Convex Hulls in 3D

Outline:

I. Algebraic definition

II. Complexity of a convex hull

III. Visible facets

IV. Conflict sets

A set $S \subseteq \mathbb{R}^n$ is **convex** if the line segment $\overline{pq} \subset S$ for any pair of points $p, q \in S$.

It is **concave** if the set does not contain all the line segments.
The **convex hull** of a set of points \( S \subseteq \mathbb{R}^n \) is the *intersection* of all convex sets containing \( S \).

Every \( x \in [x_1, x_2] \) satisfies

\[
x = \lambda_1 x_1 + \lambda_2 x_2
\]

where \( \lambda_1, \lambda_2 \geq 0 \)

\( \lambda_1 + \lambda_2 = 1 \)

\( \lambda_1, \lambda_2 \): barycentric coordinates

\[
\lambda_1 = \frac{x_2 - x}{x_2 - x_1} \quad \lambda_2 = \frac{x - x_1}{x_2 - x_1}
\]
Line Segment

\[ S = \{p_1, p_2\} \]

A point \( p \) on the segment \( p_1p_2 \)

\[ p = \lambda_1 p_1 + \lambda_2 p_2 \]

where \( \lambda_1, \lambda_2 \geq 0 \)

\[ \lambda_1 + \lambda_2 = 1 \]

\[ \lambda_1 = \frac{||p - p_2||}{||p_2 - p_1||} \]

\[ \lambda_2 = \frac{||p - p_1||}{||p_2 - p_1||} \]
Three Non-Collinear Points in 2D

A point \( p \) in the convex hull (bounded by triangle \( \Delta p_1p_2p_3 \)):

\[
p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3
\]

where \( \lambda_1, \lambda_2, \lambda_3 \geq 0 \)

\[
\lambda_1 + \lambda_2 + \lambda_3 = 1
\]

In fact, let \( A = \text{area}(\Delta p_1p_2p_3) \)

\[
\lambda_1 = \frac{\text{area}(\Delta p_2p_3p)}{A} \quad \lambda_2 = \frac{\text{area}(\Delta p_3p_1p)}{A} \quad \lambda_3 = \frac{\text{area}(\Delta p_1p_2p)}{A}
\]
$n$ Points in the Plane

$n$ points $p_1, p_2, \ldots, p_n$

A point $p$ in the convex hull has

$$p = \sum_{i=1}^{n} \lambda_i p_i$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$$

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are not uniquely determined when $n > 3$. 
Vertices of the Convex Hull

\[ p = \sum_{j=1}^{k} \mu_j p_{ij} \]

where \( \mu_1, \mu_2, \ldots, \mu_k \geq 0 \)

\( \mu_1 + \mu_2 + \cdots + \mu_k = 1 \)

\( \mu_1, \mu_2, \ldots, \mu_k \) are not uniquely determined when \( k > 3 \).
Non-Coplanar Points in 3D

Convex polyhedron (for $n$ points)

$$p = \sum_{i=1}^{n} \lambda_i p_i$$
where $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$
$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$$

Tetrahedron (for 4 points)

$$p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \lambda_4 p_4$$
where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$
$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$$
II. Faces vs Facets

Faces are features of all dimensions on a polyhedron.

- 0-faces: vertices
- 1-faces: edges
- 2-faces (facets): polygonal faces

A dodecahedron has
- 20 vertices
- 30 edges
- 12 facets

- The generalization of a polyhedron in the \(d\)-dimensional \((d\text{-D})\) space is called a polytope.
- An \(d\text{-D}\) polytope \(P\) has 0-faces, 1-faces, ..., \((d - 1)\)-faces.
- The facets of \(P\) are its \((d - 1)\)-faces.
Platonic Solids

- Convex polyhedra with equivalent faces composed of congruent convex regular polygons.
- Also called the *regular solids* or *regular polyhedra*.
- Only five types (proved by Euclid).

Image from [Platonic solid - Wikidata](https://commons.wikimedia.org/wiki/Platonic_solid).
See [Platonic Solid -- from Wolfram MathWorld](http://mathworld.wolfram.com/PlatonicSolid.html) for more.
Complexity of a Convex Hull in 3D

\( S: \) a set of \( n \) points \quad \( P: \) convex hull of \( S \) (a convex polyhedron)

**Theorem** \( \#\text{edges} \leq 3n - 6 \) and \( \#\text{facets} \leq 2n - 4 \)

**Proof** The surface of a convex polyhedron can be seen as a planar graph.

\[
\begin{align*}
\text{facet} & \leftrightarrow \text{face} \\
\text{top facet} & \leftrightarrow \text{unbounded face}
\end{align*}
\]

Apply Euler’s formula:

\[
n_v - n_e + n_f = 2
\]
Proof (cont’d)

Every facet of the polyhedron has \( \geq 3 \) edges.

Every face of the planar graph has \( \geq 3 \) edges.

Every edge is adjacent to two faces.

\[
2n_e \geq 3n_f
\]

\[
n_v - n_e + n_f = 2
\]

\[
n_v + n_f - 2 = n_e \geq \frac{3}{2}n_f
\]

\[
n \geq n_v
\]

\[
n + n_f - 2 \geq \frac{3}{2}n_f
\]

\[
n_f \leq 2n - 4
\]

\[
n_e \leq n + n_f - 2
\]

\[
n_e \leq 3n - 6
\]
Simplicial Polytope

**Corollary**  The complexity of the convex hull of $n$ points in 3D is $O(n)$.

A *simplicial polytope* has every facet as a triangle.

\[
\begin{align*}
2n_e &= 3n_f \\
n_v &= n \\
\end{align*}
\]

\[
\begin{align*}
\text{Proof of the theorem} \quad &\quad \left\{ \begin{array}{l}
2n_e = 3n_f \\
n_v = n \\
\end{array} \right. \quad \left\{ \begin{array}{l}
n_e = 3n - 6 \\
n_f = 2n - 4 \\
\end{array} \right.
\end{align*}
\]
III. Computing a Convex Hull

Randomized incremental construction

♦ Choose four points \( p_1, p_2, p_3, p_4 \in S \) that are not co-planar. \( O(n) \)

Their convex hull is a tetrahedron.

♦ Compute a random permutation \( p_5, p_6, \ldots, p_n \).

\[
P_r = \{ p_1, p_2, \ldots, p_r \} \quad r \geq 1
\]

♦ For \( r \geq 5 \), add \( p_r \) to the convex hull \( CH(P_{r-1}) \).

• \( p_r \) inside \( CH(P_{r-1}) \) or on its boundary.

\[
CH(P_r) = CH(P_{r-1})
\]
Visible Facets

• $p_r$ outside $CH(P_{r-1})$.

• Visible facets form a connected region on the surface of $CH(P_{r-1})$.

• Boundary of this visible region is called the *horizon* of $CH(P_{r-1})$.

**Observation**

A facet $f$ is visible from $p_r$ if $p_r$ and $CH(P_{r-1})$ lie on opposite sides of the plane containing $f$. 
Hull Update

Strategy:

- Keep all invisible facets.
- Replace visible facets with facets connecting $p_r$ to its horizon.
Check if $p_r$ lies in the plane of a facet of $CH(P_{r-1})$. 

$p_r$ coplanar with $\Delta p_i p_j p_k$
Data Structure

Doubly-connected edge list (DCEL)

because convex hull can be interpreted as a planar graph.

• Every vertex represents a point in space.

• Every edge represents an edge on the convex hull.

• Transforming $\text{DCEL}_{r-1}$ for $\text{CH}(P_{r-1})$ to $\text{DCEL}_r$ for $\text{CH}(P_r)$ takes time linear in the total complexity of the visible facets.
IV. Finding Visible Facets

Which facets of $CH(P_{r-1})$ are visible to $p_r$?

Slow strategy

Test every facet $f$ whether $p_r$ and $CH(P_{r-1})$ are on the opposite sides of the plane $\Pi$ containing $f$.

- $O(1)$ for each facet.
- $O(n)$ for all facets.

Algorithm runs in $O(n^2)$ time.
Faster Testing – the Conflict Graph

Heuristic  Maintain additional information related to $CH(P_{r-1})$.

- A *conflict graph* for $CH(P_{r-1})$ is a bipartite graph $G$:

  - Vertices are from two sets:
    - $\{p_r, \ldots, p_n\}$  // points yet to be added
    - facets of $CH(P_{r-1})$

  - Every edge connects a point and a facet.

  An edge $\langle p_t, f \rangle$ exists if $f$ is visible from $p_t$, $r \leq t \leq n$. 
Example of the Conflict Graph

\[ r = 5 \]

\( \Gamma \): 
- \( p_1p_2p_4 \)
- \( p_2p_3p_4 \)
- \( p_1p_3p_4 \)
- \( p_1p_2p_3 \)

\( f_1 \): \( \Delta p_1p_2p_4 \)
- \( f_2 \): \( \Delta p_2p_3p_4 \)
- \( f_3 \): \( \Delta p_1p_3p_4 \)
- \( f_4 \): \( \Delta p_1p_2p_3 \)
Conflict Graph for \( CH(P_{r-1}) \)

\[ G: \]

Points to be added

\[ p_r \]
\[ p_{r+1} \]
\[ p_t \]
\[ p_n \]

Facets in \( CH(P_{r-1}) \)

\[ f \]
\[ f_1 \]
\[ f_2 \]
\[ f_3 \]
\[ f_4 \]

\[ p_1 \]
\[ p_2 \]
\[ p_3 \]
\[ p_4 \]
\[ p_5 \]
\[ p_6 \]
Conflict Sets

\[ G : \]

\[ P_{conflict}(f) = \{ p_t \mid r \leq t \leq n \text{ and } f \text{ visible from } p_t \} \]

Set of nodes adjacent to \( f \) in \( G \).

\[ F_{conflict}(p_t) = \{ \text{facets of } CH(P_{r-1}) \text{ visible from } p_t \} \]

Set of nodes adjacent to \( p_t \) in \( G \).
Visible Facets

When inserting $p_r$ into $CH(P_{r-1})$, look up $F_{\text{conflict}}(p_r)$ to get the visible facets.

$G$: facets in $CH(P_{r-1})$

- $f$ can be seen from every $p_t \in P_{\text{conflict}}(f)$.
- Once we add the first such $p_t$, $f$ must be deleted.

visibility $\iff$ conflict
Graph Initialization & Updates

Initialize $G$:

- $P_4 = \{p_1, p_2, p_3, p_4\}$ is a tetrahedron.
- Check every point $p_i$, $5 \leq i \leq n$, which of the four facets are visible.

$O(n)$

Update $G$ after adding $p_r$:

1. Discard neighbors (all visible facets) of $p_r$ in $G$.
2. Delete the node representing $p_r$.
3. Add nodes for the new facets (which connect $p_r$ to the horizon).
Updating the Conflict Sets of New Faces

4. Construct $P_{\text{conflict}}(f)$ for every new facet $f$.
\[
\{p_t \mid r < t \leq n \text{ and } f \text{ visible from } p_t\}
\]

Suppose a point $p_t$ can see $f$.

Then it can see $e$, which is an edge in $CH(P_{r-1})$ that bounds $f$.

Then $e$ must have been visible from $p_t$ in $CH(P_{r-1}) \subseteq CH(P_r)$

$f_1$ or $f_2$ is visible from $p_t$.

Test all points $p_t \in P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2)$, where $r < t \leq n$, add $\langle p_t, f \rangle$ to $G$ if $f$ is visible from $p_t$.

5. $F_{\text{conflict}}(p_t)$ thus gets updated with the bipartite graph $G$ for $r < t \leq n$. 