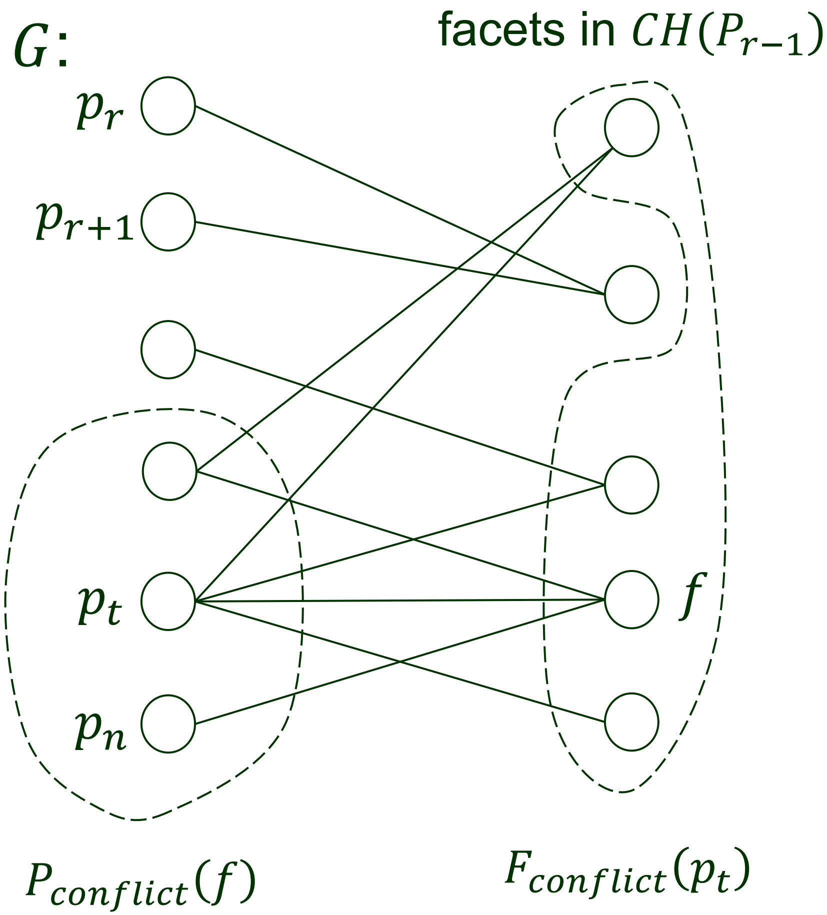
Set of nodes adjacent to $f$ in $G$.

$F_{conflict}(p_t)$

$= \{$ facets of $CH(P_{r-1})$ visible from $p_t\}$

Set of nodes adjacent to $p_t$ in $G$.

# Visible Facets




visibility $\Longleftrightarrow$ conflict

- $f$ can be seen from every $p_t \in P_{conflict}(f)$.
- Once we add the first such $p_t$, $f$ must be deleted.

When inserting $p_r$ into $CH(P_{r-1})$ , look up $F_{conflict}(p_r)$ to get the visible facets.

# Graph Initialization & Updates

Initialize $G$:

- $P_4 = \{p_1, p_2, p_3, p_4\}$ is a tetrahedron.
- Check every point $p_i, 5 \leq i \leq n$, which of the four facets are visible.

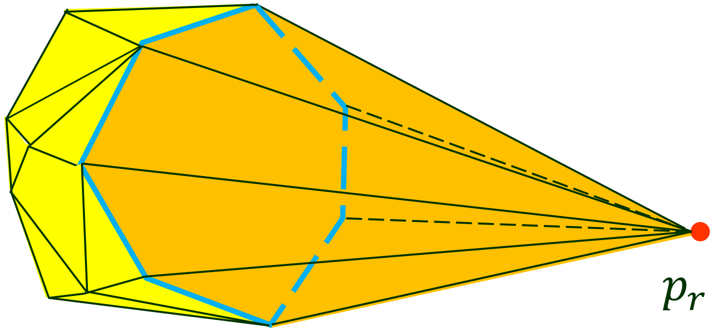
$O(n)$

Update $G$ after adding $p_r$:

1. Discard neighbors (all visible facets) of $p_r$ in $G$.

2. Delete the node representing $p_r$.

3. Add nodes for the new facets (which connect $p_r$ to the horizon).

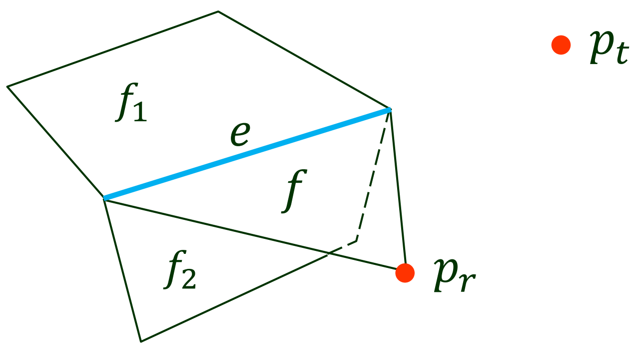


# Updating the Conflict Sets of New Faces

4. Construct $P_{conflict}(f)$ for every new facet $f$.

||

$\{p_t \mid r < t \leq n$ and $f$ visible from $p_t\}$




$e$: edge of $f$ on the horizon and opposite to $p_r$

$f_1, f_2$: facets in $CH(P_{r-1})$ that are incident on $e$.

Suppose a point $p_t$ can see $f$.

⇩

Then it can see $e$, which is an edge in $CH(P_{r-1})$ that bounds $f$ .

⇩

$e$ must have been visible from $p_t$ in $CH(P_{r-1}) \subseteq CH(P_r)$

⇩

$f_1$ or $f_2$ is visible from $p_t$.

Test all points $p_t \in P_{conflict}(f_1) \cup P_{conflict}(f_2)$, where $r < t \leq n$, add $\langle p_t, f \rangle$ to $G$ if $f$ is visible from $p_t$.

5. $F_{conflict}(p_t)$ thus gets updated with the bipartite graph $G$ for $r < t \leq n$.