Yan-Bin Jia, Iowa State University

Line Arrangements

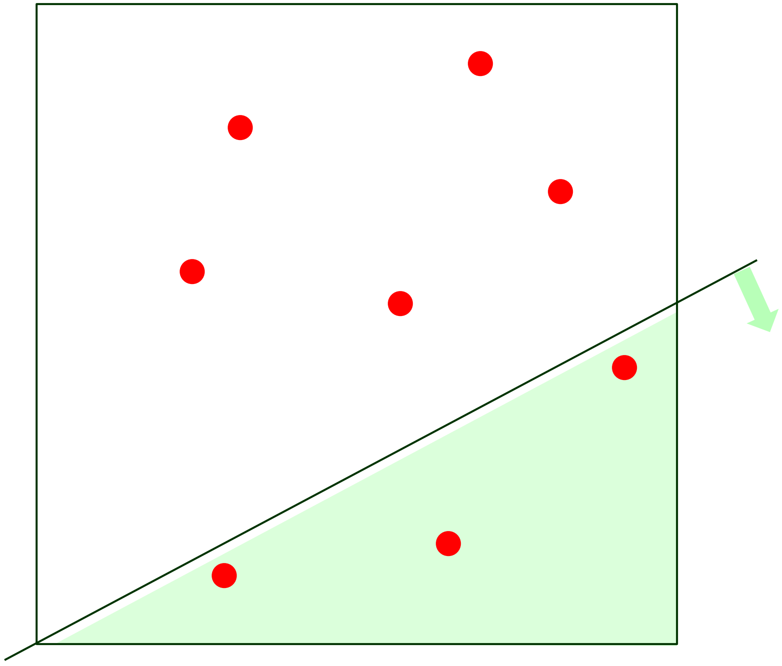
Outline:

I. Geometric complexity of a line arrangement

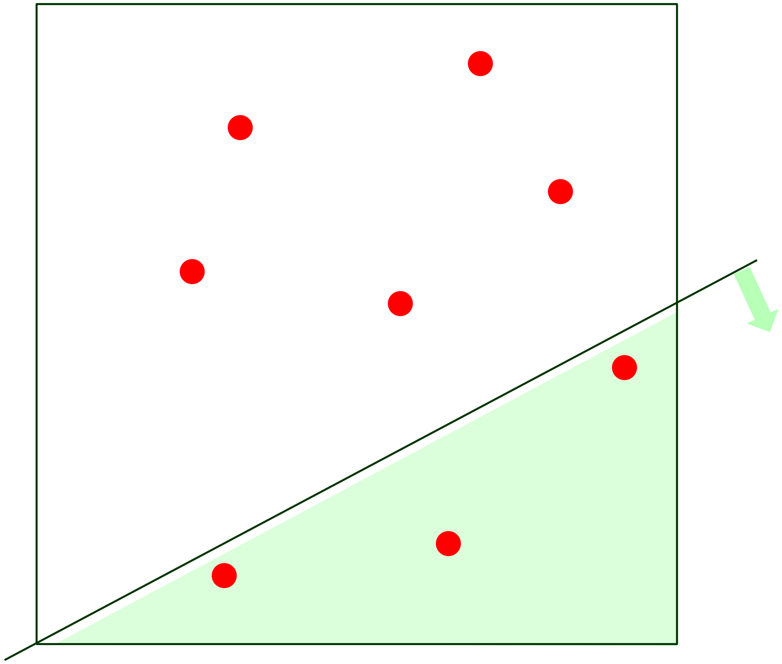
II. Incremental construction

III. Computation of the discrete discrepancy measure

The Discrepancy Problem

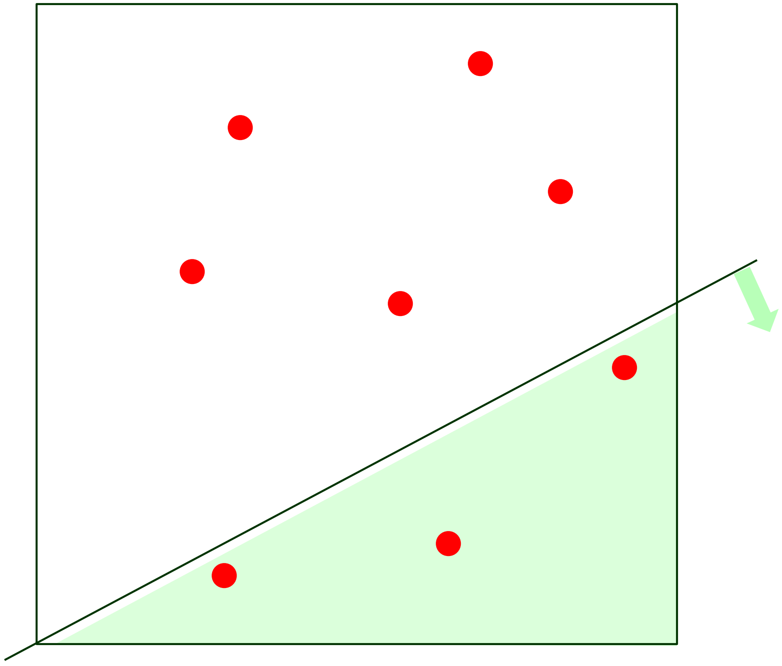


The Discrepancy Problem



Continuous measure: $\mu(h) = \frac{1}{4}$

The Discrepancy Problem



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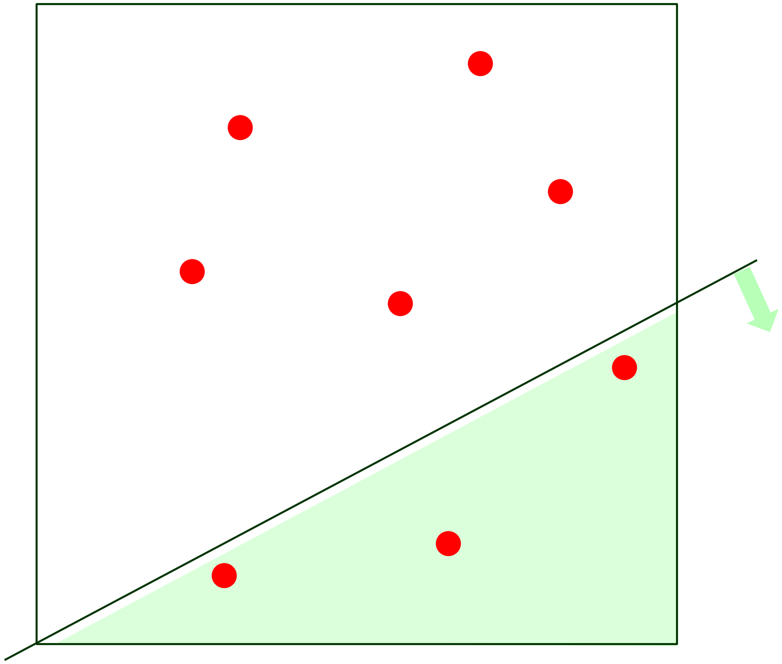

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Maximize $|\mu_s(h) - \mu(h)|$

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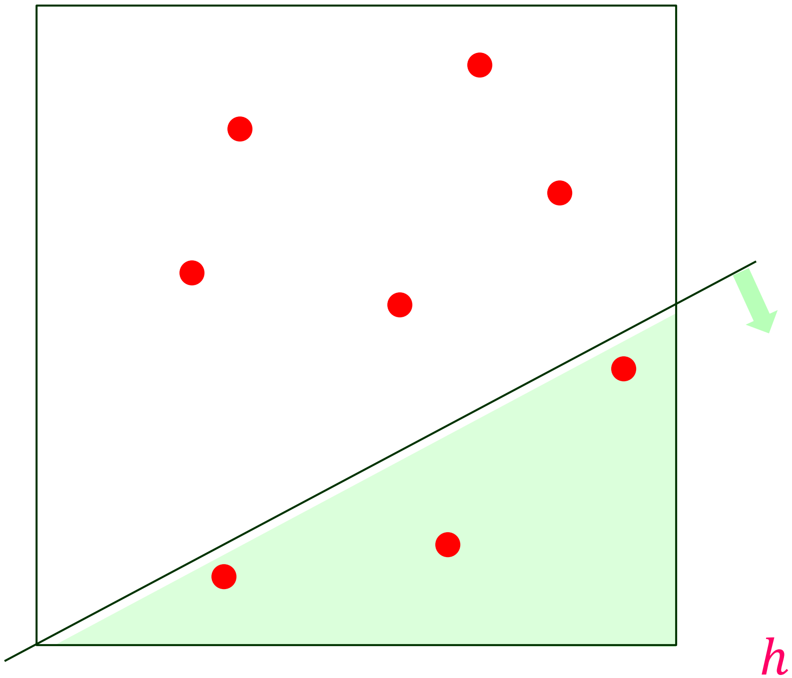

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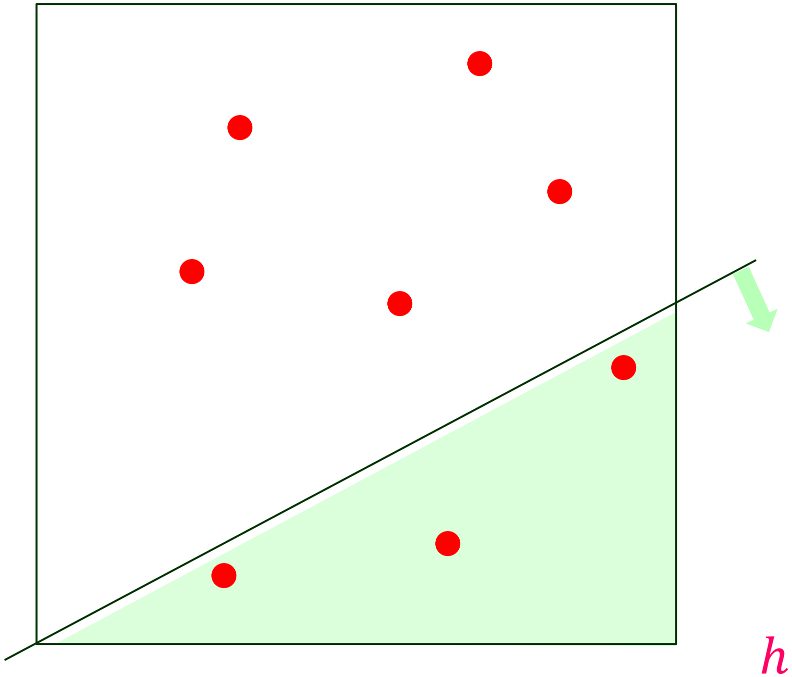

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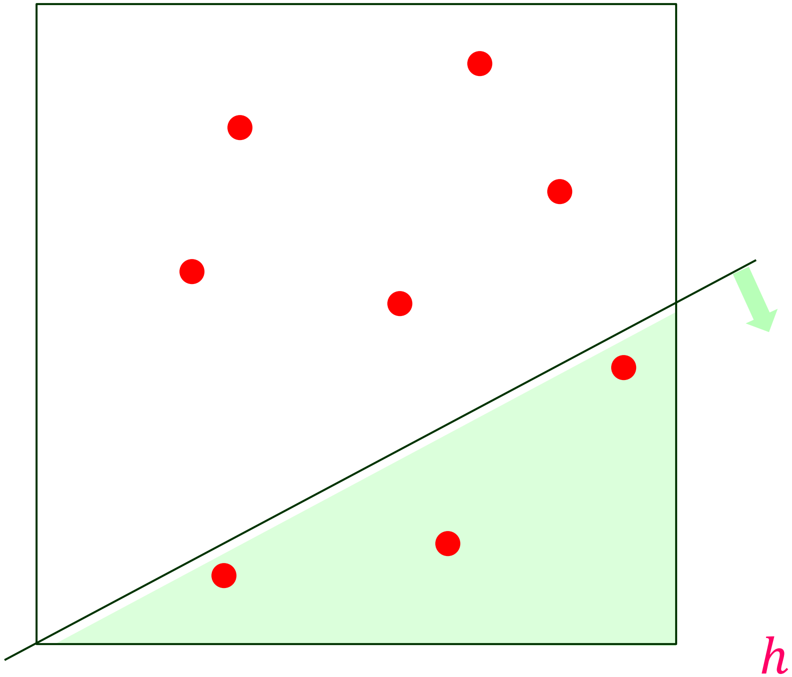

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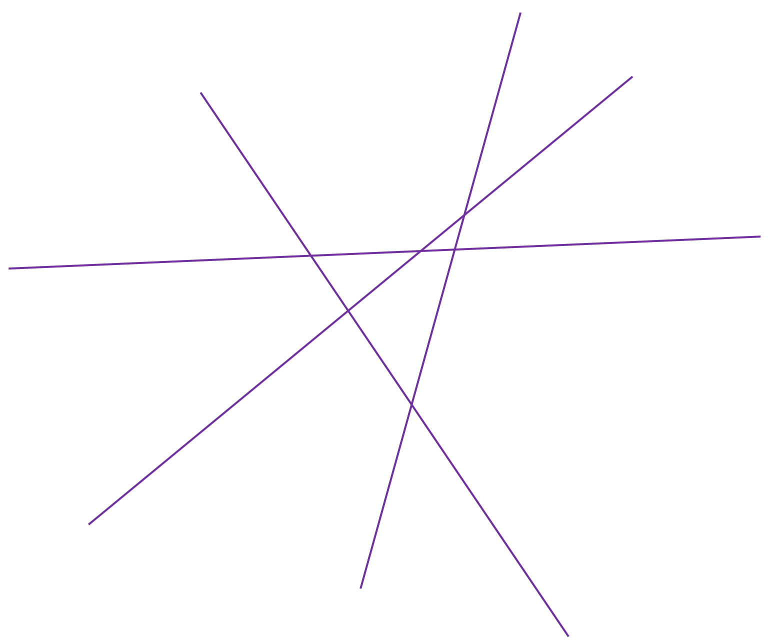
☆ exactly one point (Type i) $\Rightarrow$ brute-force method $O(n^2)$

☆ at least two points (Type ii) $\Rightarrow$ apply duality + line arrangement

I. Arrangement of Lines

L: a set of n lines.



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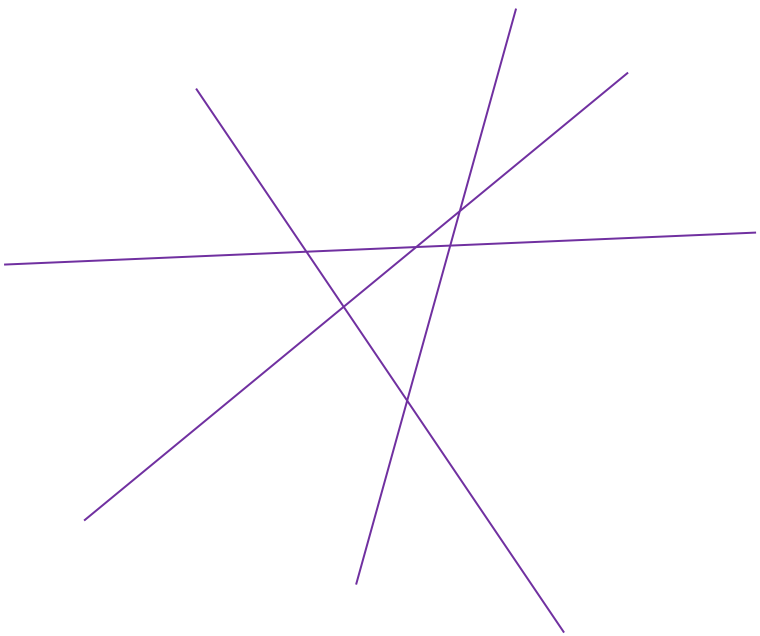

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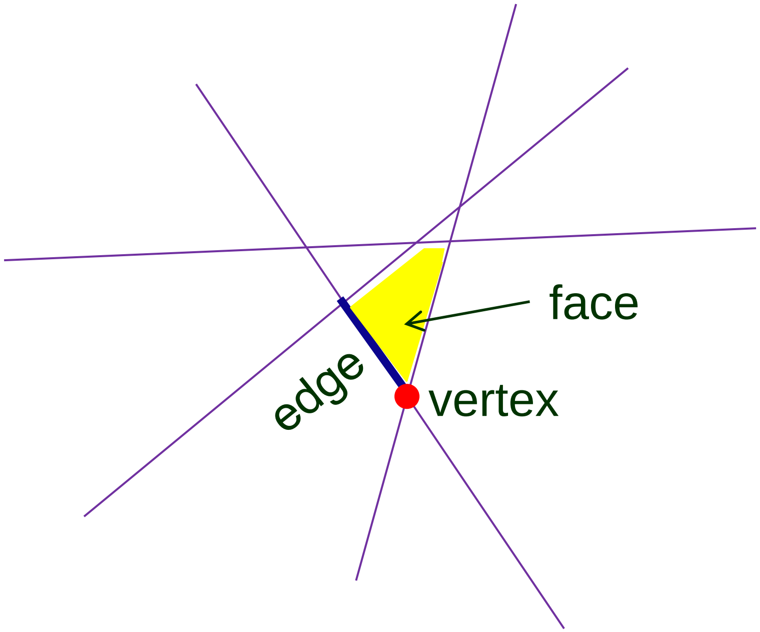

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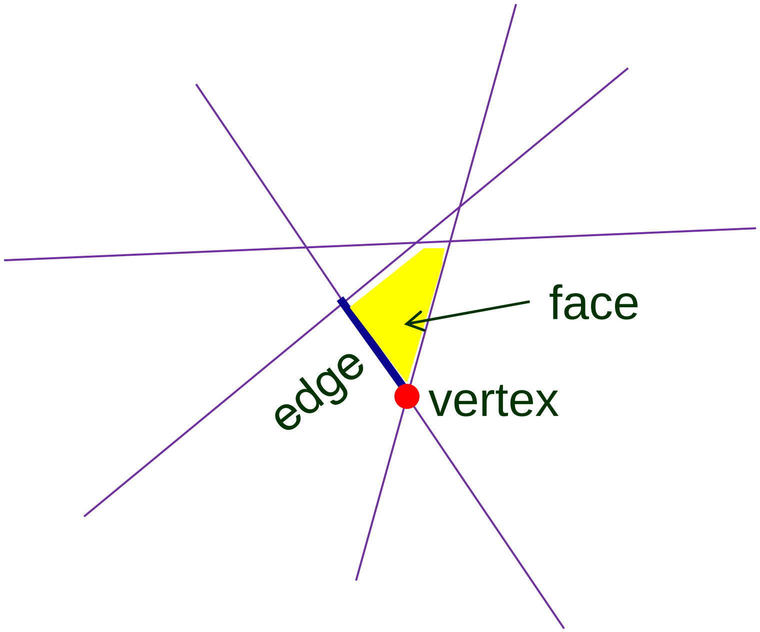
- with unbounded edges and faces



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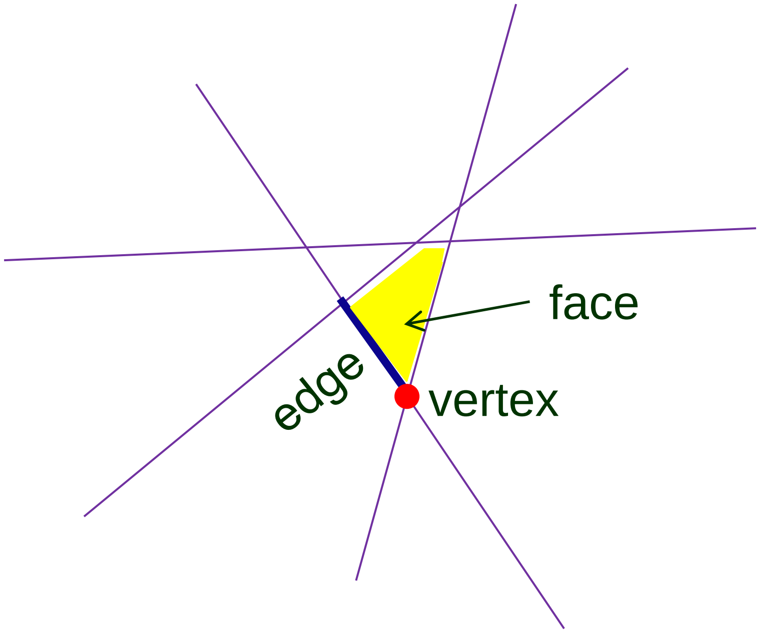
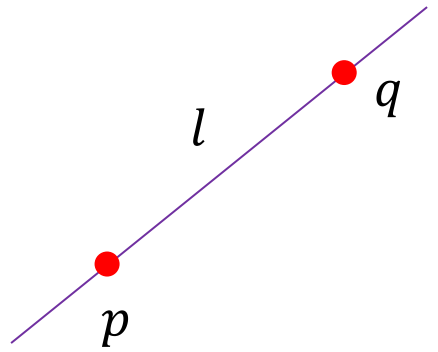
Simple arrangement if

- no three lines are concurrent;
- no two lines are parallel.

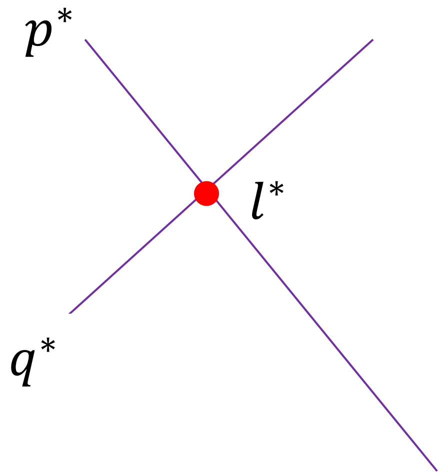


Reduction to Line Arrangment

Problem on points $\Longrightarrow$ problem on an arrangement of dual lines.



primal plane



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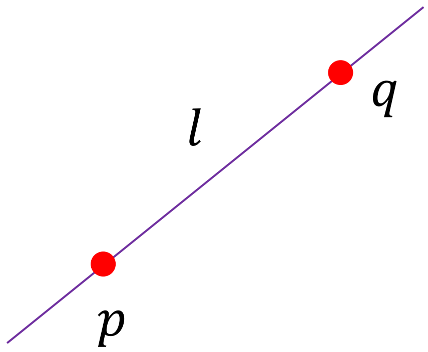

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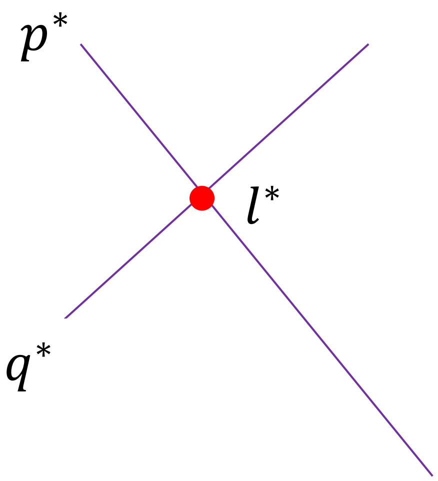
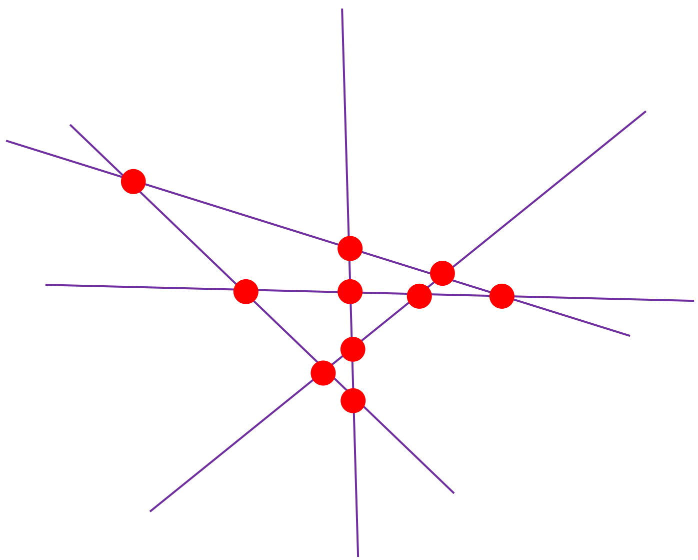


dual plane

Structure of a line arrangement is ***more apparent*** than that of a point set.

Combinatorial Complexity

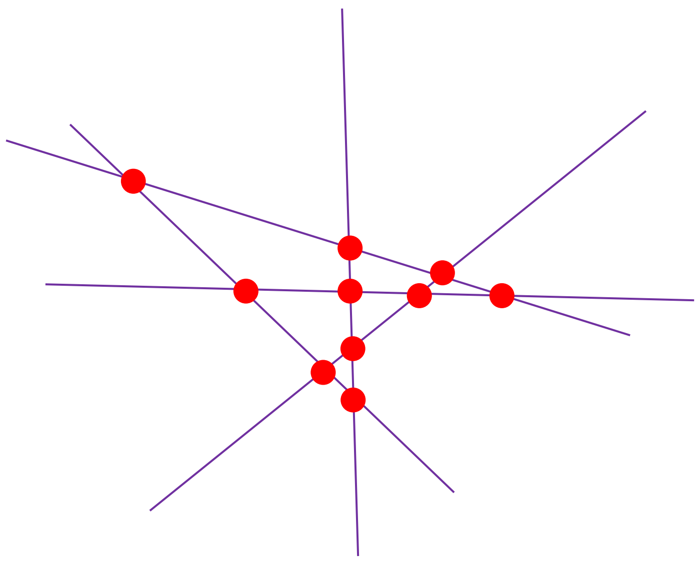
#vertices + #edges + #faces



Combinatorial Complexity

#vertices + #edges + #faces

Theorem #vertices $\leq \frac{n(n-1)}{2}$



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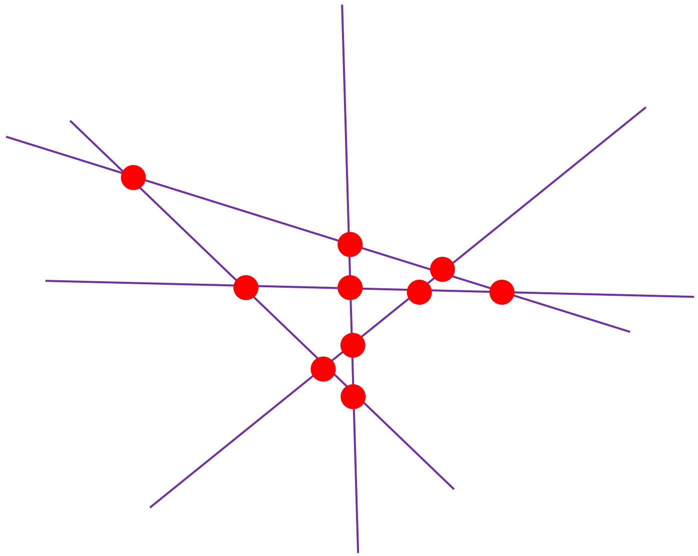

Combinatorial Complexity

#vertices + #edges + #faces

Theorem #vertices $\leq \frac{n(n-1)}{2}$

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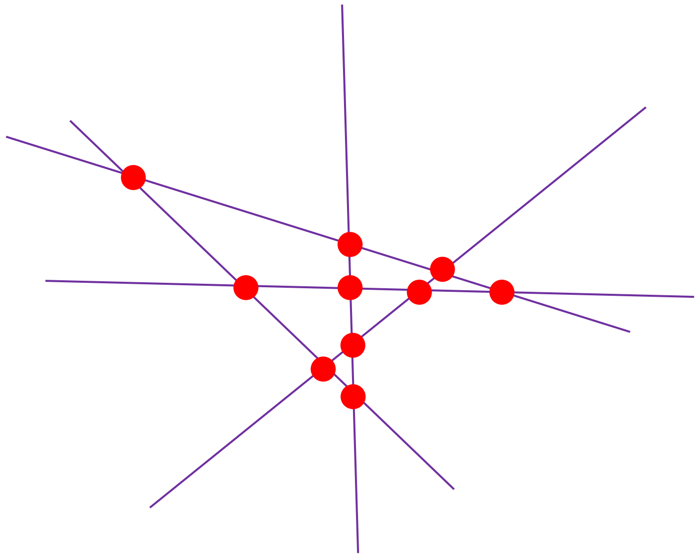
#faces $\leq \frac{n^2}{2} + \frac{n}{2} + 1$



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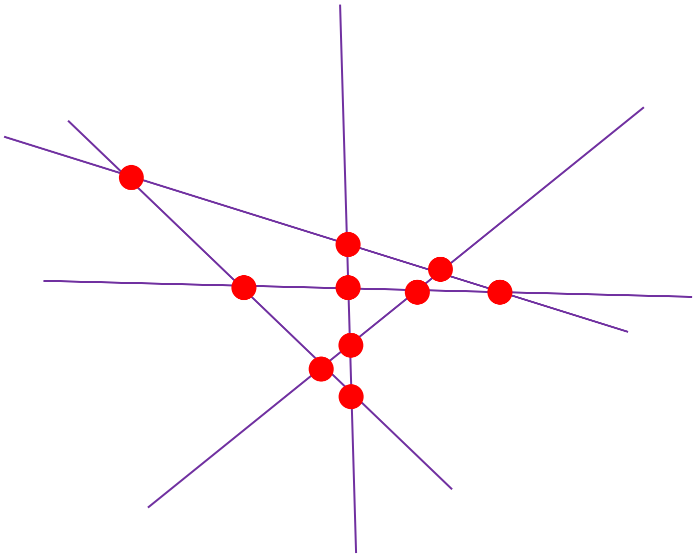

Equality holds if and only if $A(L)$ is simple.

Proof of Complexity

We first show that #vertices, #edges, #faces are maximal when $A(L)$ is simple (no parallel or ≥ 3 concurrent lines) and not otherwise.

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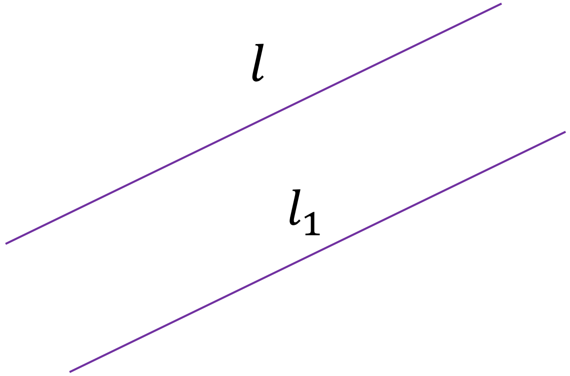

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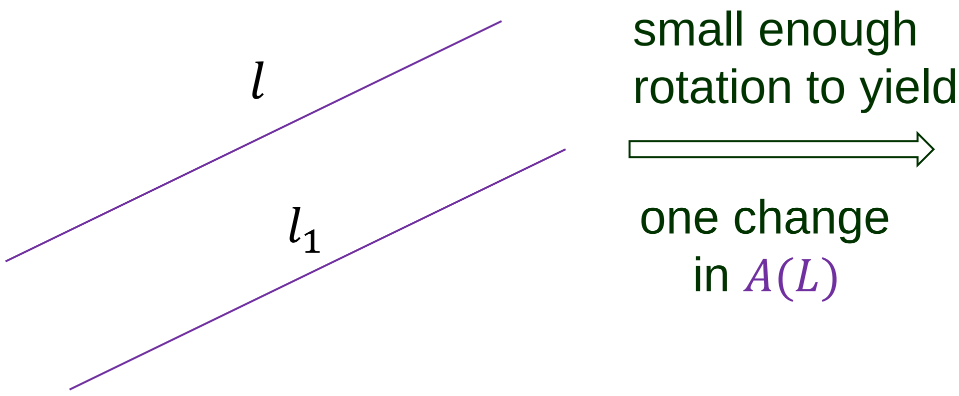

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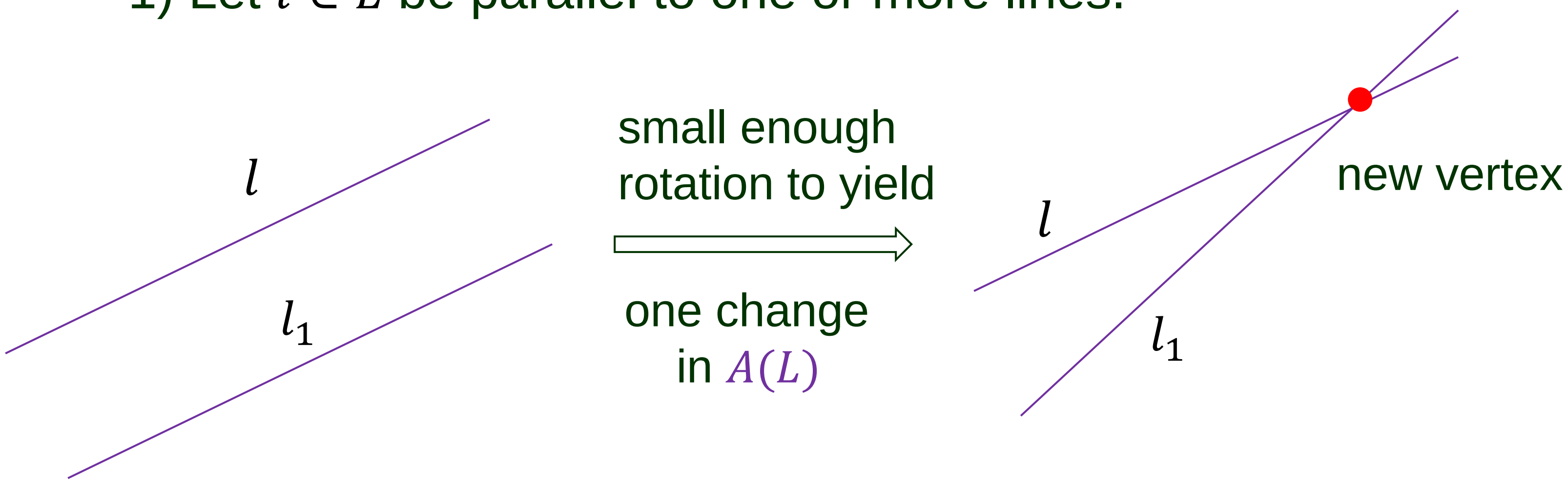

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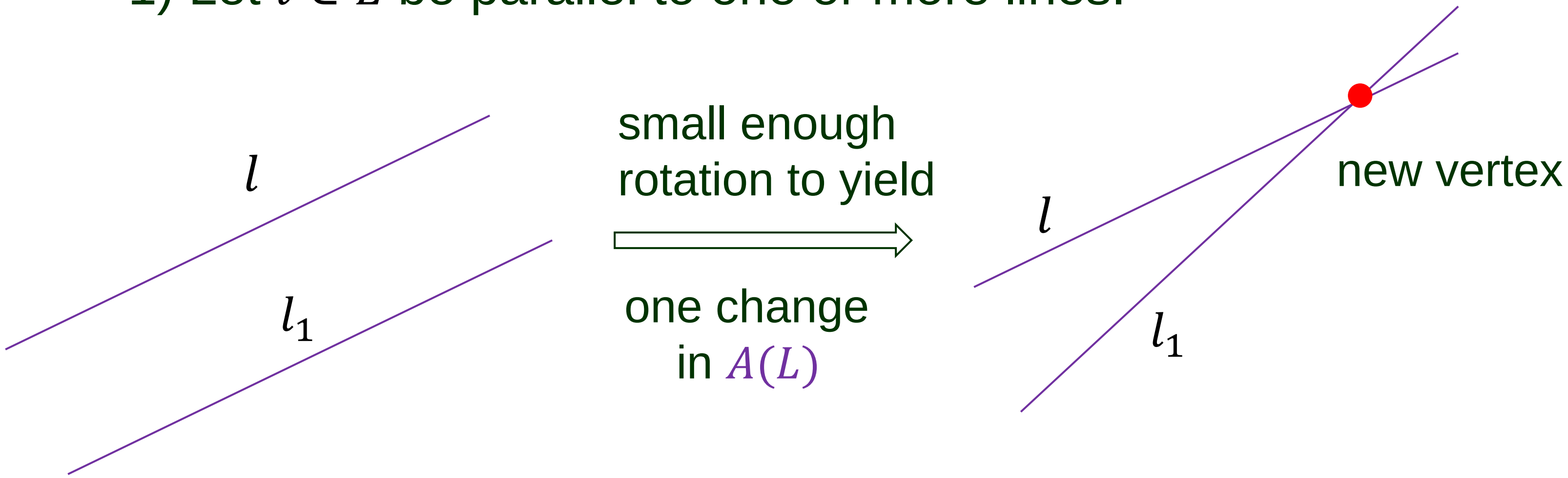
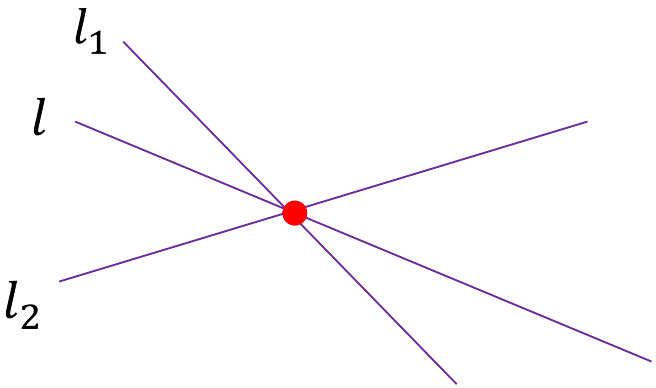


Complexity increases in this case. (Hence such configuration cannot be maximal.)

Proof (Cont'd)

2) Suppose l passes through a vertex.



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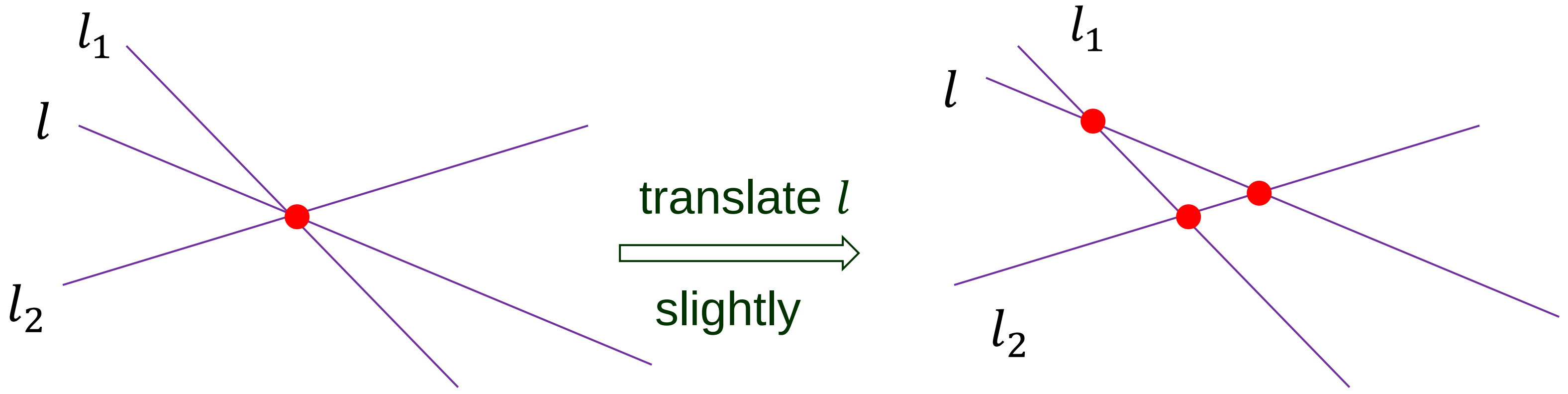

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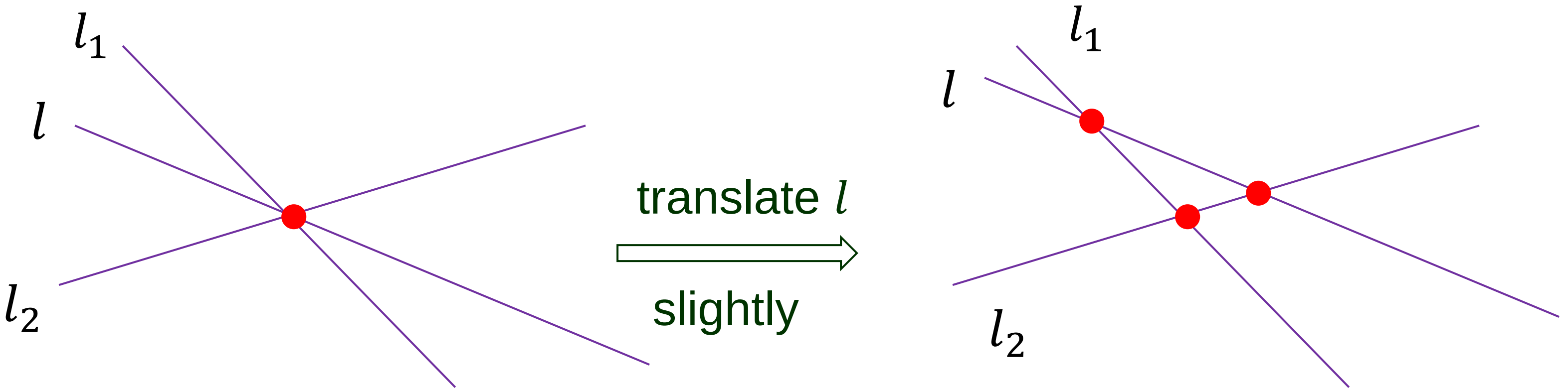

2 new vertices
3 new edges
1 new face

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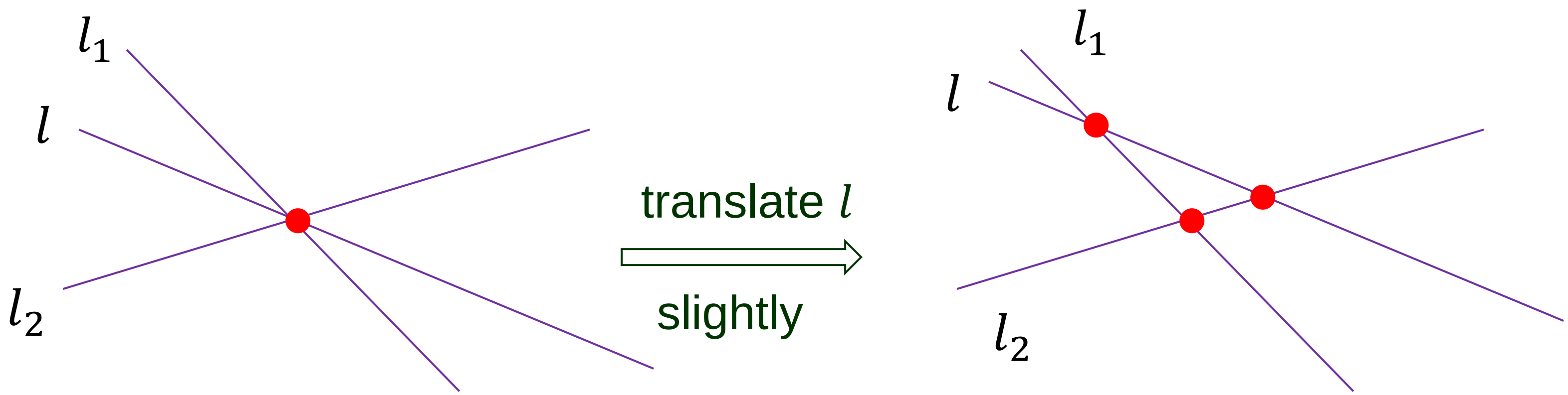

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Since complexity increases, such configuration cannot be maximal either.

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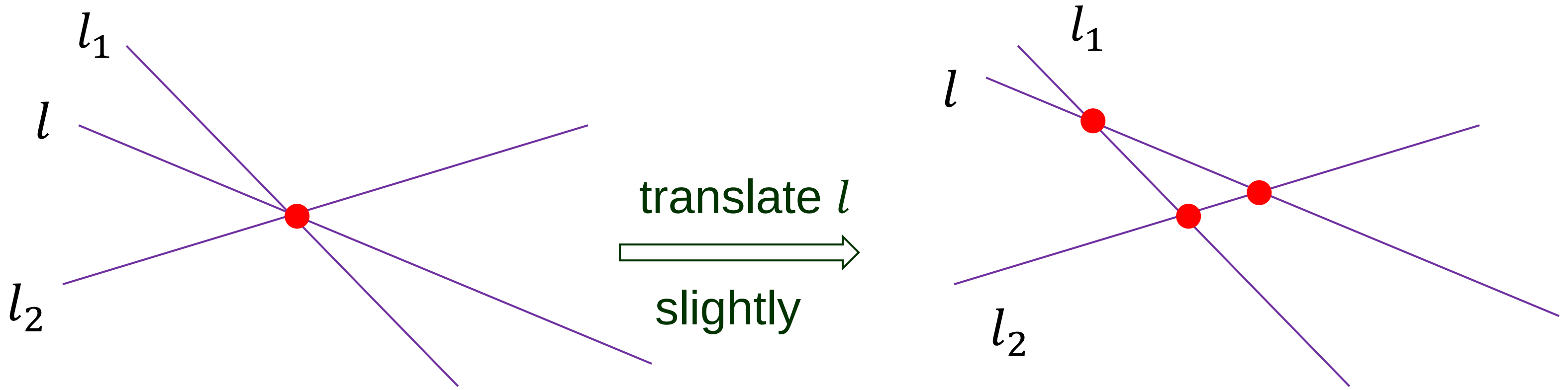

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The arrangement with maximal complexity must be simple.

Exact Size of a Simple Arrangement

$n_v =$ # vertices

$n_e =$ # edges

$n_f =$ # faces

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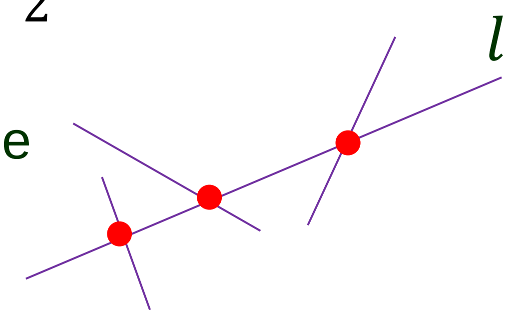

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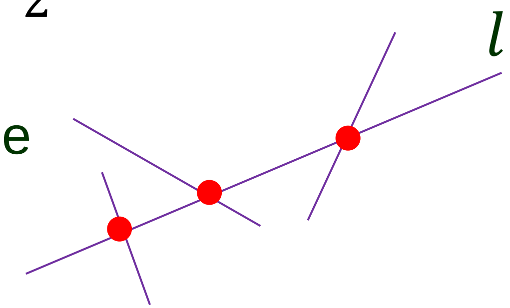

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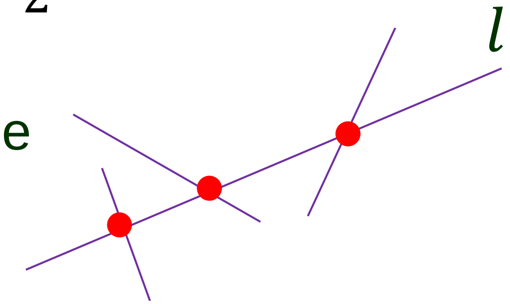
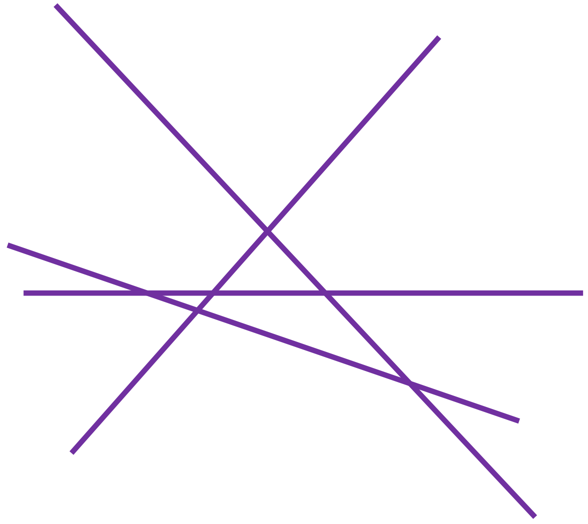


$$\Longrightarrow \quad n_e = n \cdot (1 + n - 1) = n^2$$

Number of Faces



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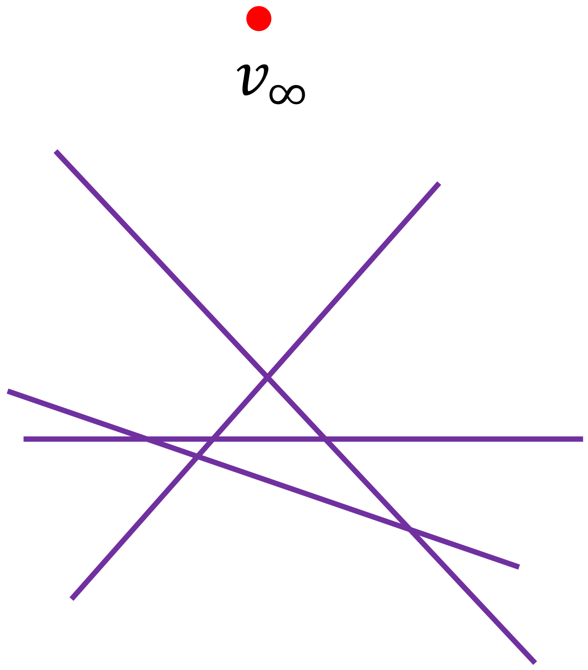
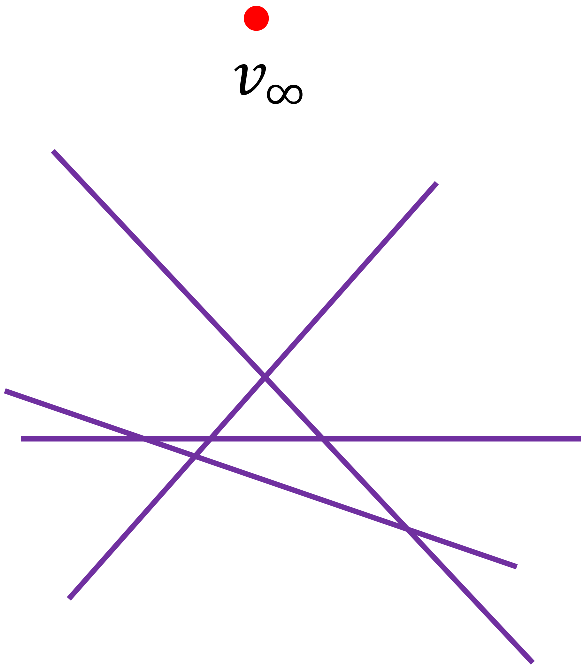


1. Add a vertex v_∞ at infinity.

Number of Faces



1. Add a vertex v_∞ at infinity.

2. Extend (and bend) every half-infinite edge to v_∞.

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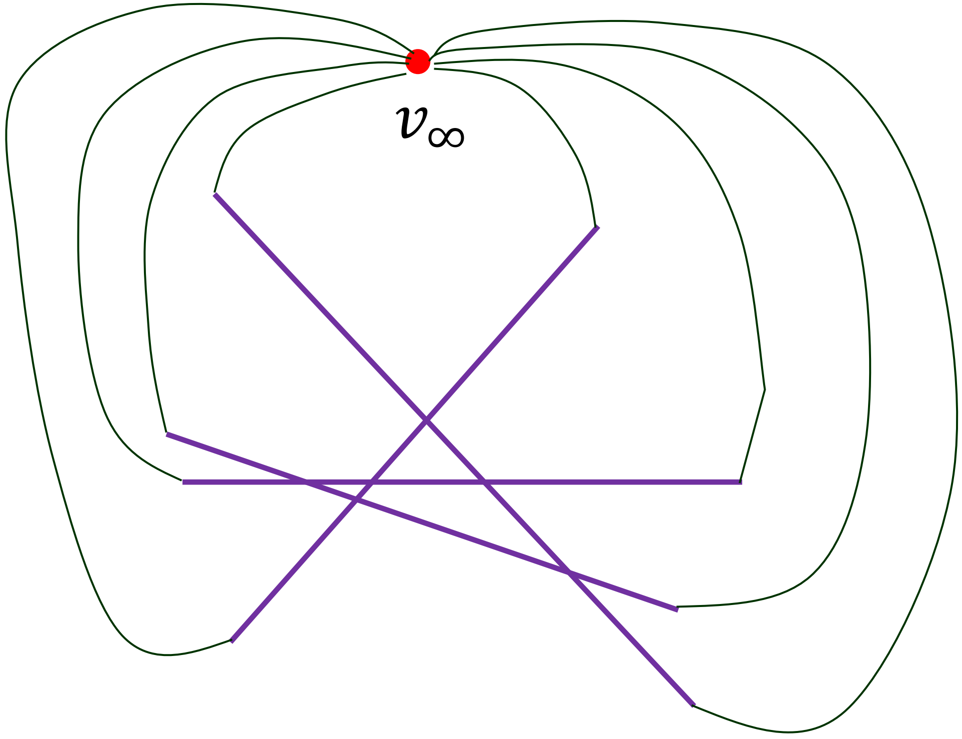

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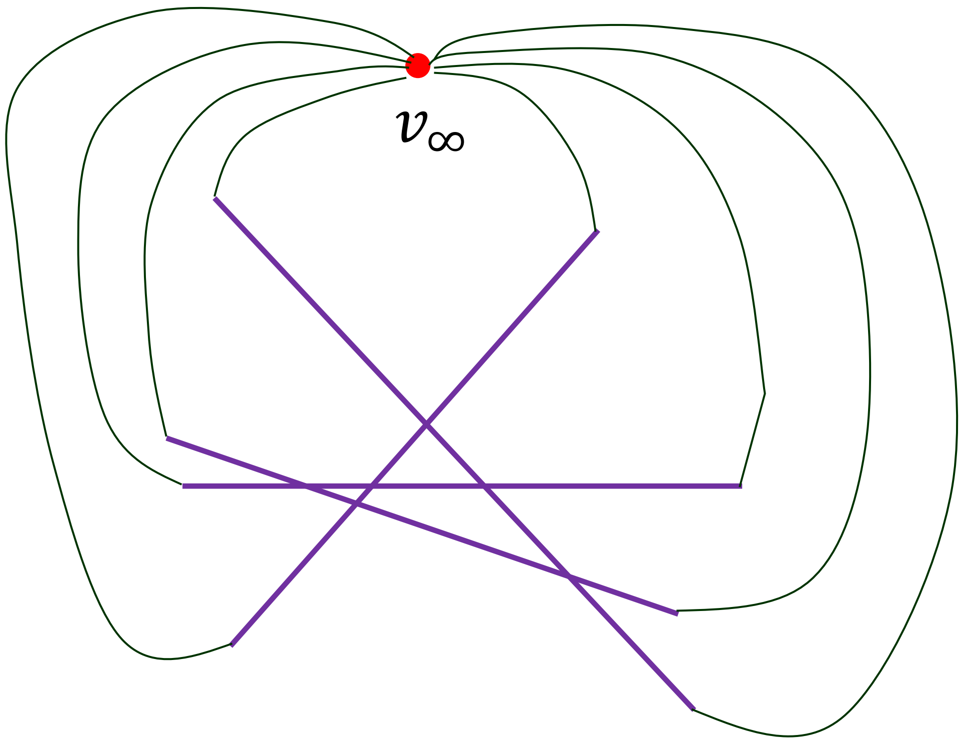

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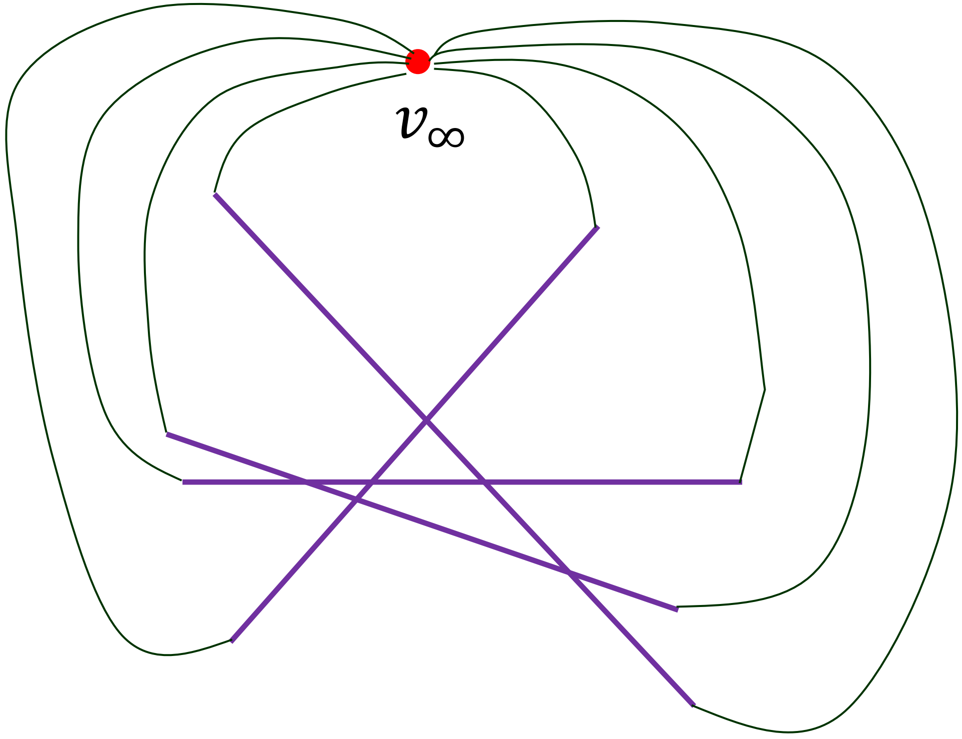
2. Extend (and bend) every half-infinite edge to v_∞.

No edge crossing

Number of Faces



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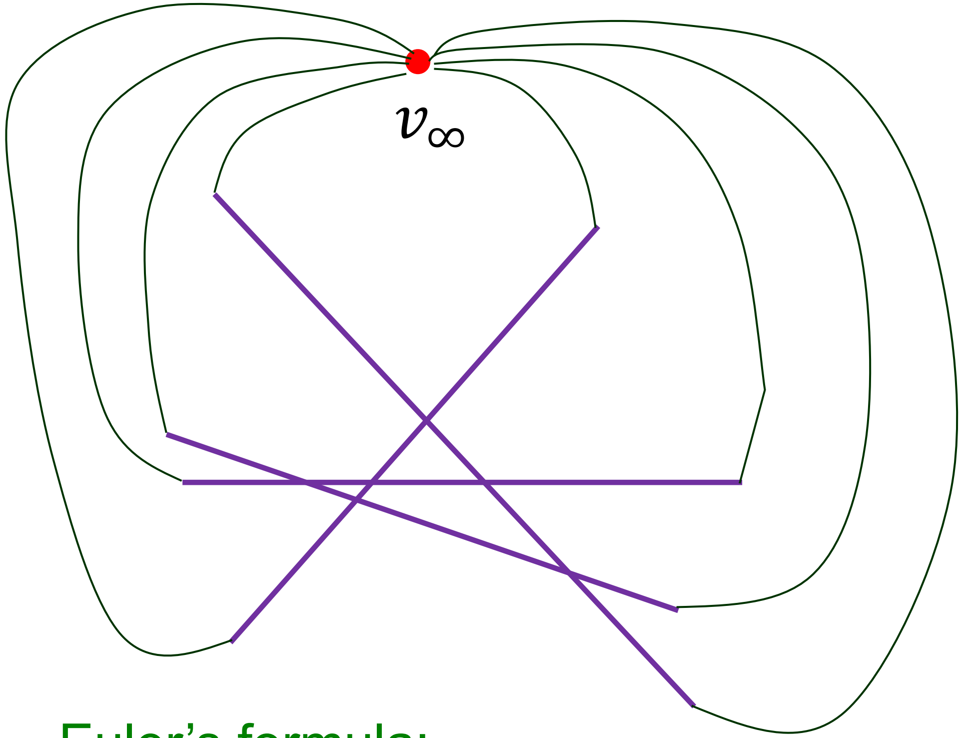
No edge crossing

⇩

Planar graph

Number of Faces



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No edge crossing

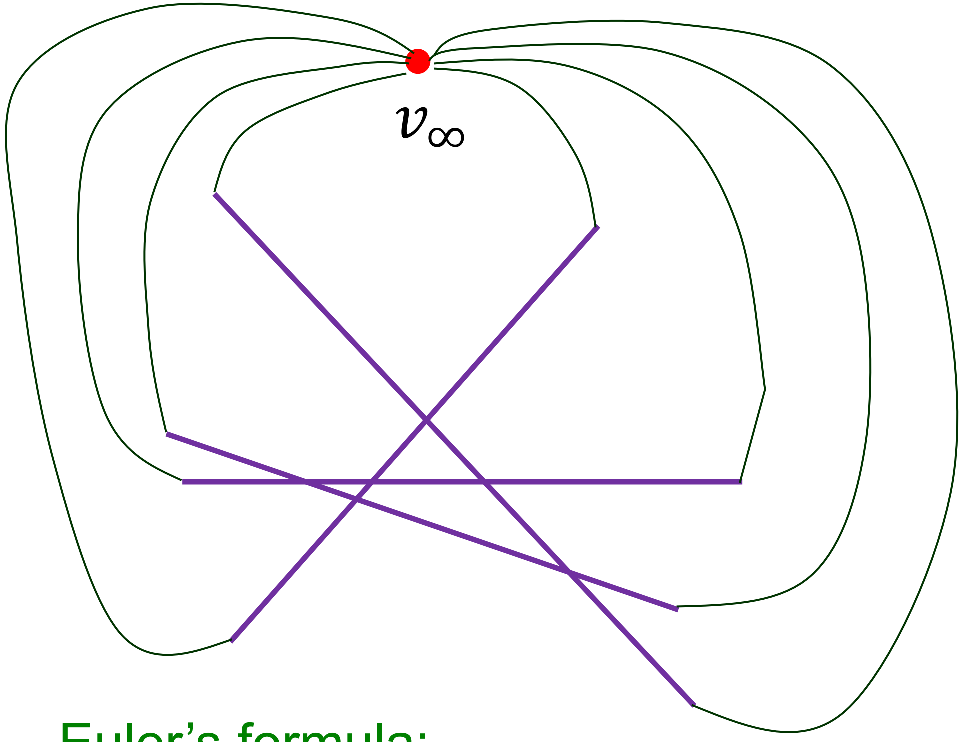
⇓

Planar graph

Euler's formula:

$$(n_v + 1) + n_f - n_e = 2$$

Number of Faces



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No edge crossing

⇓

Planar graph

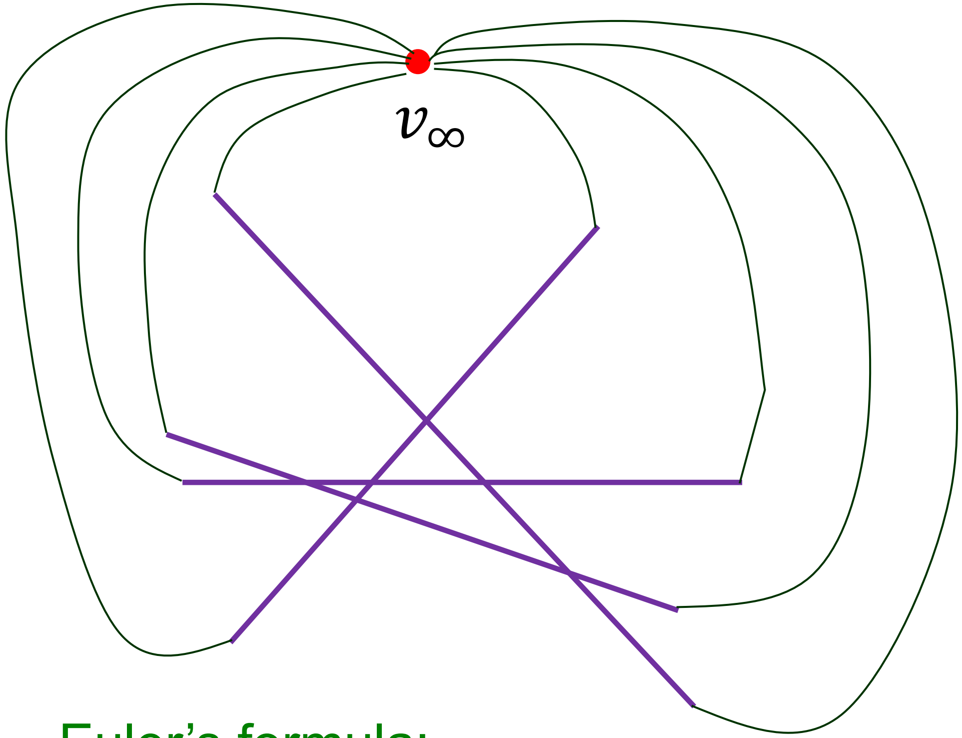
Euler's formula:

$$(n_v + 1) + n_f - n_e = 2$$

⇒

$$n_f = 2 - (n_v + 1) + n_e$$

Number of Faces



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2. Extend (and bend) every half-infinite edge to v_∞.

No edge crossing

⇓

Planar graph

Euler's formula:

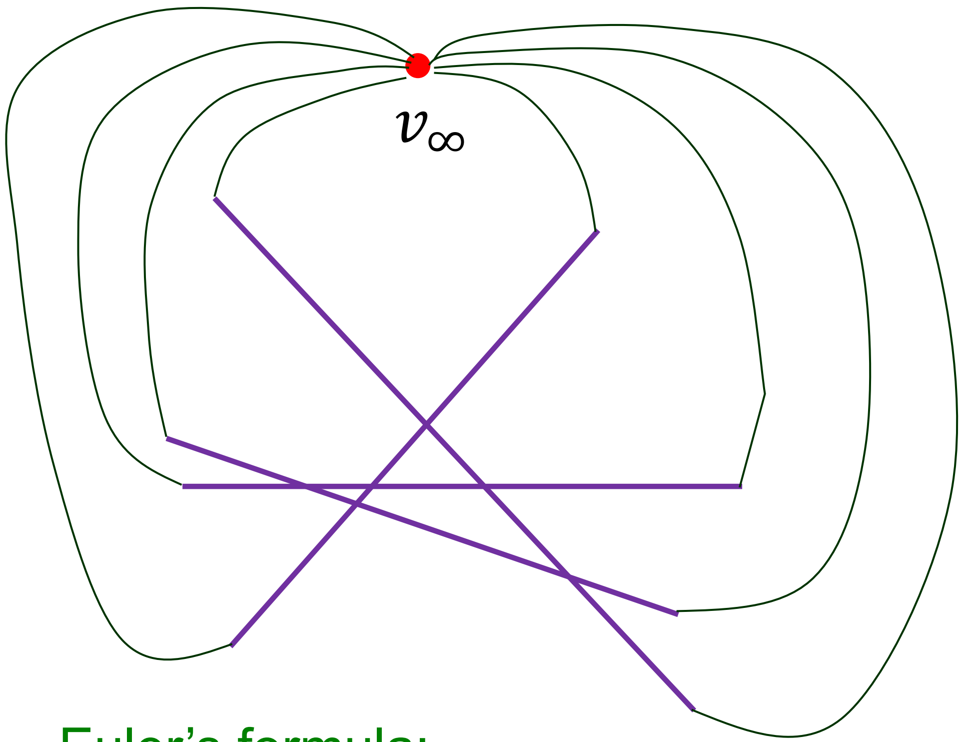
$$(n_v + 1) + n_f - n_e = 2$$

⇒

$$n_f = 2 - (n_v + 1) + n_e$$

$$= 2 - \left(\frac{n(n-1)}{2} + 1\right) + n^2$$

Number of Faces



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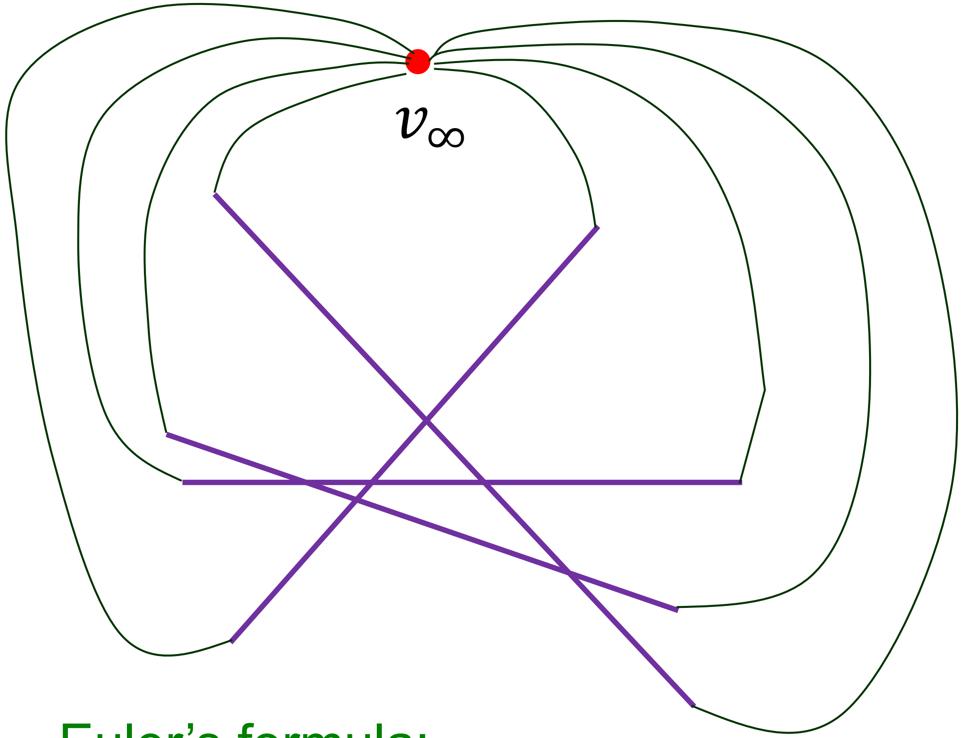
$$\boxed{= \frac{n^2}{2} + \frac{n}{2} + 1}$$

Number of Faces



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No edge crossing

$\Downarrow$

Planar graph

Euler's formula:

$$(n_v + 1) + n_f - n_e = 2$$

$$\Rightarrow$$

$$n_f = 2 - (n_v + 1) + n_e$$

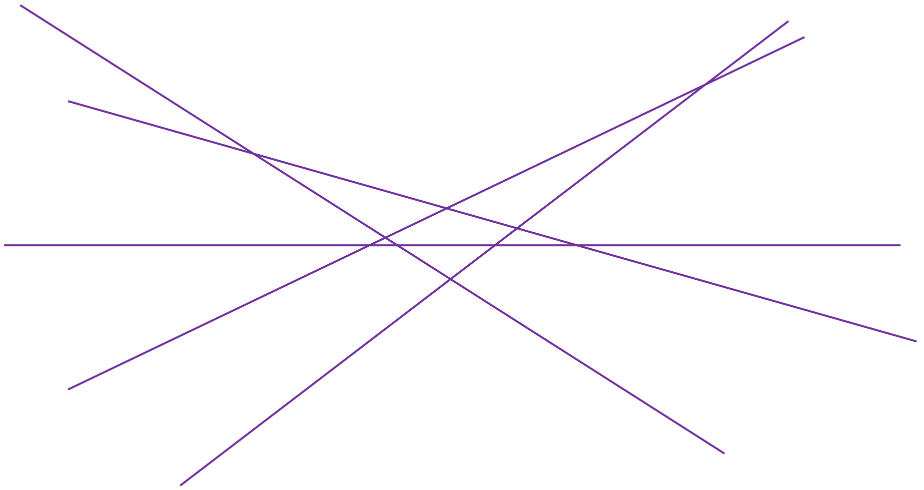
$$= 2 - \left(\frac{n(n-1)}{2} + 1\right) + n^2$$

$$\boxed{= \frac{n^2}{2} + \frac{n}{2} + 1}$$

□

II. Storage of Line Arrangement

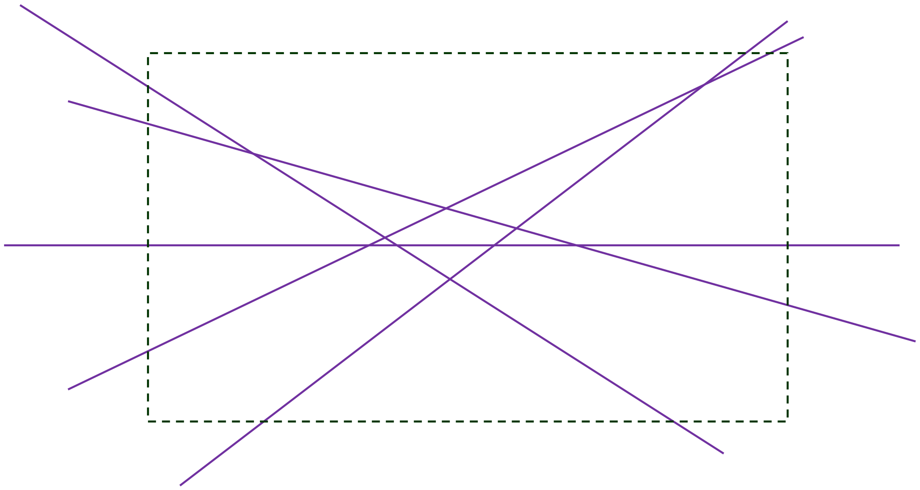
Doubly-connected edge list.



II. Storage of Line Arrangement

Doubly-connected edge list.

Add a bounding box to contain all vertices in interior.



Construction of DCEL for $A(L)$

Plane sweep?

$$O(n \log n + I \log n)$$

Construction of DCEL for $A(L)$

Plane sweep?

$$O(n \log n + I \log n)$$

pairwise intersection

Construction of DCEL for $A(L)$

Plane sweep?

$$O(n \log n + I \log n)$$

pairwise intersection $\Longrightarrow$ $I = n(n-1)/2$

Construction of DCEL for $A(L)$

Plane sweep?

$$O(n \log n + I \log n) = O(n^2 \log n)$$

pairwise intersection $\Rightarrow$ $I = n(n-1)/2$

Construction of DCEL for $A(L)$

Plane sweep?

$$O(n \log n + I \log n) = O(n^2 \log n)$$

pairwise intersection $\Rightarrow$ $I = n(n-1)/2$

Not optimal!

Incremental Algorithm

Preprocessing

- Compute the bounding box $B(L)$.

Incremental Algorithm

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  - $n(n-1)/2$ intersections.

Incremental Algorithm

Preprocessing

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  - leftmost, rightmost, top, bottom intersections.

Incremental Algorithm

Preprocessing

- Compute the bounding box $B(L)$.
  - $n(n-1)/2$ intersections.
  - leftmost, rightmost, top, bottom intersections.
  - each line yielding two intersections with $B(L)$.

Incremental Algorithm

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- Compute the bounding box $B(L)$.
  - $n(n-1)/2$ intersections.
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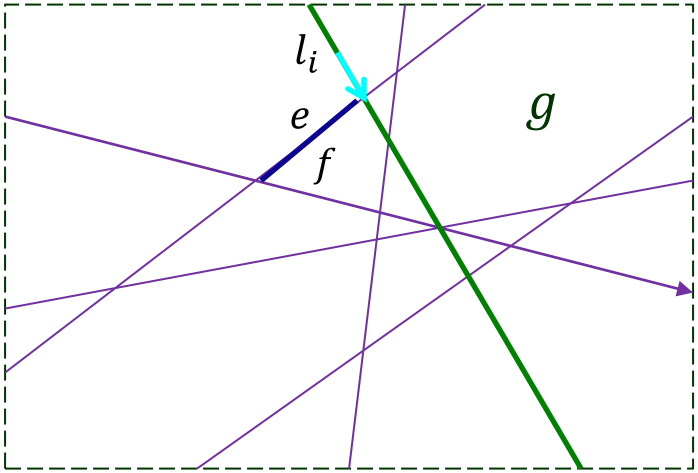
Updating the Subdivision

A_i: subdivision due to $\{l_1, l_2, \ldots, l_i\}$



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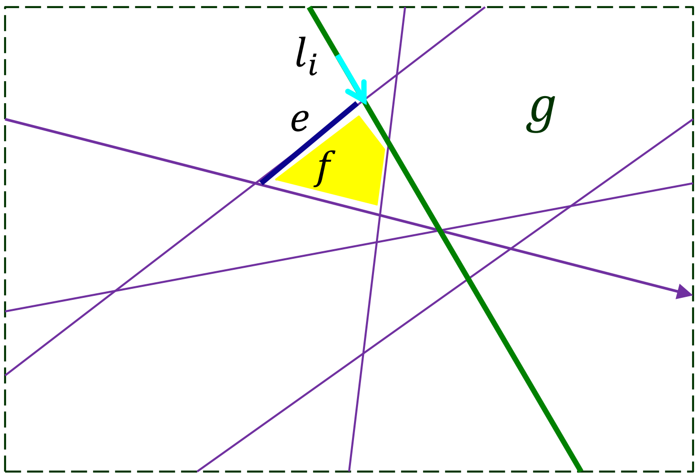

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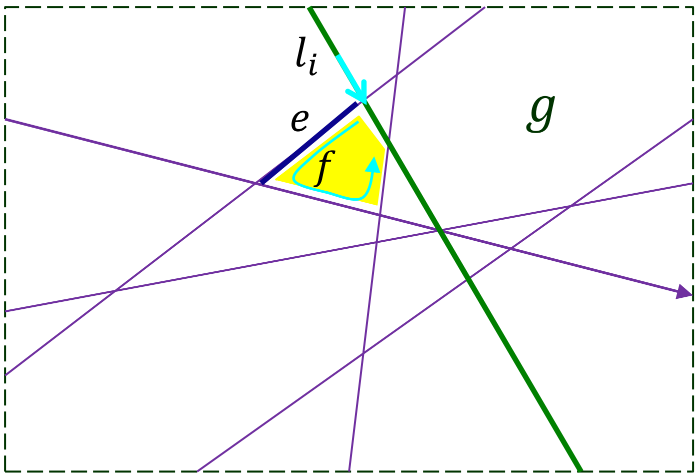

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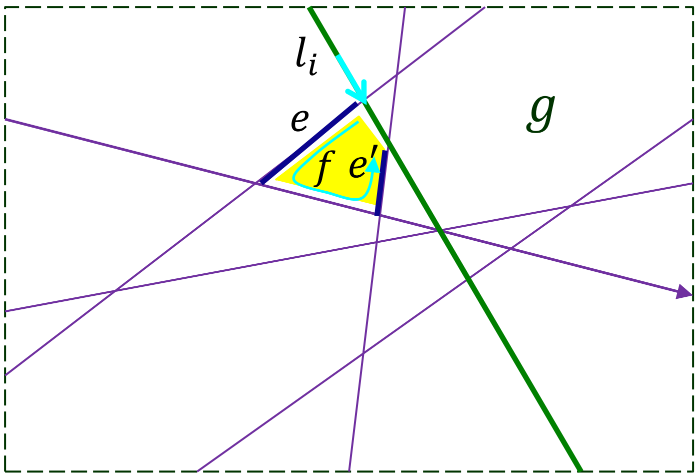

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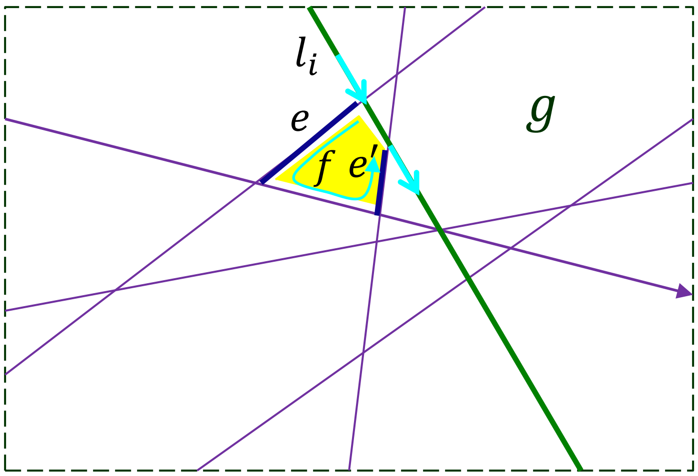

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- Use its Twin() pointer to enter face g.

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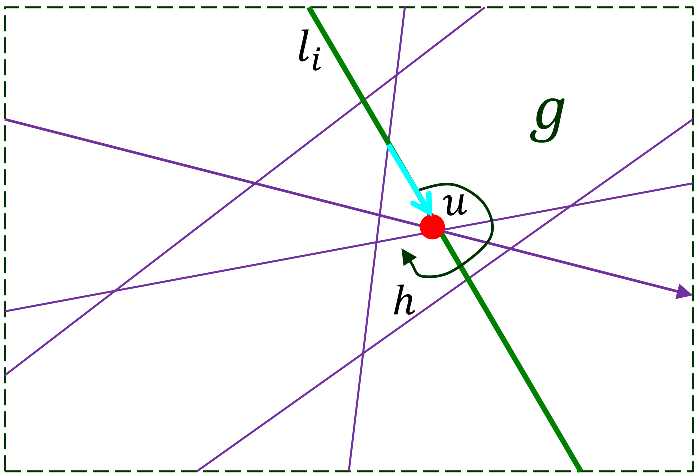
Case 2 Leave a face (g) through a vertex (u).



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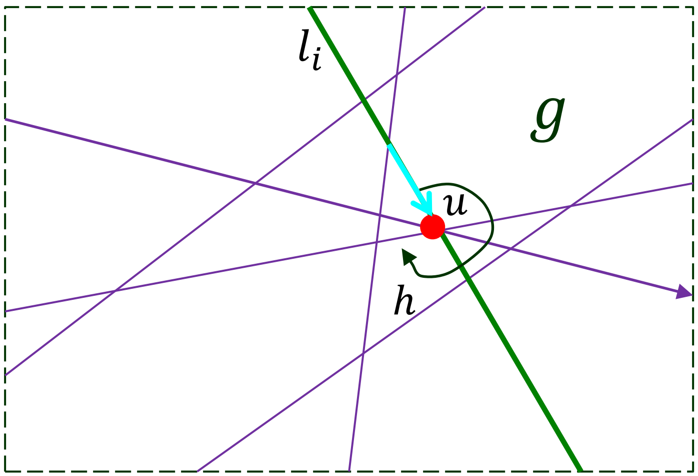

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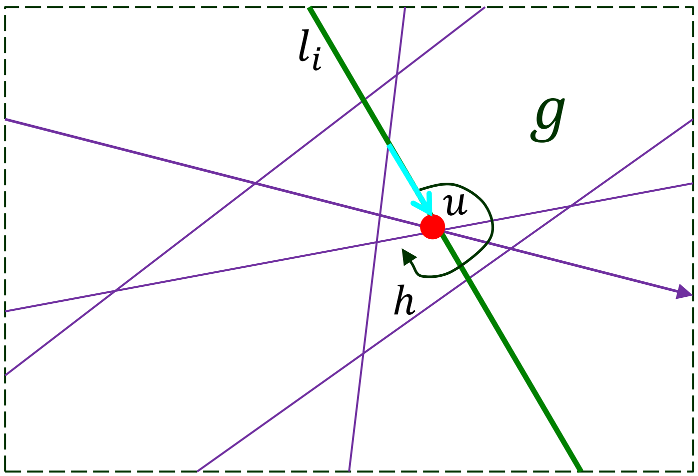

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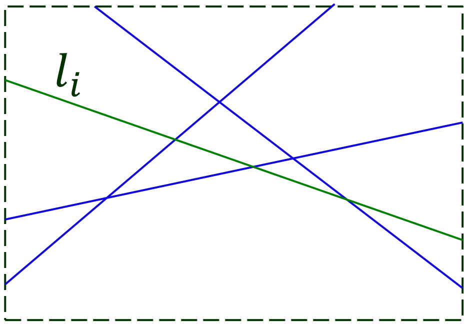
Alternatively use Next() and Twin() pointers.

First Edge of Intersection

How to find the first edge (leftmost edge) intersected by l_i?



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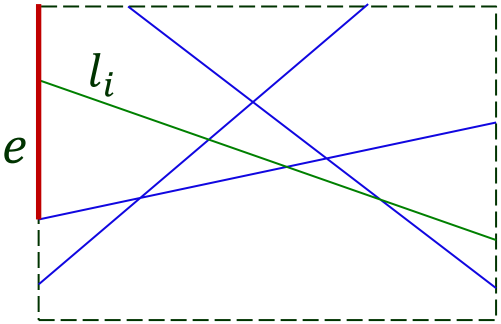

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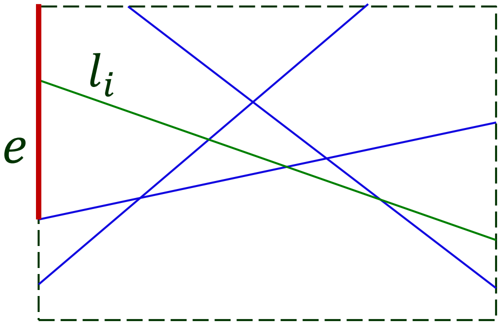

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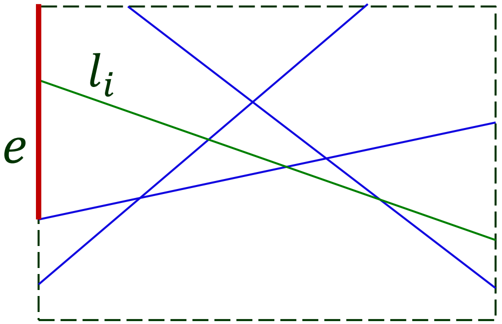

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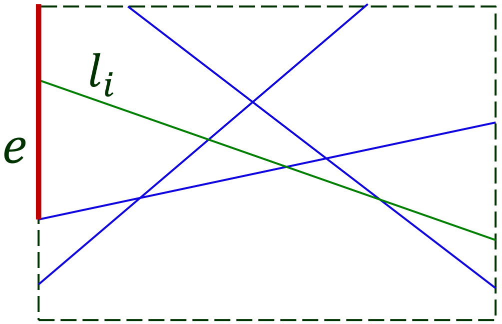

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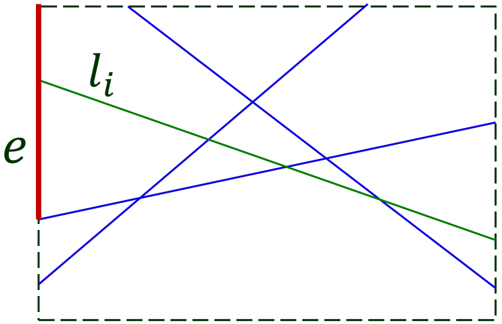

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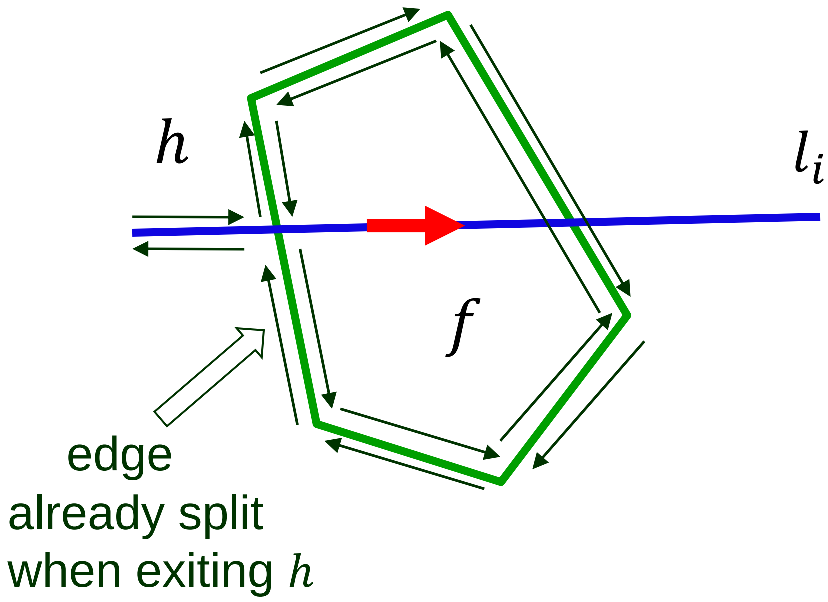
First intersection edge can be found in $O(i)$ time.

Splitting a Face



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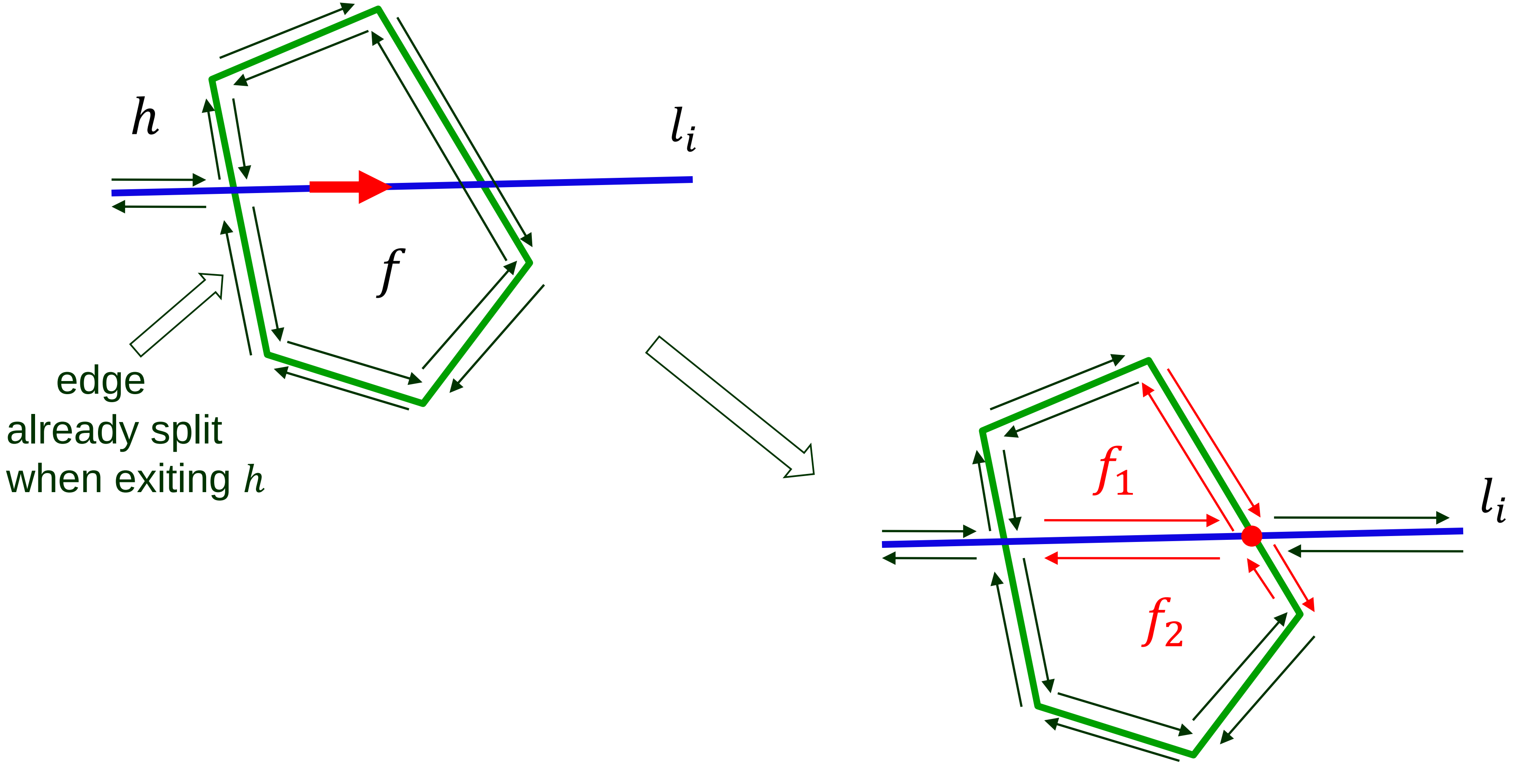

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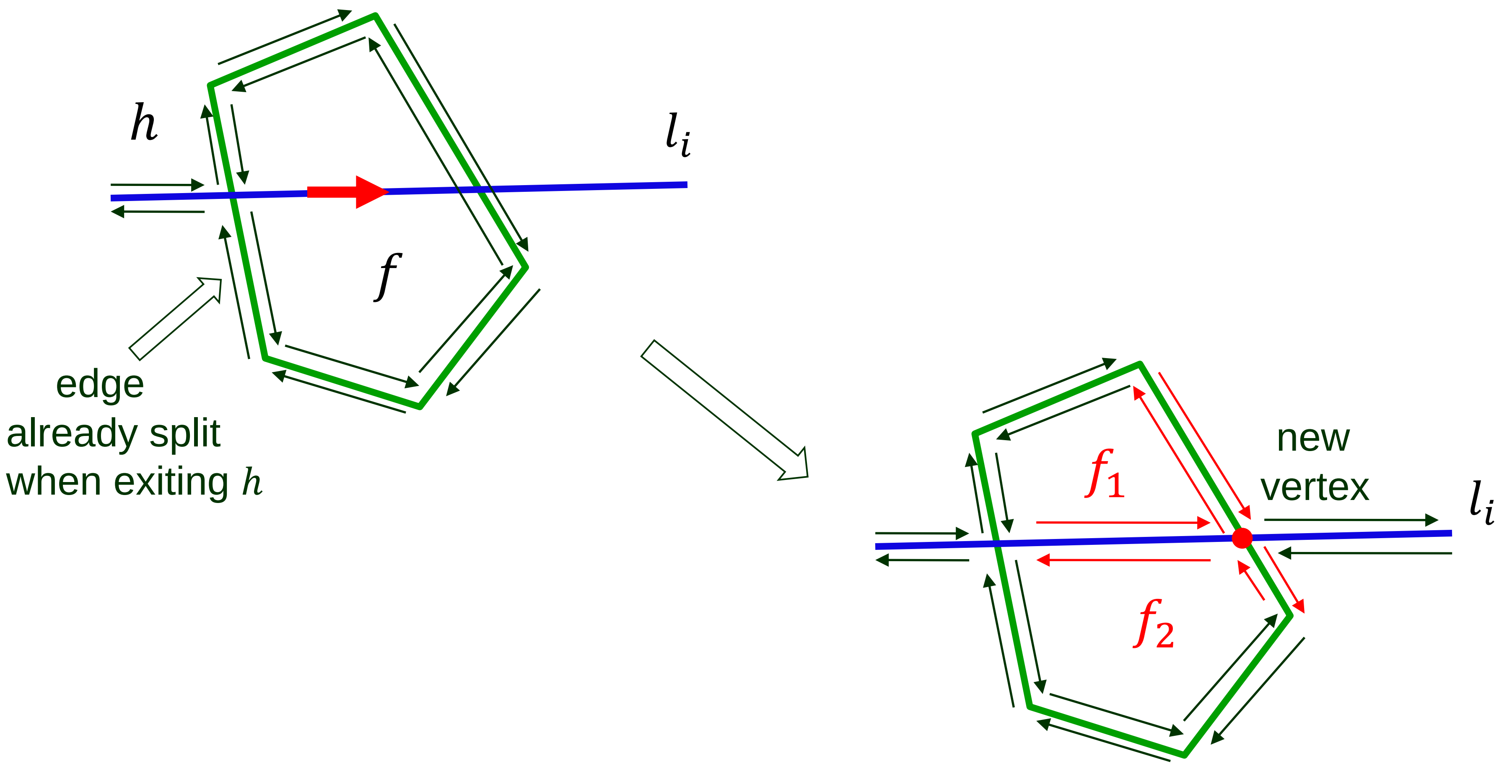

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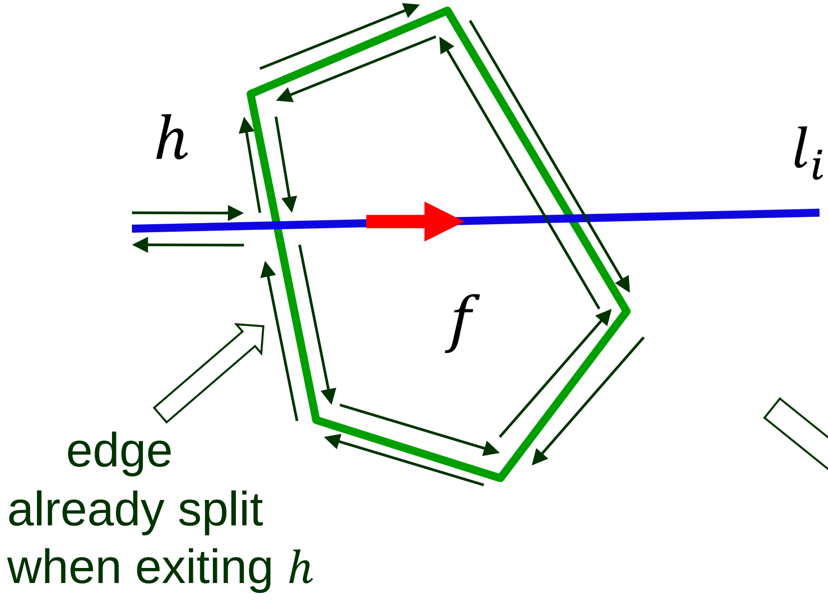
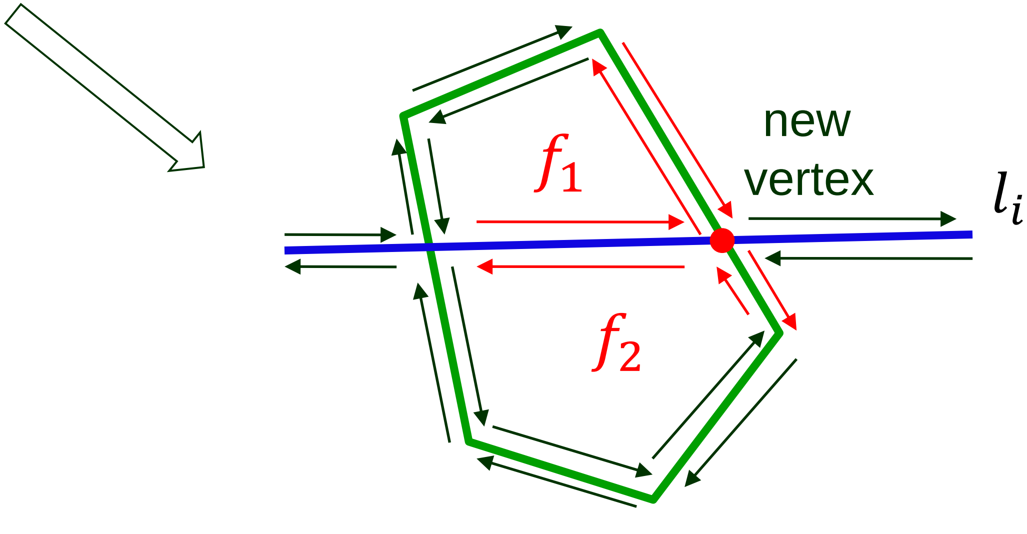


2 new faces
1 new vertex
6 new half-edges



The Algorithm

```
ConstructArrangement(L)

1.  Compute a bounding box B(L) to contain all vertices of A(L)// O(n^2)
2.  Initialize DCEL for B(L)  // O(1)
3.  for i ← 1 to n
4.      do e ← edge on B(L) that first intersects with l_i
5.         f ← bounded face incident on e      // O(i)
6.         while f is bounded
7.             split f
8.             f ← next intersected face
```

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    $O(\text{complexity of } f)$

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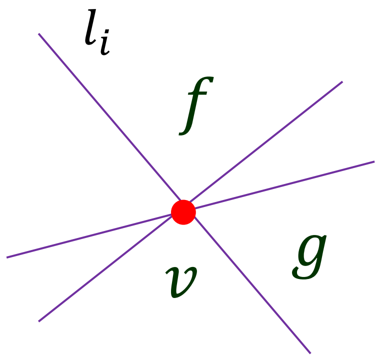

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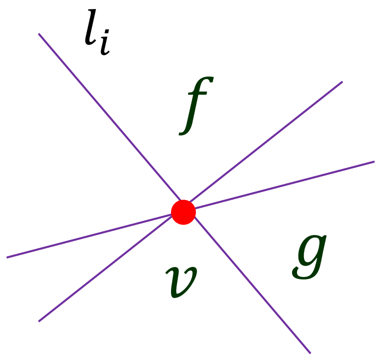

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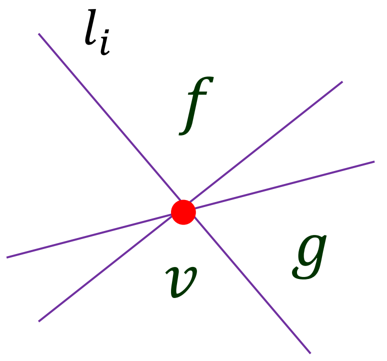

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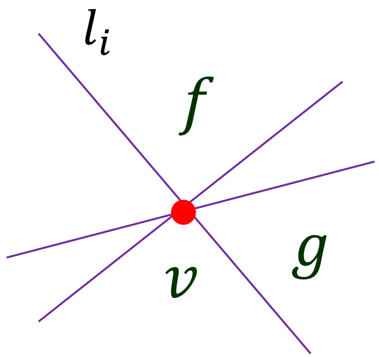
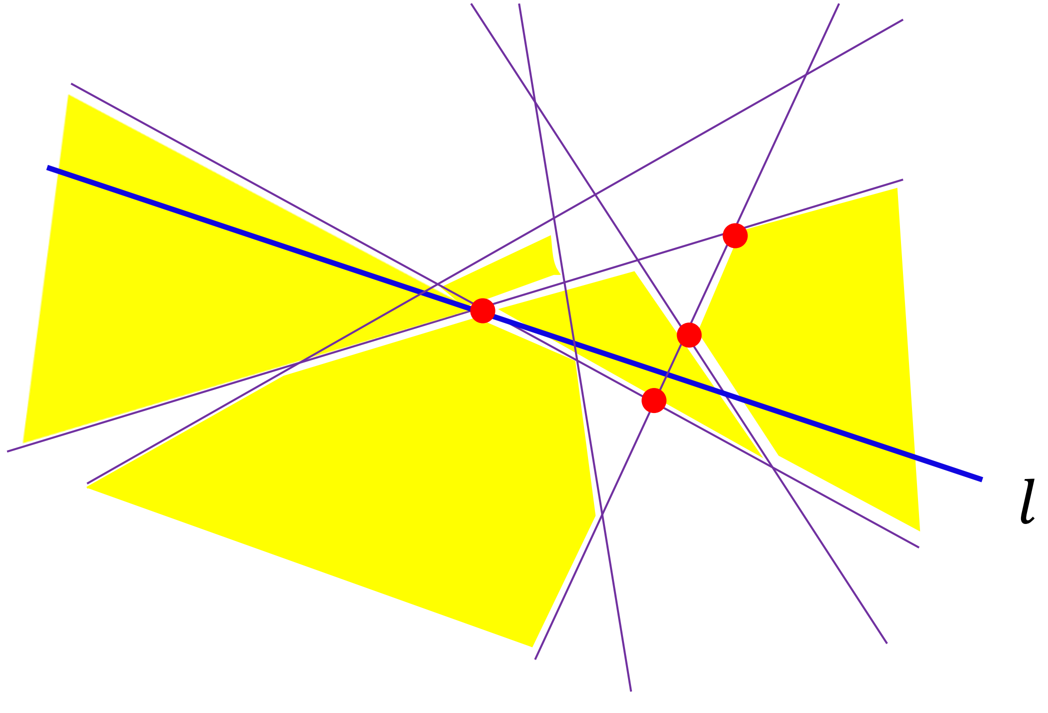


Zone

$Z(l)$ = { f | f is a face of $A(L)$ intersected by l }



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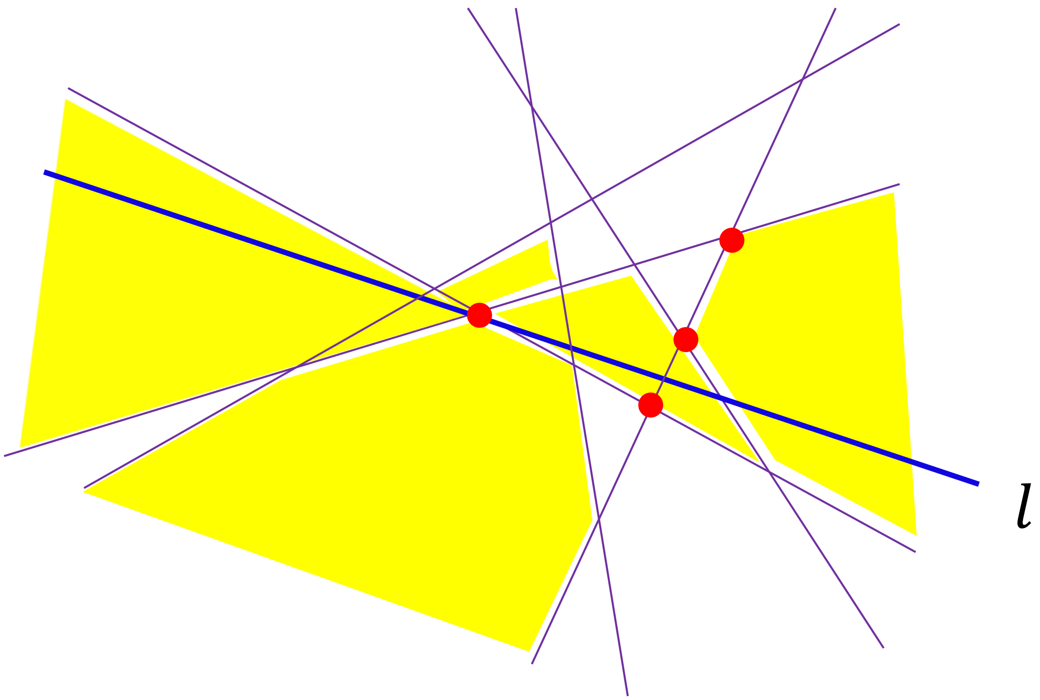

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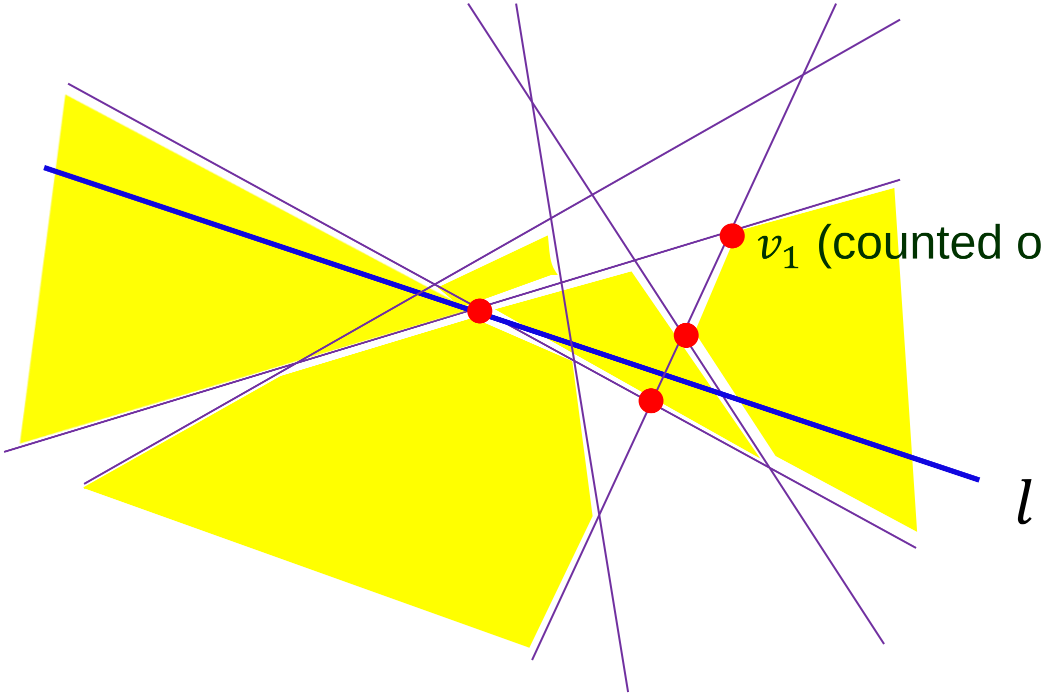

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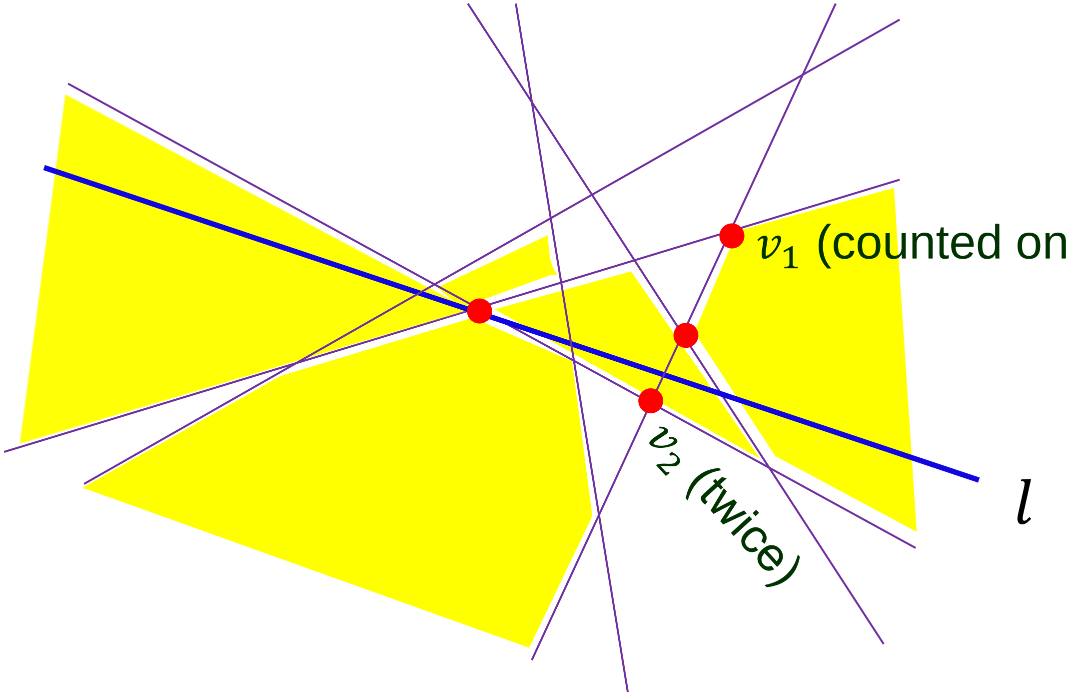

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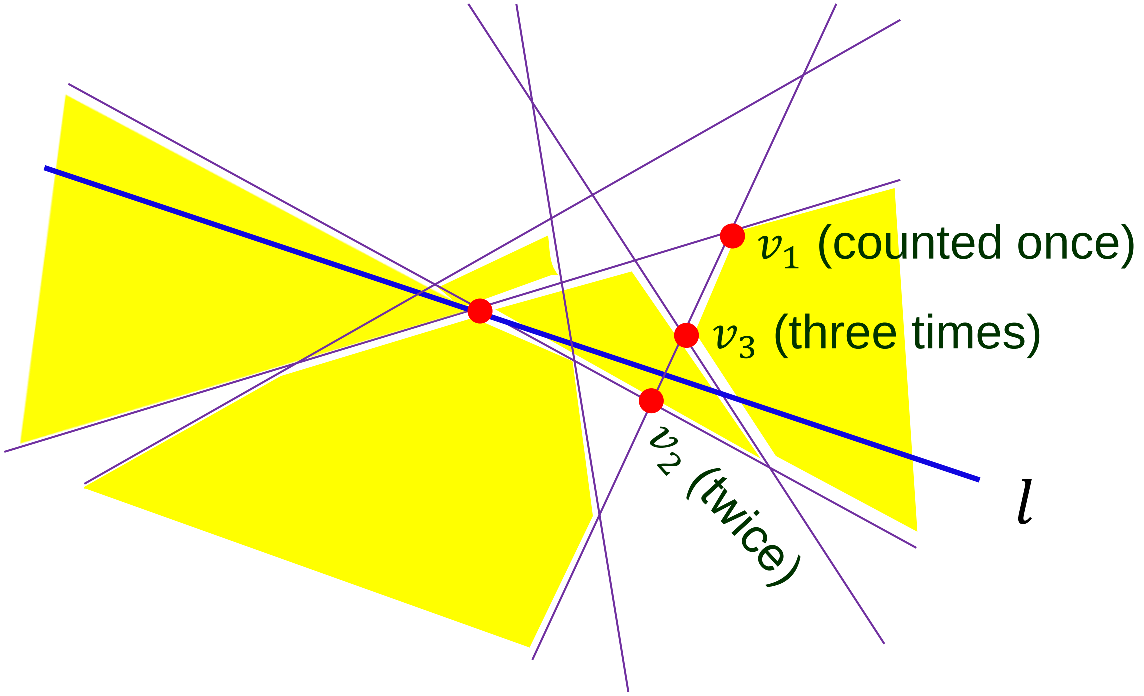

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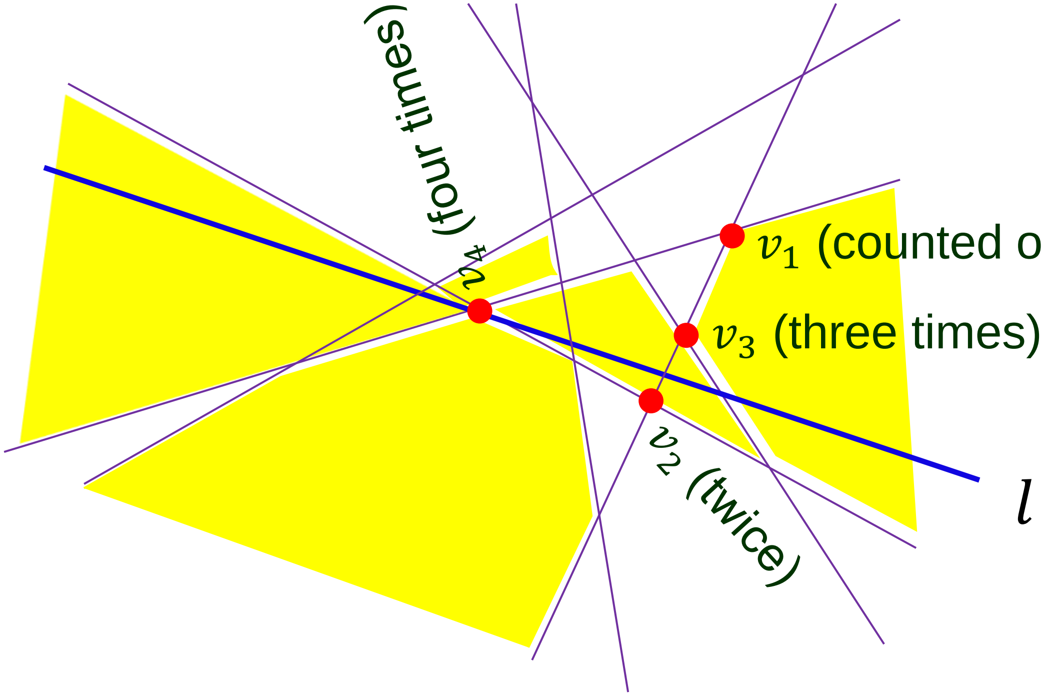

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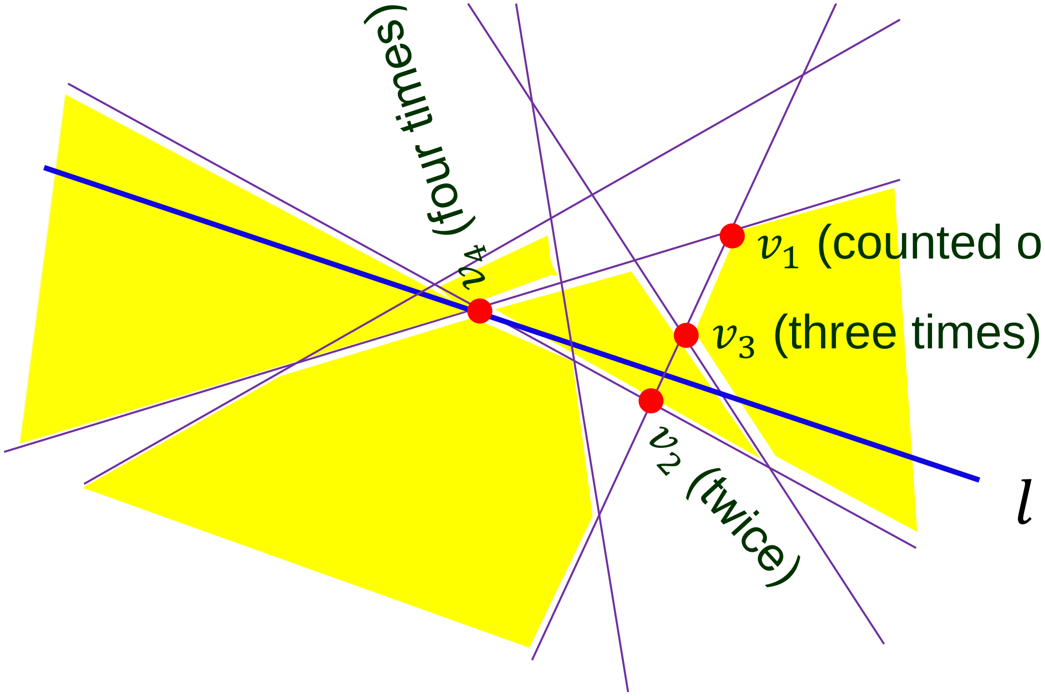

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A vertex may be counted ≤ 4 times.

Time of Arrangement Construction

Zone Theorem Complexity of zone in an arrangement of m lines is $O(m)$.

Proof By induction.

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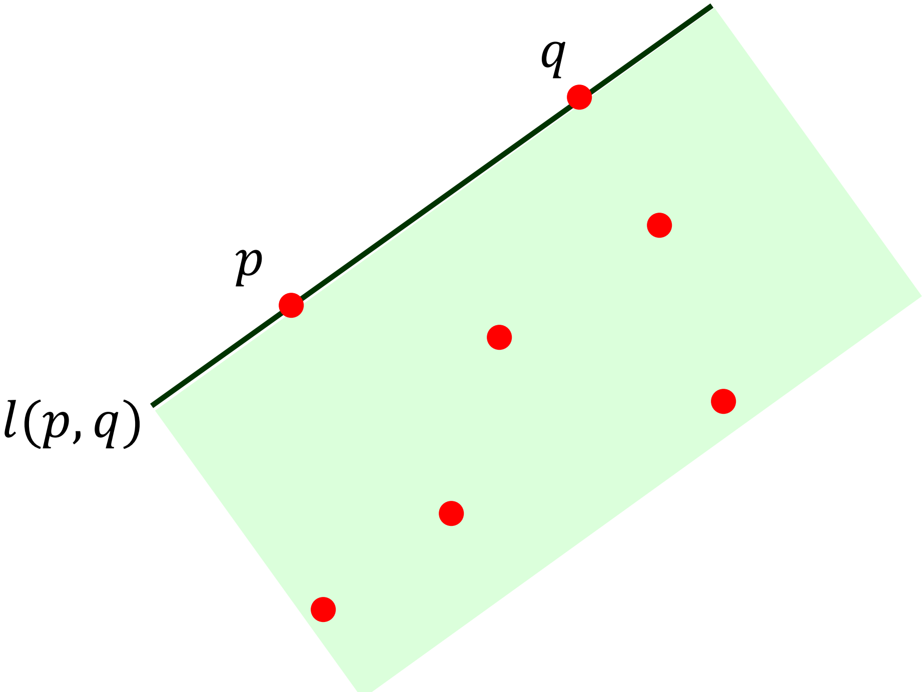
Time to insert all lines, and thus to construct line arrangement:

$$\sum_{i=1}^{n} O(i) = O(n^2)$$

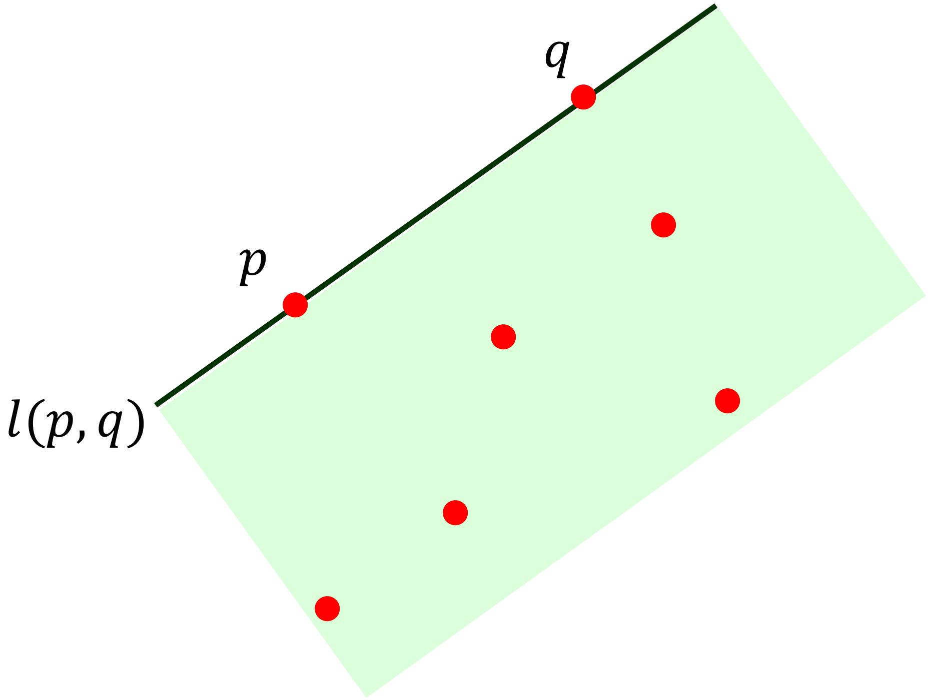
III. Discrete Measure

Primal plane



III. Discrete Measure

Primal plane



points below $l(p,q)$

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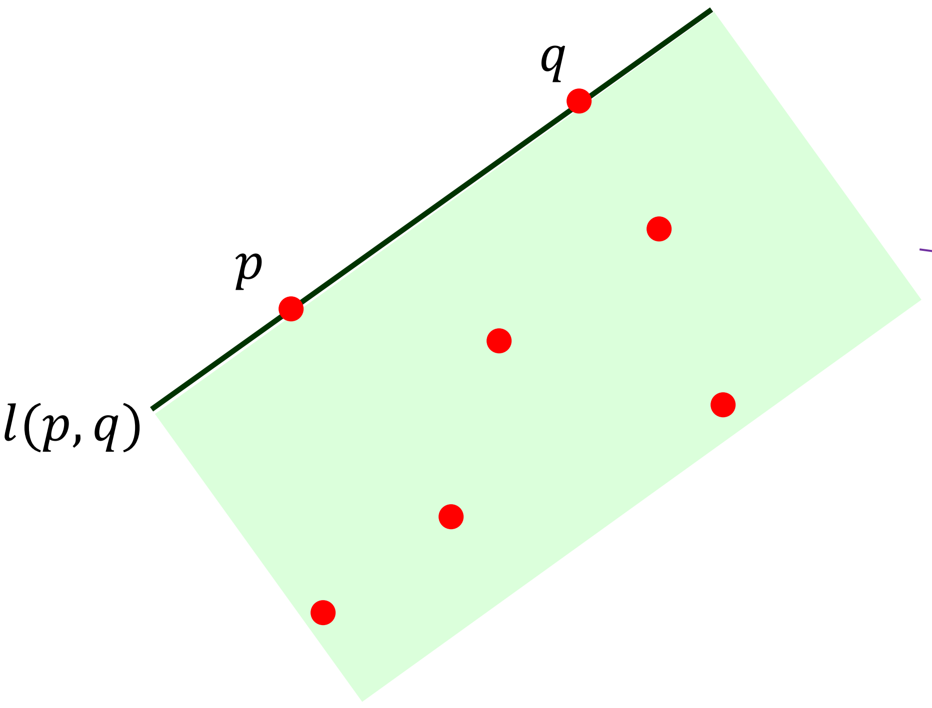
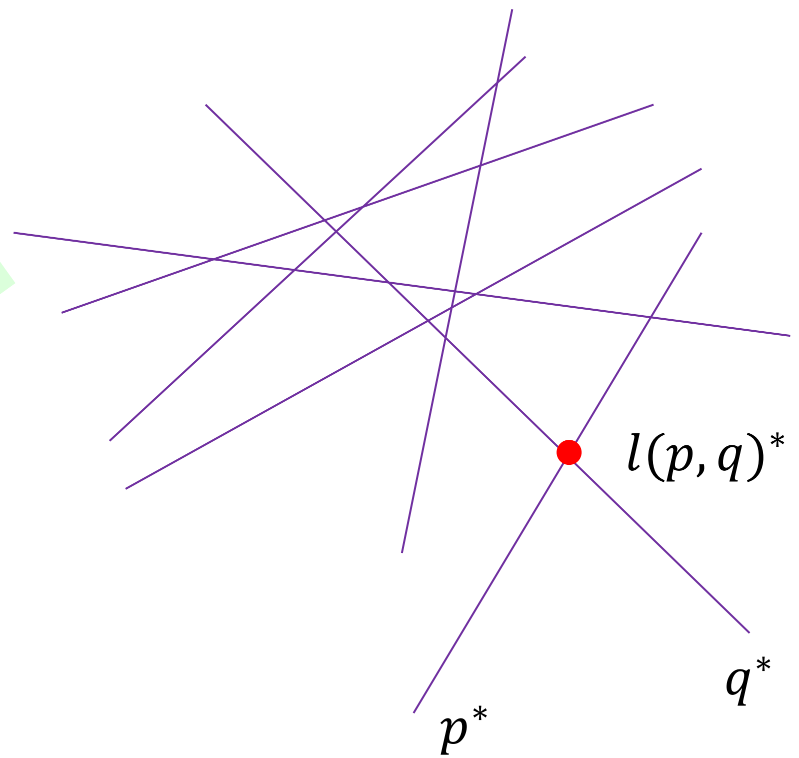


Dual plane



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$\Leftrightarrow$ lines strictly above dual point $l(p,q)^*$

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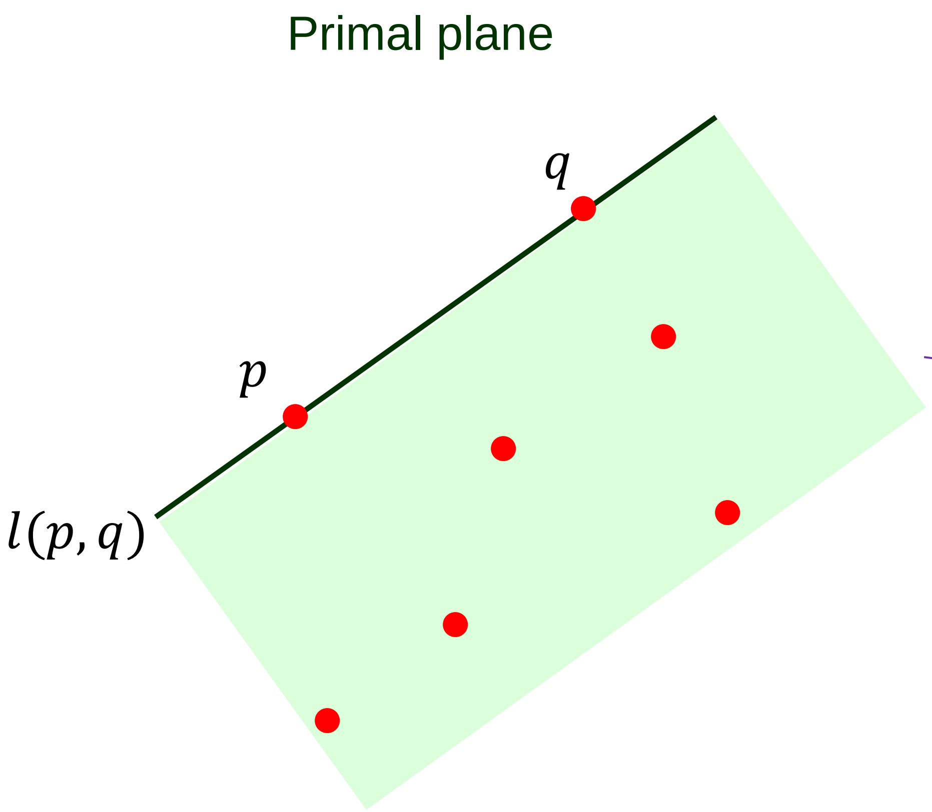

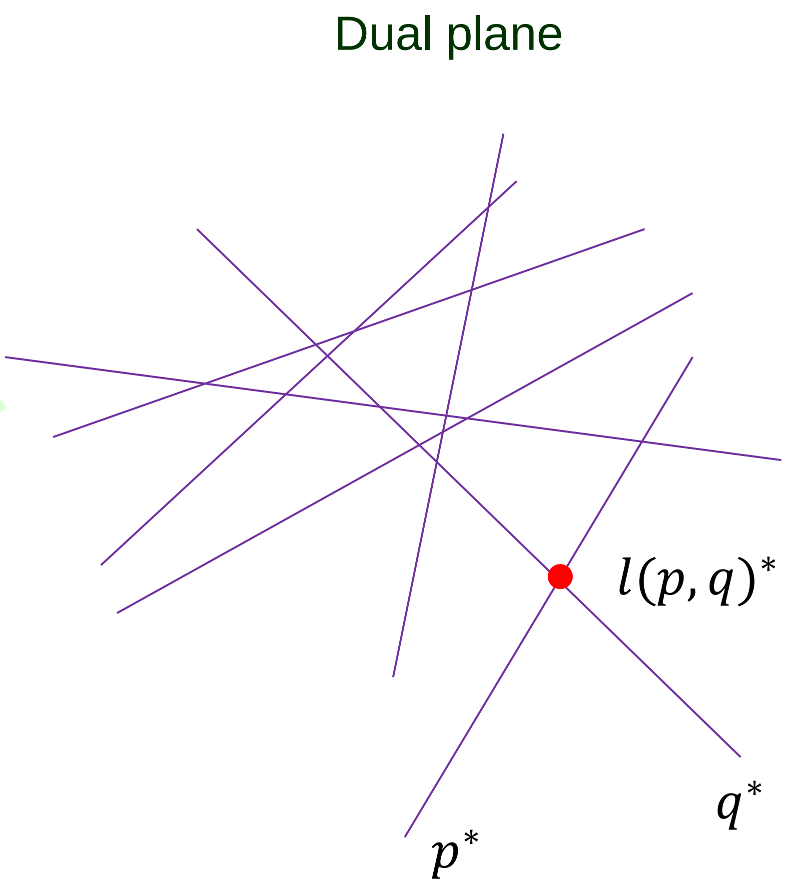

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Efficient algorithm exists!

How to Use Duality?

A set S of n sample points

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Because p lies above l iff l^* lies above p^*, the discrepancy problem reduces to the following:

Problem For every vertex in $A(S^*)$, compute the numbers of lines above it, passing through it, and below it.

Reduction

$n_a =$ # lines above a vertex

$n_b =$ # lines below the vertex

$n_o =$ # lines through the vertex

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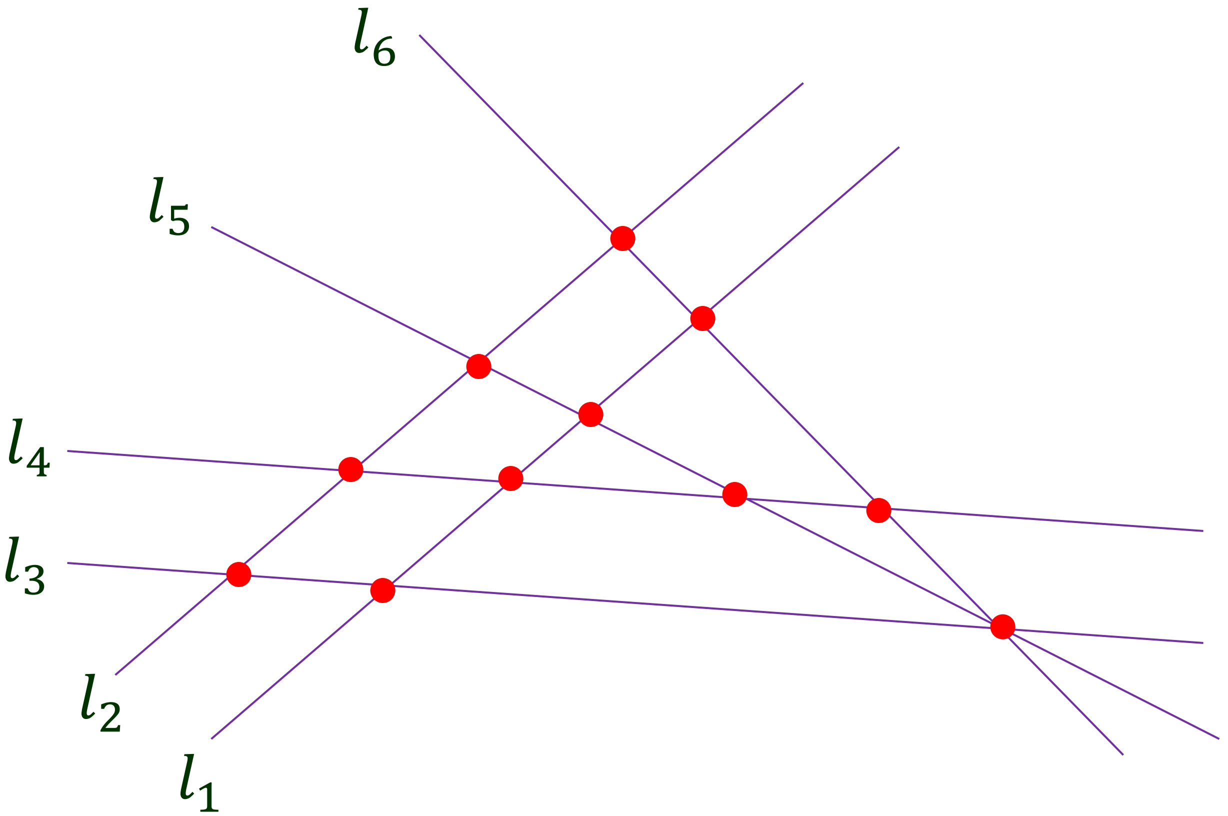
Need only compute n_a of every vertex in $A(S^*)$.

Levels of Vertices in an Arrangement

level of a point = # lines strictly above it.

A line l is *above* a point p if its intersection with the vertical line through p is above p.



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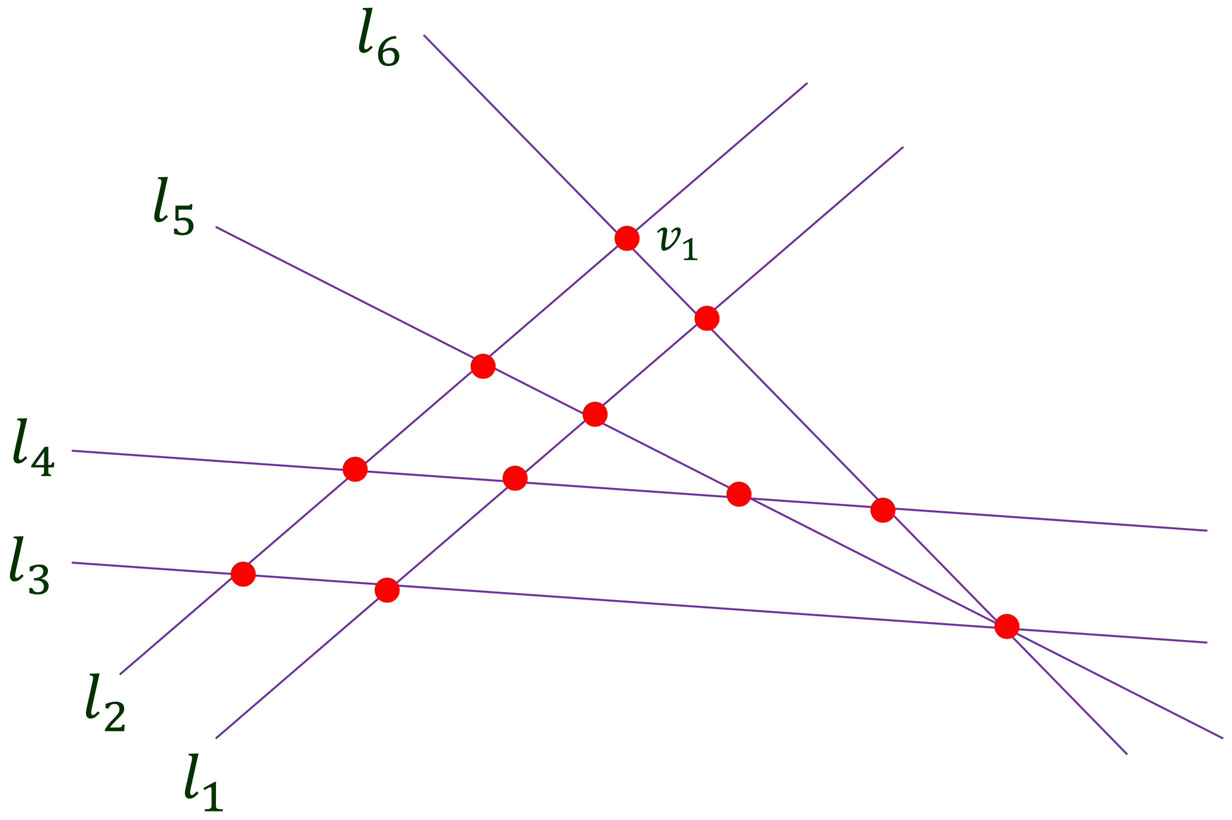

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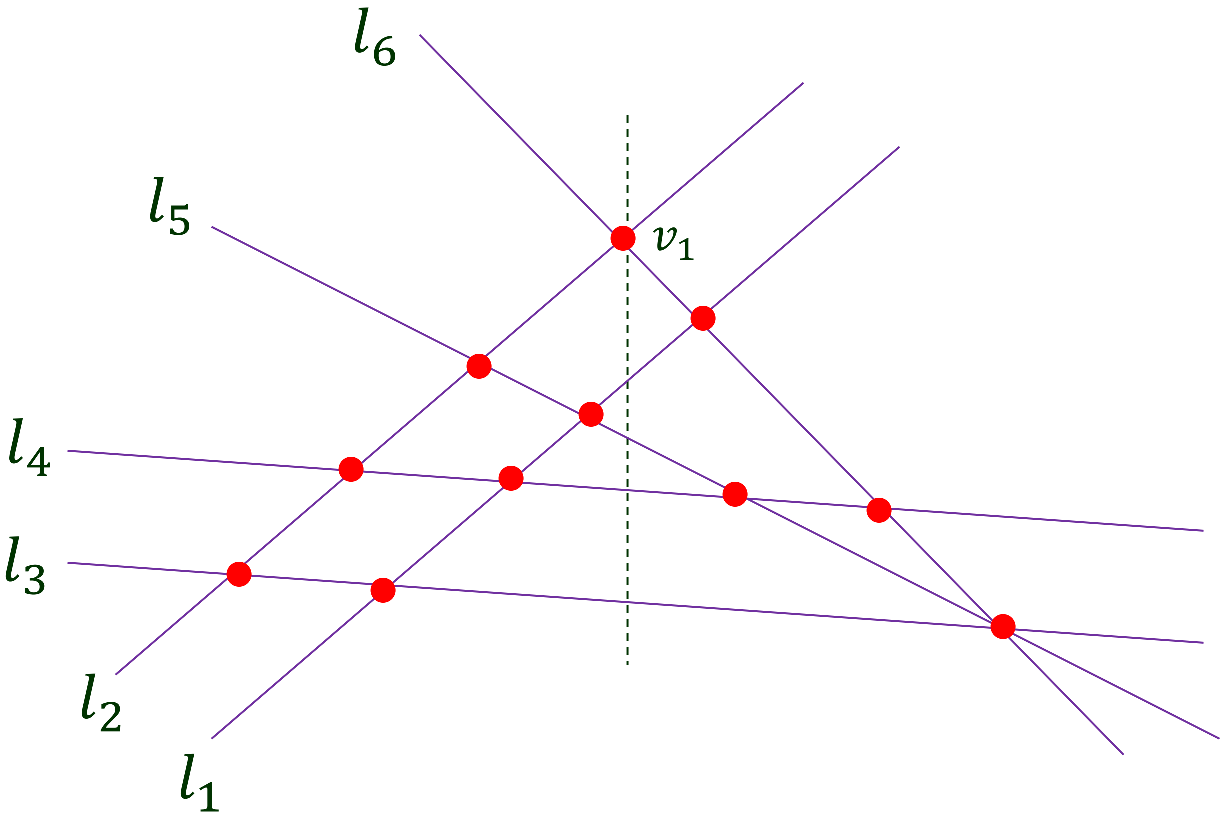

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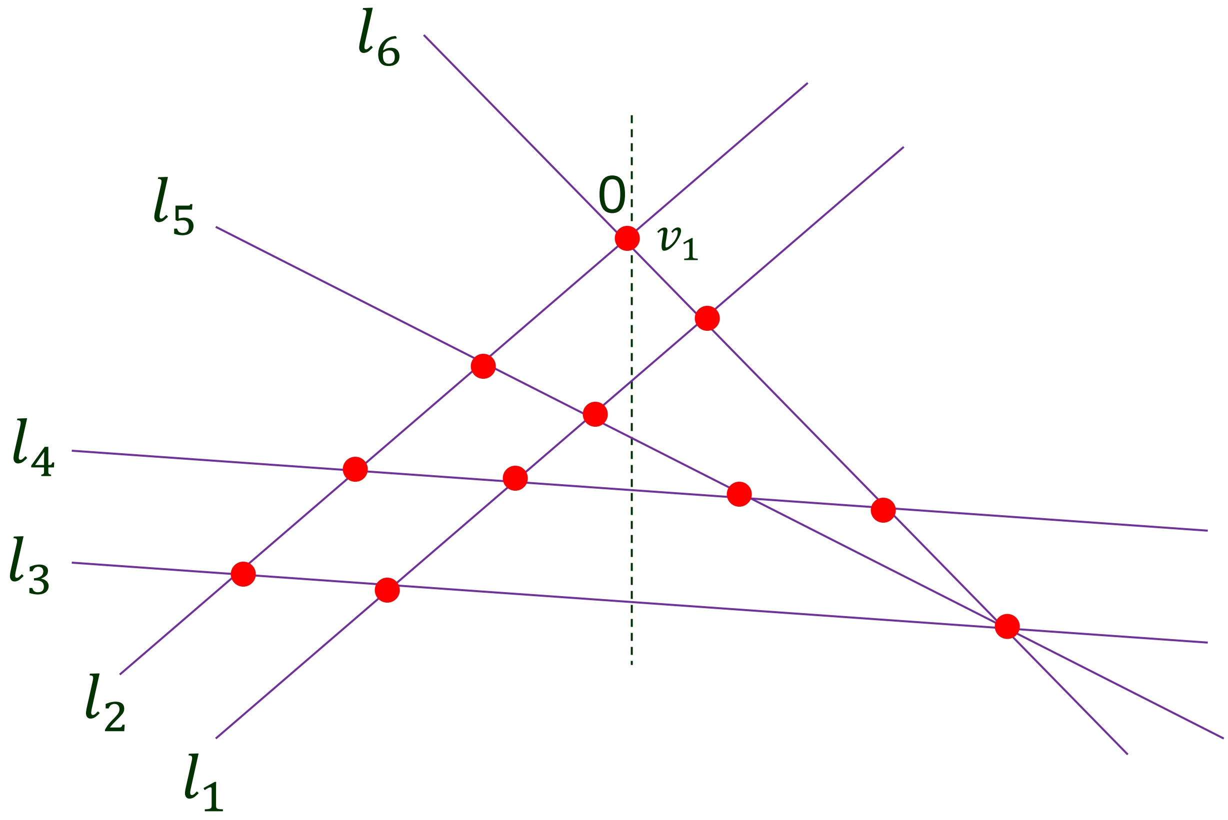

Levels of Vertices in an Arrangement

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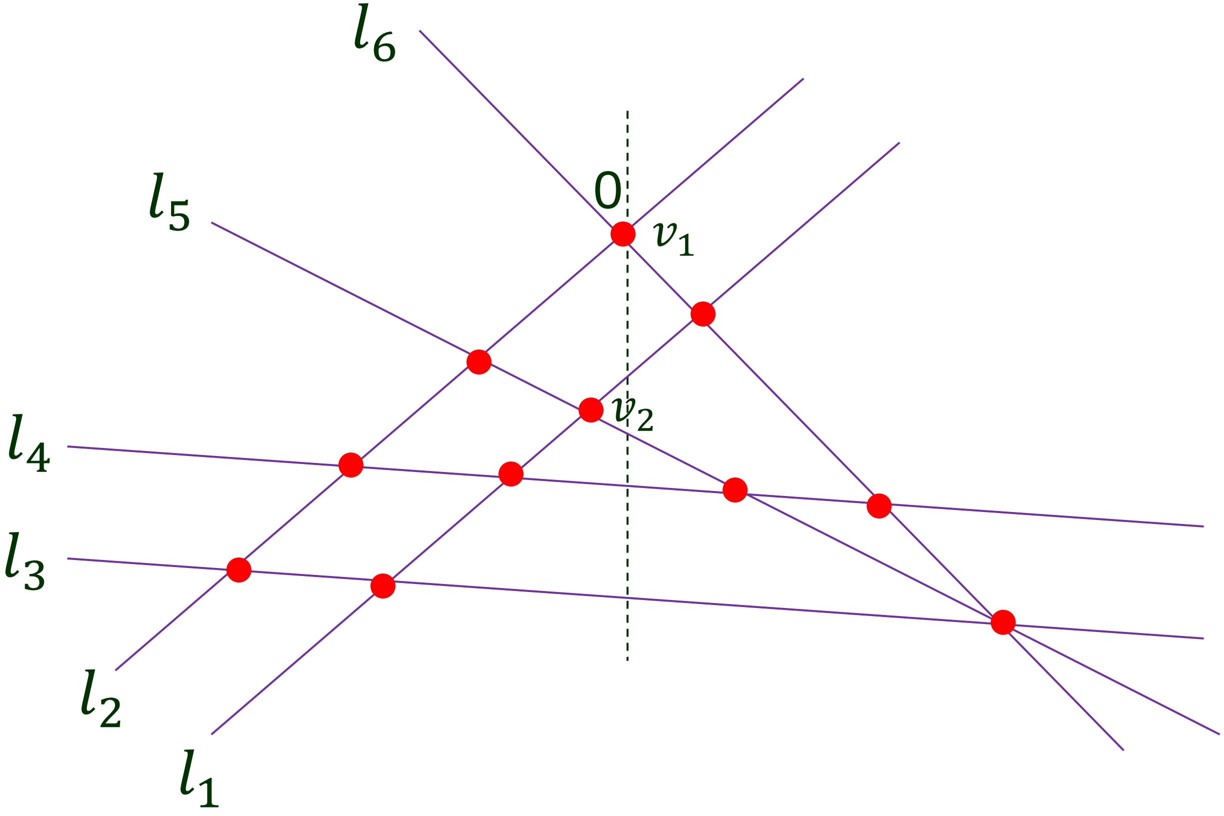

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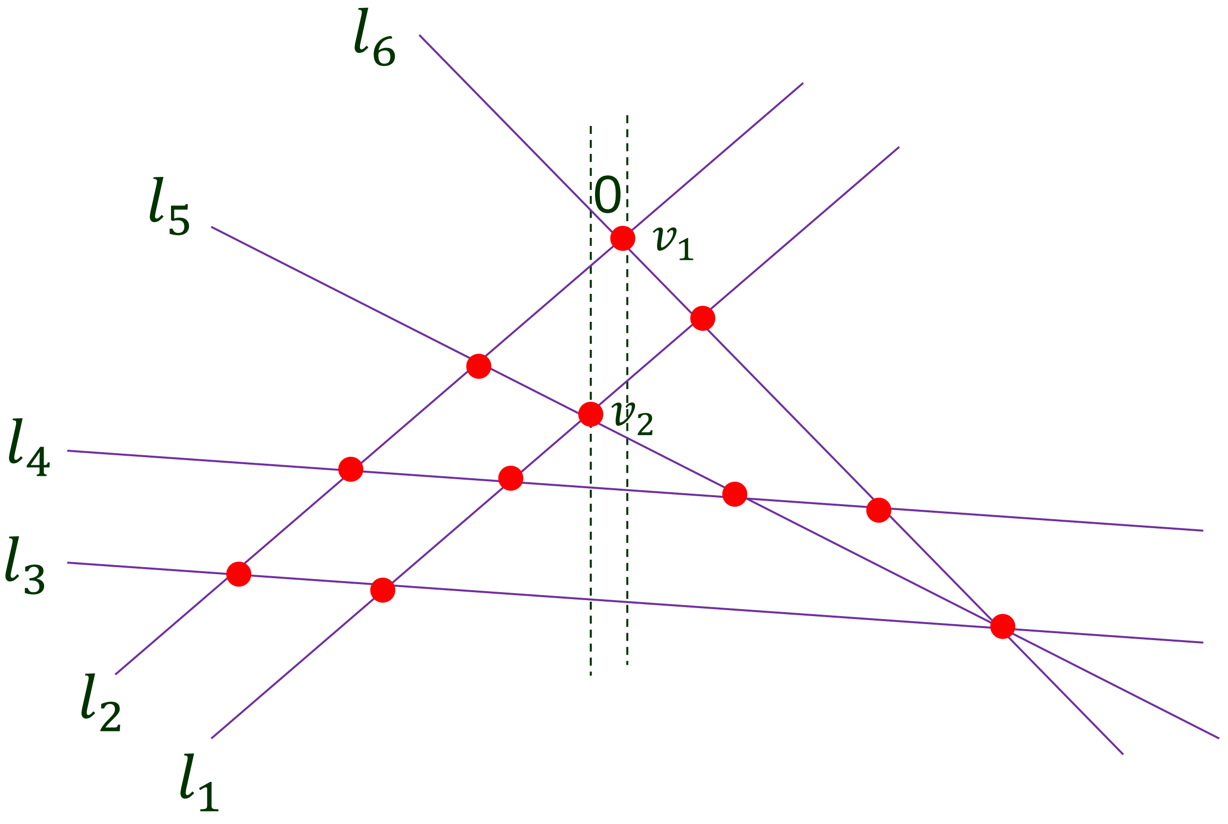

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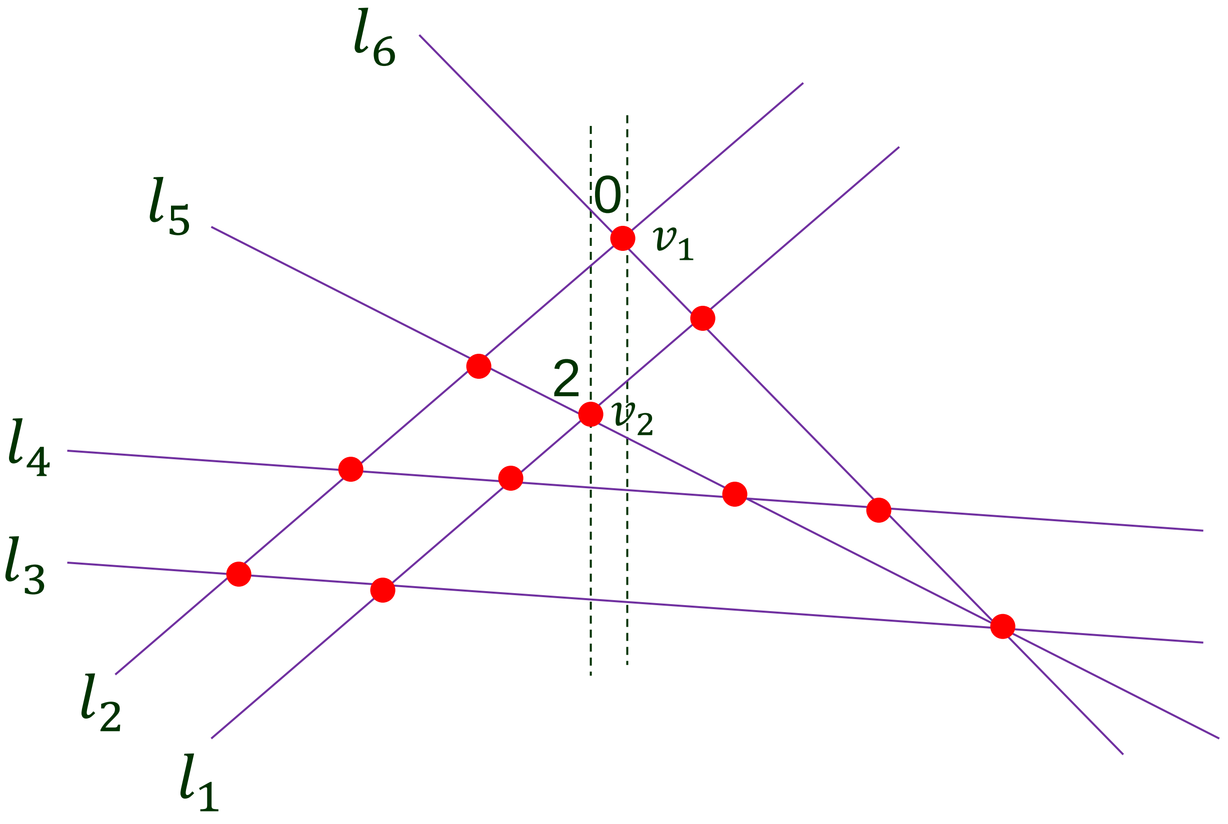

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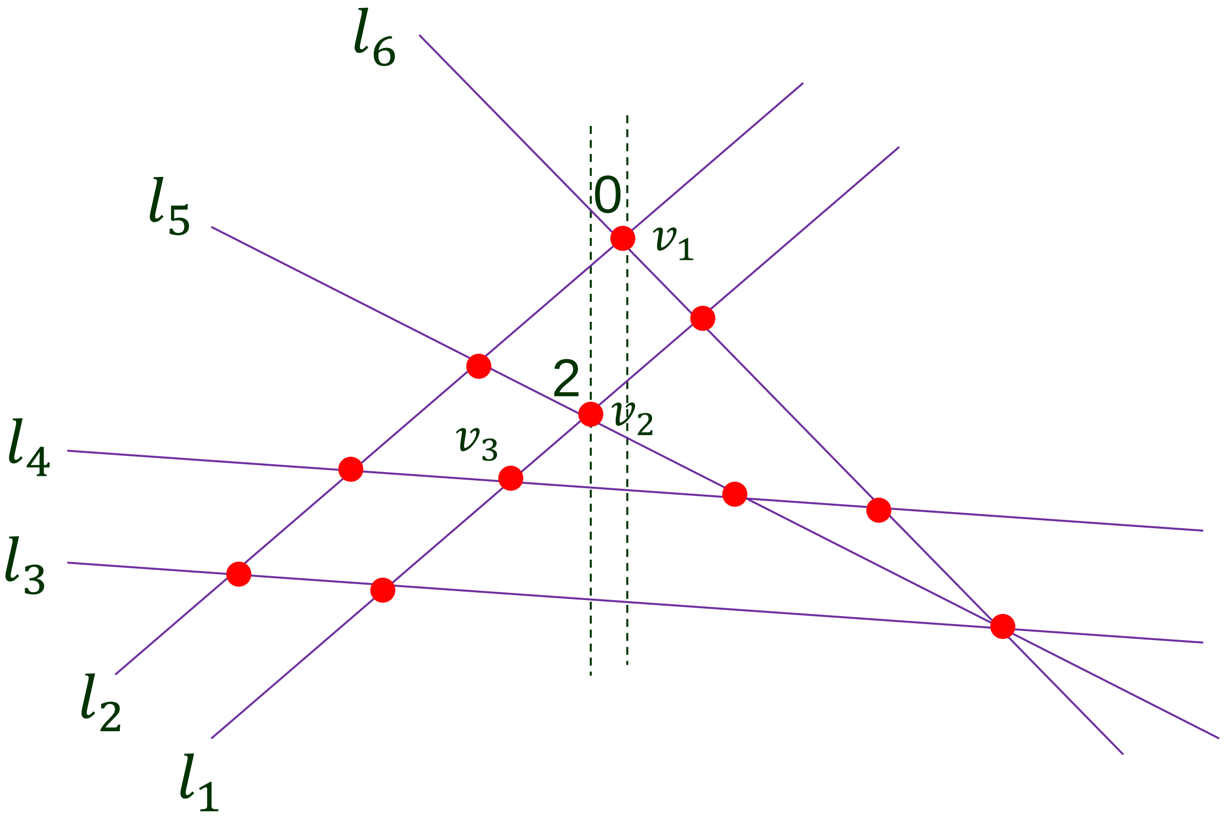

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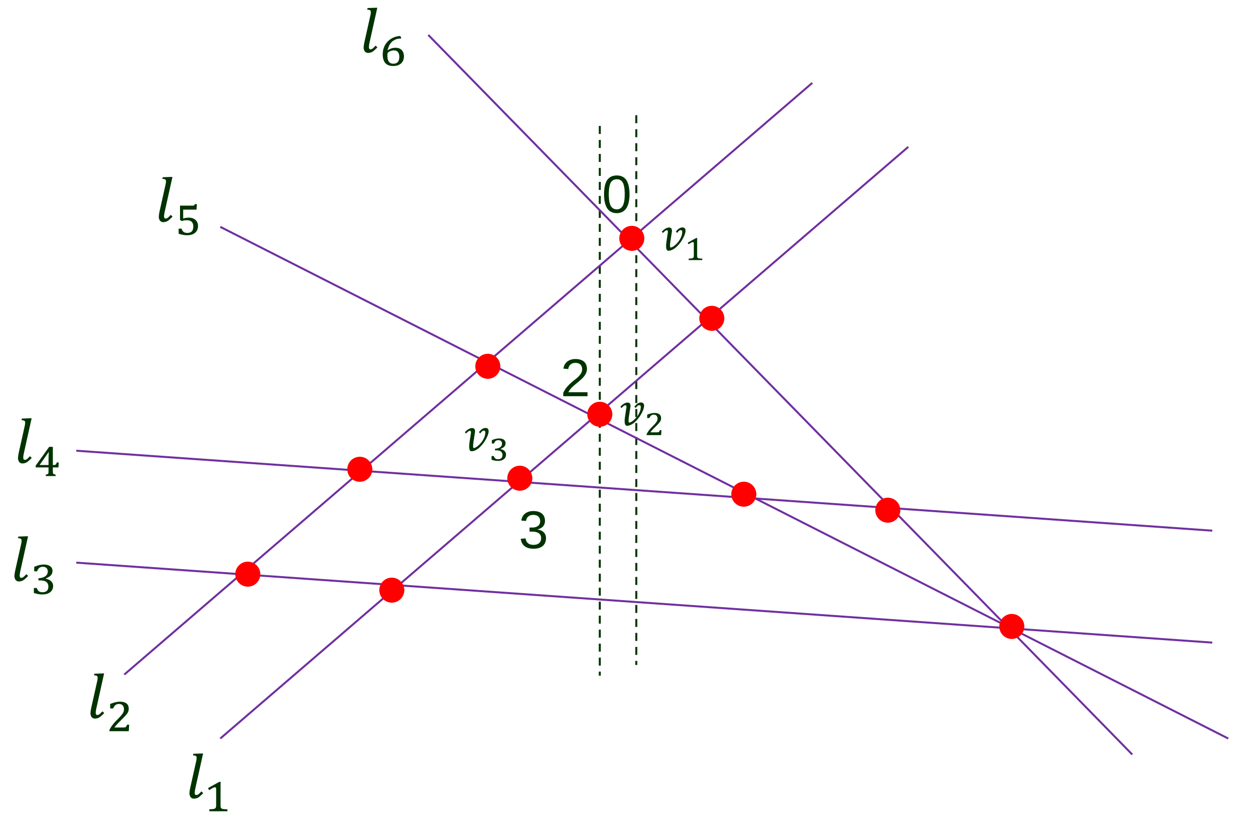

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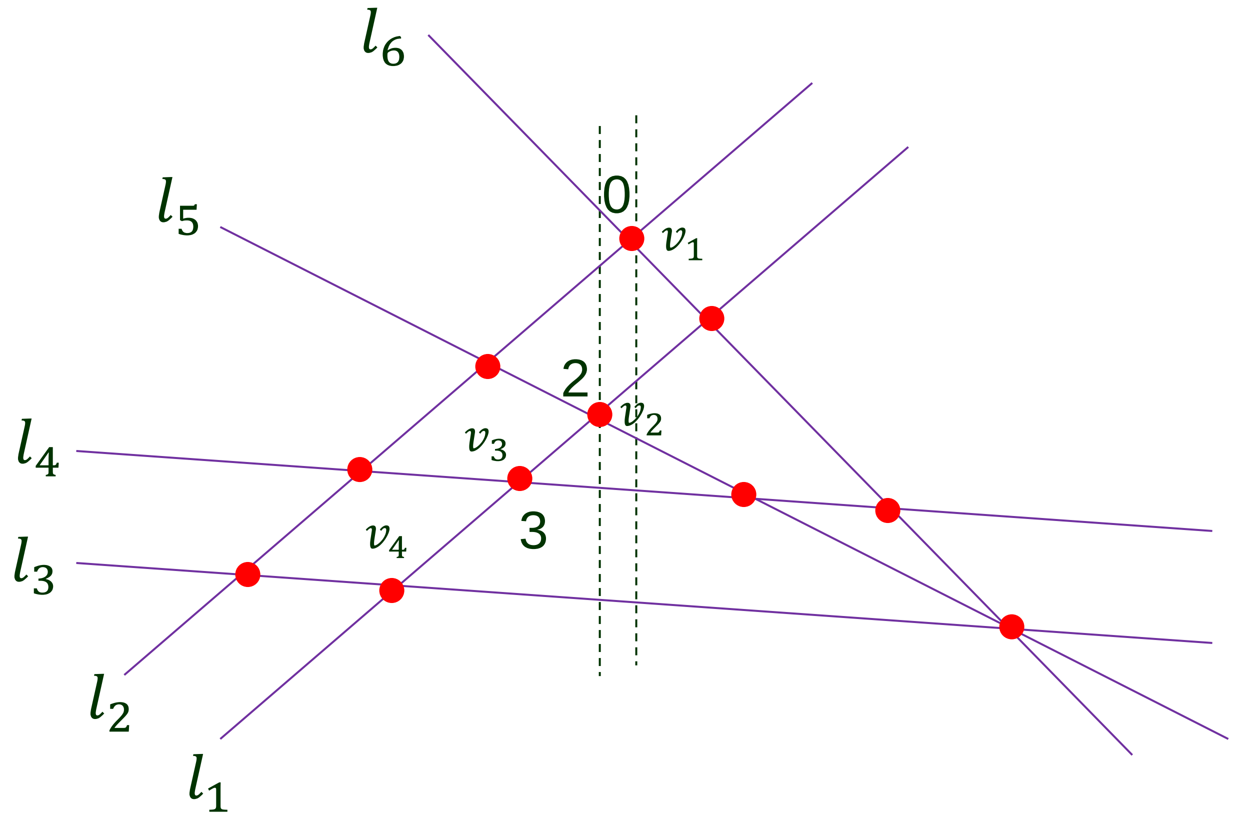

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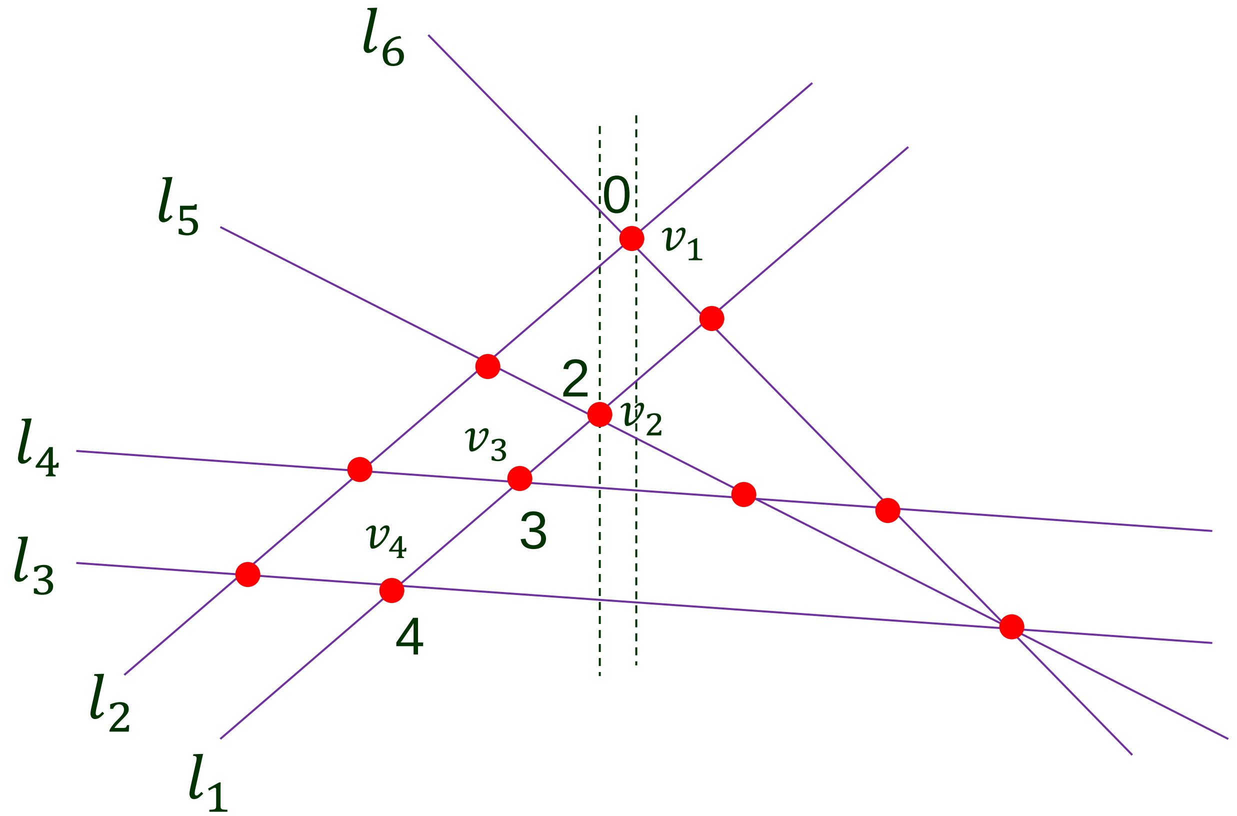

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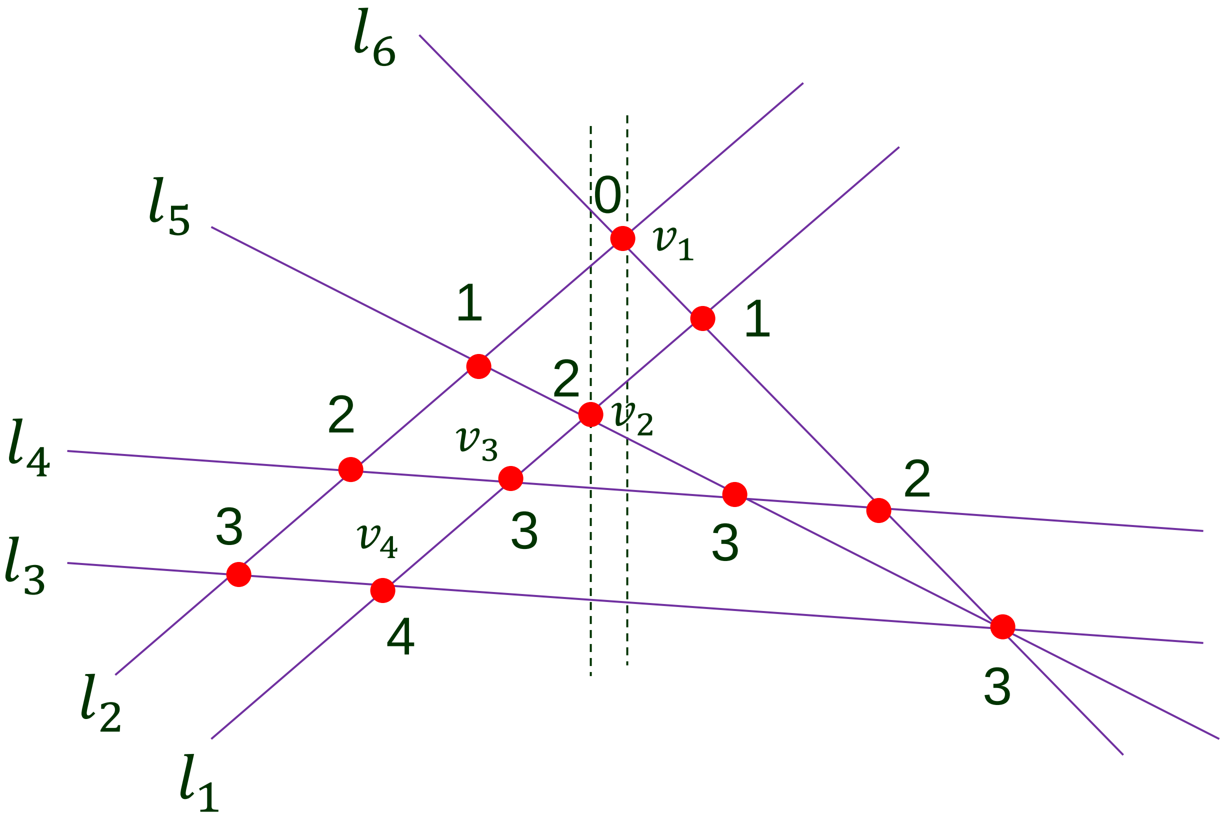
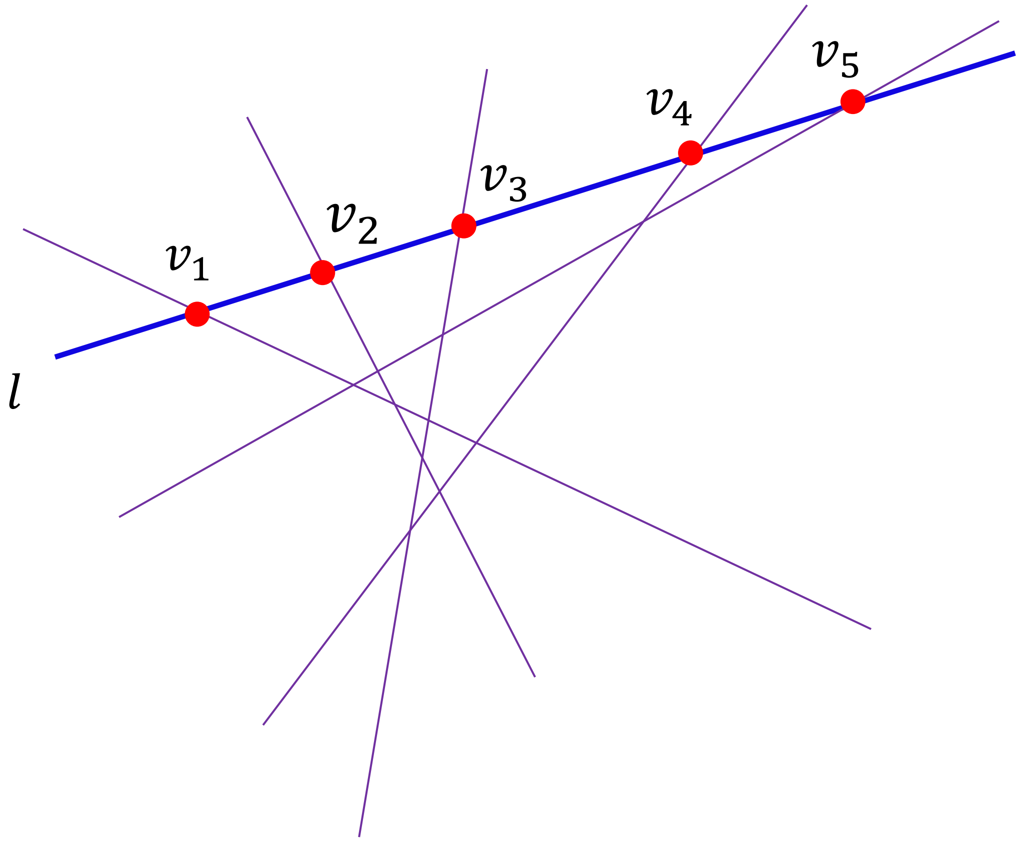


Counting Levels of Vertices



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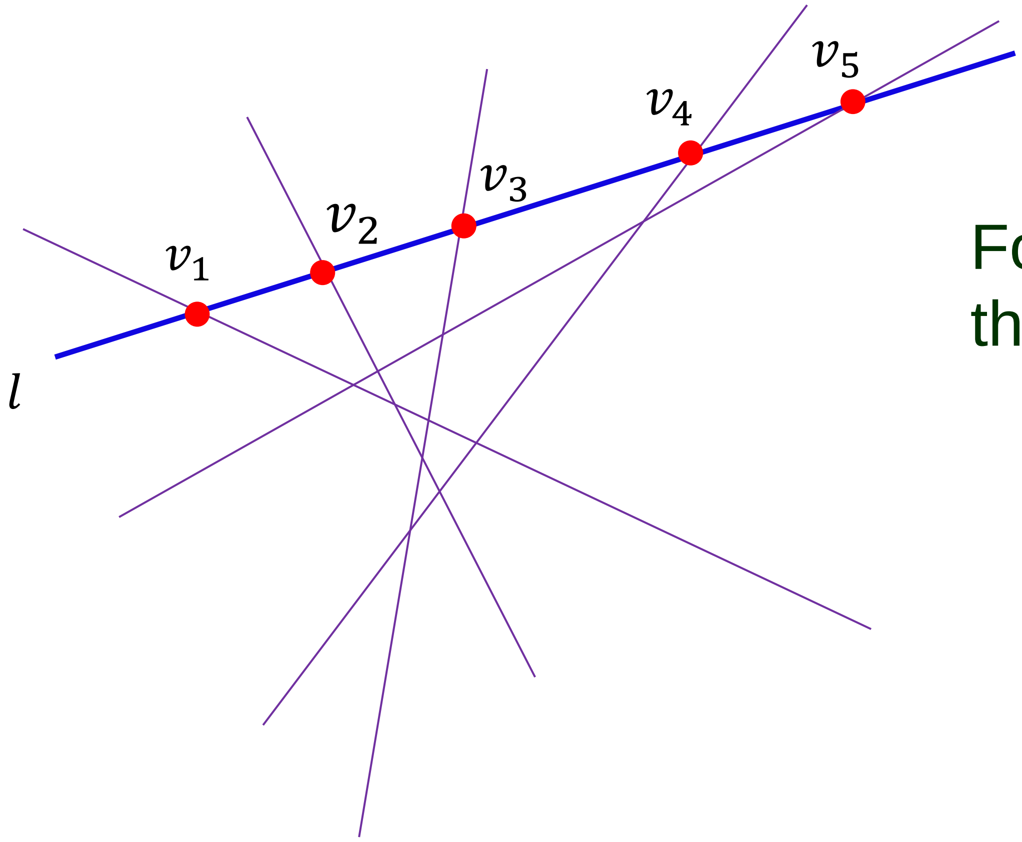

For each line l, compute the levels of vertices on it.

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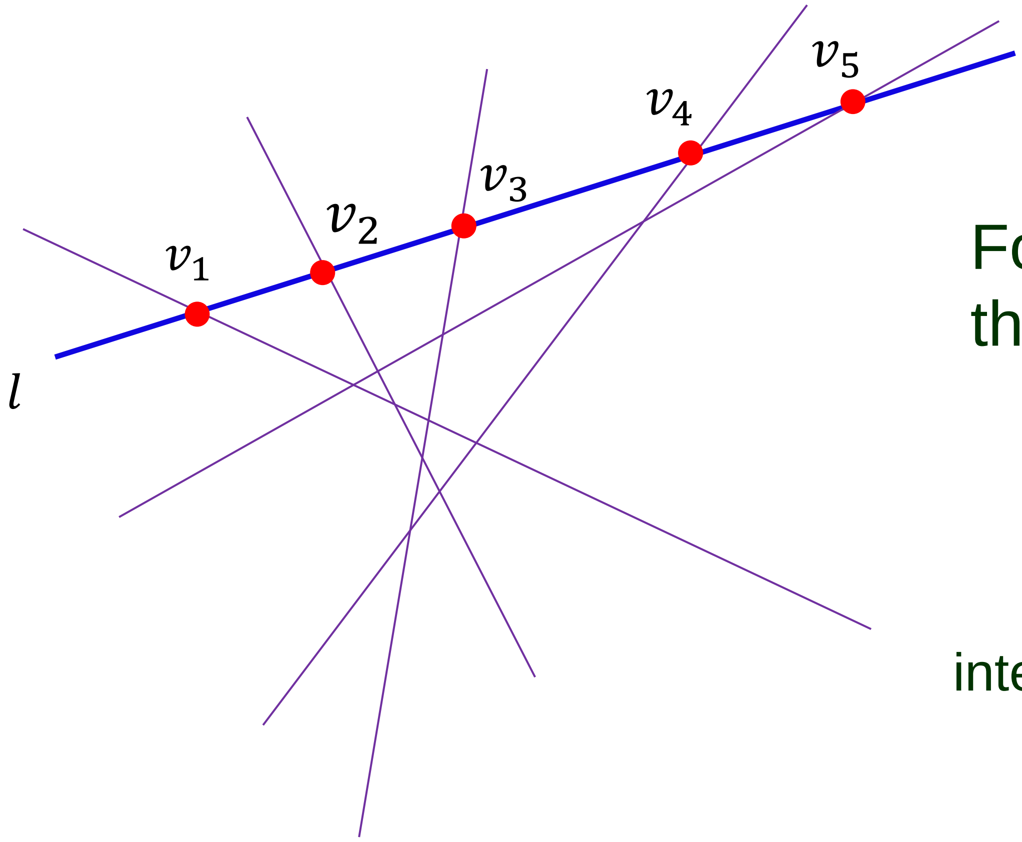

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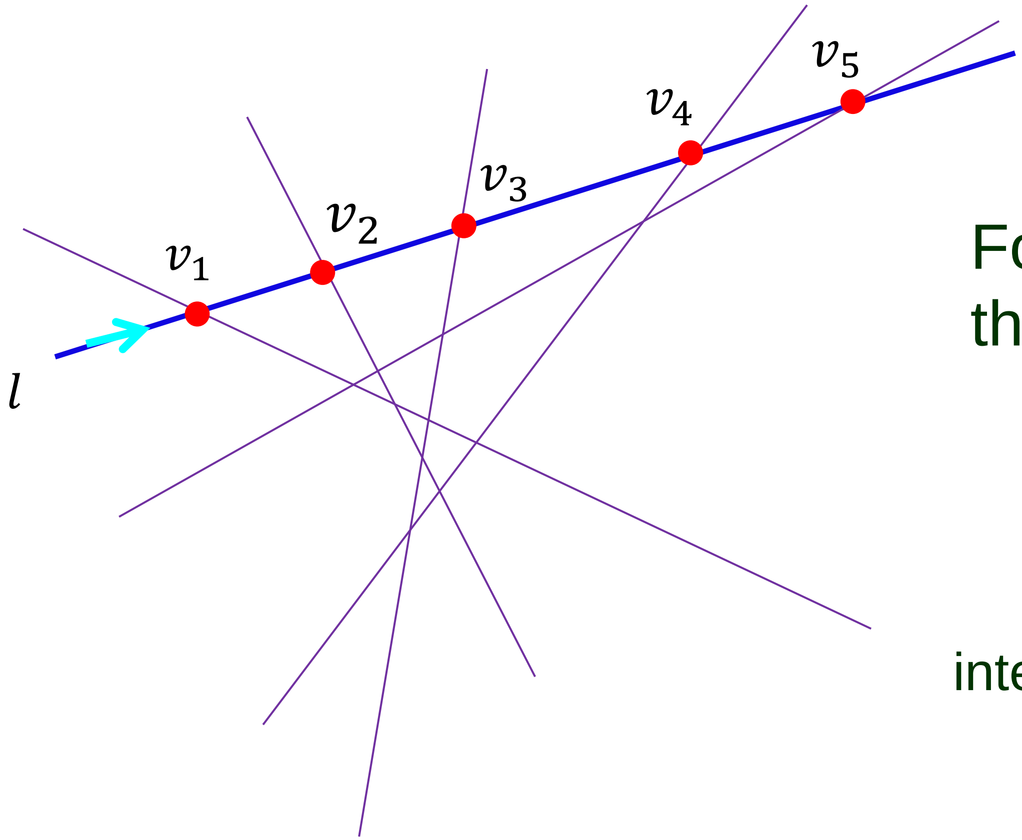

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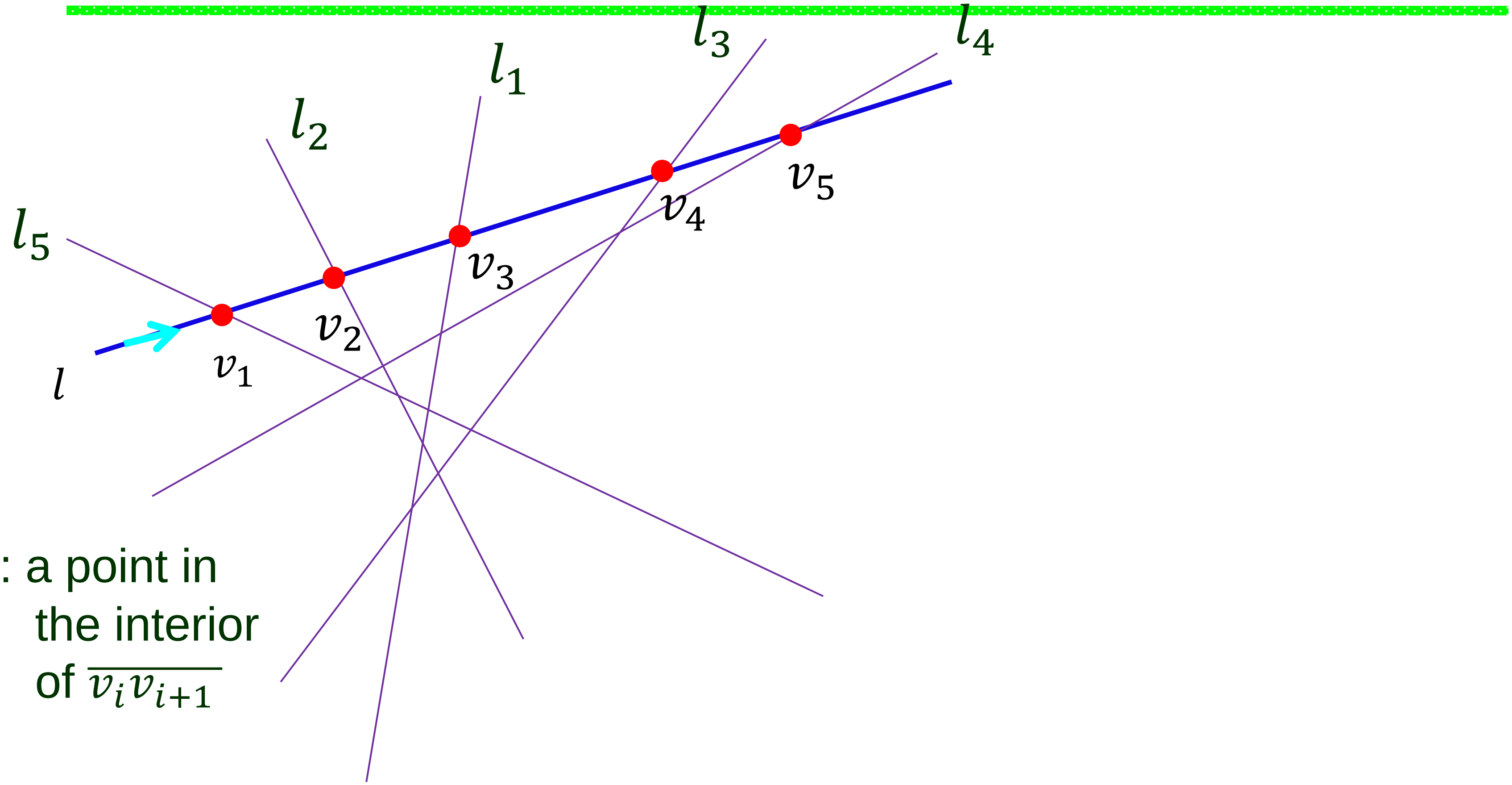
Strategy: walk along l from left to right.

Counting Levels of Vertices



v_i^+: a point in the interior of $\overline{v_i v_{i+1}}$

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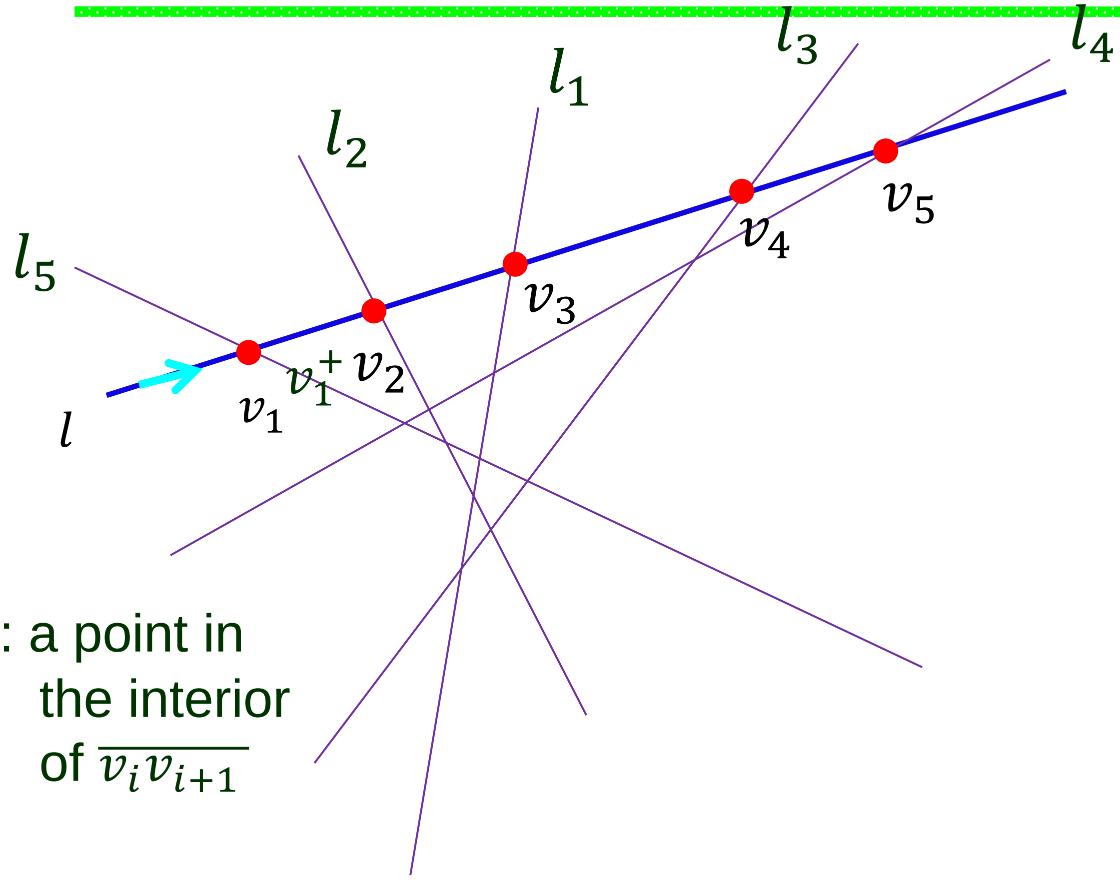

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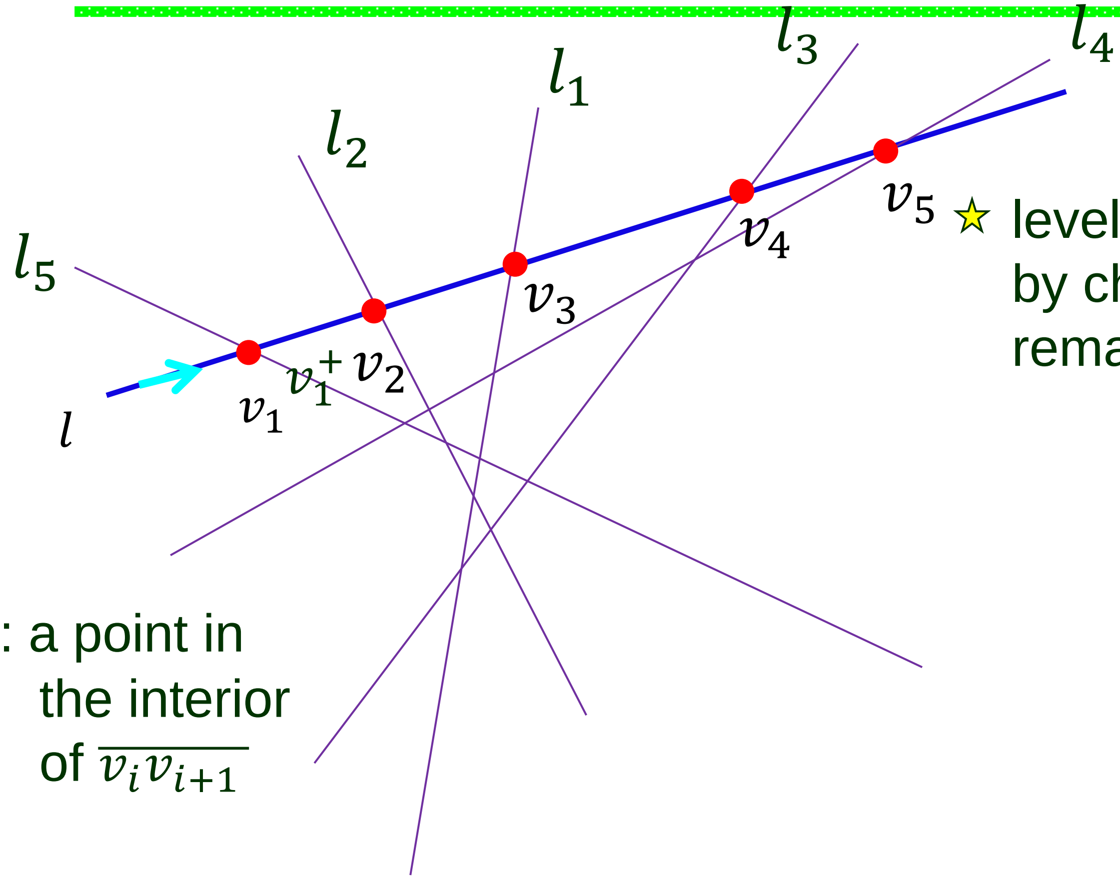

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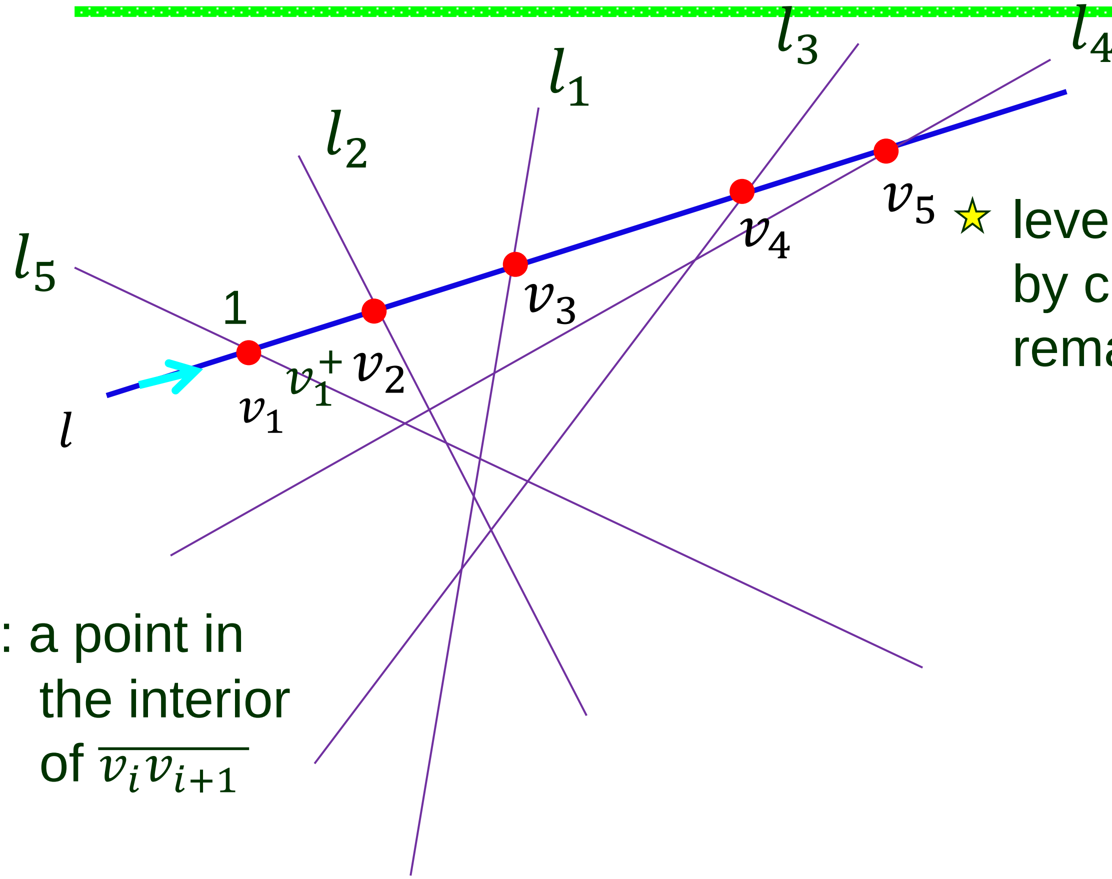

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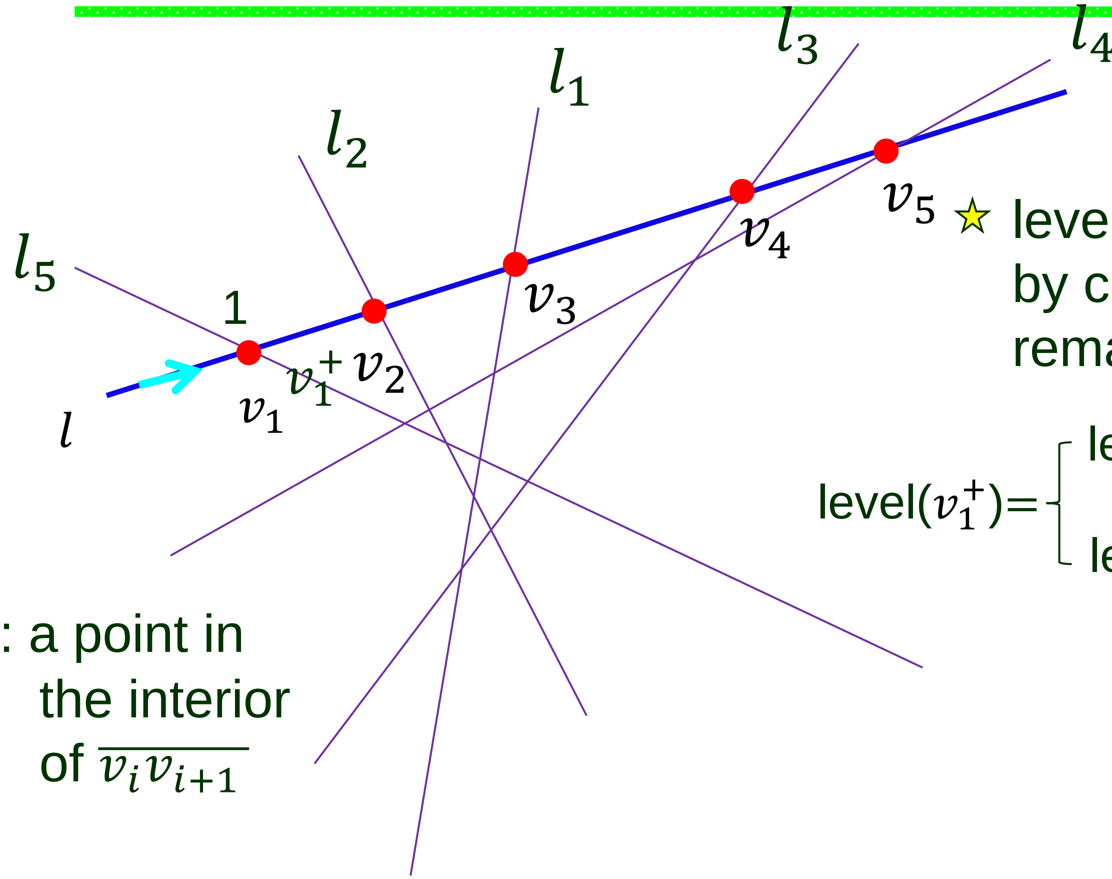

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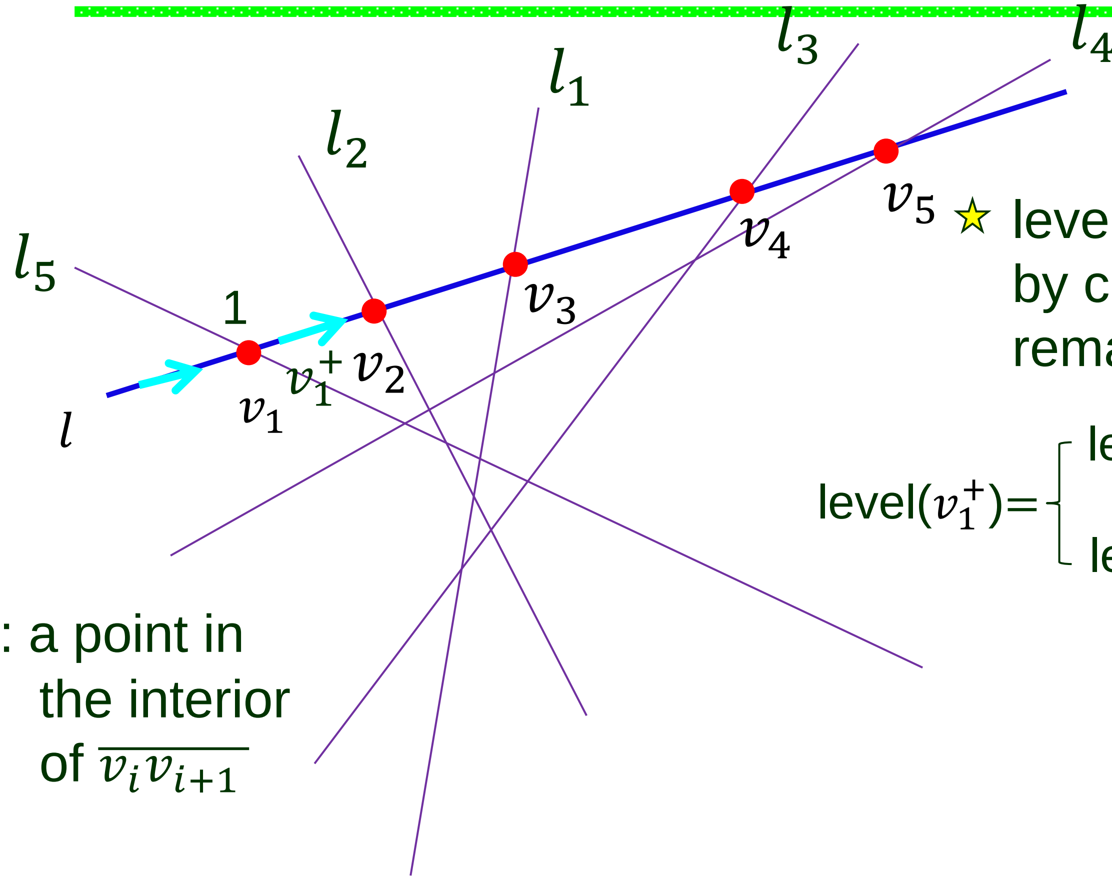

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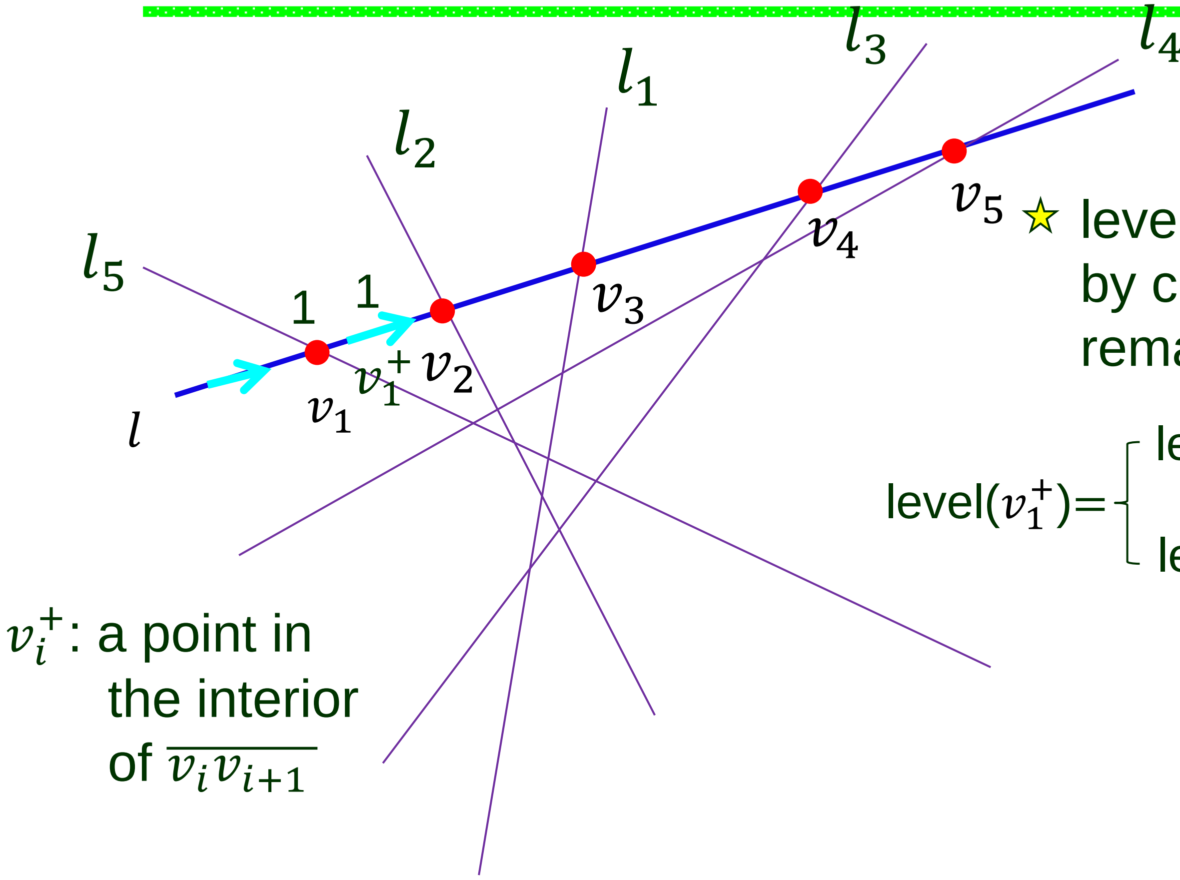

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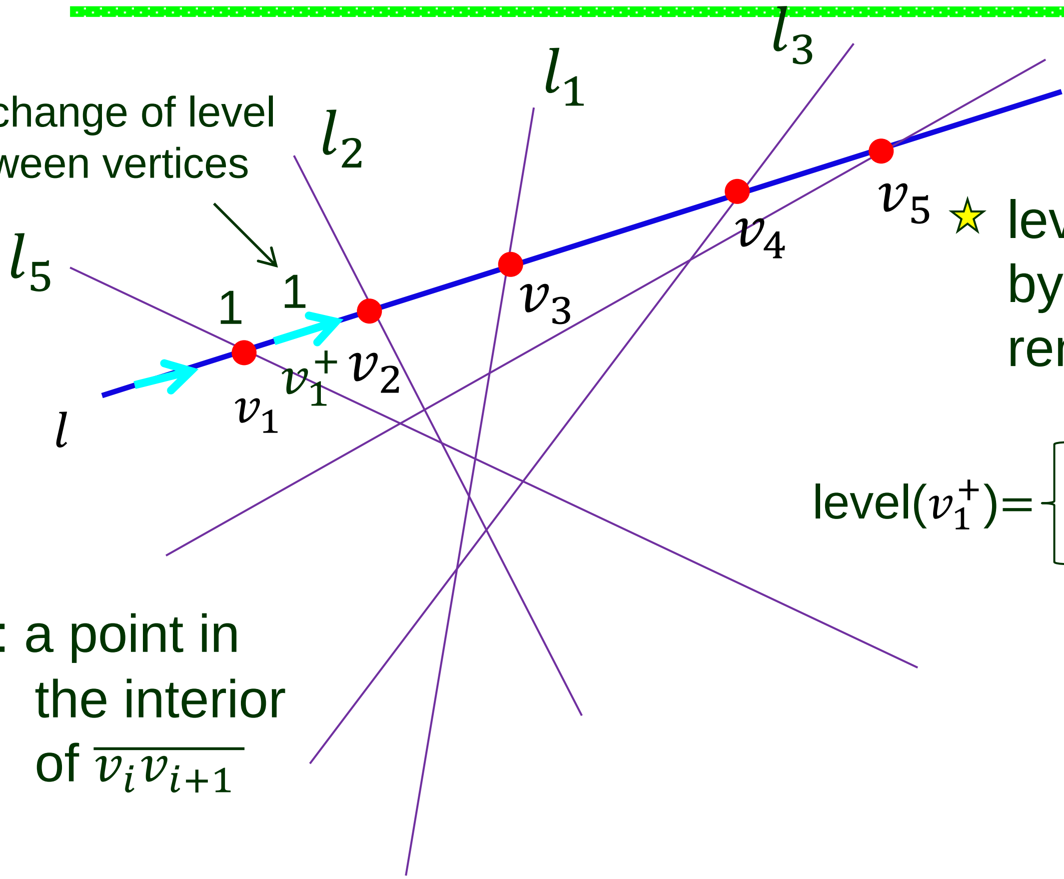
Counting Levels of Vertices

no change of level between vertices



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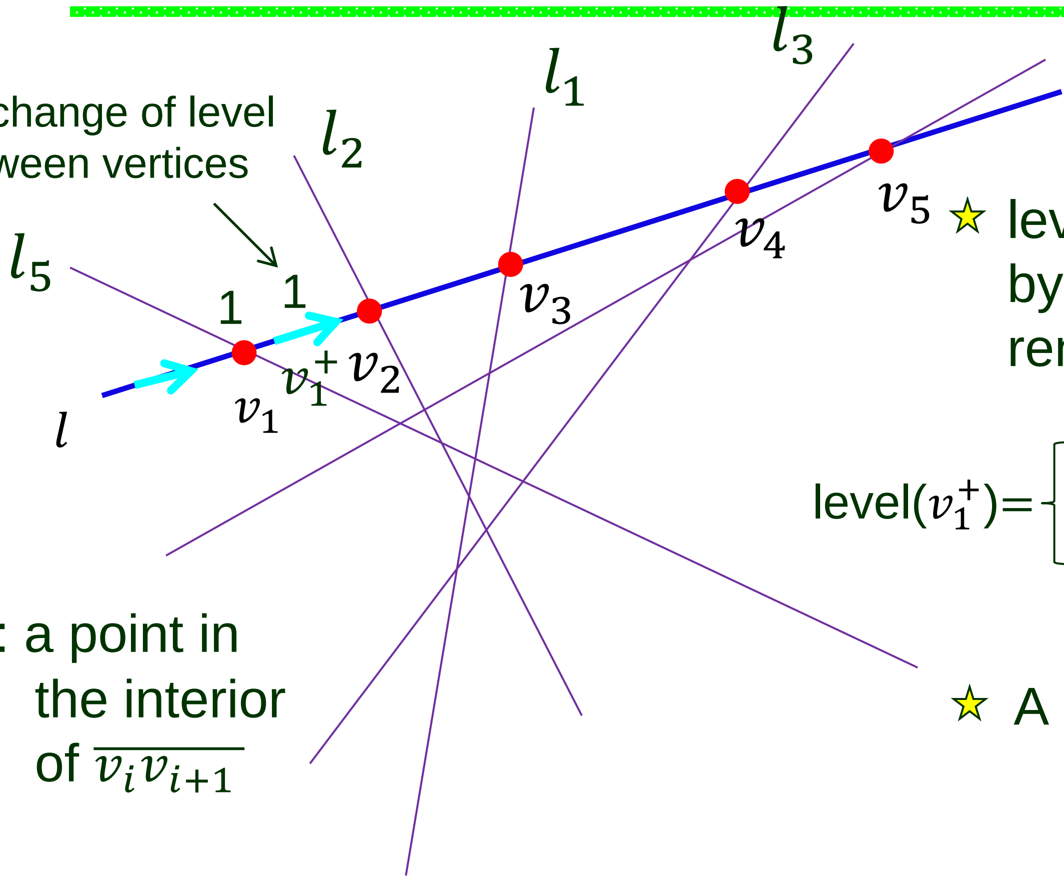

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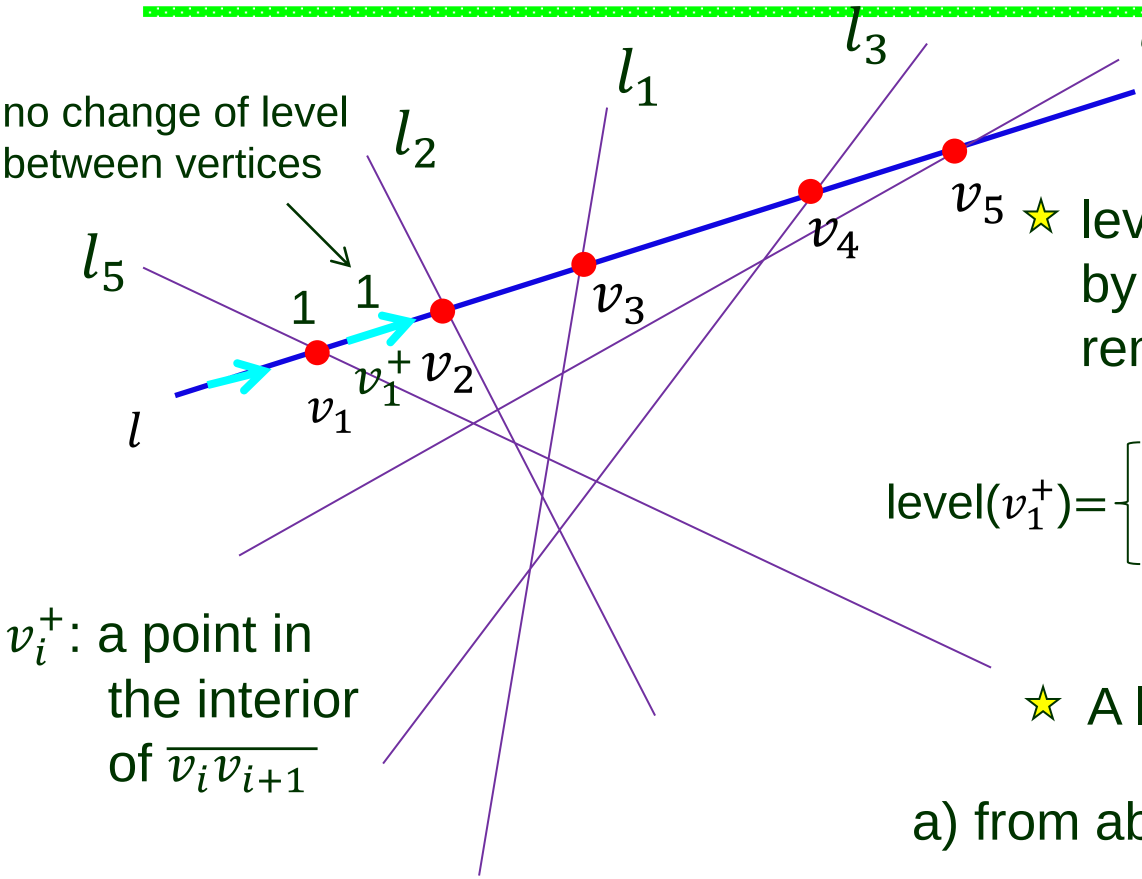
Counting Levels of Vertices



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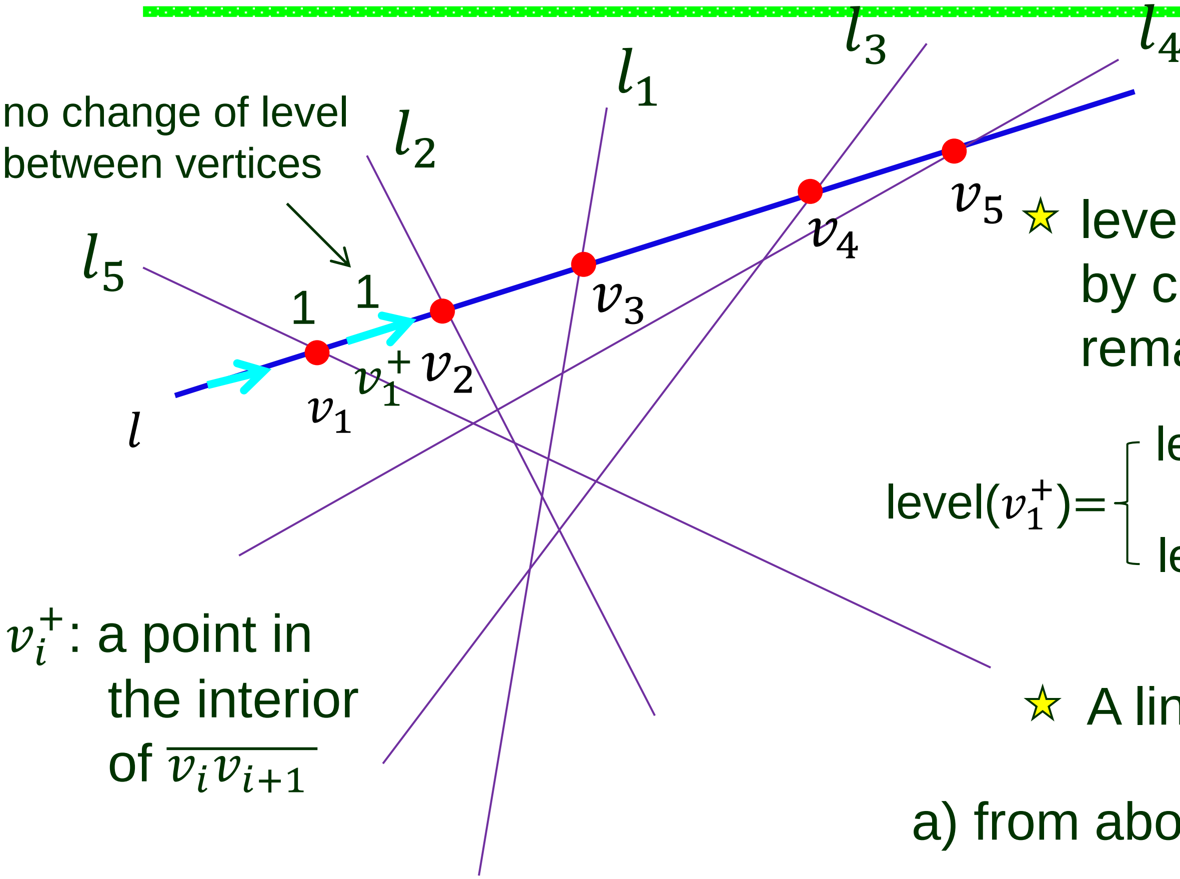
★ A line crossing v_i, $i > 1$, is coming

a) from above

b) from below

Counting Levels of Vertices



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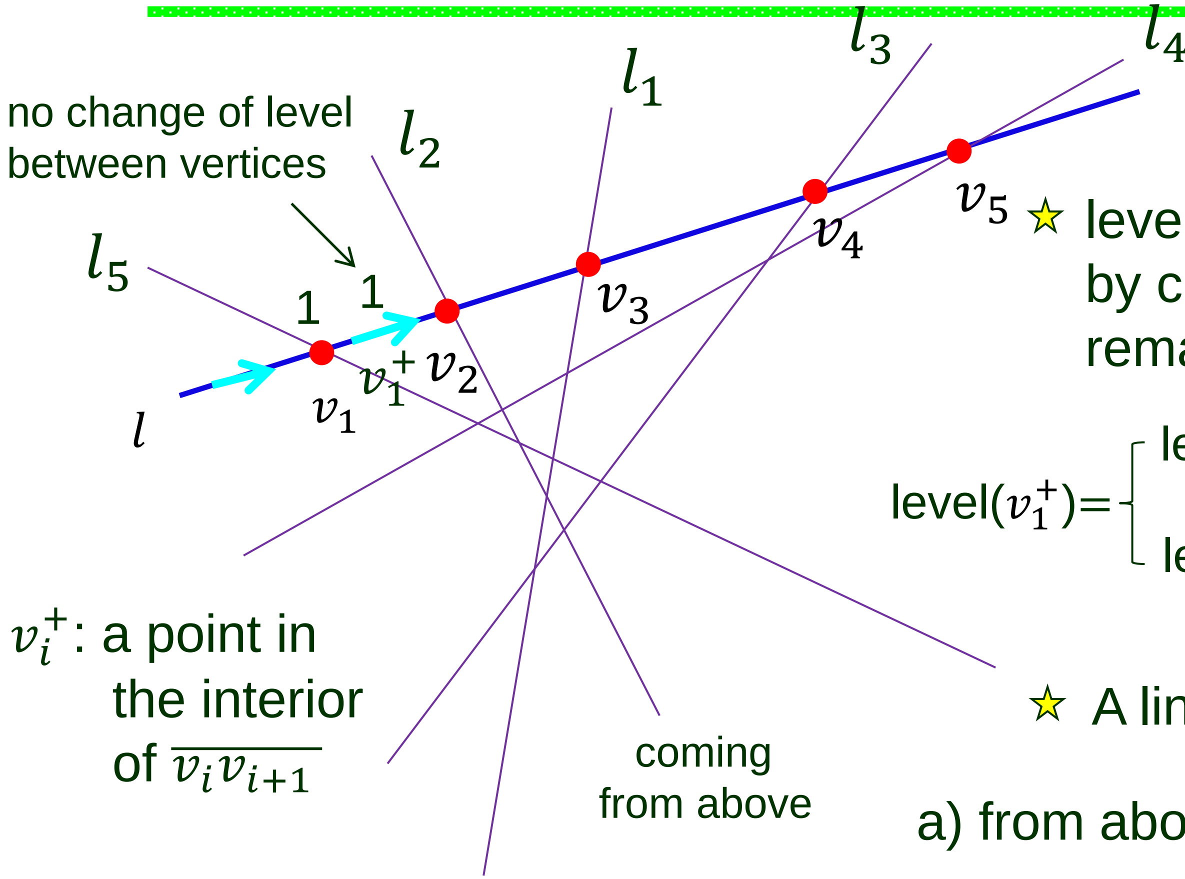

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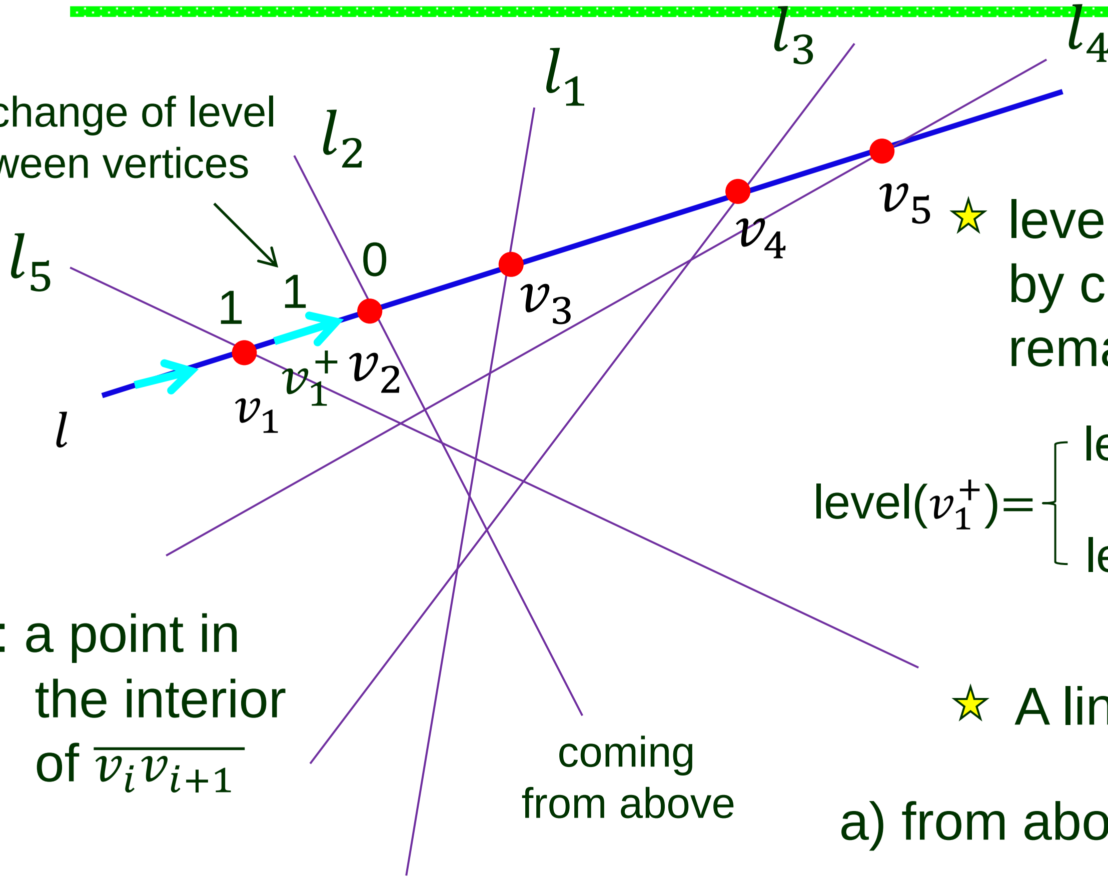
Counting Levels of Vertices

no change of level between vertices



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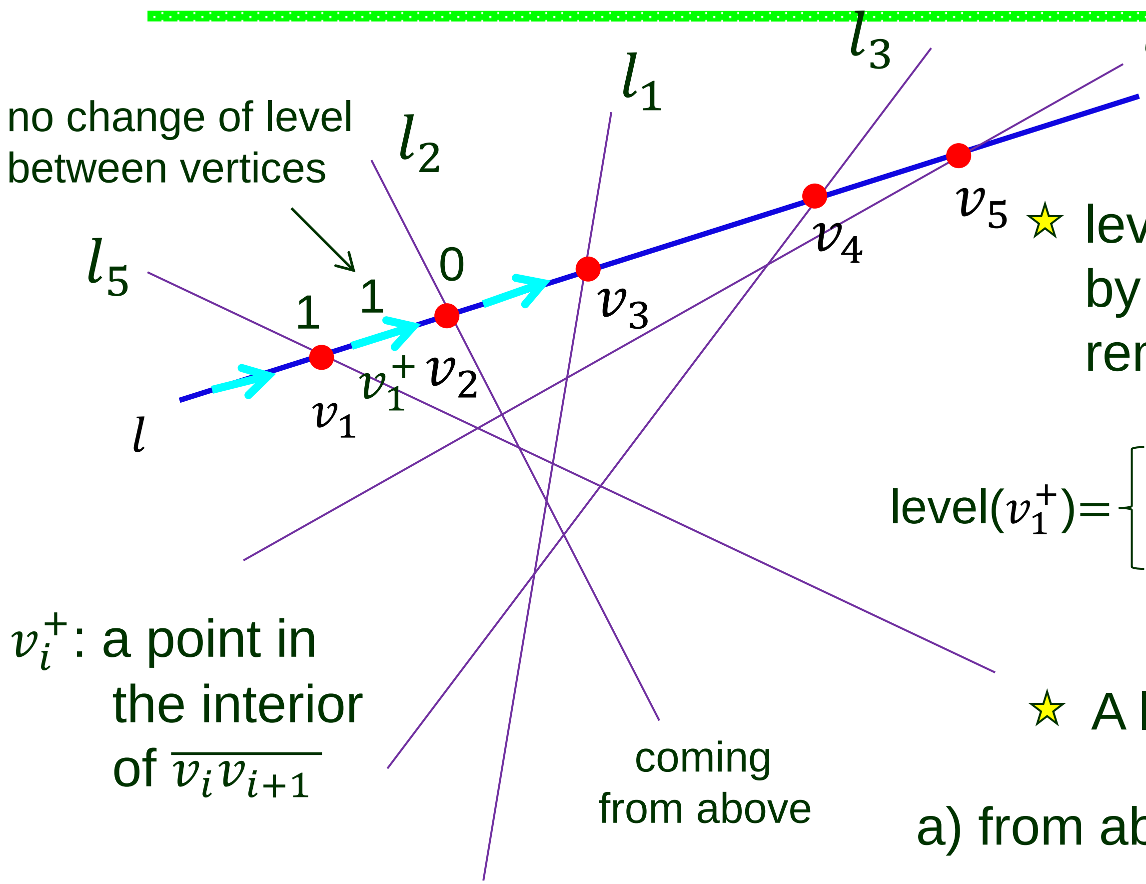

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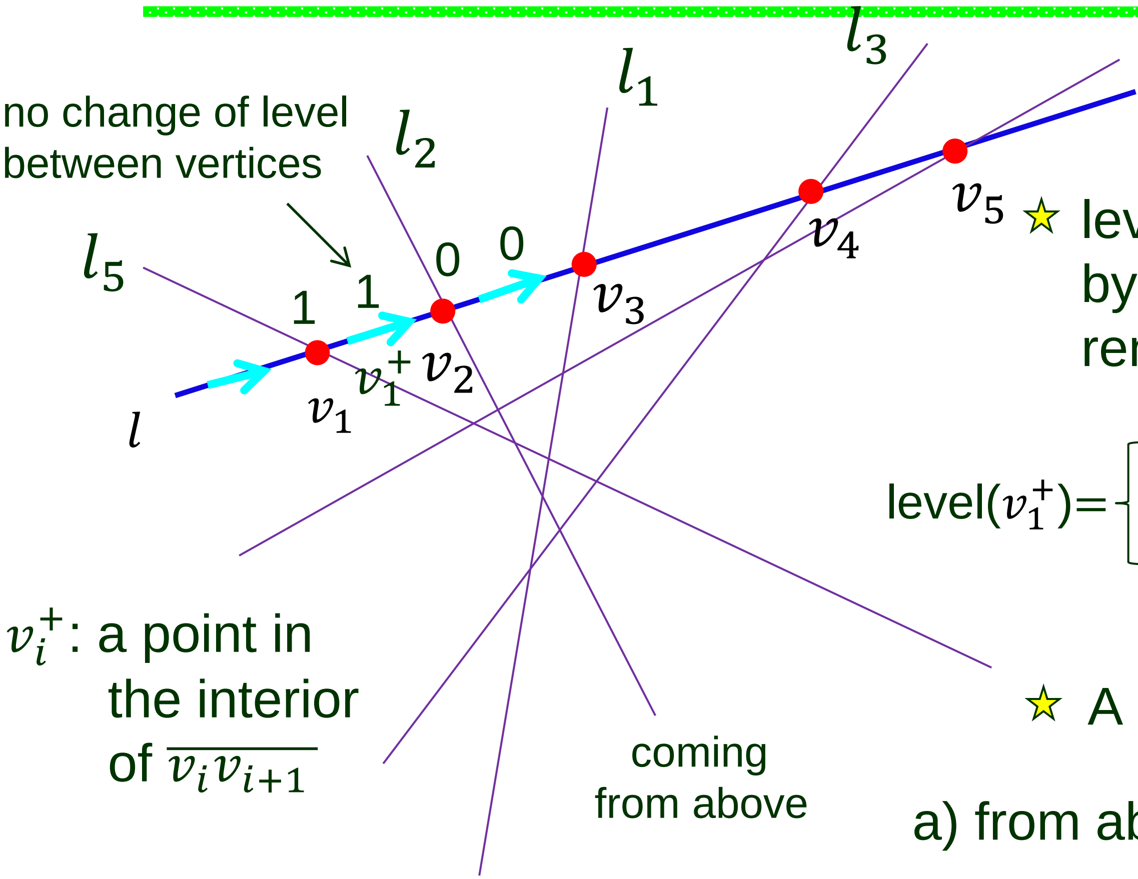

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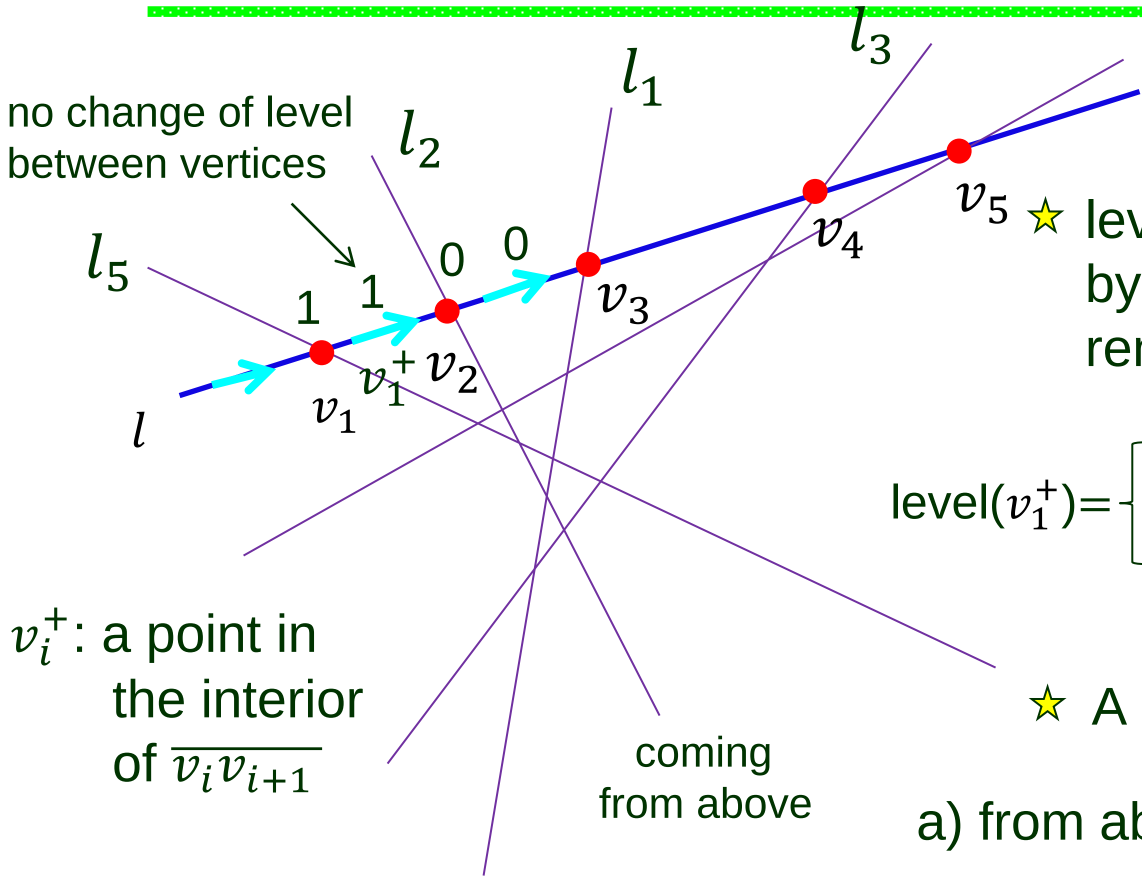

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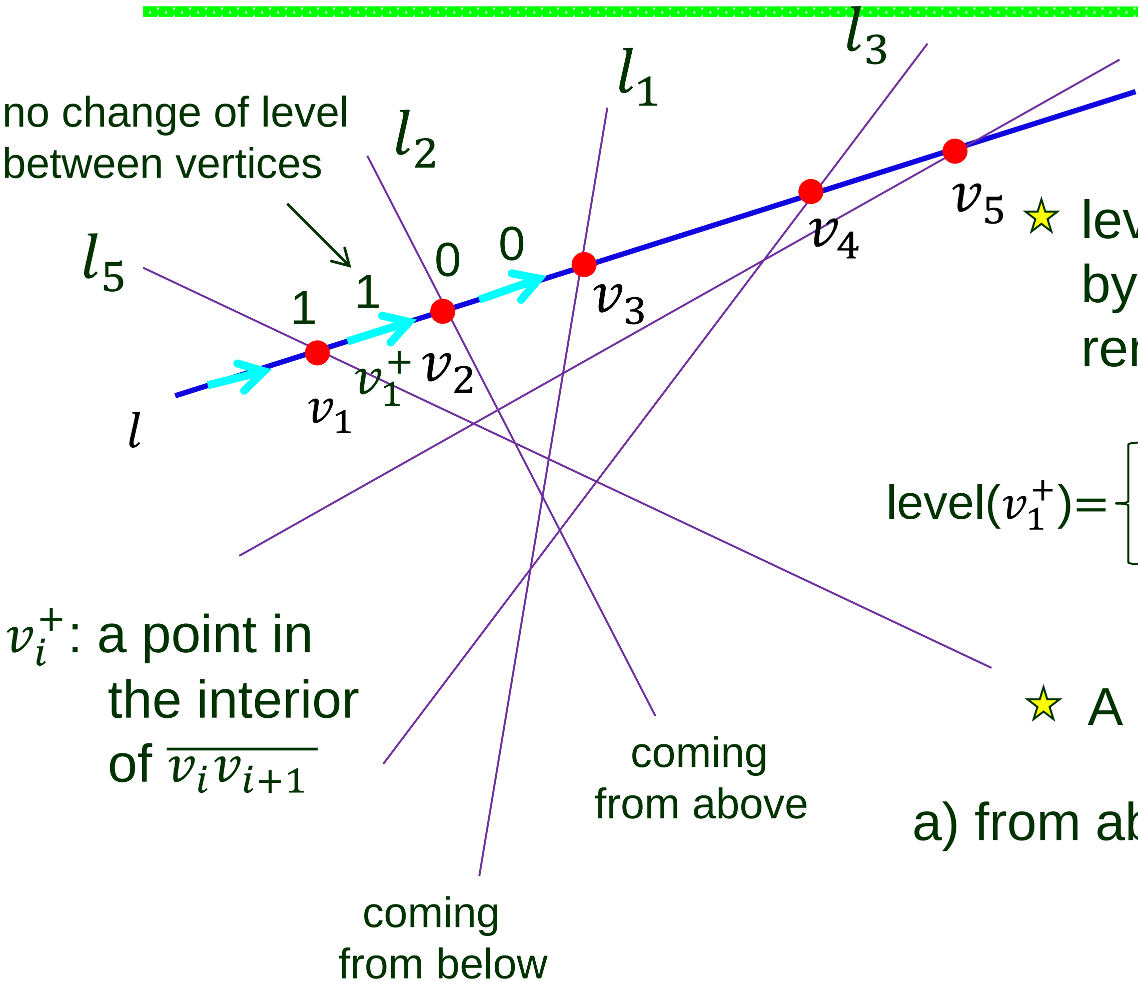

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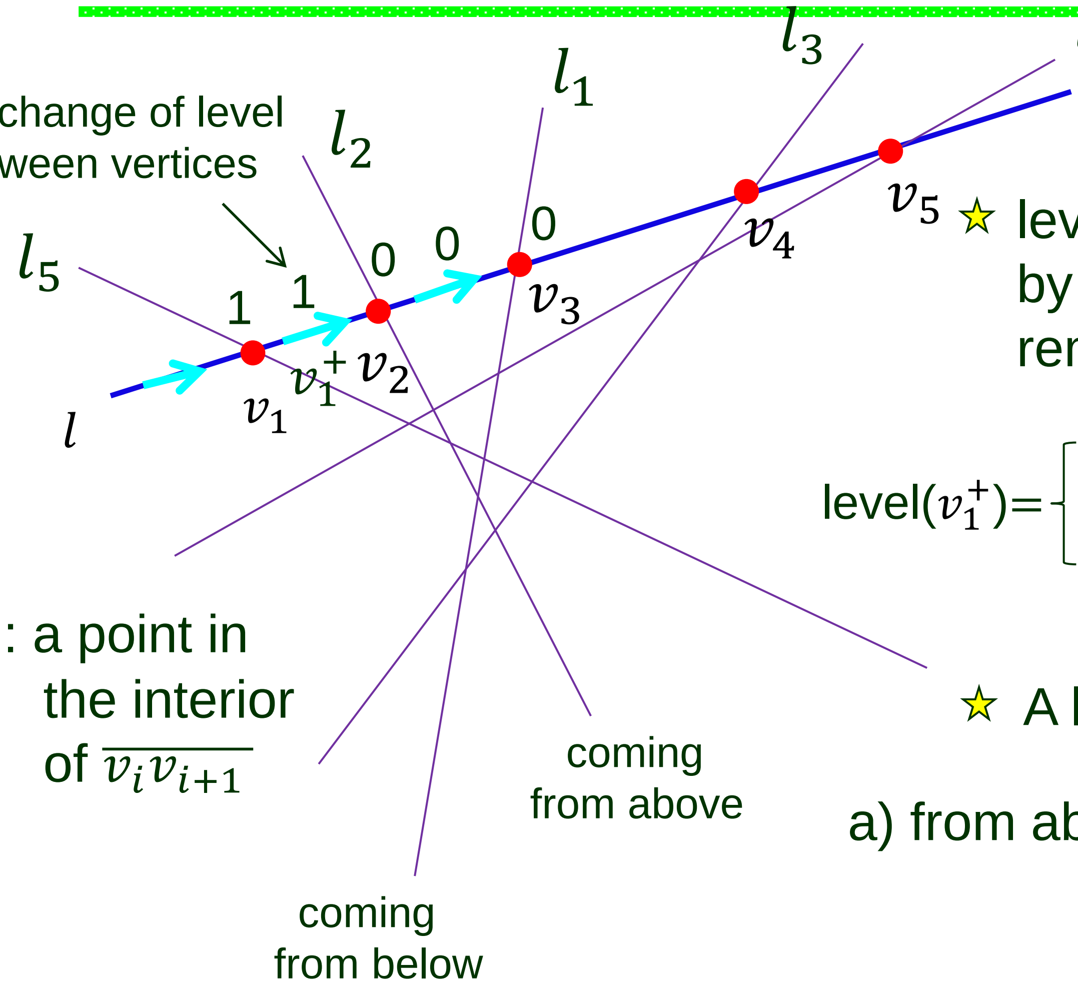

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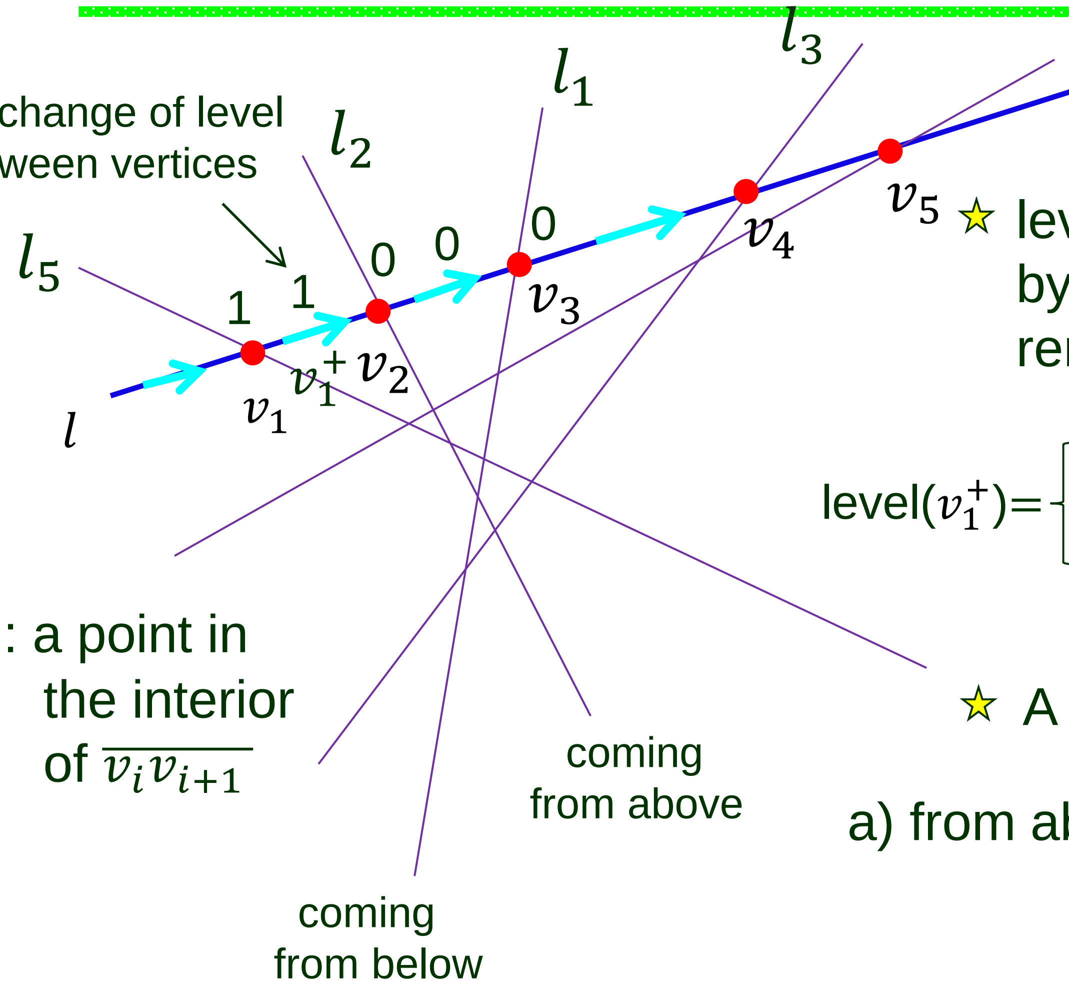

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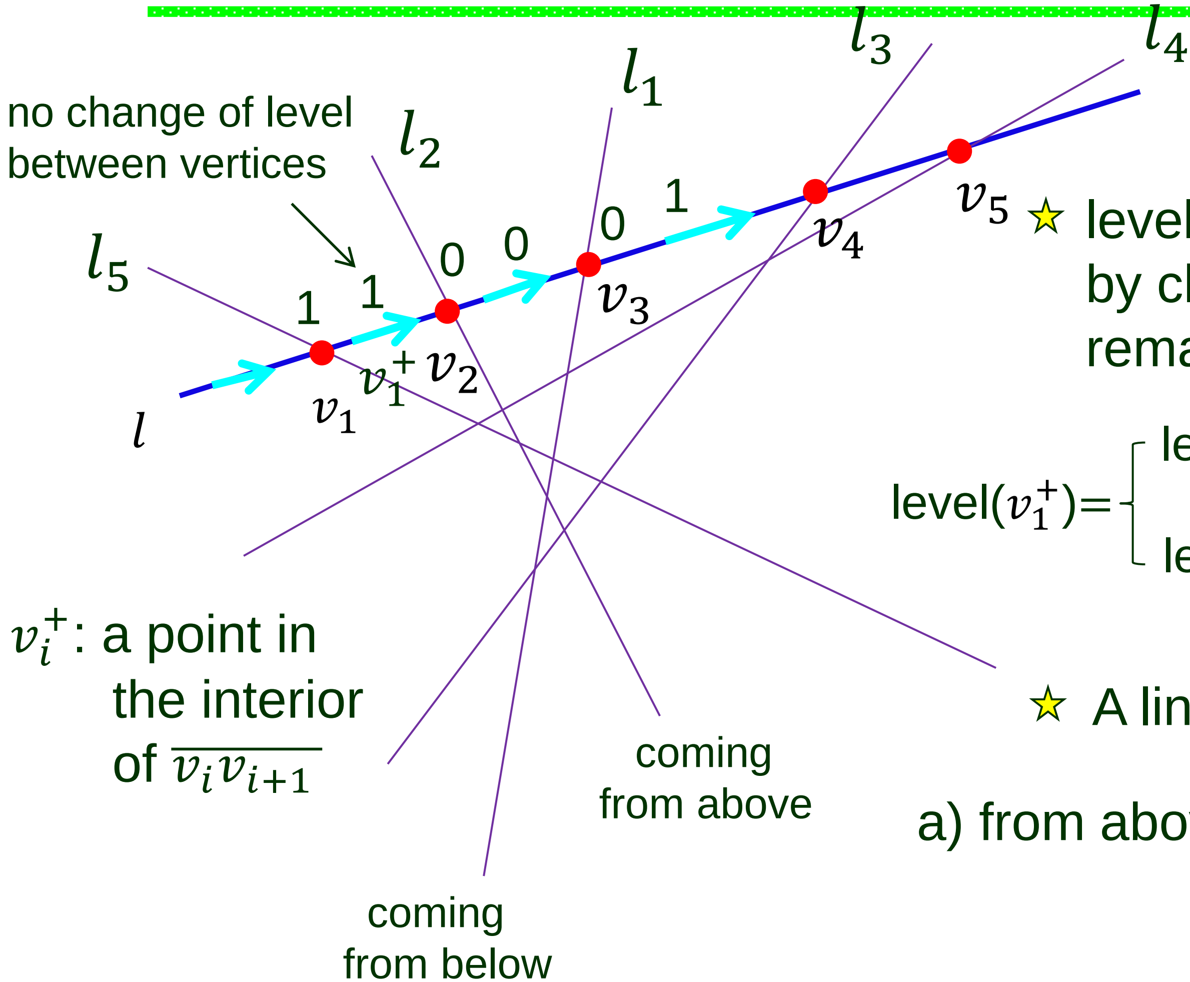

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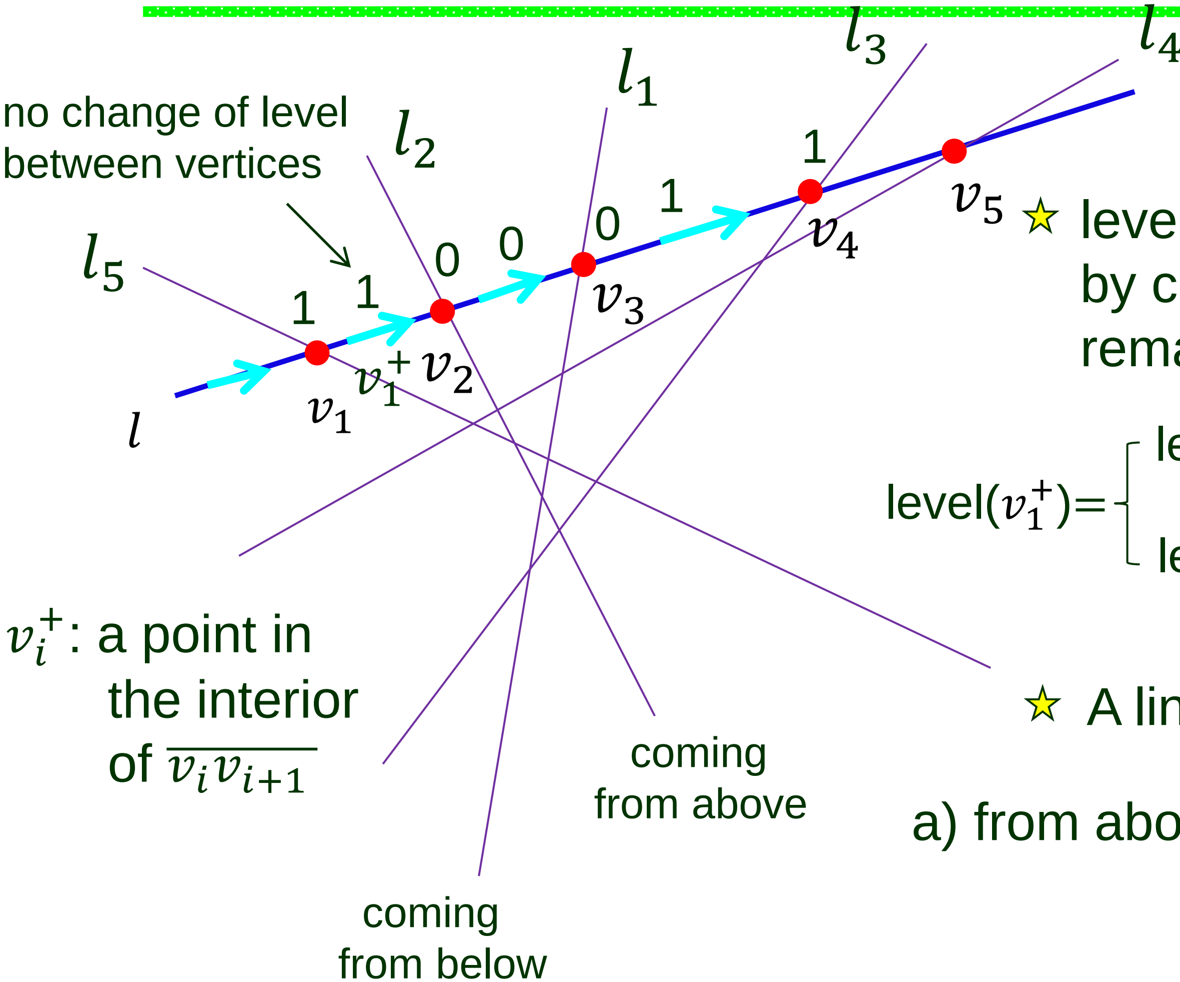

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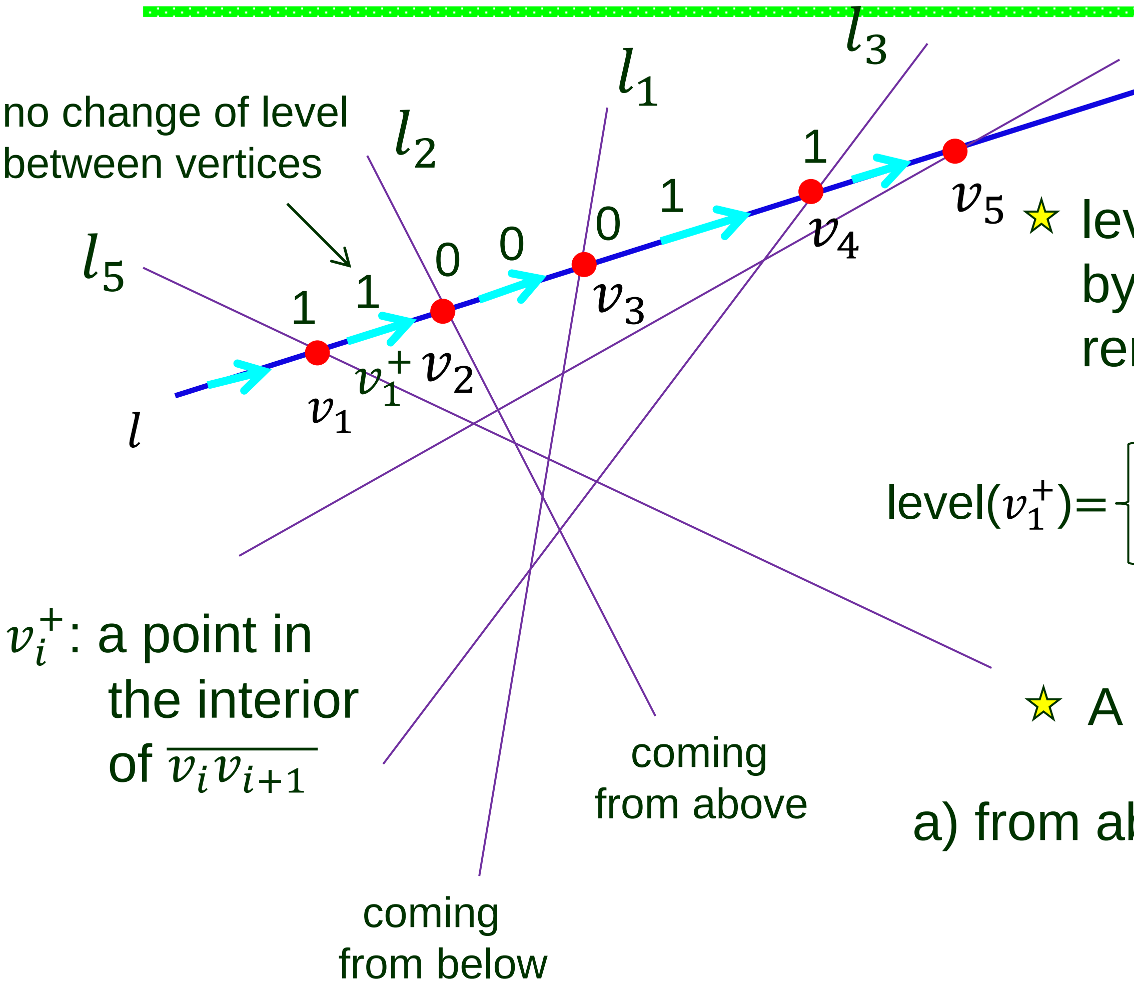

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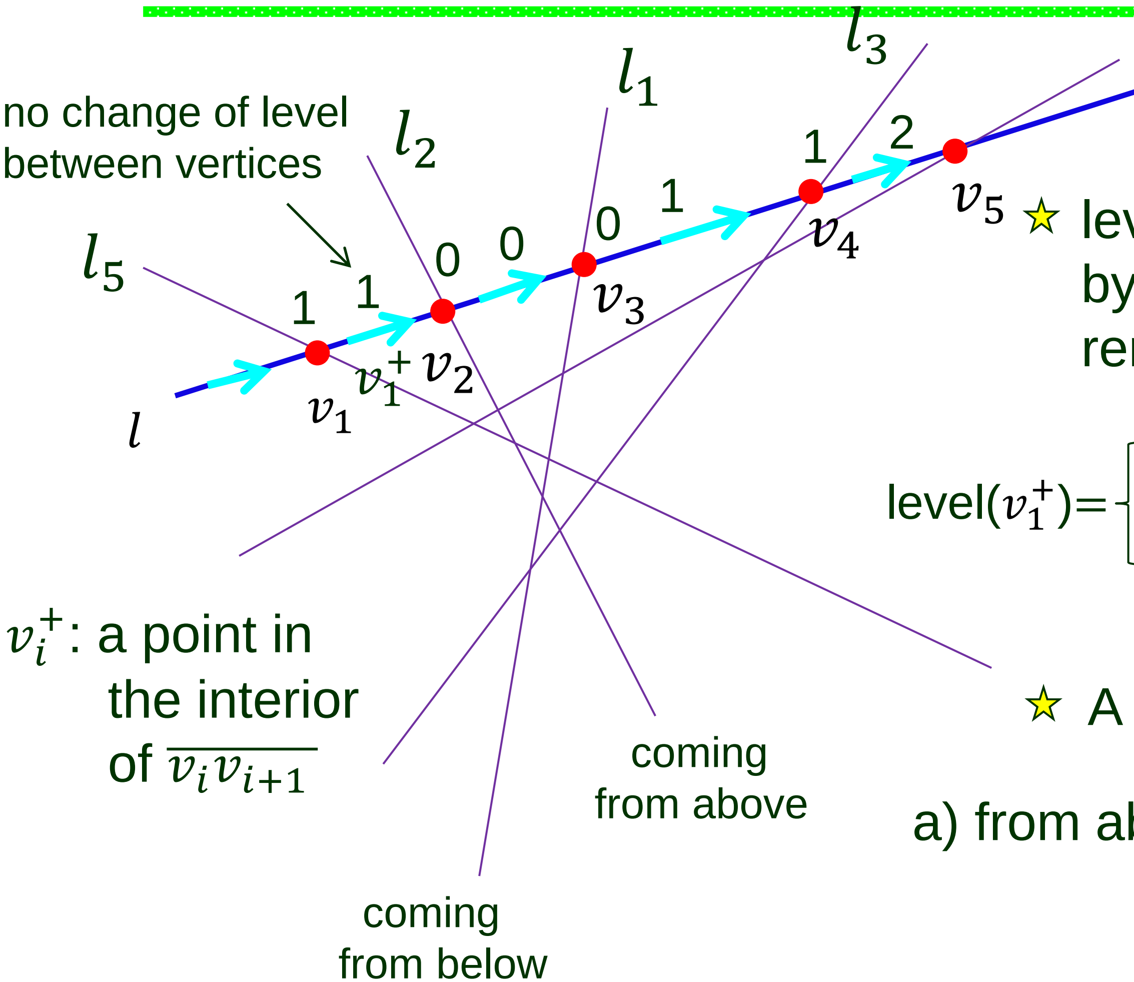
Counting Levels of Vertices



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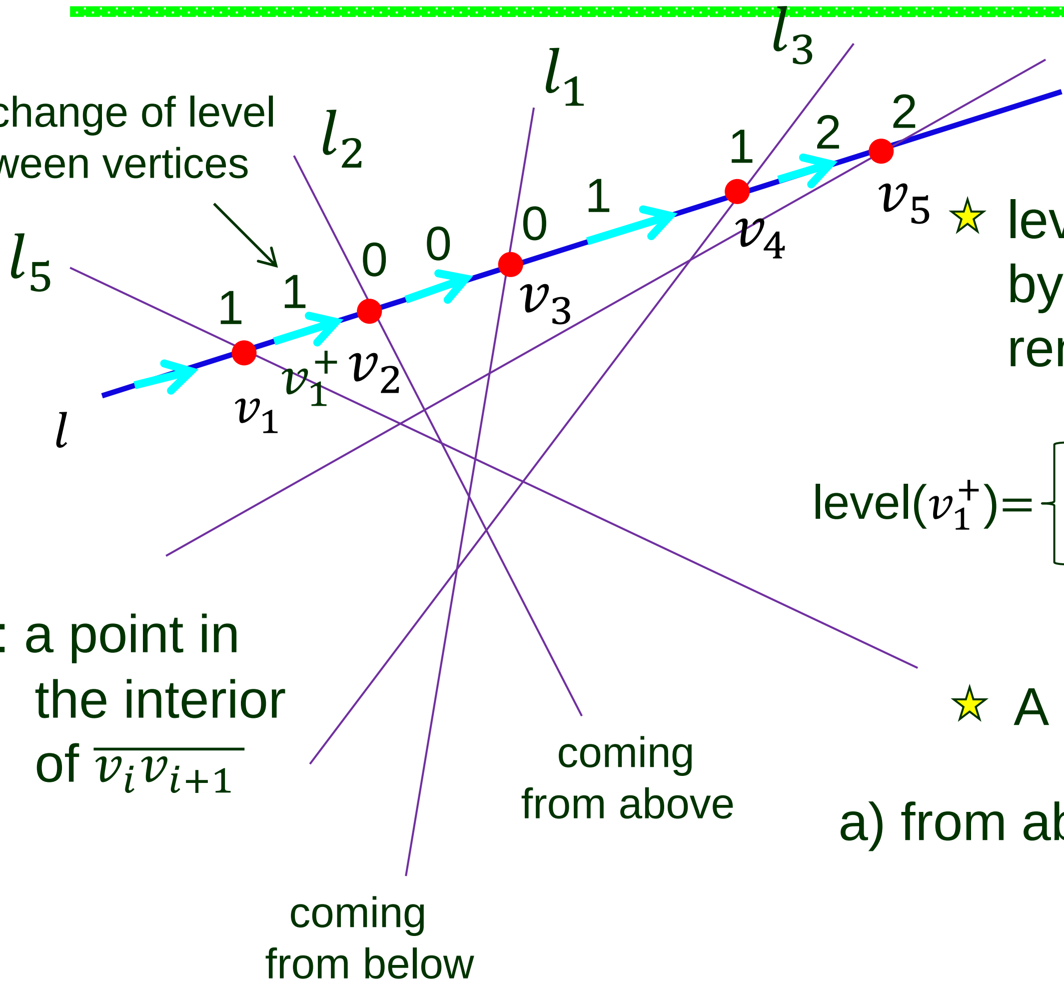

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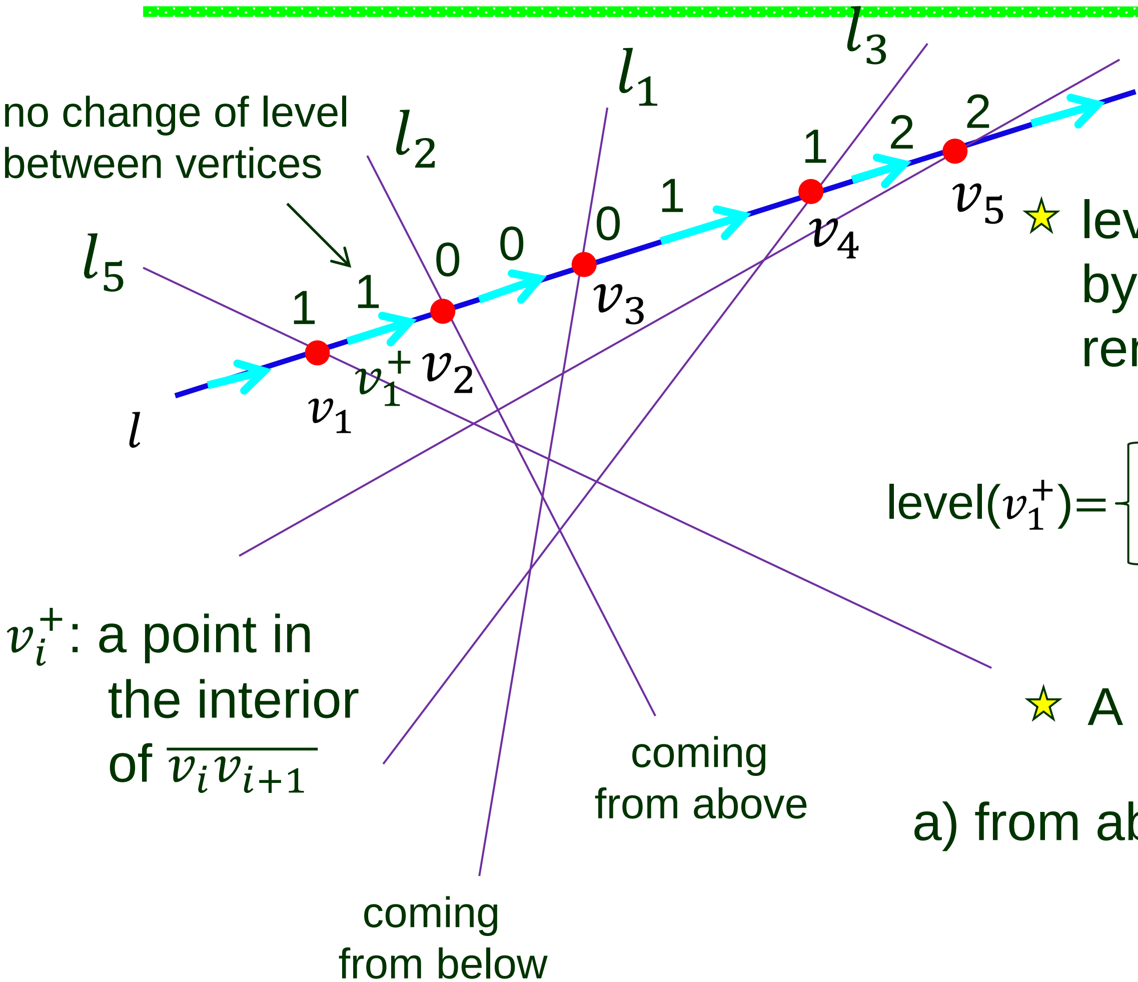

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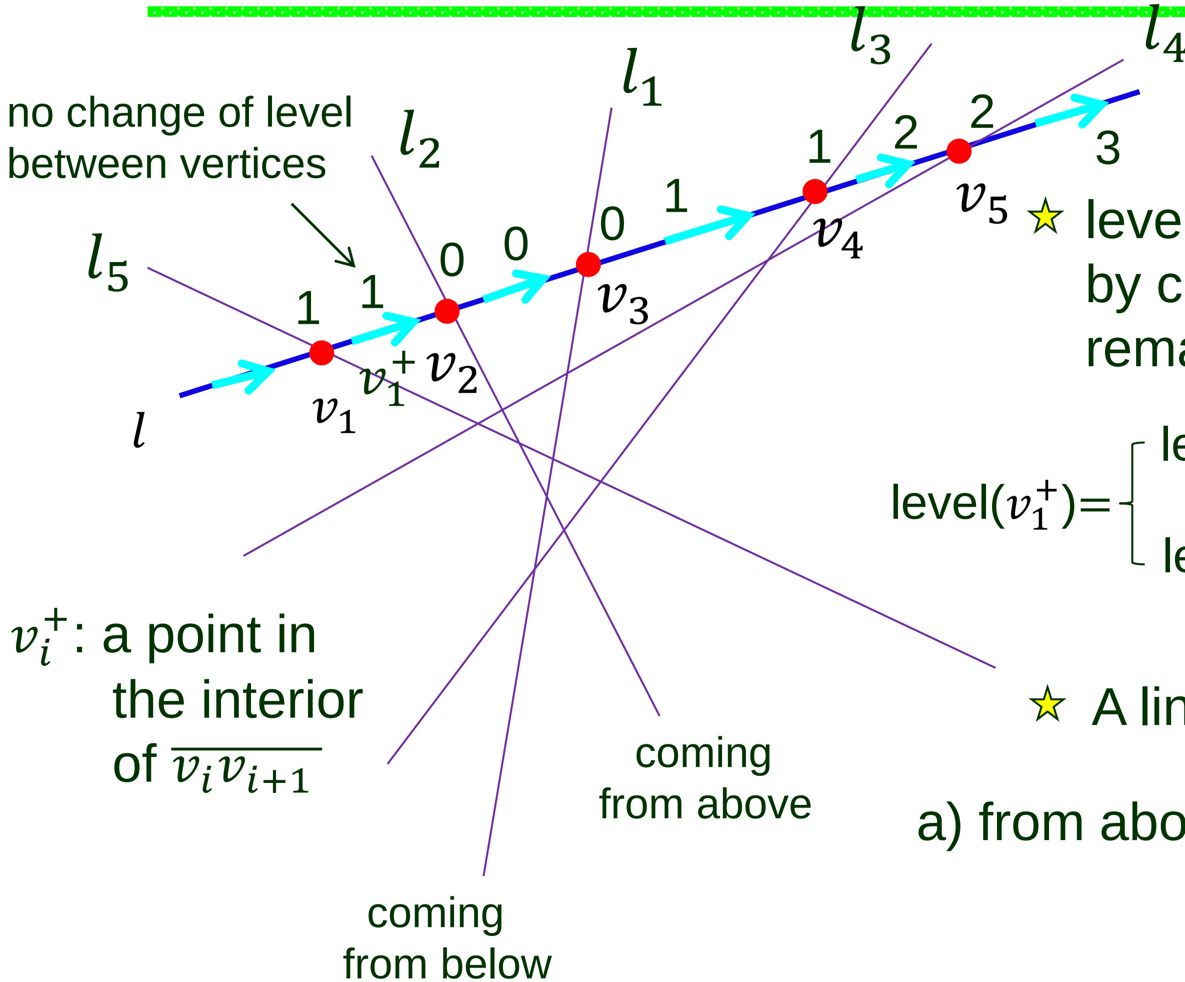

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Running Times

Levels of vertices along a line computable in $O(n)$ time.

Levels of all vertices in a line arrangement can be computed in $O(n^2)$ time.

Discrete Measures & Degeneracy

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Maximum discrepancy due to half-planes:

$$\max_h \Delta_S(h) = |\mu(h) - \mu_S(h)|$$

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The boundary line of the maximizing h must pass either i) one point or ii) ≥ 2 points.

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Total time: $O(n^2)$