FOL Models and Usage

Outline

I. Quantifiers

II. Model for FOL

III. Assertions & queries in FOL

IV. Natural numbers & sets

* Figures are from the textbook site unless a source is specifically cited.

I. Scope of a Quantifier

- Quantifiers ∀ and ∃ have the lowest precedence.

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- Each of ∀ and ∃ quantifies the remaining scope of the innermost pair of parentheses containing it.

$$\cdots\ (\cdots\ (\cdots\ (\cdots \forall x \quad \cdots \quad)\cdots)\cdots)\cdots$$

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different bound variable

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Nested Quantifiers

$\forall x \exists y$ *Student*$(x) \wedge$ *Course*(y) $\wedge$ *Enrolled*(x, y)

$\forall x \exists y$ *Brother*$(x, y) \Rightarrow$ *Sibiling*(x, y)

Order matters for quantifiers of different types:

$\forall x \exists y$ *Loves*(x, y) // Everybody (x) loves somebody (y)

$\exists x \forall y$ *Loves*(y, x) // There is someone (x) whom everyone (y) loves.

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but not for those of the same type and appearing next to each other:

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Connections Between ∀ and ∃ Through ¬

$$\forall x \ \neg Likes(x, Parsnips) \equiv \neg \exists x \ Likes(x, Parsnips)$$

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// Zeus has at least
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$$\wedge\ (\forall z\ \textit{Brother}(z, \text{Zeus}) \Rightarrow (z = x) \vee (z = y))$$
// Zeus has no other brothers
// besides x, y.

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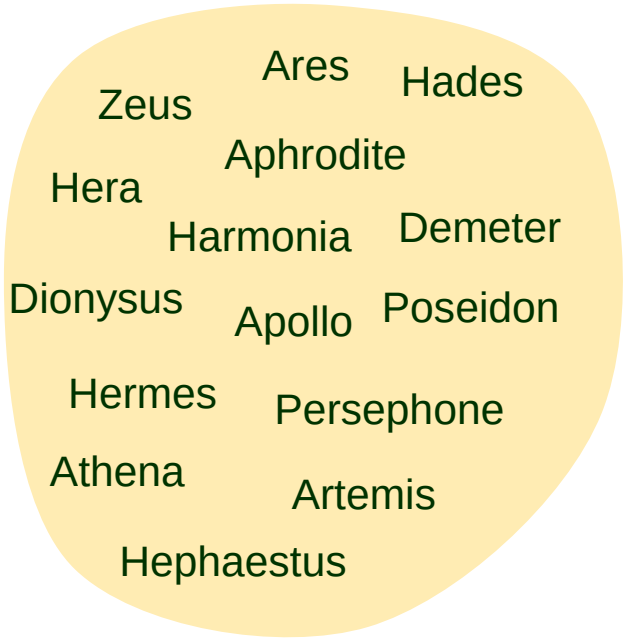
- Each function $f(x_1, \ldots, x_k)$ is mapped to a function from D^k to $D \cup \{o\}$, where o is some invisible object.

Model Example

Model for the family relationships of the Greek gods (incomplete).



domain D

Father(Zeus, Hermes)

Mother(Hera, Ares)

Mother(Aphrodite, Harmonia)

Father(Zeus, Athena)

⋮

predicates

Weapon(Zeus) // ≡ Thunderbolt

Weapon(Apollo) // ≡ BowAndArrows

Carry(Hermes) // ≡ Flute

Carry(Aphrodite) // ≡ Apple

⋮

functions

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Quantifiers allow us to express properties of a collection of objects instead of enumerating them by name.

∀ (universal): "for all"

∃ (existential): "there exists"

Truths with Quantifications

- $\forall x\ P(x)$ is true in a model M iff $P(x)$ is true with x assuming every object in the model

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$\exists x \neg Likes(x, \text{sushi})$ — *true* (in a model that includes all the people in the world)

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$\exists x\ \mathit{Mother}(x, \text{Ares}) \wedge \mathit{Mother}(x, \text{Harmonia})$ — *false* (in the model of Greek mythology)

III. Using FOL

Knowledge engineering represents information about the world in a form that can be utilized by a computer to solve complex tasks such as:

- medical diagnosis
- dialog in a natural language
- etc.

♦ Knowledge representation (logical rules, semantic nets, frames, etc.)

♦ Automated reasoning (inference engines, theorem provers, etc.)

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A *domain* is some part of the world about which we wish to express some knowledge.

Assertions and Queries in FOL

- Add sentences, called *assertions*, to a KB using TELL.

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Any query is entailed by the KB should be answered affirmatively.

Substitution

Suppose another KB has the following predicates:

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  substitution or *binding list*

The Kinship Domain

Kinship relations are represented by binary predicates.

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$\forall m, c \; \mathit{Mother}(c) = m \Leftrightarrow \mathit{Female}(m) \wedge \mathit{Parent}(m, c)$

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These are *definitions* in the form of $\forall x, y\ P(x, y) \Leftrightarrow \ldots$ and built upon a basic set of predicates *Child*, *Male*, *Female*, etc.

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$\forall x\ \mathit{Person}(x) \Rightarrow \cdots$ // partial specification of

$\forall x\ \cdots \Rightarrow \mathit{Person}(x)$ // properties

IV. Domain of Natural Numbers

Describe the theory of natural numbers using merely:

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♦ Axioms to constrain the successor function:

$\forall n\ \ 0 \neq S(n)$

$\forall m, n\ \ m \neq n \Rightarrow S(m)\ \neq S(n)$

Defining Addition

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Syntactic sugar: $\forall m \forall n$ according to syntax

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  $s_1 \cap s_2$ $s_1 \cup s_2$ $Add(x, s)$

  the set resulting from add element x to set s

Eight Axioms for Sets

- A set is either an empty set or made by adding something to a set.

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must be true at some recursion level

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Equality

Axiomatize equality: write down sentences about equality in the *KB.*

- One for each basic axioms.

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$\forall x, y \;\; x = y \Leftrightarrow y = x$ // symmetric

$\forall x, y, z \;\; x = y \wedge y = z \Rightarrow x = z$ // transitive

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- One for each function.

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