Yan-Bin Jia, Iowa State University

Duality

Outline:

I. Point-line duality

II. Dual of a line segment

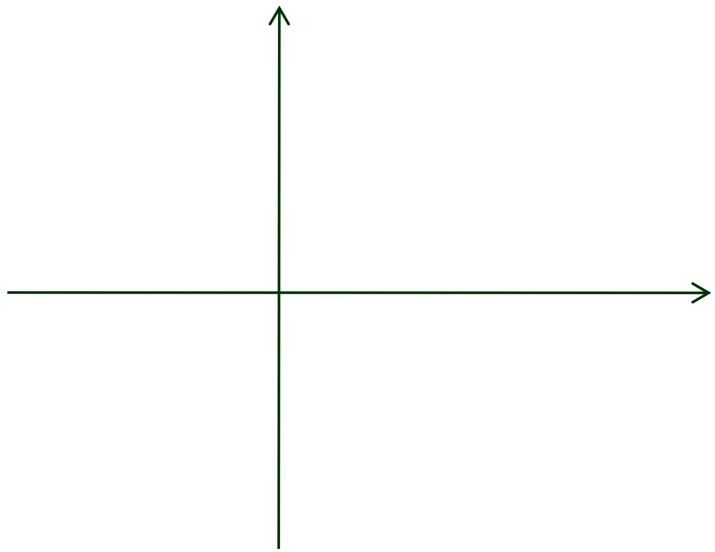
III. Duality through a parabola

IV. Inversion via the unit paraboloid

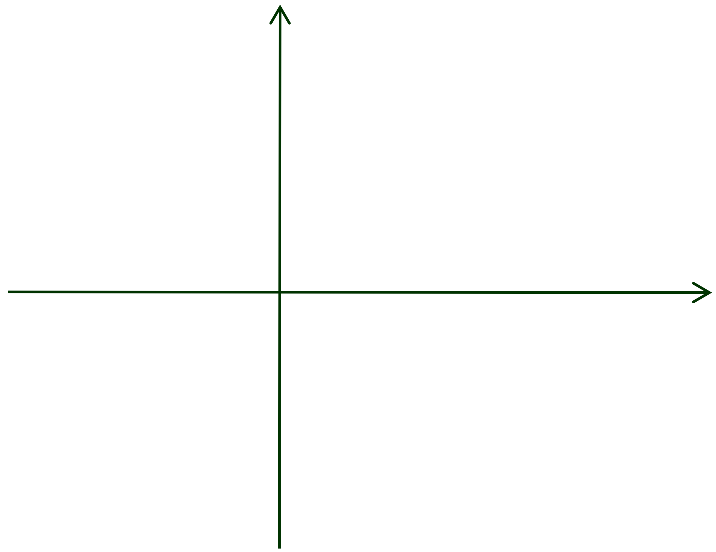
Textbook: *Computational Geometry: Algorithms and Applications (*3rd ed.) by M. de Berg et al., Springer-Verlag, 2008.

Point-Line Duality

Point $p = (p_x, p_y)$



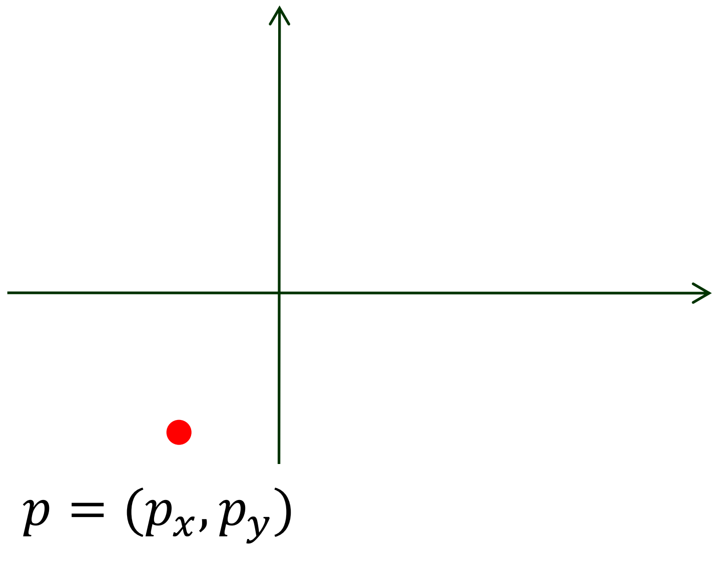
primal plane



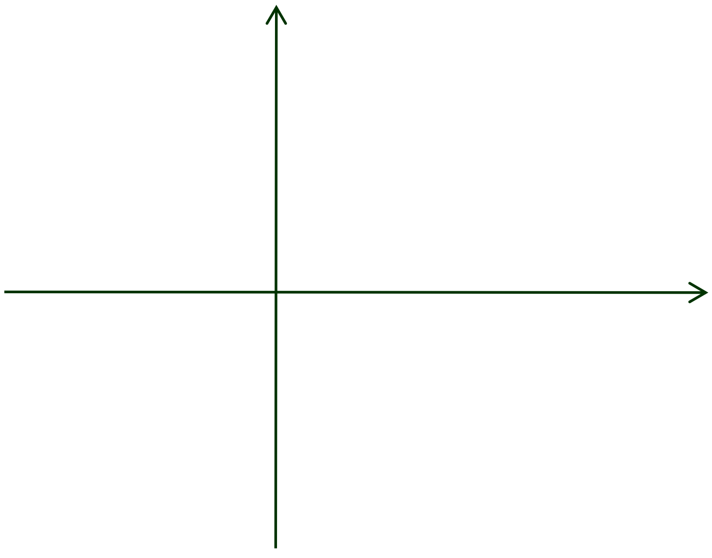
dual plane

Point-Line Duality

Point $p = (p_x, p_y)$



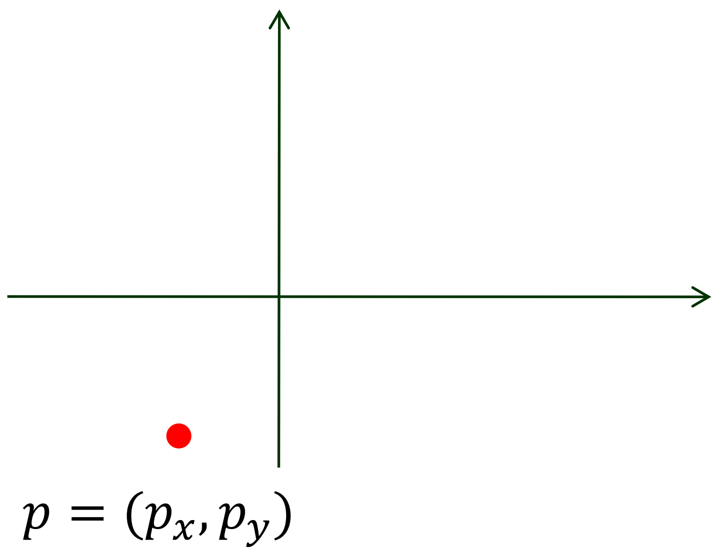
primal plane



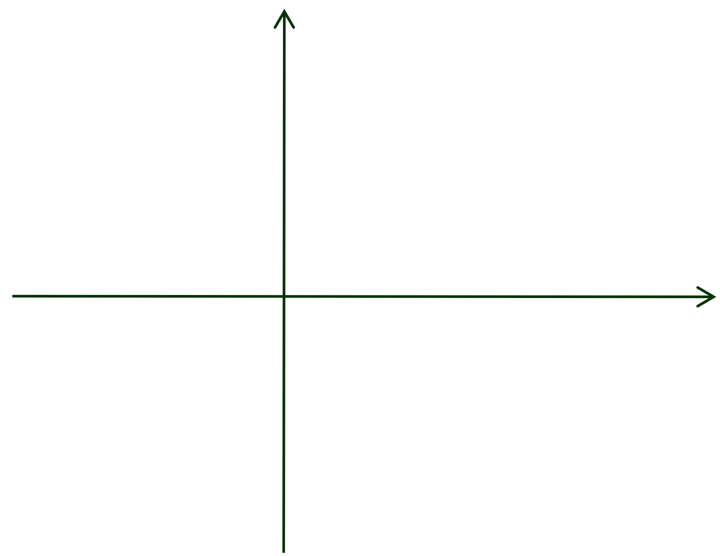
dual plane

Point-Line Duality

Point $p = (p_x, p_y)$ $\Longrightarrow$ Line $p^*: y = p_x x - p_y$



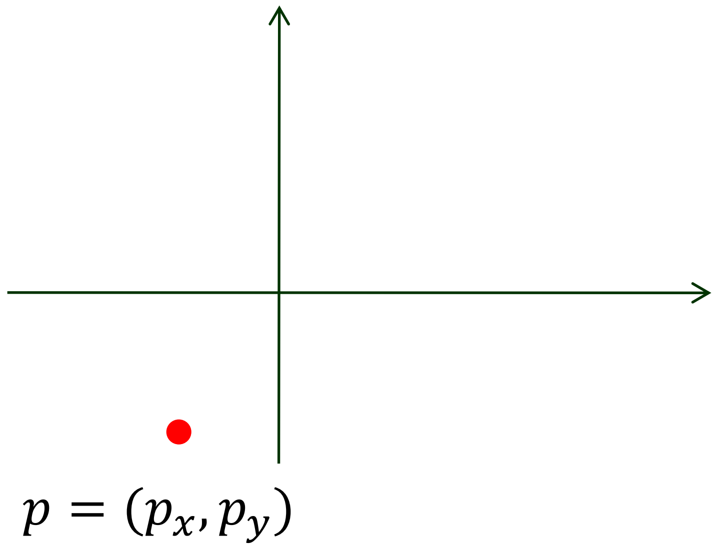
primal plane



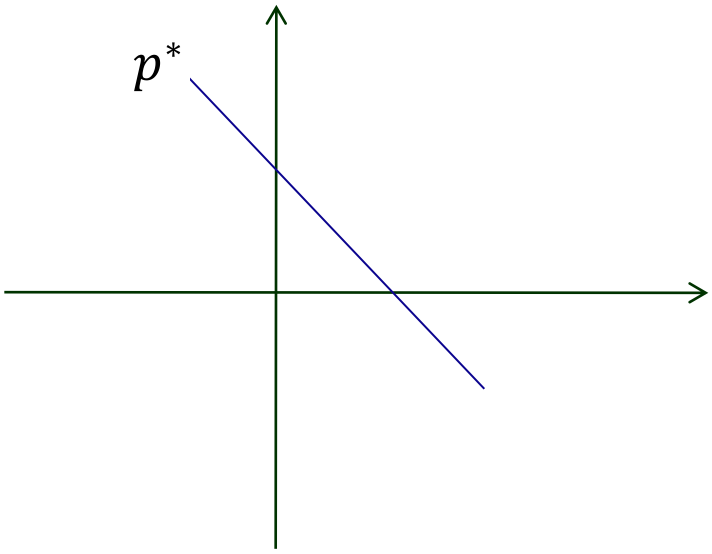
dual plane

Point-Line Duality

Point $p = (p_x, p_y)$ $\Longrightarrow$ Line $p^*: y = p_x x - p_y$



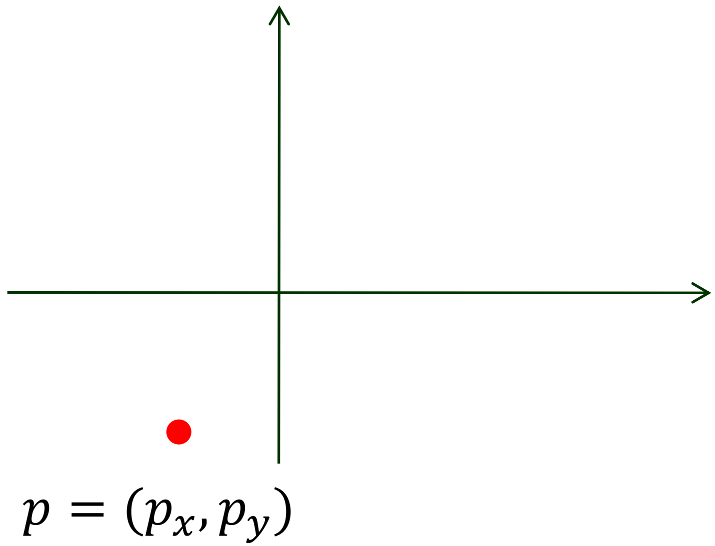
primal plane

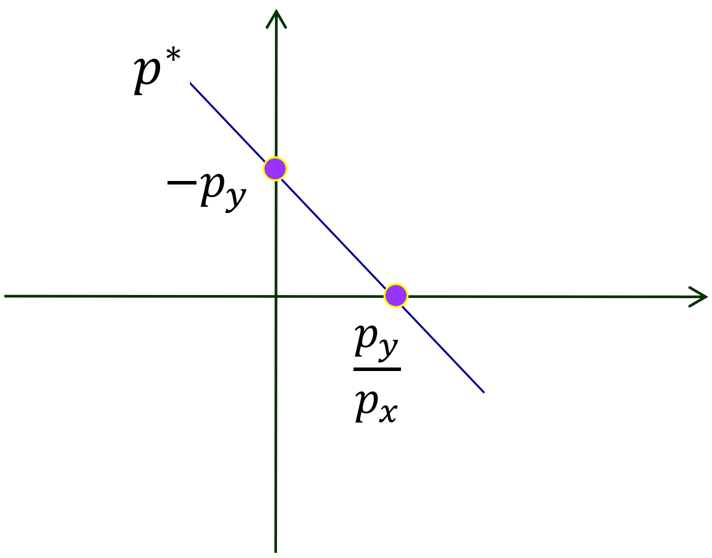


dual plane

Point-Line Duality

Point $p = (p_x, p_y)$ $\Longrightarrow$ Line $p^*: y = p_x x - p_y$





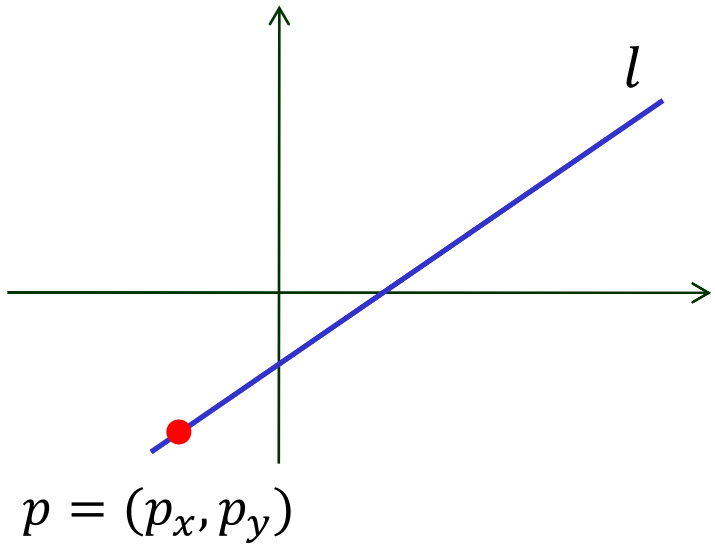
primal plane

dual plane

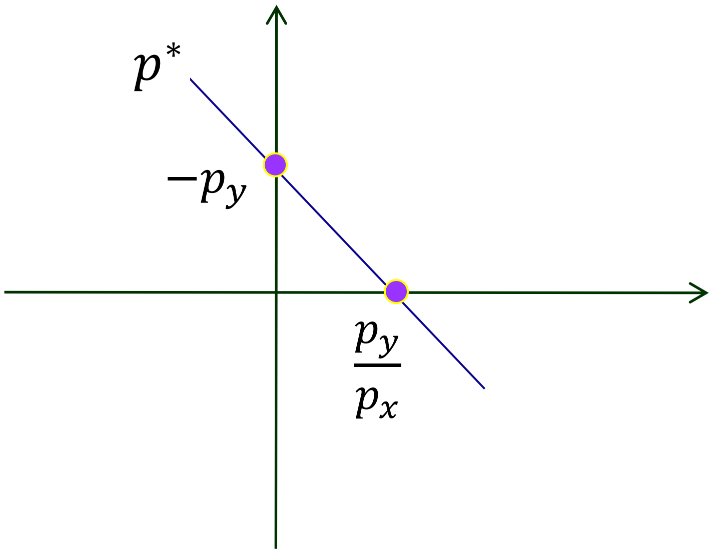
Point-Line Duality

Point $p = (p_x, p_y)$ $\Longrightarrow$ Line $p^*: y = p_x x - p_y$

Line $l: y = mx + b$



primal plane

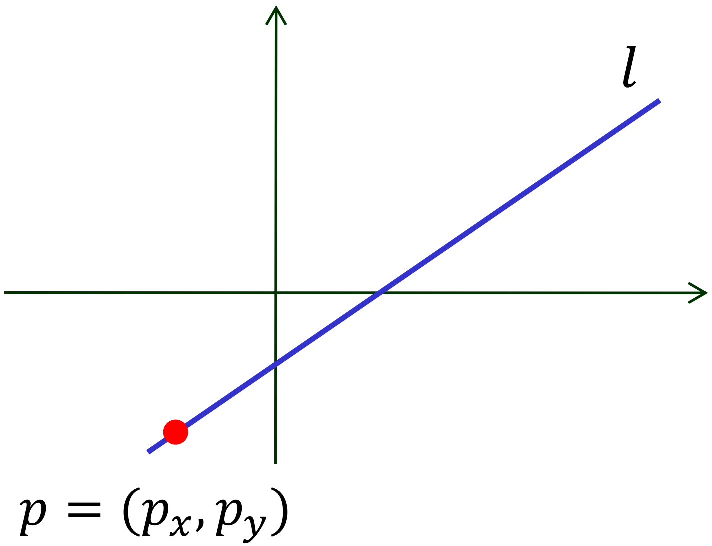


dual plane

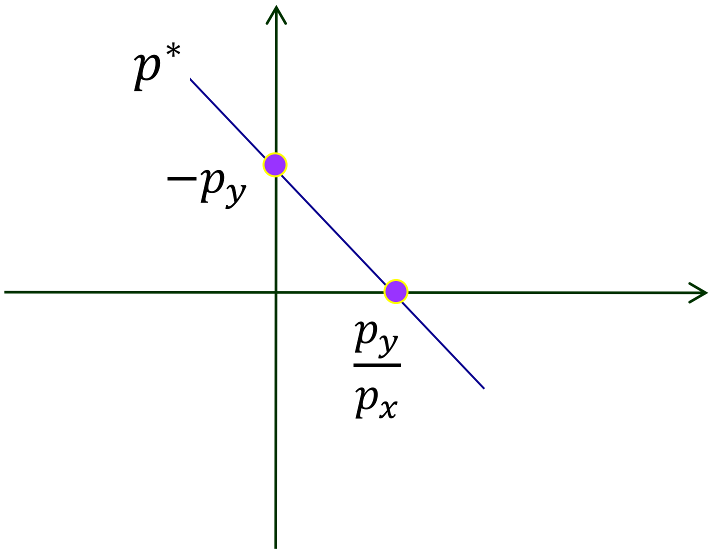
Point-Line Duality

Point $p = (p_x, p_y)$ $\Longrightarrow$ Line $p^*: y = p_x x - p_y$

Line $l: y = mx + b$ $\Longrightarrow$ Point $l^* = (m, -b)$



primal plane

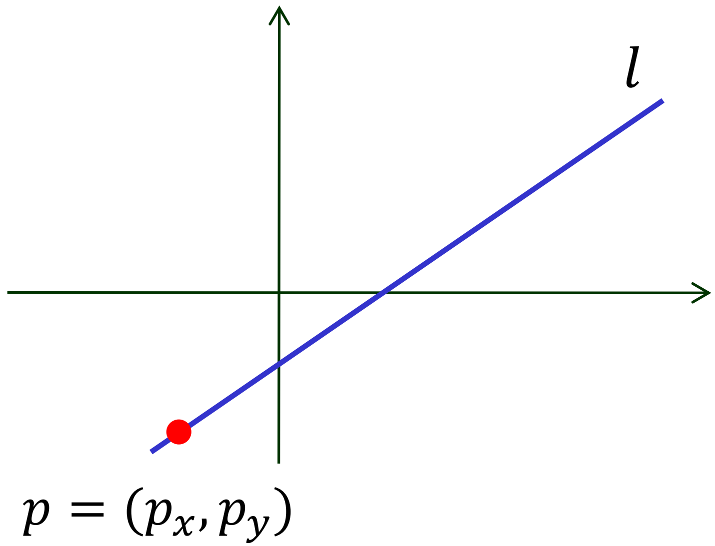


dual plane

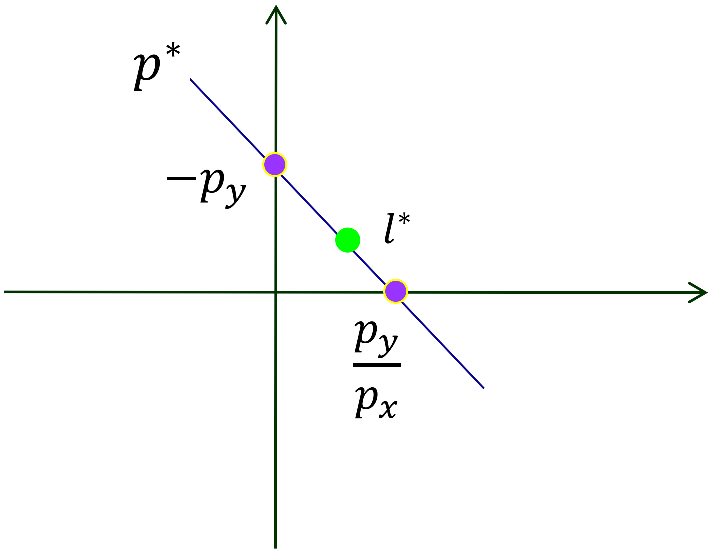
Point-Line Duality

Point $p = (p_x, p_y)$ $\Longrightarrow$ Line $p^*: y = p_x x - p_y$

Line $l: y = mx + b$ $\Longrightarrow$ Point $l^* = (m, -b)$



primal plane

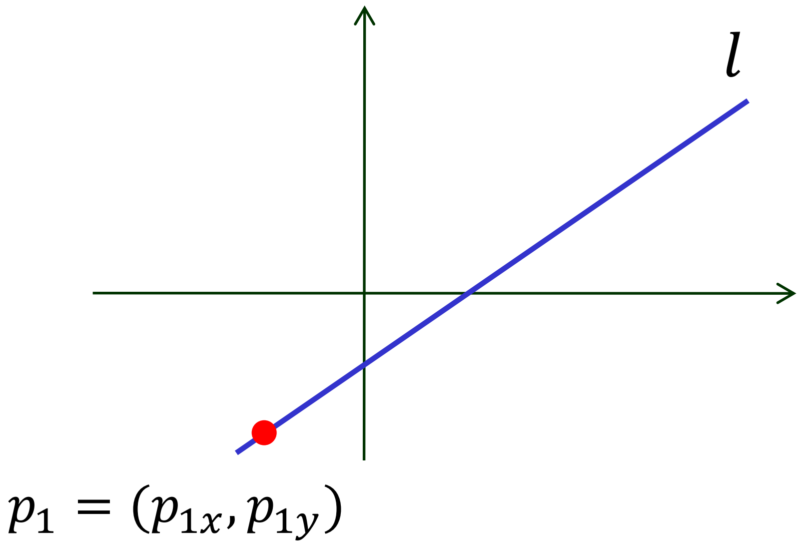


dual plane

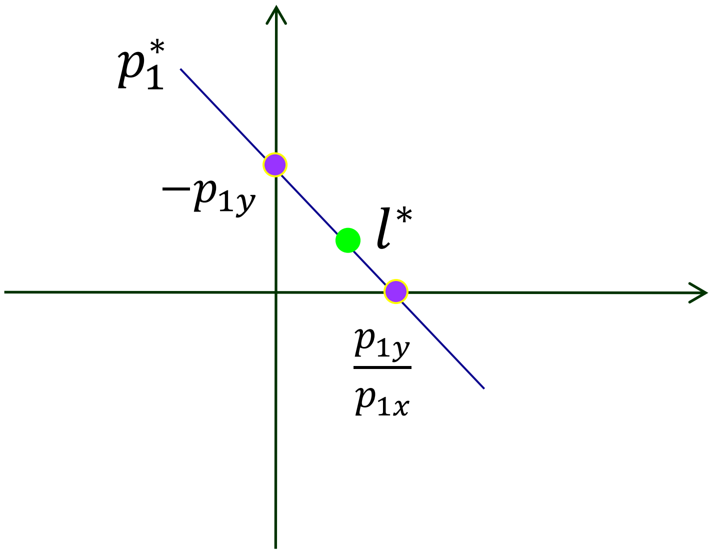
Dual of the Dual & Incidence

$$p = (p_x, p_y) \mapsto p^*: \ y = p_x x - p_y$$

$$l: \ y = mx + b \ \mapsto l^* = (m, -b)$$



primal plane



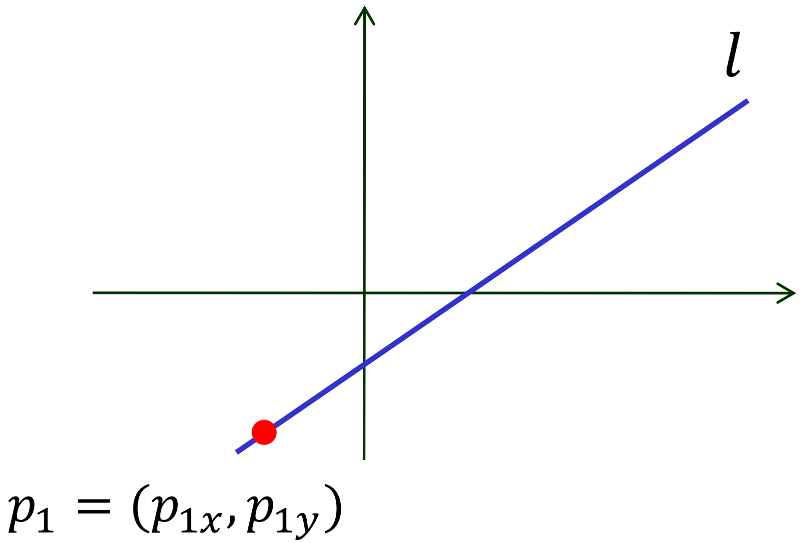
dual plane

Dual of the Dual & Incidence

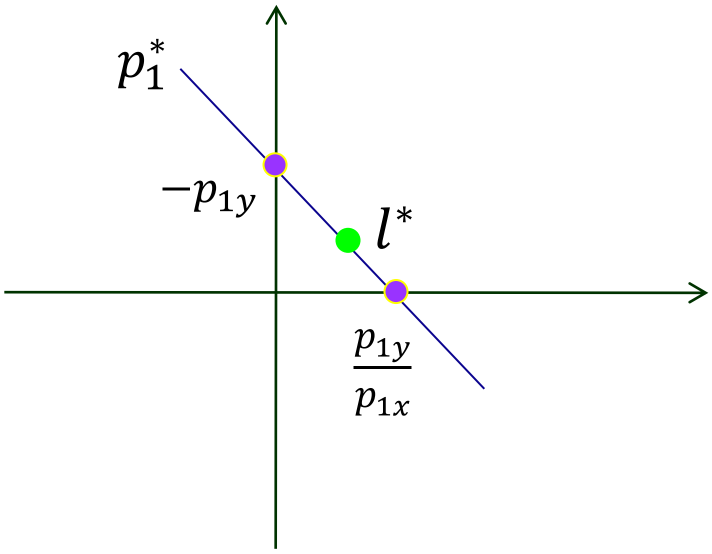
$p = (p_x, p_y) \longmapsto p^*\colon\ y = p_x x - p_y$

$l\colon\ y = mx + b \ \longmapsto l^* = (m, -b)$

- $(p^*)^* = p$



primal plane



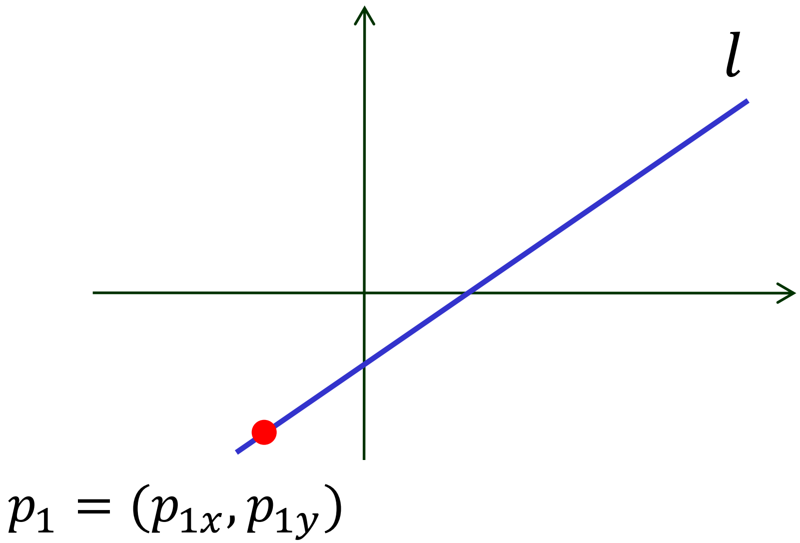
dual plane

Dual of the Dual & Incidence

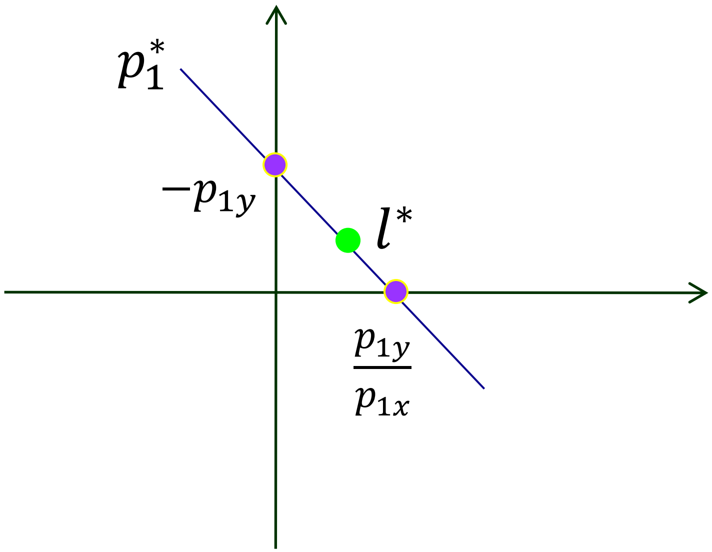
$p = (p_x, p_y) \mapsto p^*: y = p_x x - p_y$

$l: y = mx + b \mapsto l^* = (m, -b)$

- $(p^*)^* = p$
- $(l^*)^* = l$



primal plane



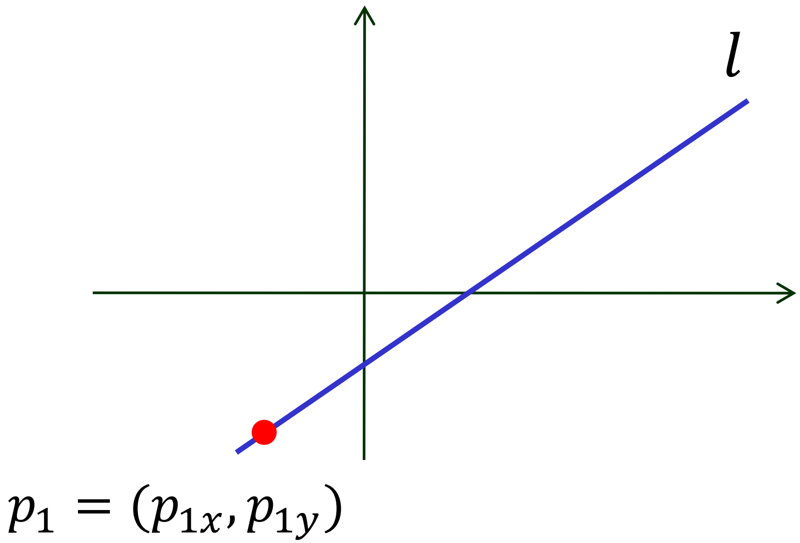
dual plane

Dual of the Dual & Incidence

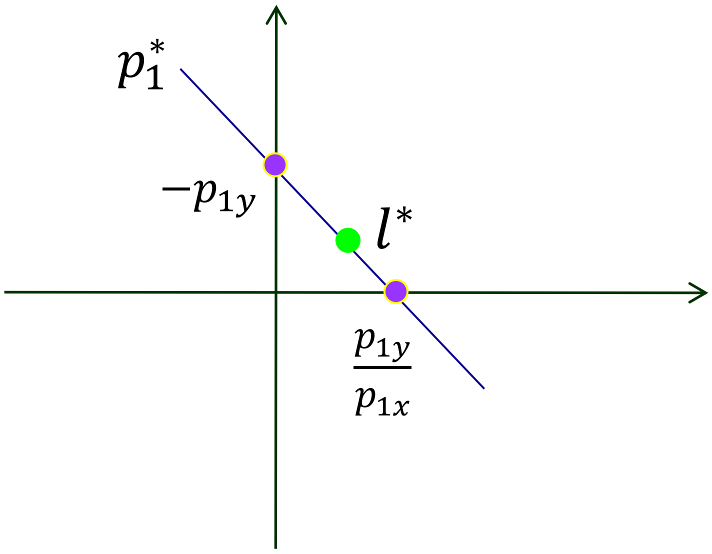
$p = (p_x, p_y) \longmapsto p^*\colon\ y = p_x x - p_y$

$l\colon\ y = mx + b \ \longmapsto l^* = (m, -b)$

- $(p^*)^* = p$
- $(l^*)^* = l$
- $p \in l \Leftrightarrow l^* \in p^*$



primal plane



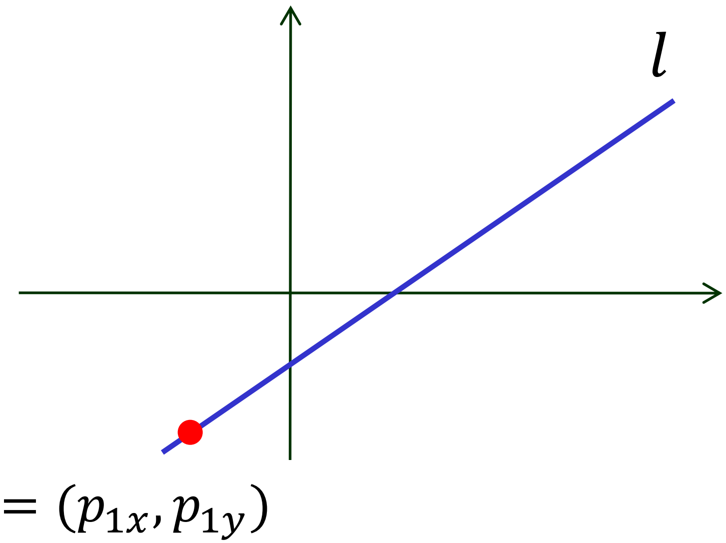
dual plane

Dual of the Dual & Incidence

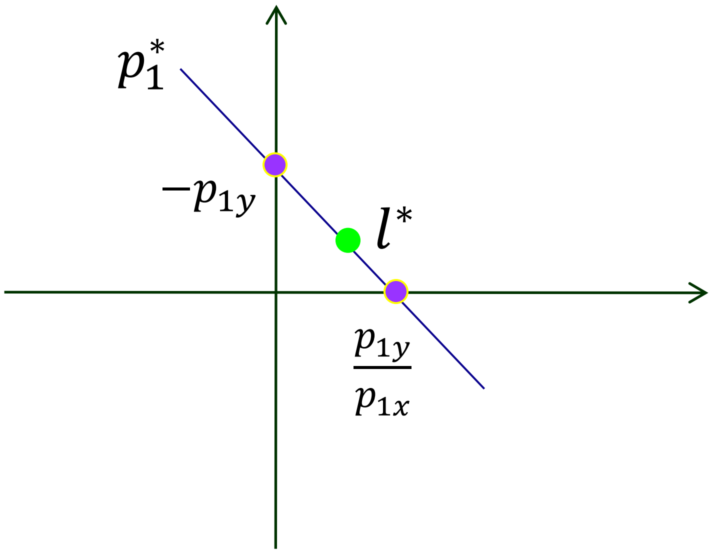
$p = (p_x, p_y) \mapsto p^*\colon\ y = p_x x - p_y$

$l\colon\ y = mx + b \ \mapsto l^* = (m, -b)$

- $(p^*)^* = p$
- $(l^*)^* = l$
- $p \in l \Leftrightarrow l^* \in p^*$?



primal plane



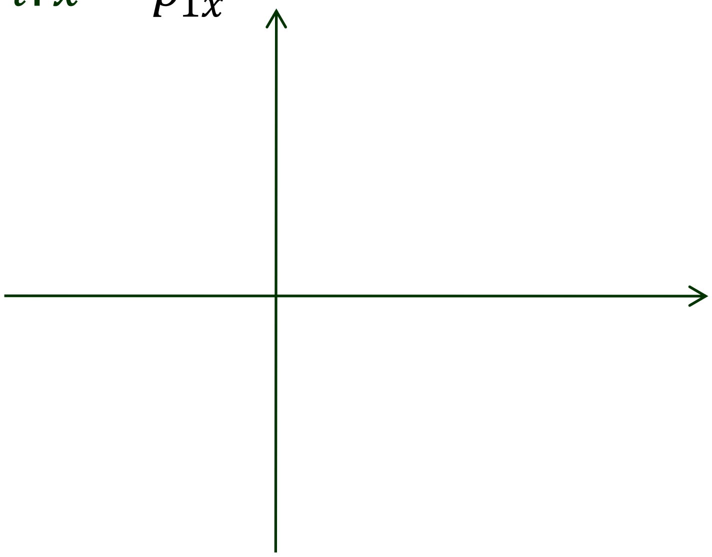
dual plane

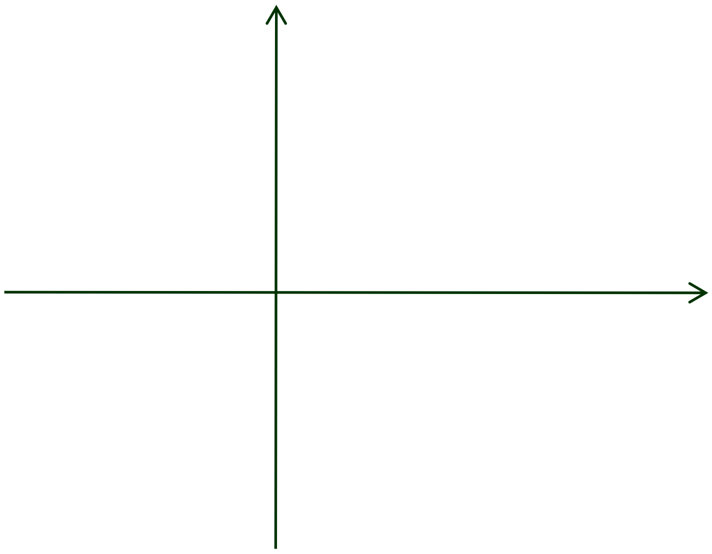
Points on a Vertical Line

$l\colon\ y = mx + b \ \longmapsto l^* = (m, -b)$

♣ No dual defined for a vertical line ($x = a$)!

$l\colon x = p_{1x}$



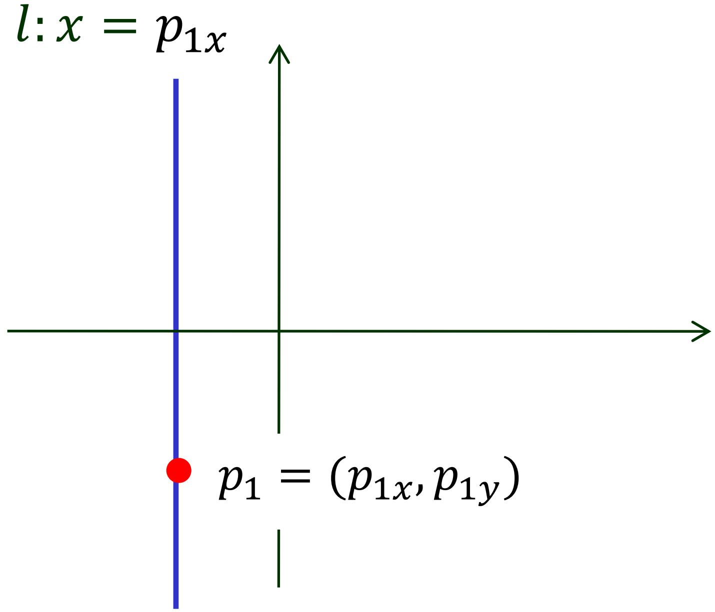


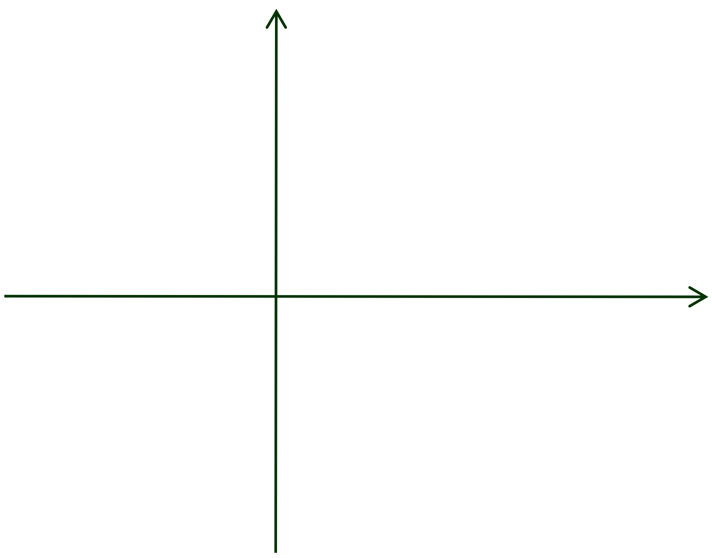
$p = (p_x, p_y) \longmapsto p^*\colon\ y = p_x x - p_y$

Points on a Vertical Line

$l\colon y = mx + b \;\mapsto\; l^* = (m, -b)$

♣ No dual defined for a vertical line ($x = a$)!



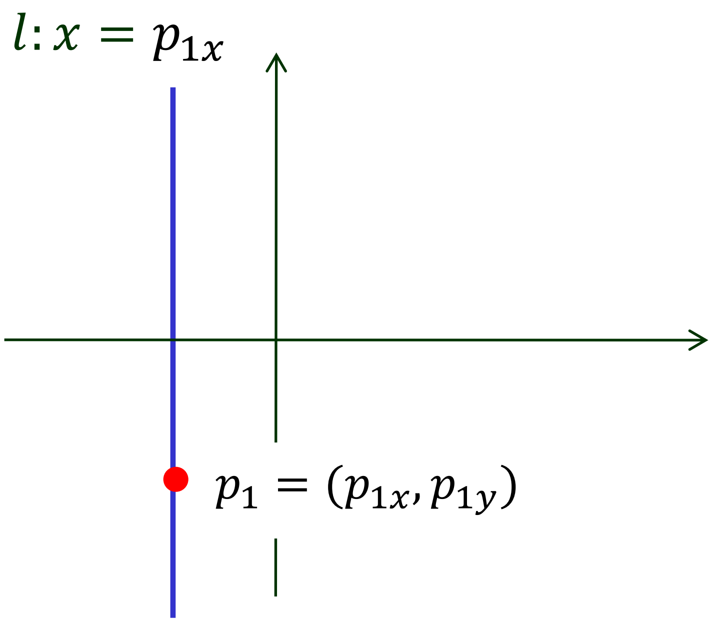


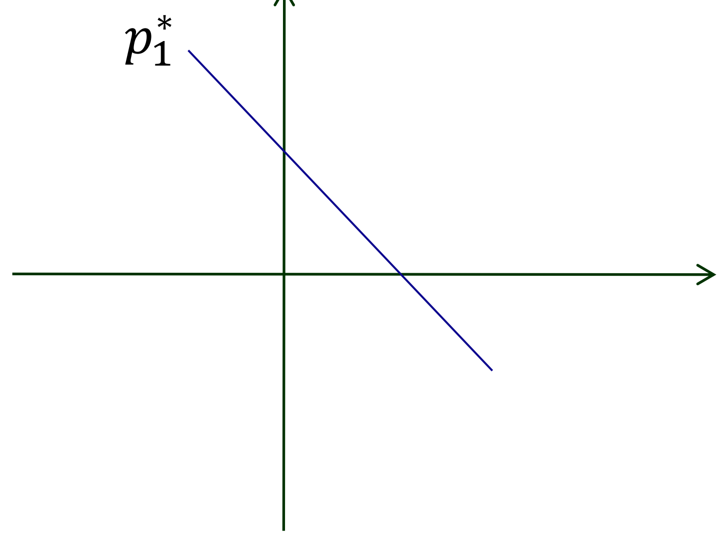
$p = (p_x, p_y) \mapsto p^*\colon\; y = p_x x - p_y$

Points on a Vertical Line

$l\colon\ y = mx + b \ \mapsto\ l^* = (m, -b)$

♣ No dual defined for a vertical line ($x = a$)!





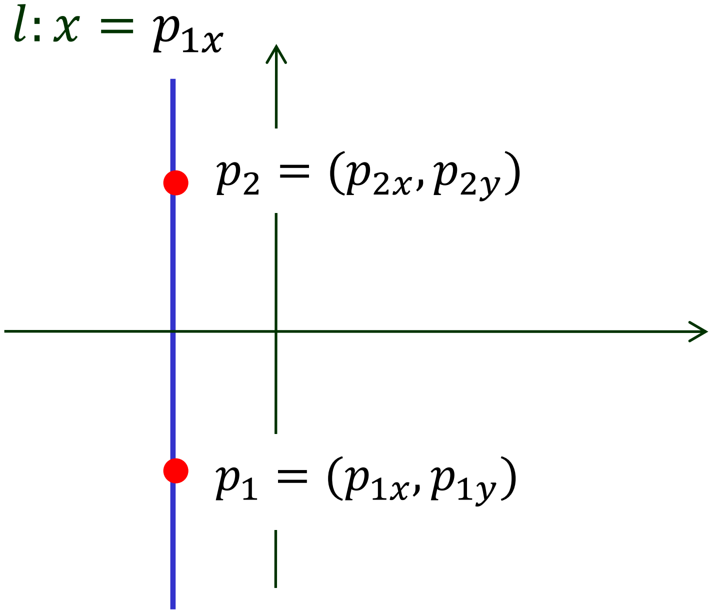
$p = (p_x, p_y) \mapsto p^*\colon\ y = p_x x - p_y$

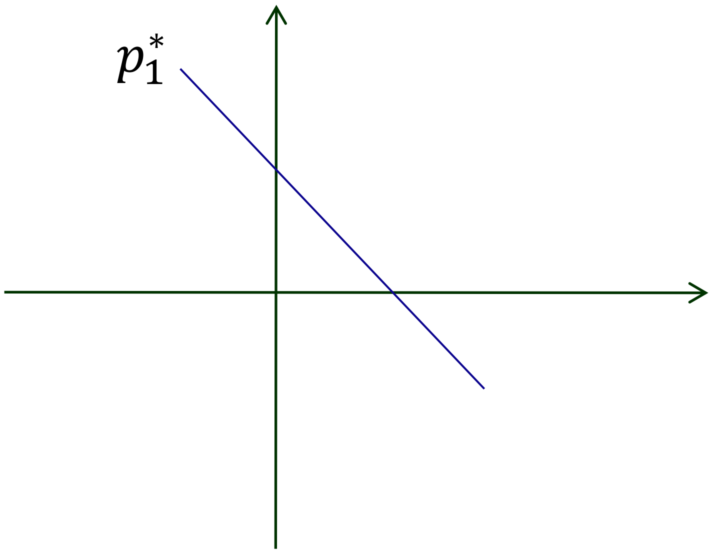
Points on a Vertical Line

l: $y = mx + b \;\longmapsto\; l^* = (m, -b)$

♣ No dual defined for a vertical line ($x = a$)!

$$p_{1x} = p_{2x}$$





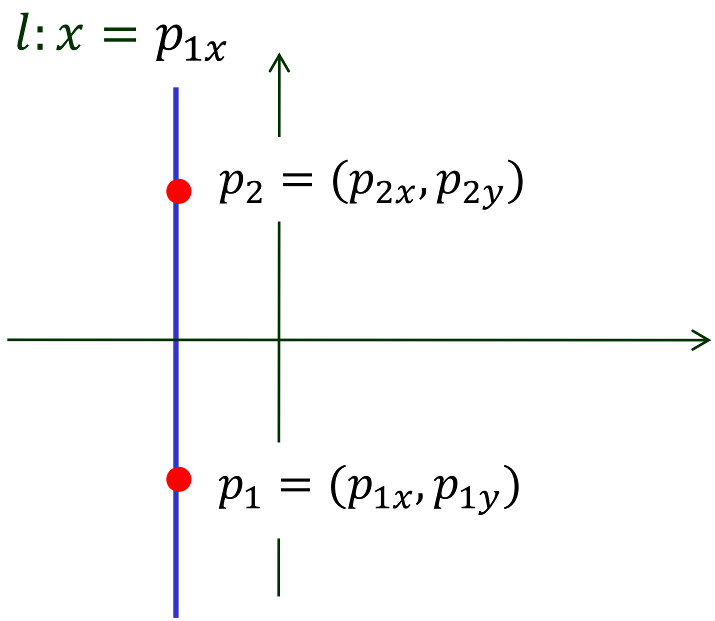
$p = (p_x, p_y) \longmapsto p^*\!: \; y = p_x x - p_y$

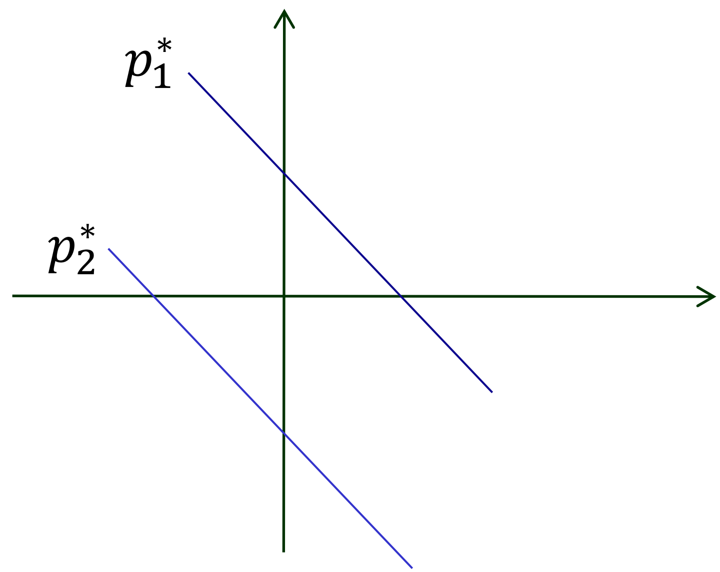
Points on a Vertical Line

$l\colon\ y = mx + b \ \mapsto\ l^* = (m, -b)$

♣ No dual defined for a vertical line ($x = a$)!

$$p_{1x} = p_{2x}$$





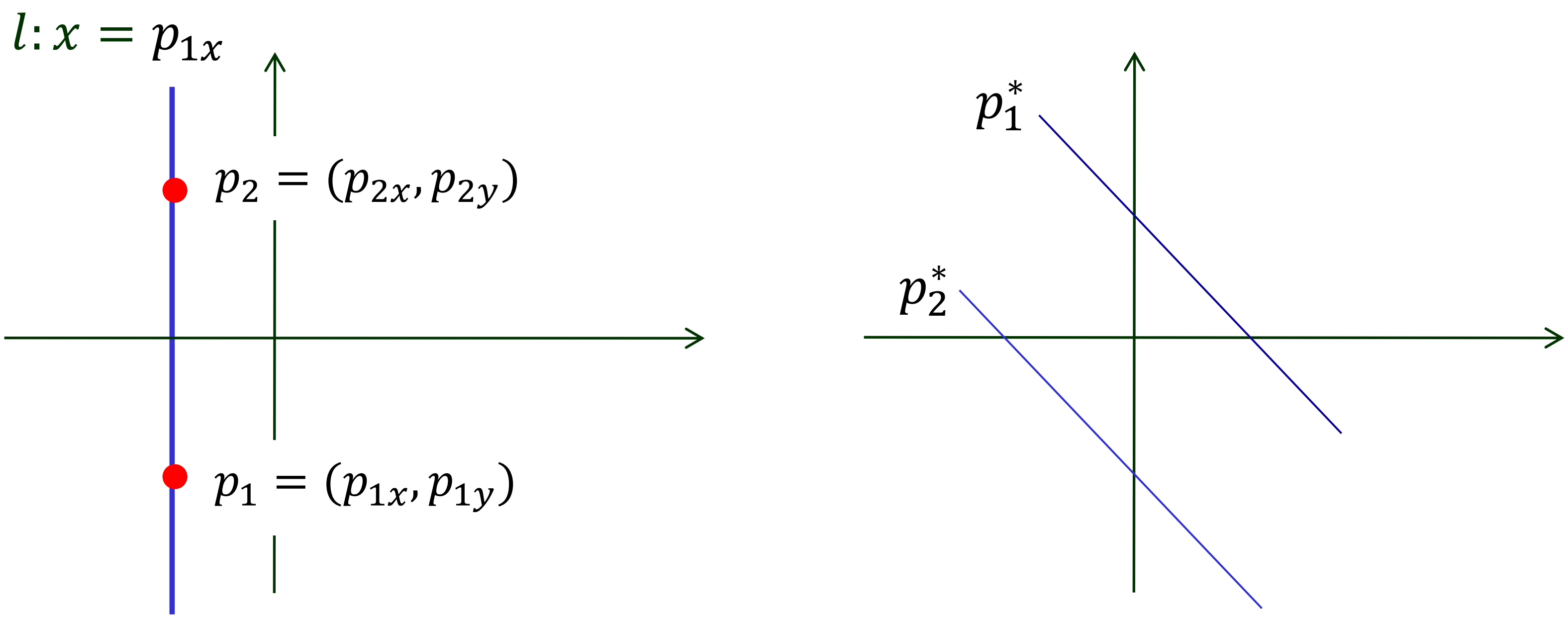
$p = (p_x, p_y) \mapsto p^*\colon\ y = p_x x - p_y$

Points on a Vertical Line

$l\colon y = mx + b \mapsto l^* = (m, -b)$

♣ No dual defined for a vertical line ($x = a$)!

$$p_{1x} = p_{2x} \implies p_1^* \parallel p_2^*$$

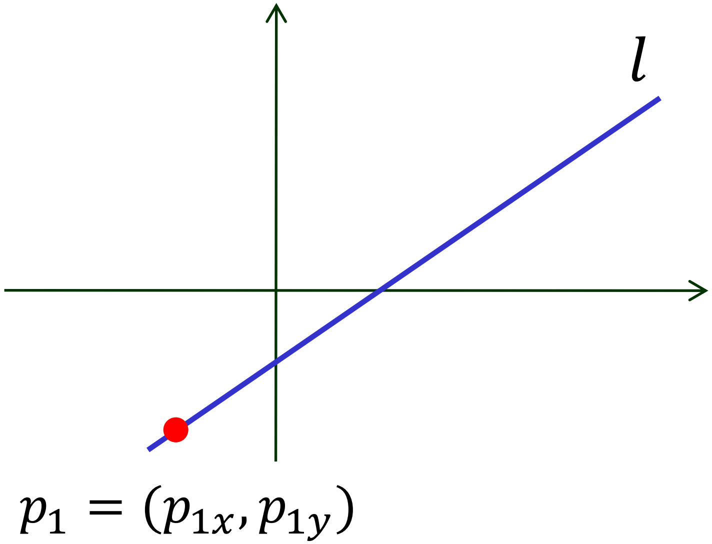


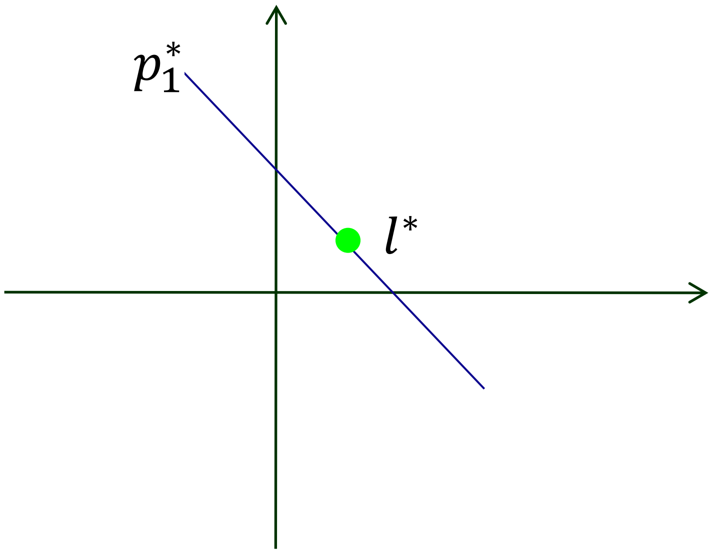
$p = (p_x, p_y) \mapsto p^*\colon y = p_x x - p_y$

Collinear Points

$$p = (p_x, p_y) \longmapsto p^*: \ y = p_x x - p_y$$

$$l: \ y = mx + b \ \longmapsto l^* = (m, -b)$$

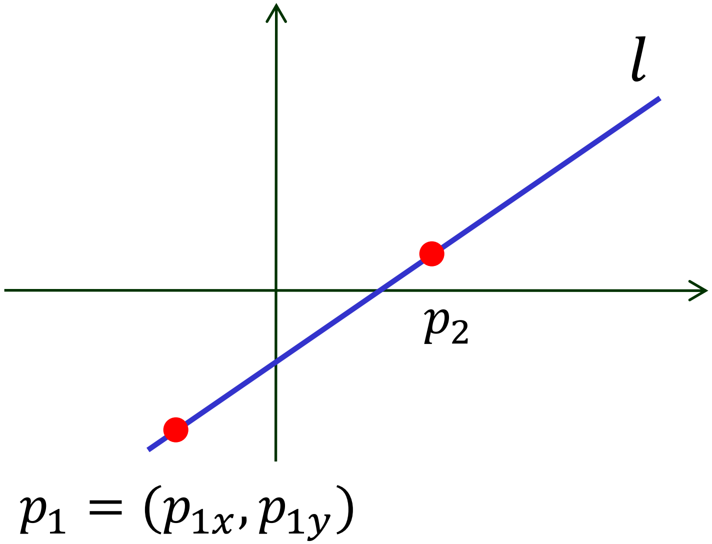


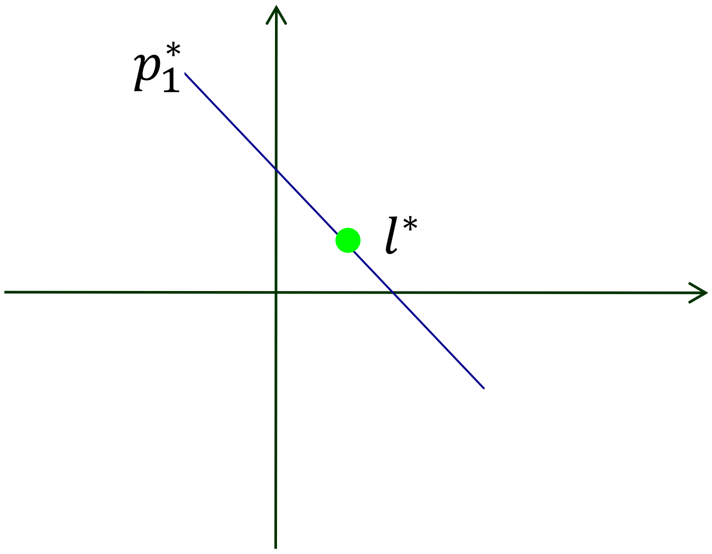


Collinear Points

$p = (p_x, p_y) \longmapsto p^*\colon\ y = p_x x - p_y$

$l\colon\ y = mx + b \ \longmapsto l^* = (m, -b)$

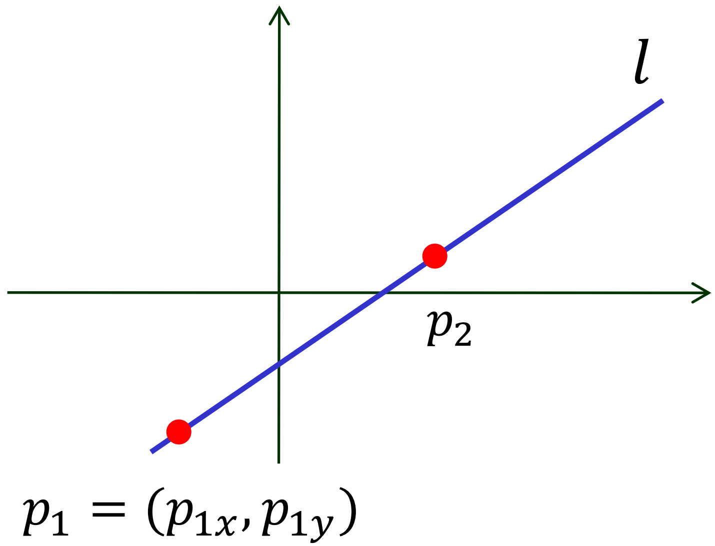


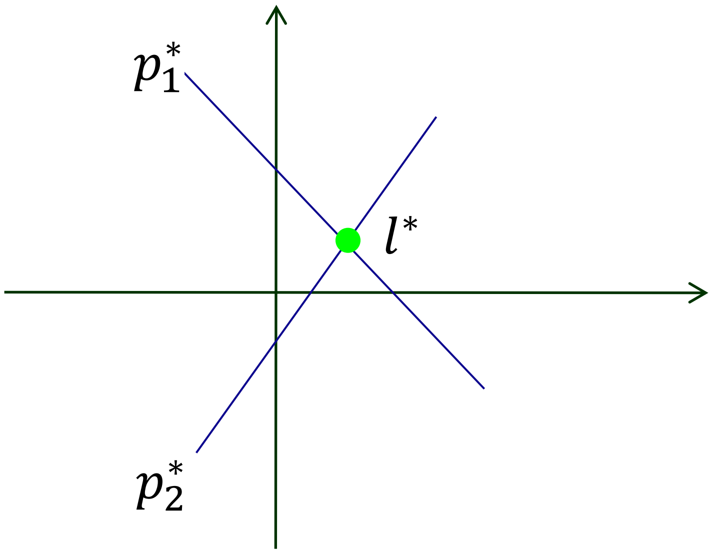


Collinear Points

$$p = (p_x, p_y) \longmapsto p^*: \; y = p_x x - p_y$$

$$l: \; y = mx + b \; \longmapsto l^* = (m, -b)$$

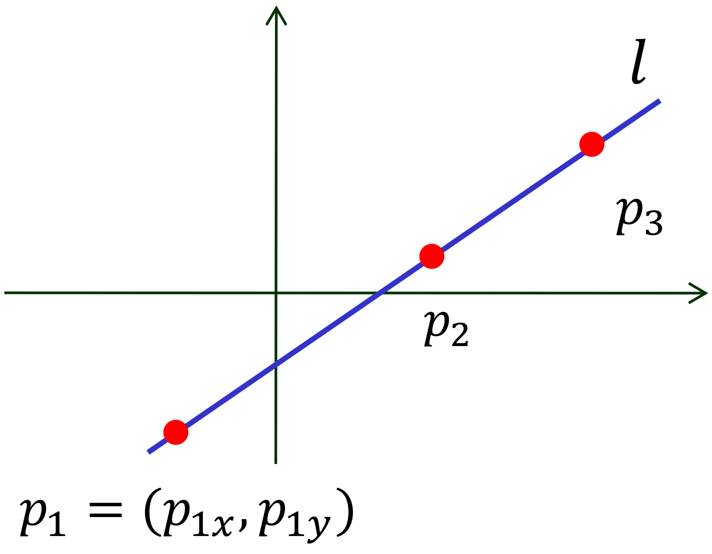


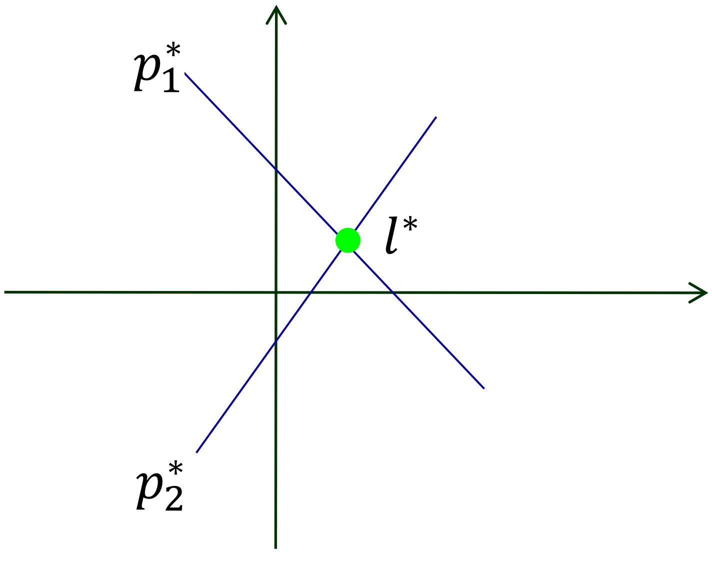


Collinear Points

$$p = (p_x, p_y) \longmapsto p^*\colon\ y = p_x x - p_y$$

$$l\colon\ y = mx + b \ \longmapsto l^* = (m, -b)$$

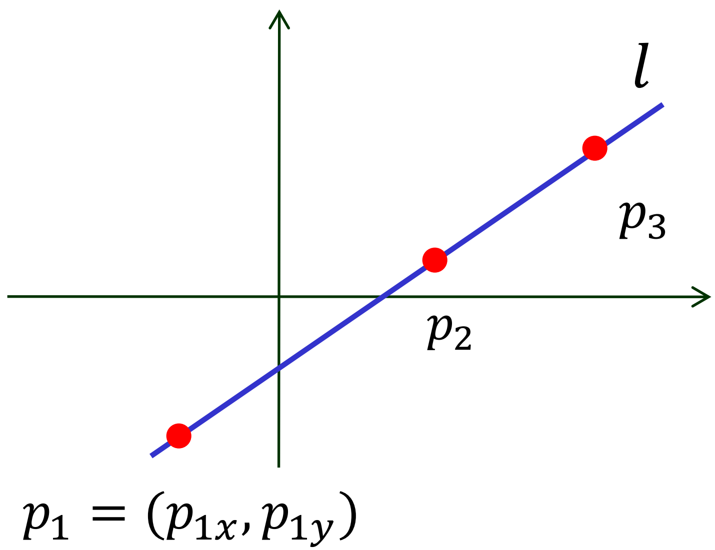


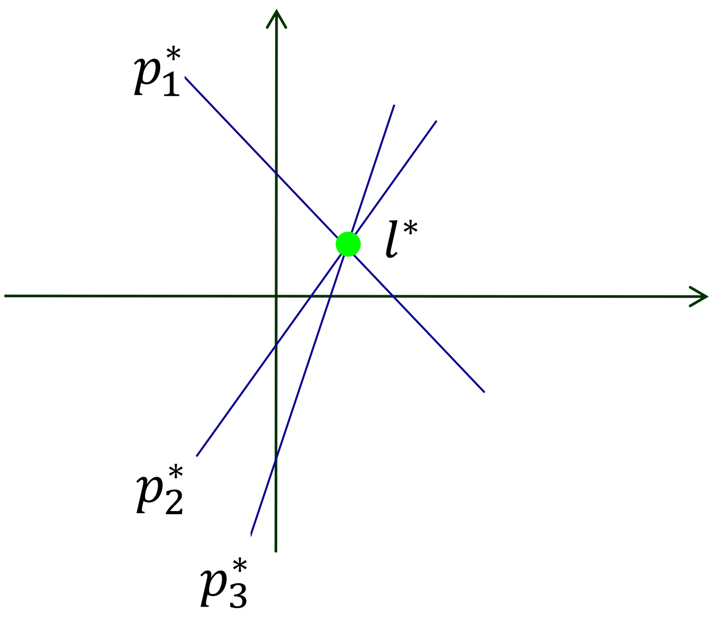


Collinear Points

$p = (p_x, p_y) \mapsto p^*\colon\ y = p_x x - p_y$

$l\colon\ y = mx + b \ \mapsto l^* = (m, -b)$



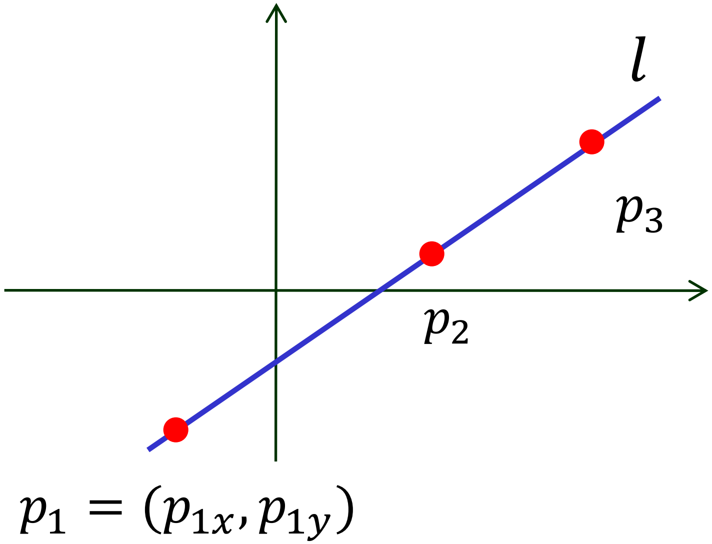


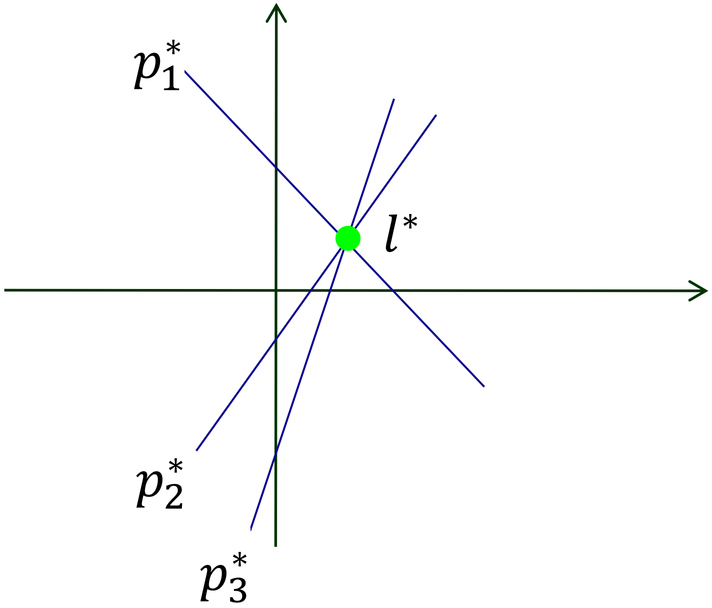
Collinear Points

p_1, p_2, p_3 *collinear* on the line l

$p = (p_x, p_y) \longmapsto p^*\colon\ y = p_x x - p_y$

$l\colon\ y = mx + b \ \longmapsto l^* = (m, -b)$





Collinear Points

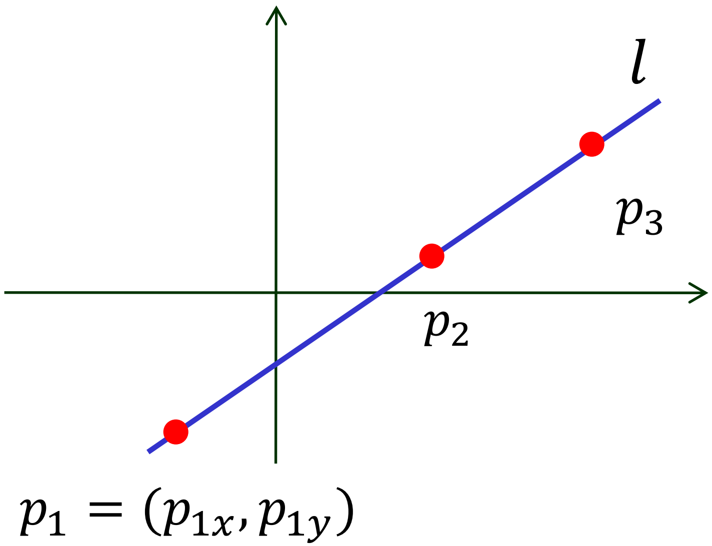
$p = (p_x, p_y) \mapsto p^*: \; y = p_x x - p_y$

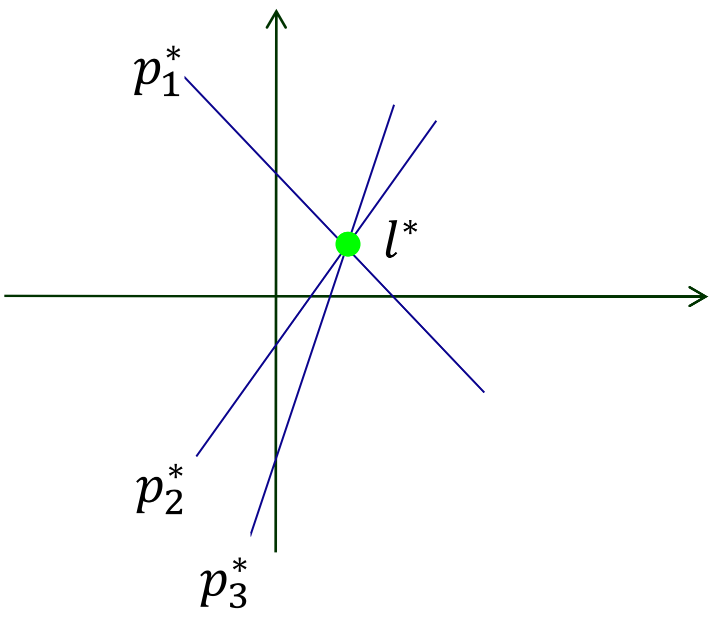
$l: \; y = mx + b \; \mapsto l^* = (m, -b)$

p_1, p_2, p_3 *collinear* on the line l

$\Downarrow$

The dual point l^* on the dual lines p_1^*, p_2^*, p_3^*





Collinear Points

p_1, p_2, p_3 *collinear* on the line l

$p = (p_x, p_y) \mapsto p^*\colon\ y = p_x x - p_y$

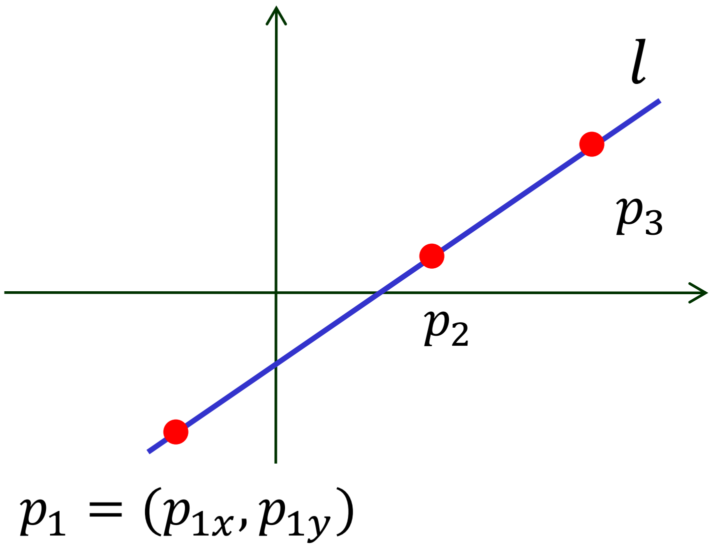
$l\colon\ y = mx + b \ \mapsto l^* = (m, -b)$

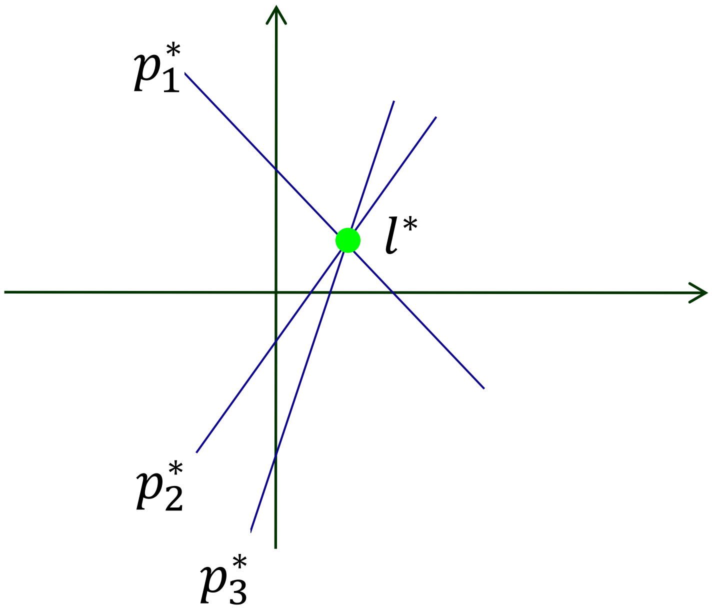
$\Downarrow$

The dual point l^* on the dual lines p_1^*, p_2^*, p_3^*

$\Downarrow$

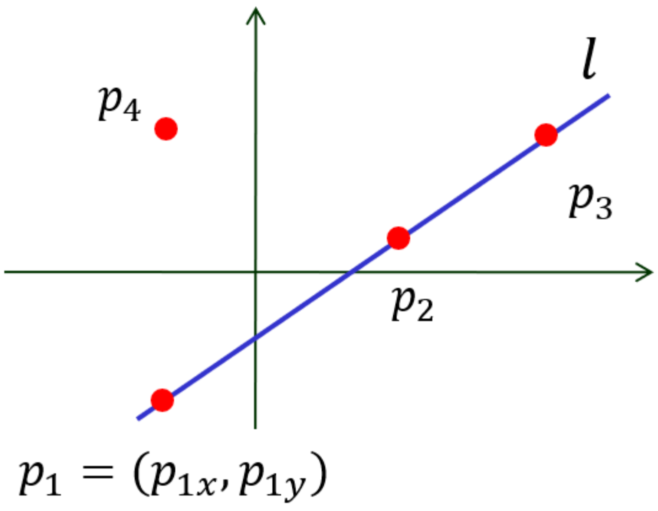
p_1^*, p_2^*, p_3^* *concurrent* at l^*

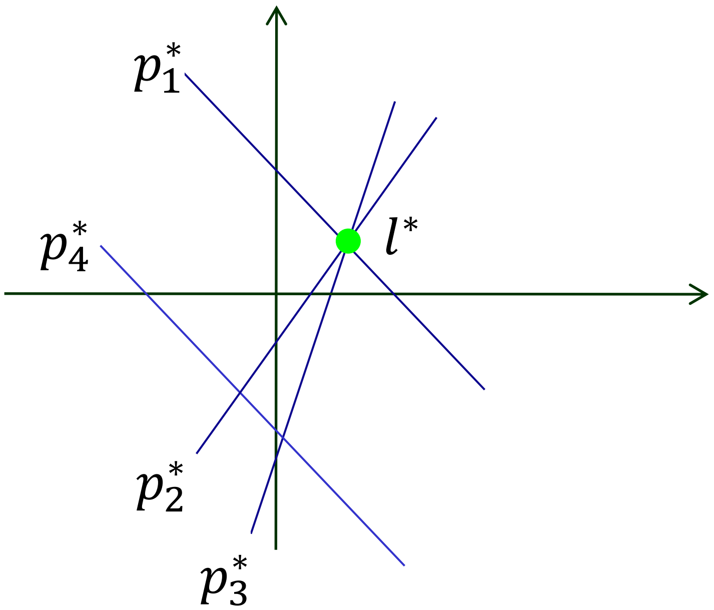




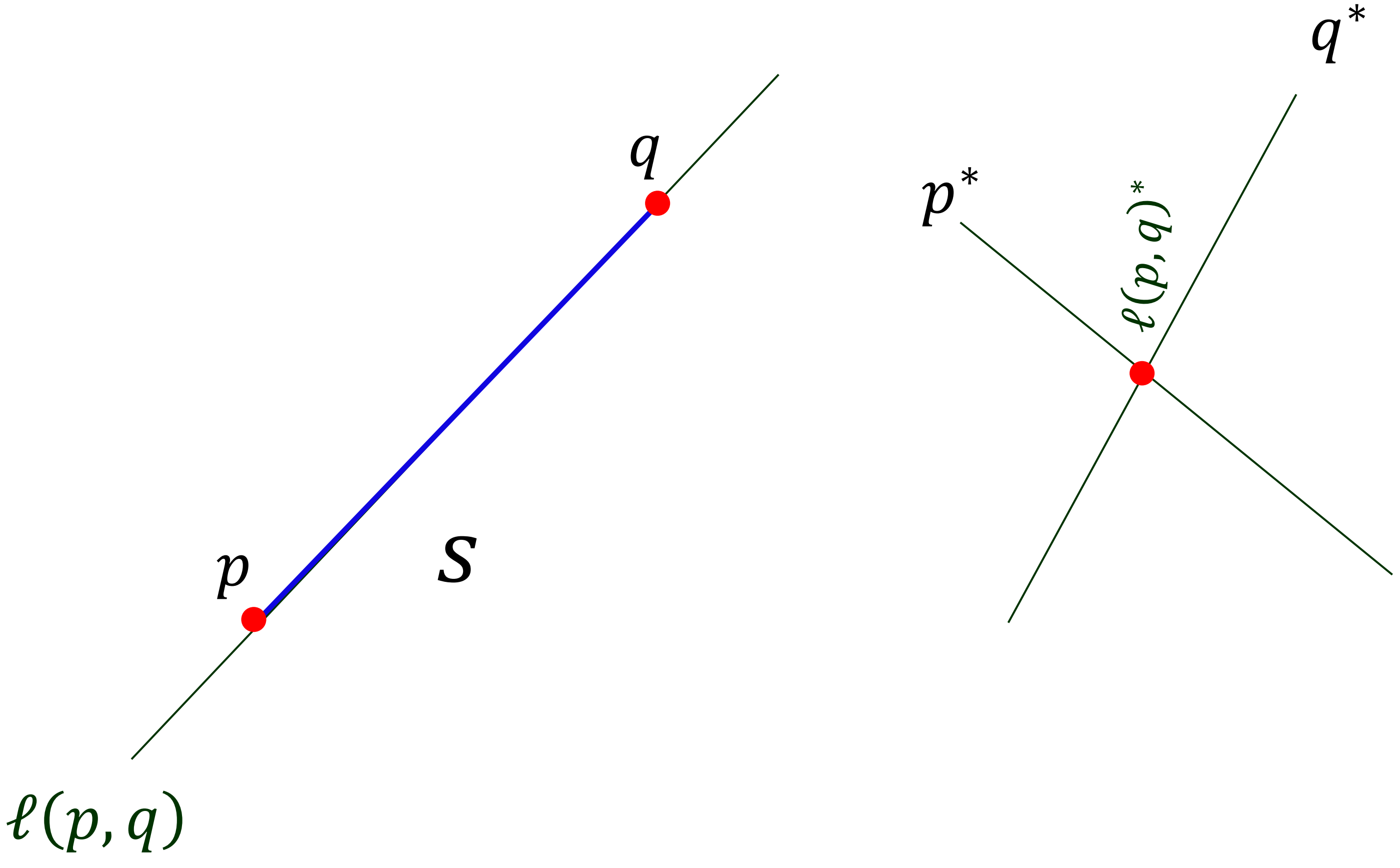
Relative Point-Line Position

- p lies above l if and only if l^* lies above p^*.

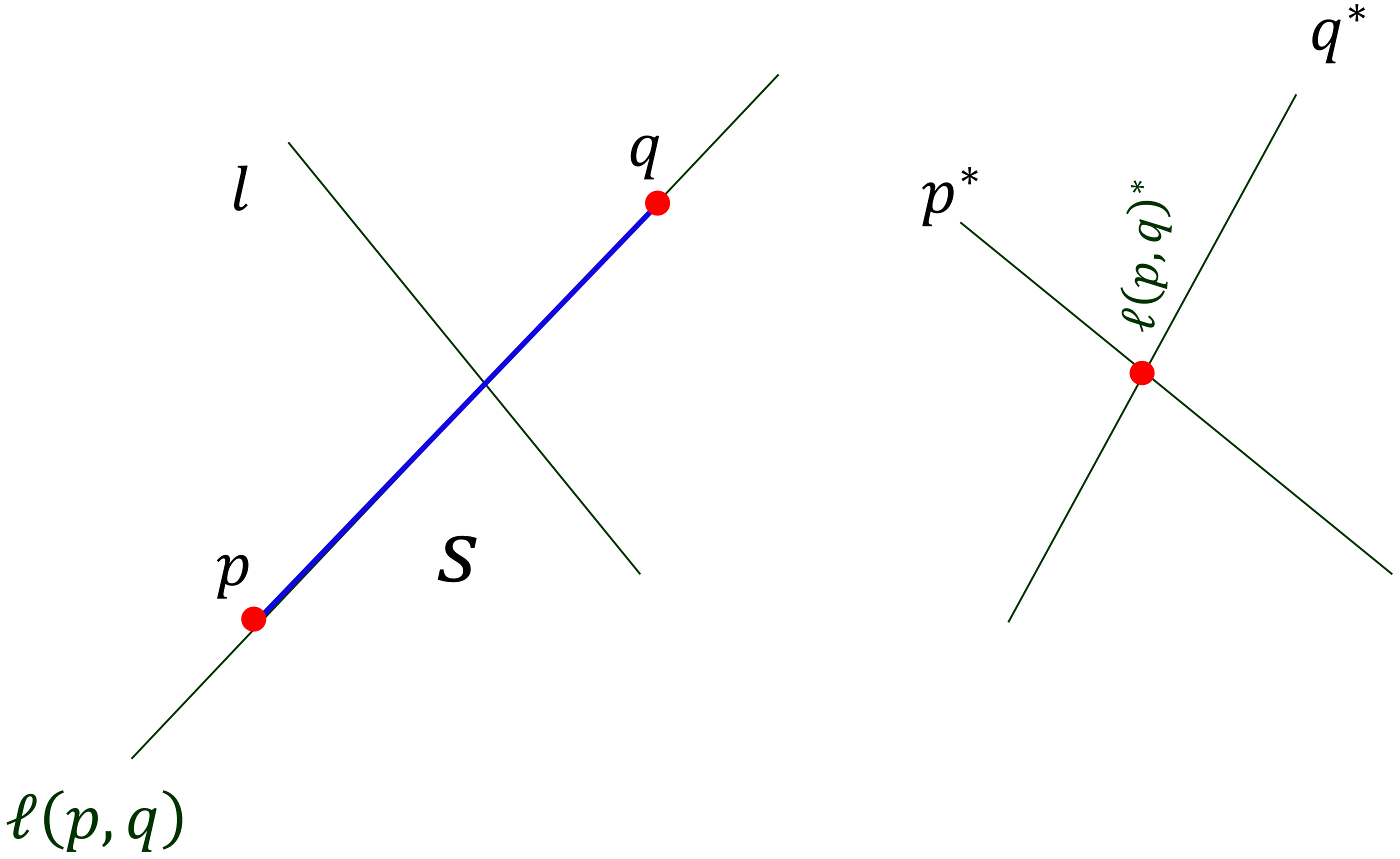




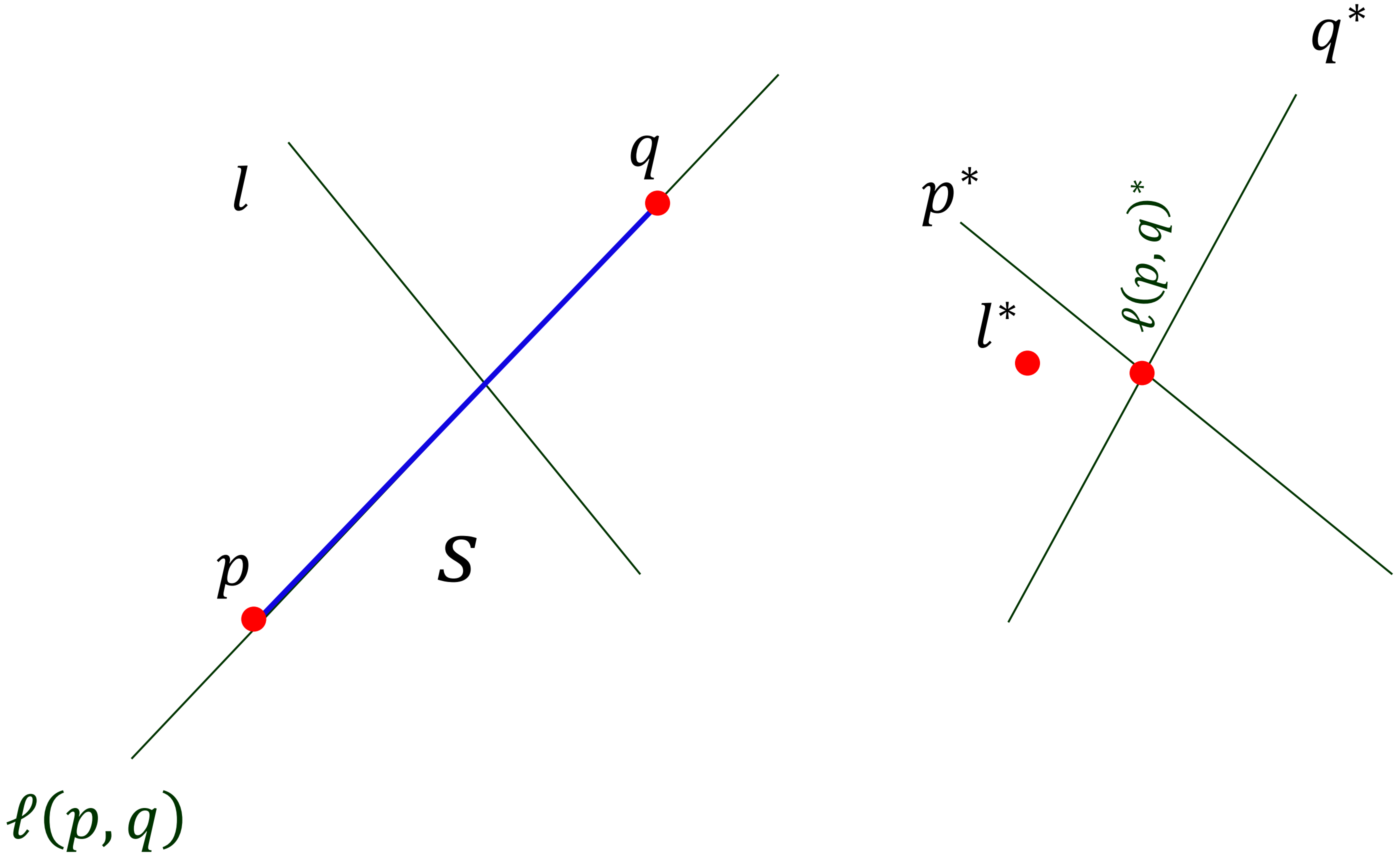
II. Duals of Points on a Line Segment?



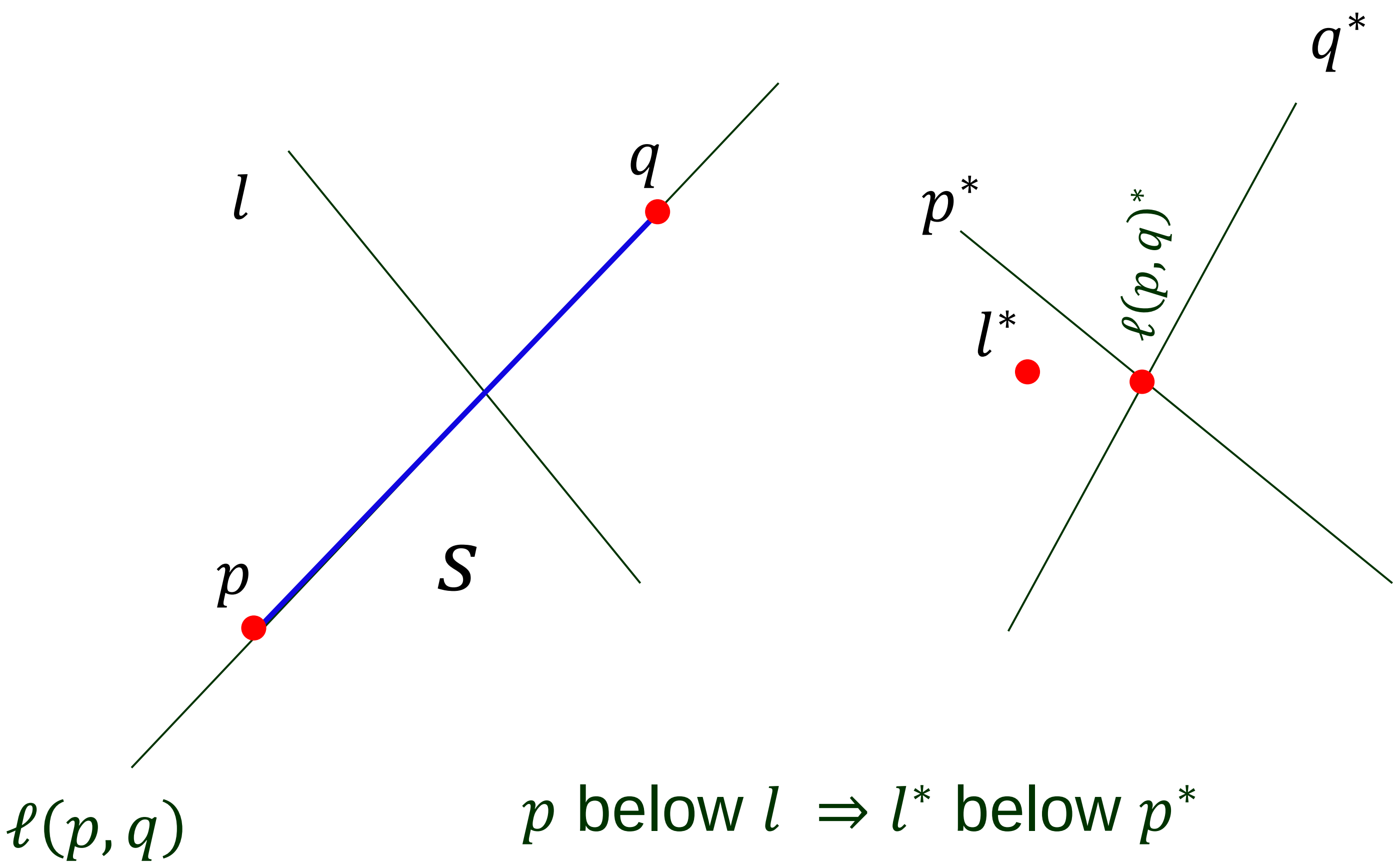
II. Duals of Points on a Line Segment?



II. Duals of Points on a Line Segment?



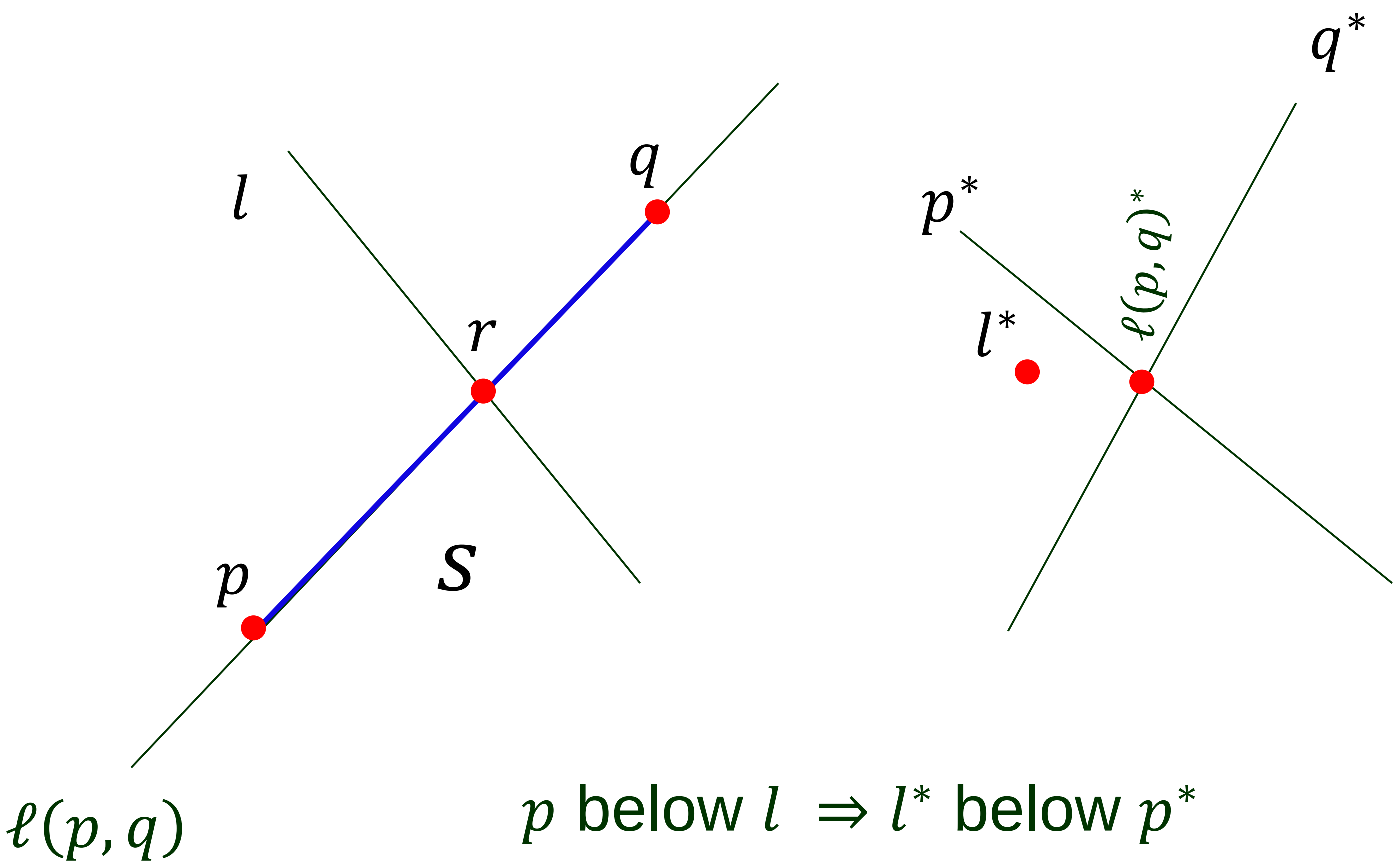
II. Duals of Points on a Line Segment?



p below $l \Rightarrow l^*$ below p^*

q above $l \Rightarrow l^*$ above q^*

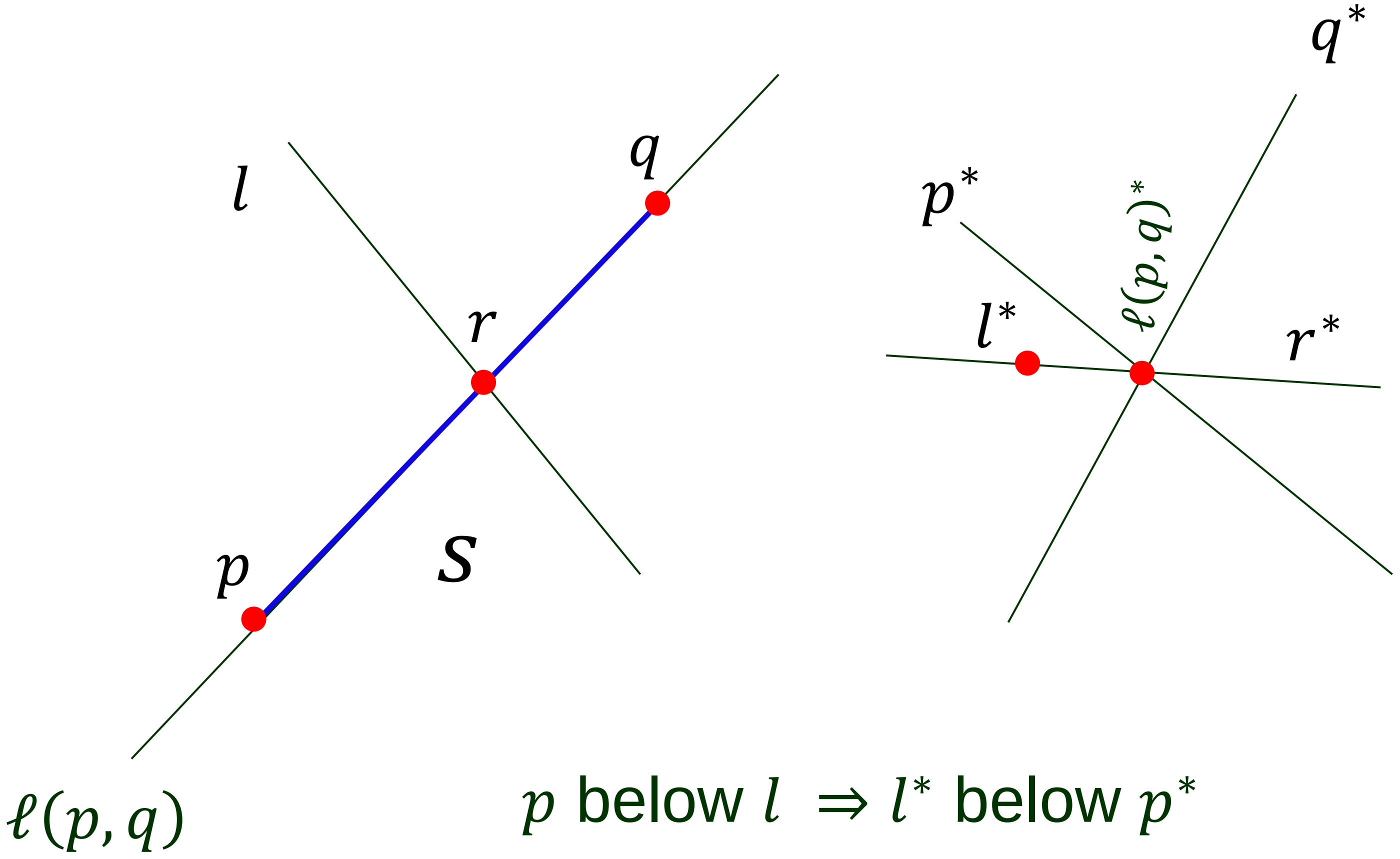
II. Duals of Points on a Line Segment?



p below l $\Rightarrow$ l^* below p^*

q above l $\Rightarrow$ l^* above q^*

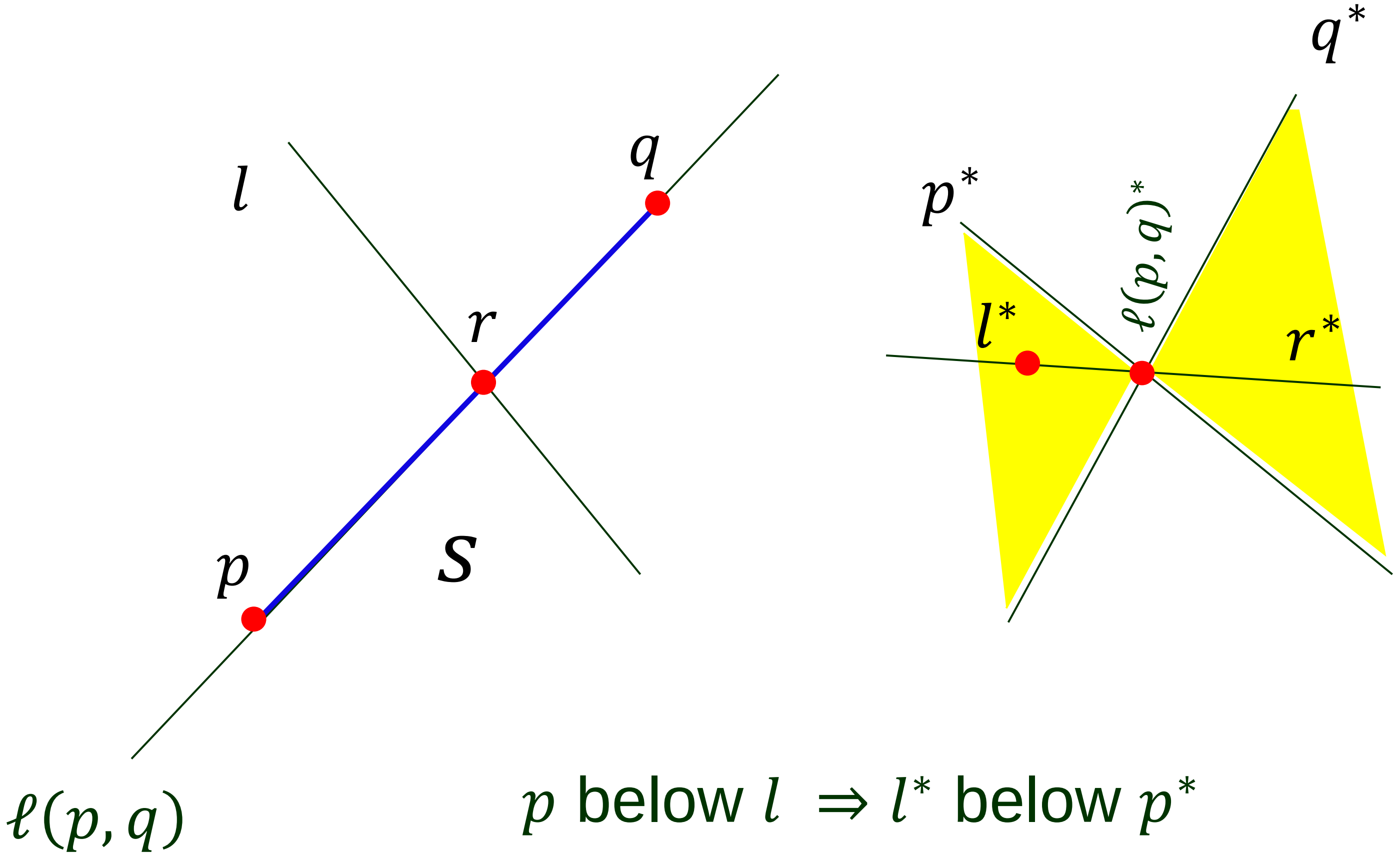
II. Duals of Points on a Line Segment?



p below $l \Rightarrow l^*$ below p^*

q above $l \Rightarrow l^*$ above q^*

II. Duals of Points on a Line Segment?

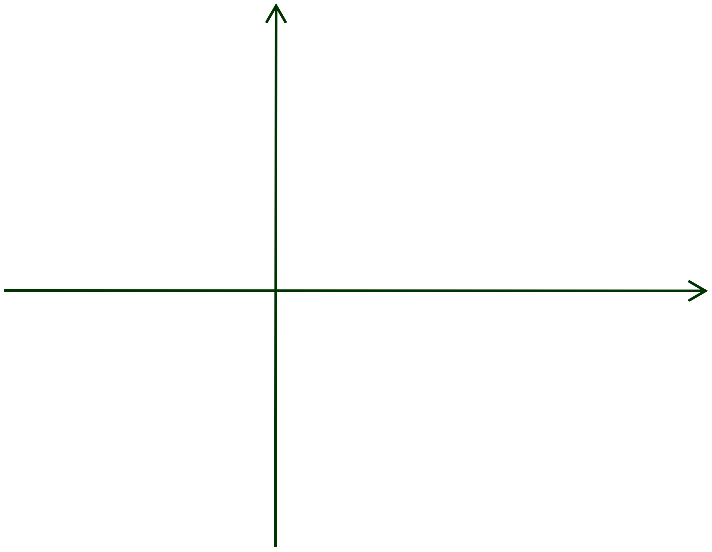


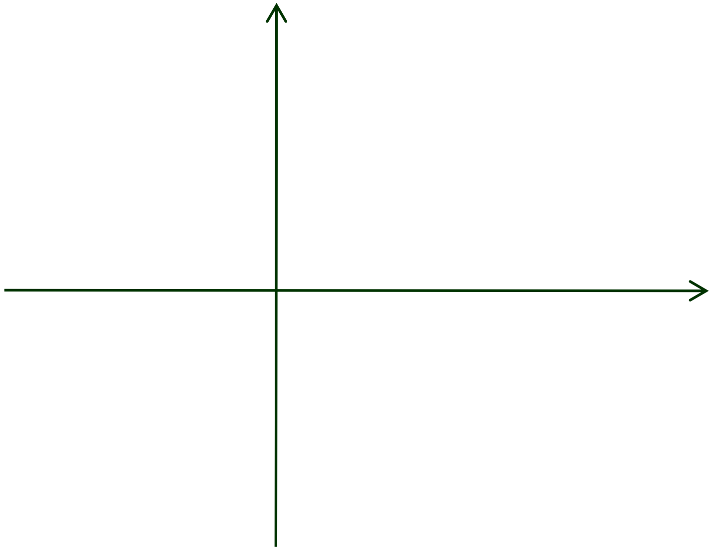
p below l $\Rightarrow$ l^* below p^*

q above l $\Rightarrow$ l^* above q^*

Dual of the x-axis

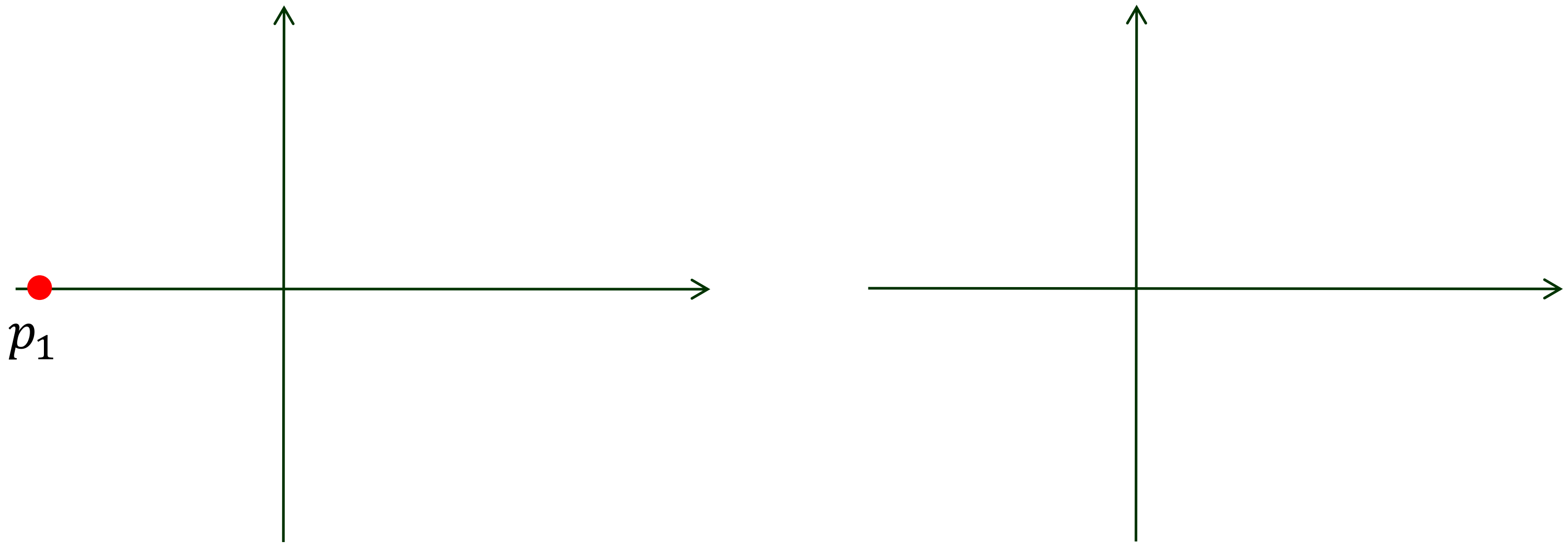
x-axis $= \{(k,0)\} \longmapsto \{y = kx\}$





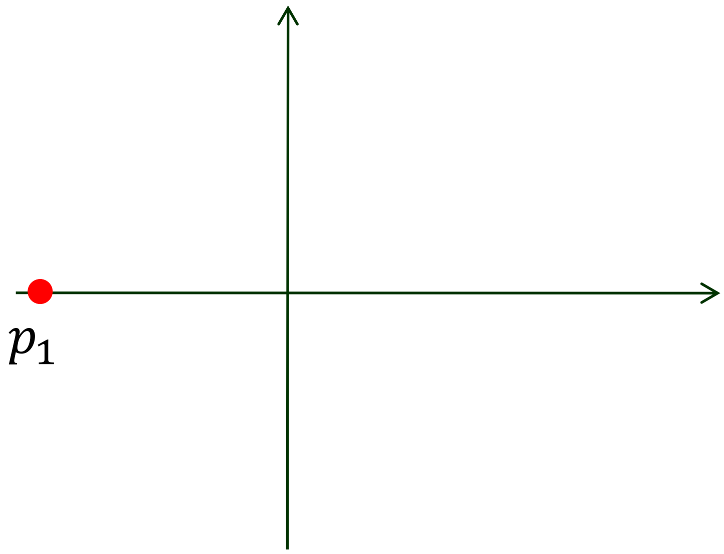
Dual of the x-axis

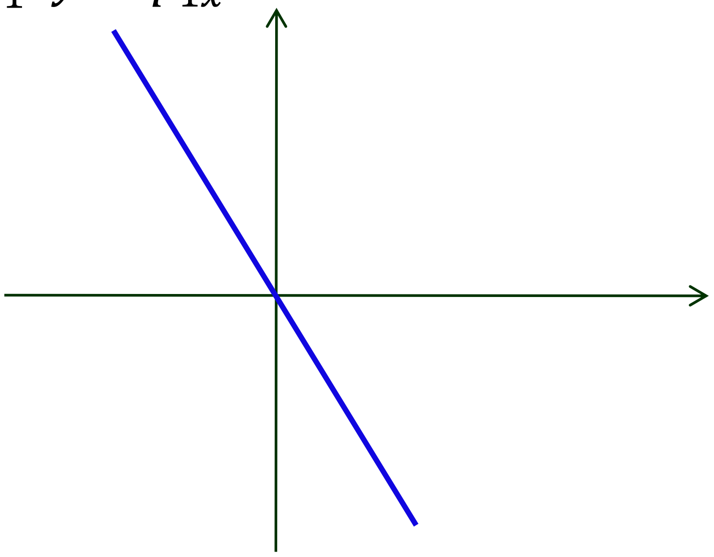
x-axis $= \{(k, 0)\} \mapsto \{y = kx\}$



Dual of the x-axis

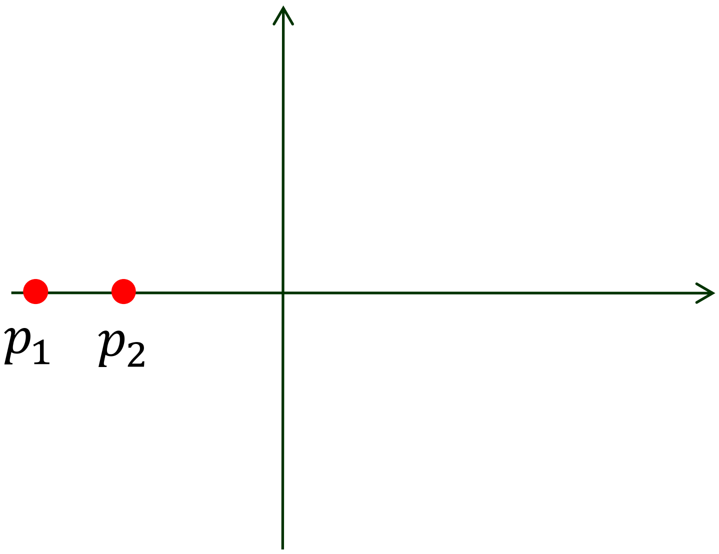
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$

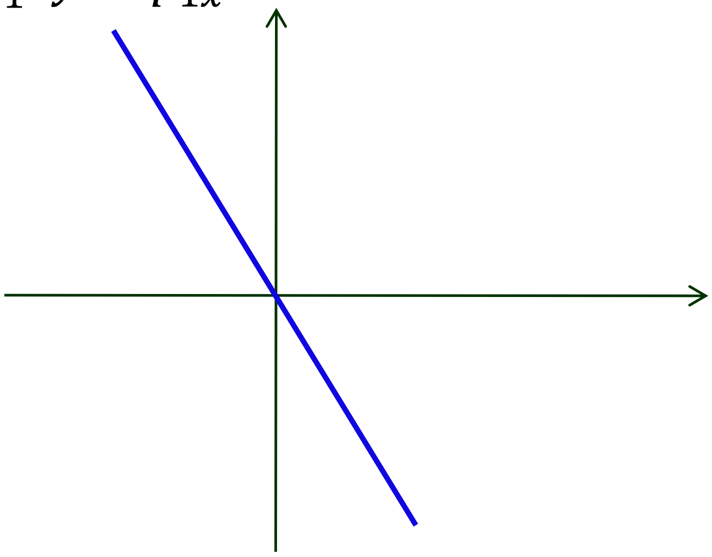




Dual of the x-axis

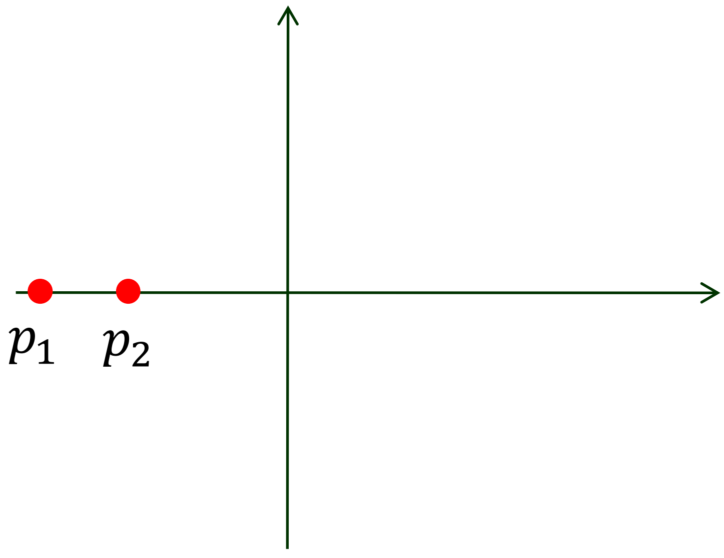
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$

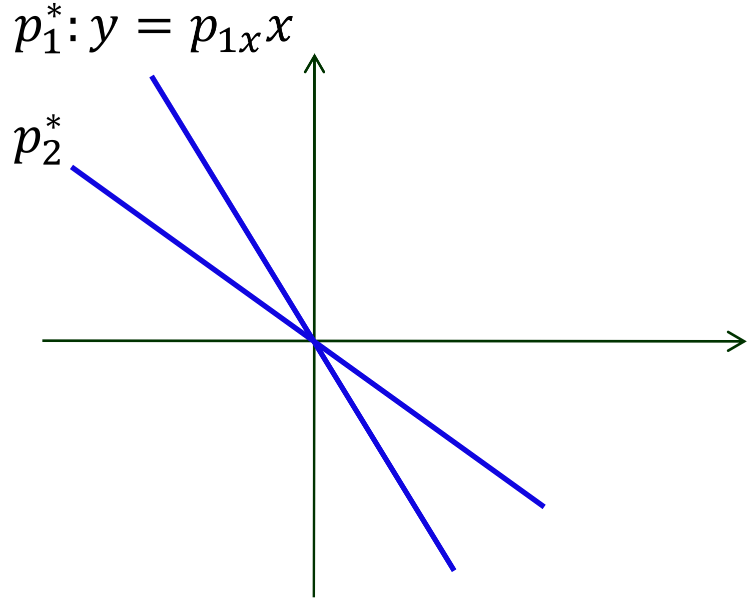




Dual of the x-axis

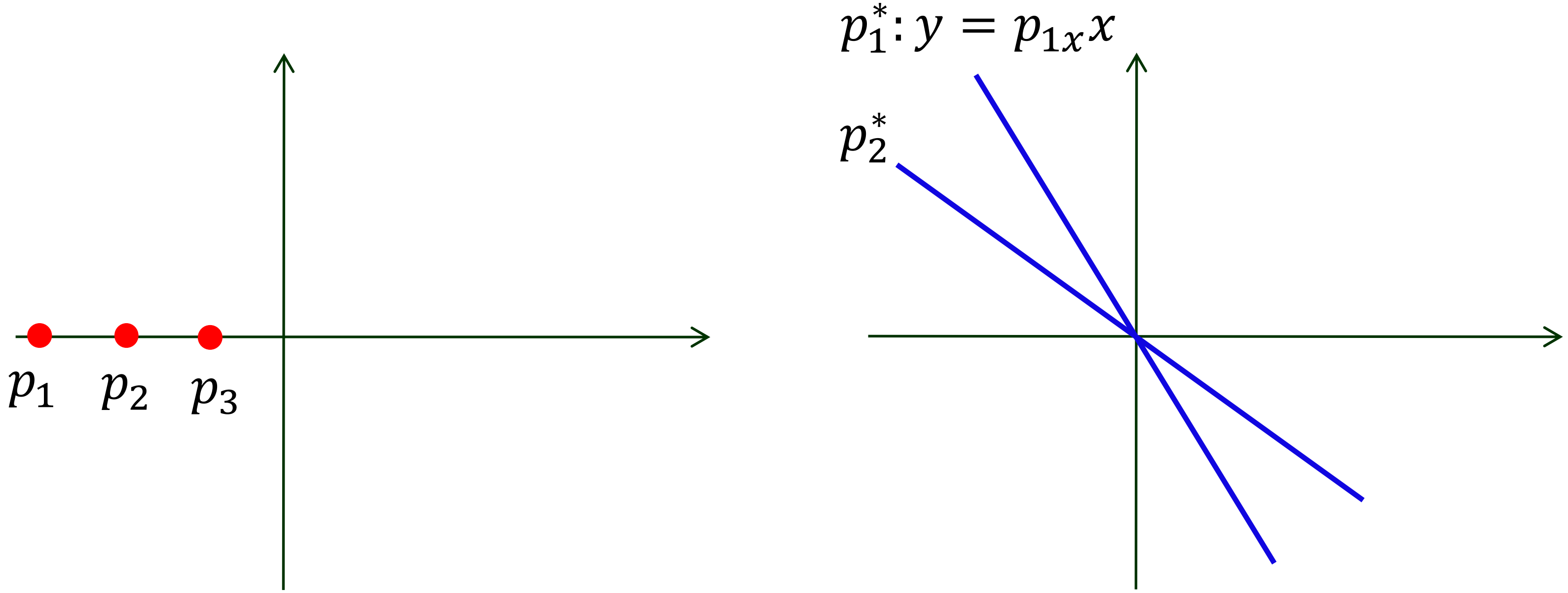
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$





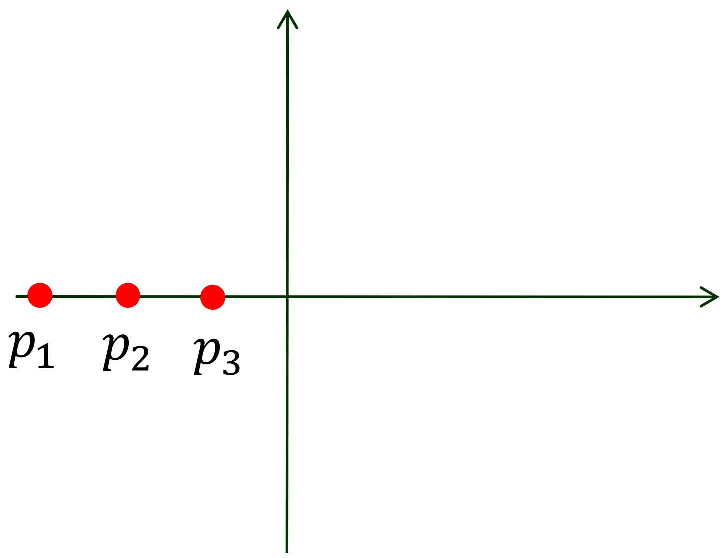
Dual of the x-axis

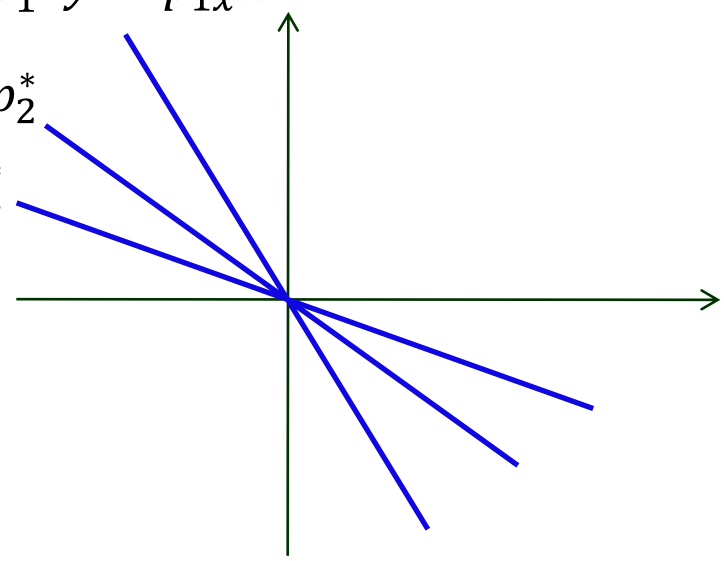
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$



Dual of the x-axis

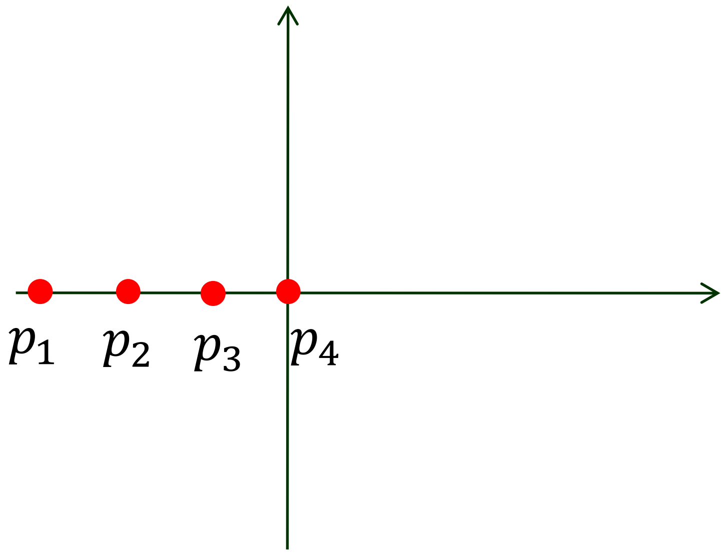
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$

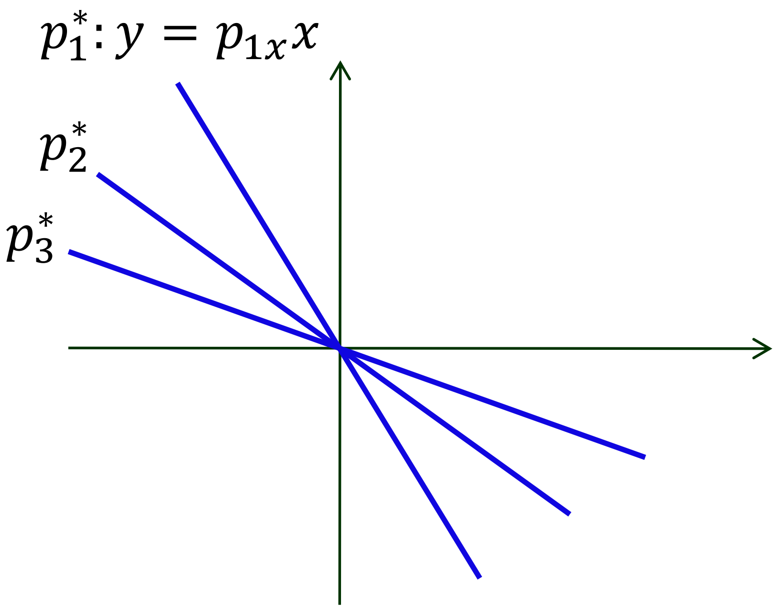




Dual of the x-axis

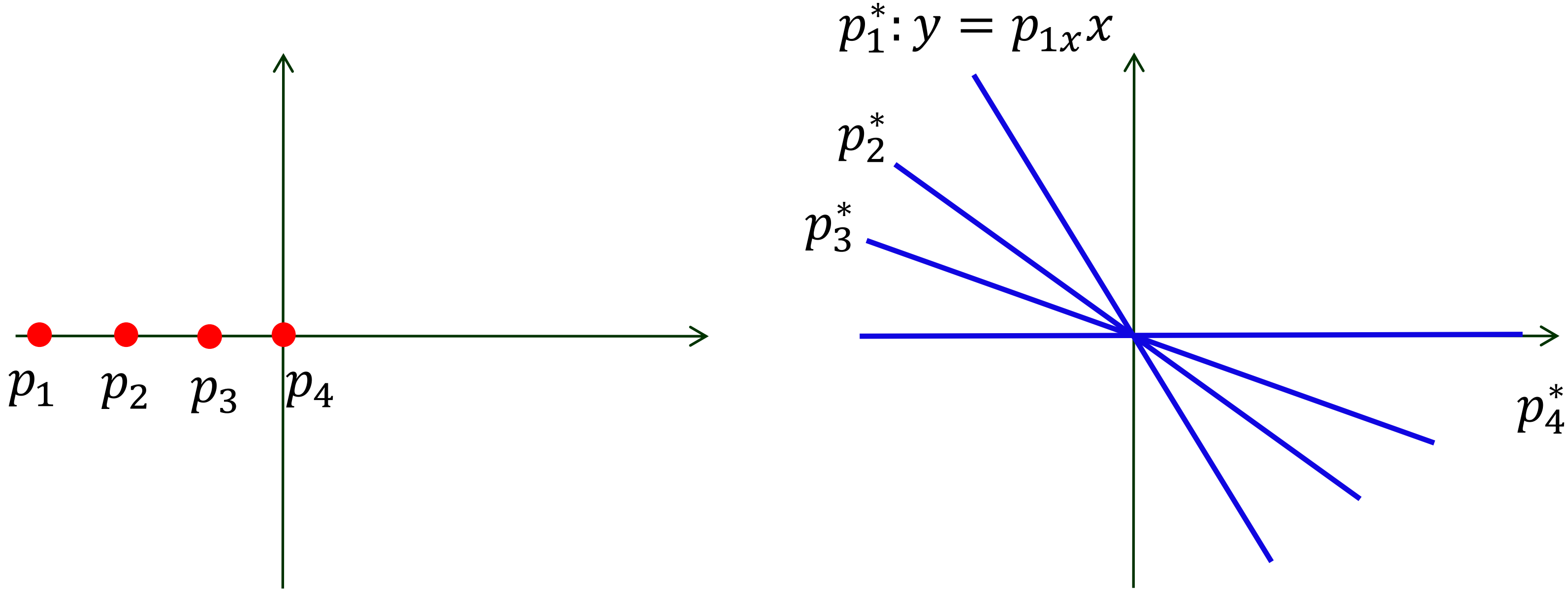
$$x\text{-axis} = \{(k, 0)\} \longmapsto \{y = kx\}$$





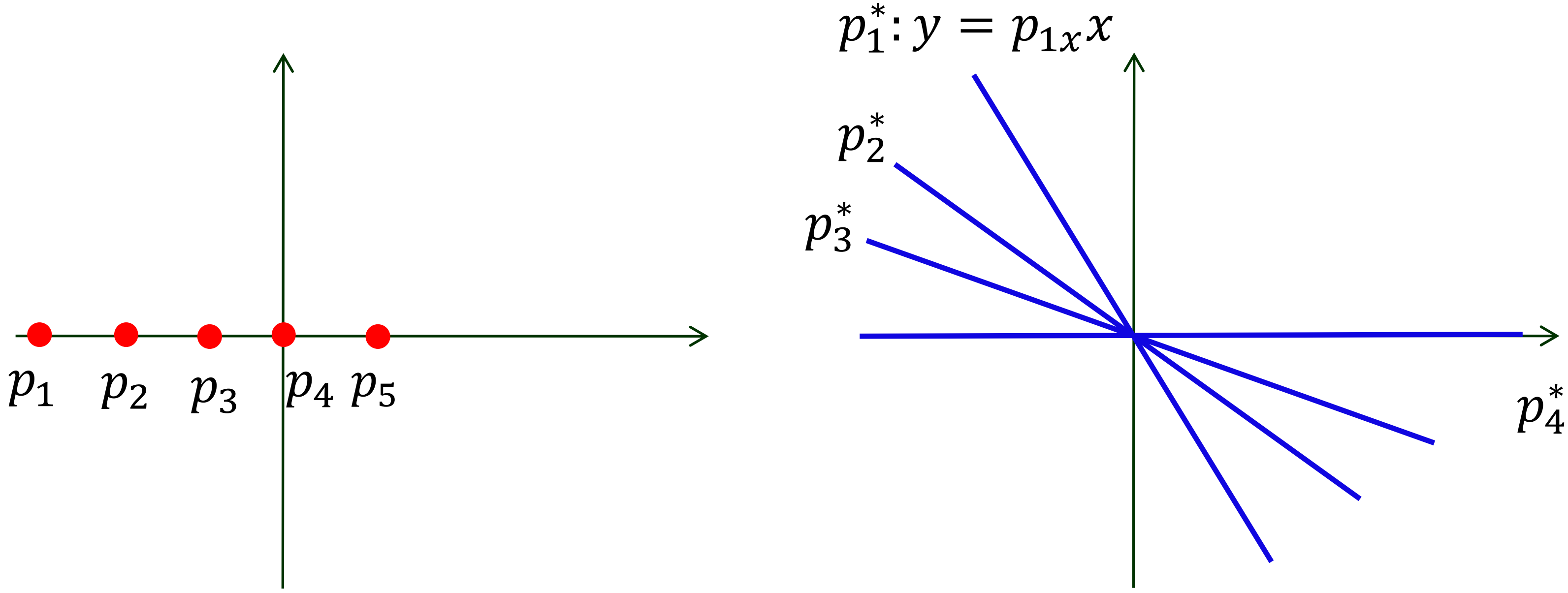
Dual of the x-axis

x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$



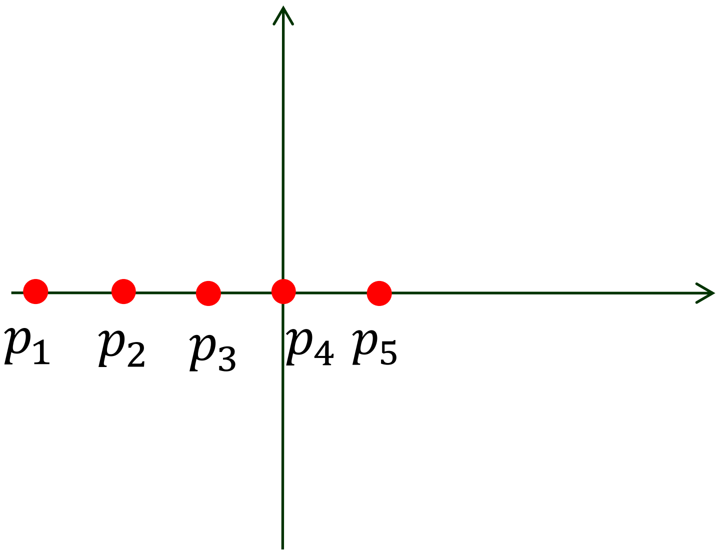
Dual of the x-axis

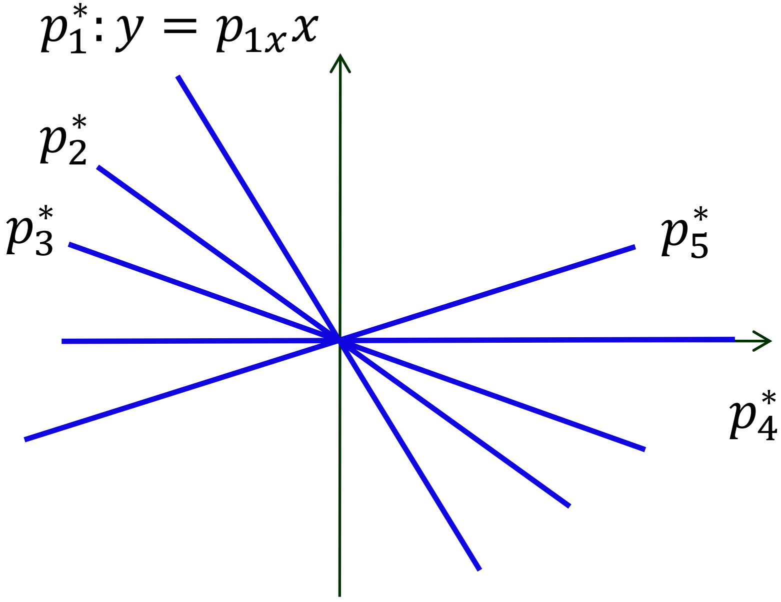
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$



Dual of the x-axis

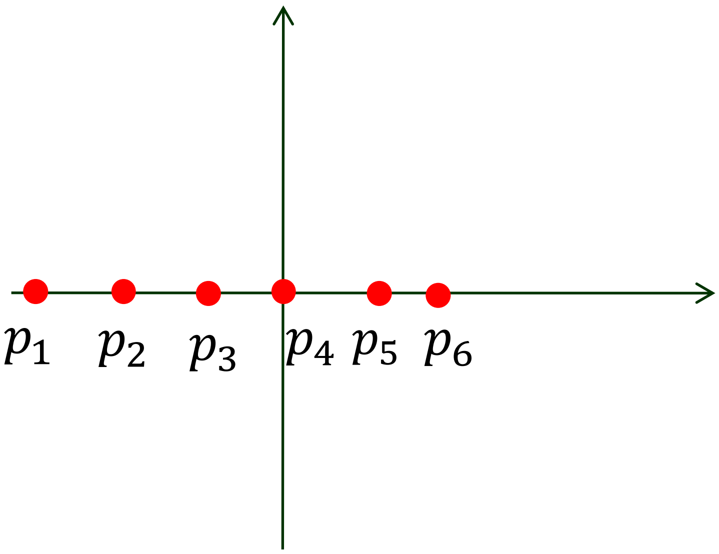
$x\text{-axis} = \{(k, 0)\} \longmapsto \{y = kx\}$

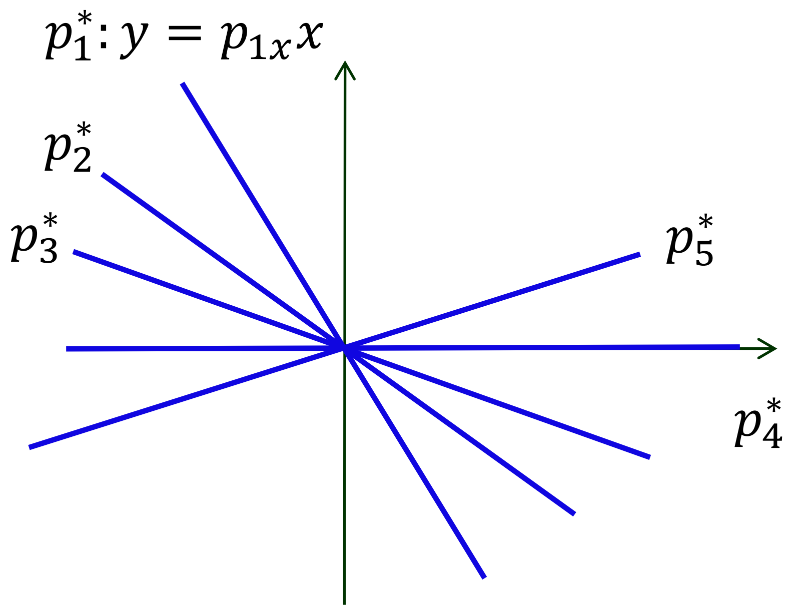




Dual of the x-axis

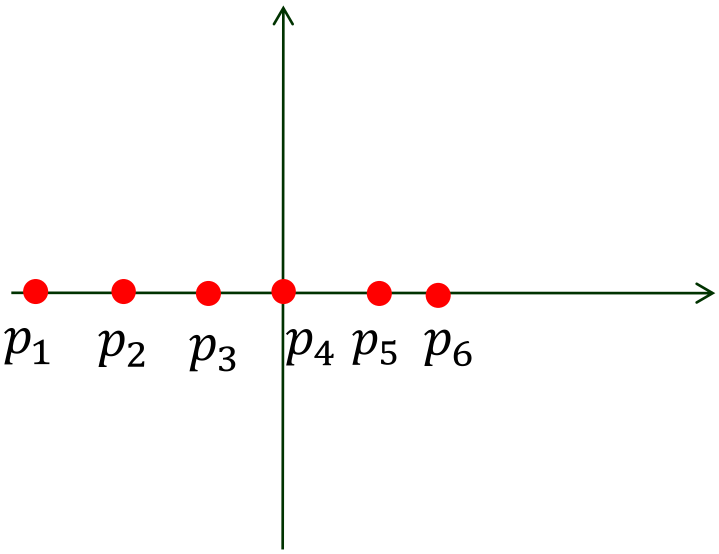
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$

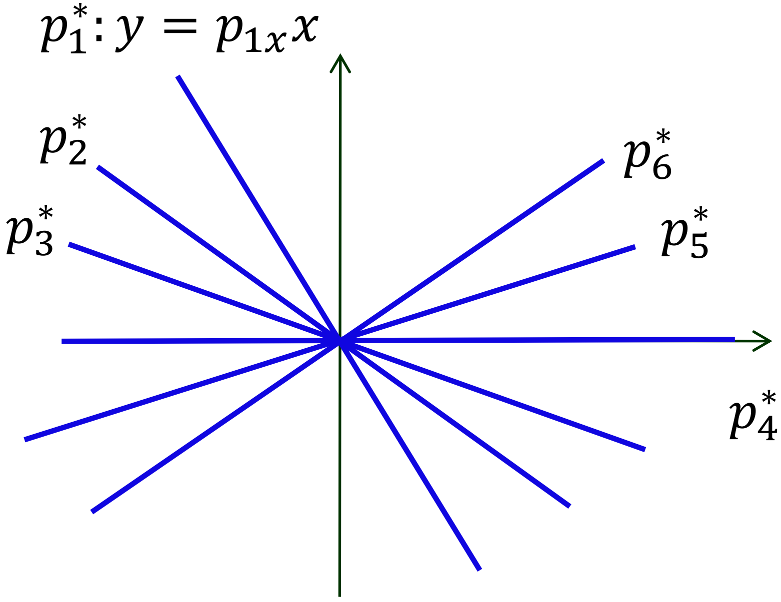




Dual of the x-axis

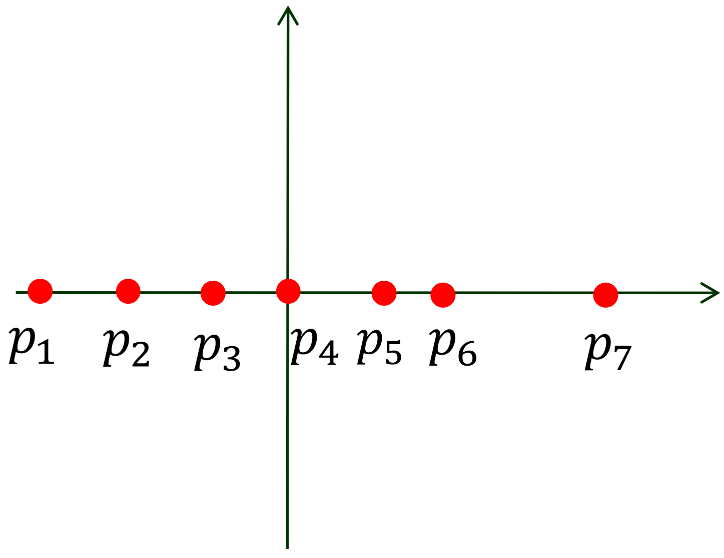
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$

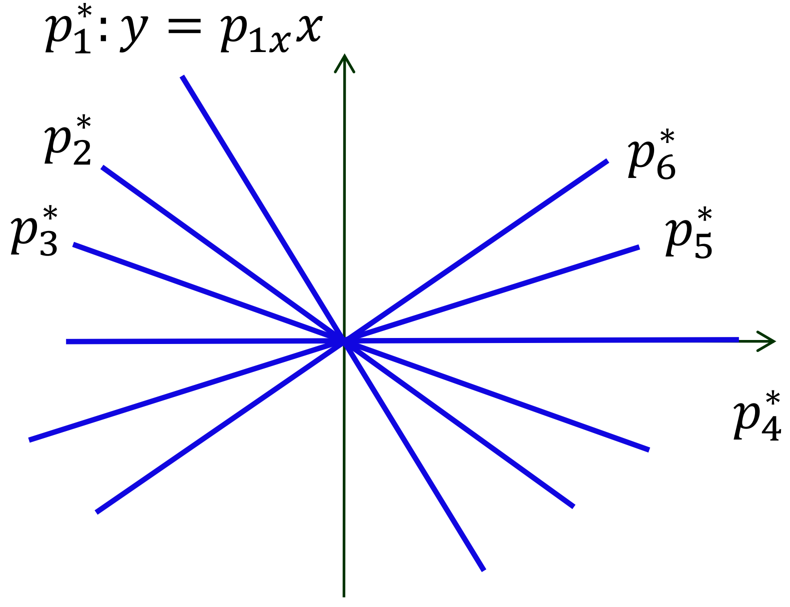




Dual of the x-axis

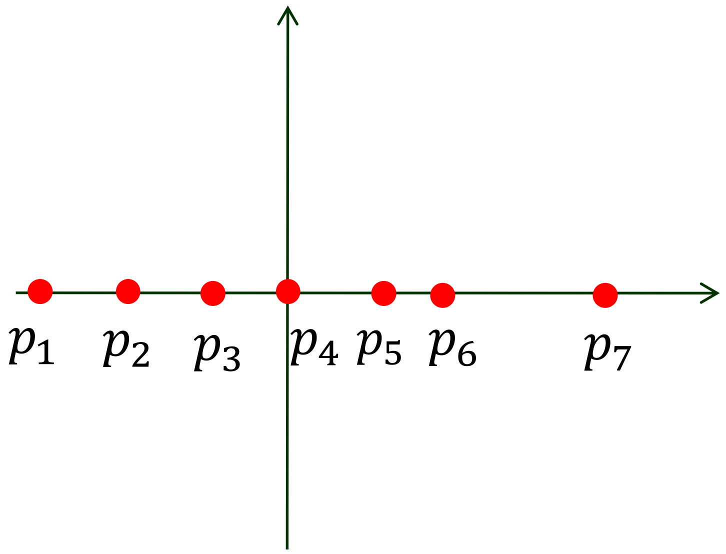
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$

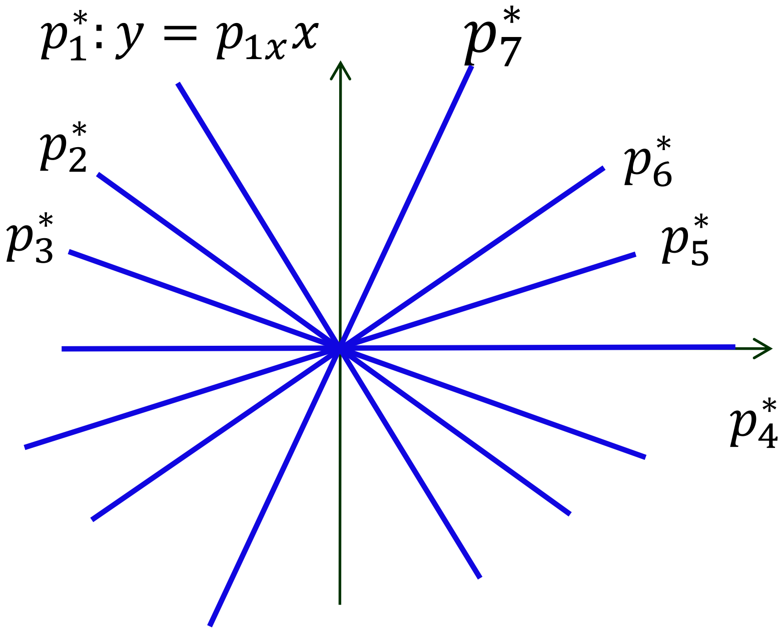




Dual of the x-axis

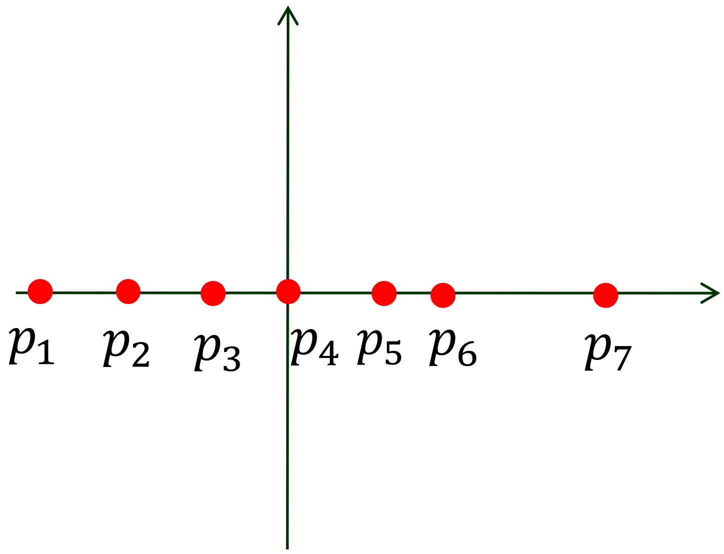
x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$

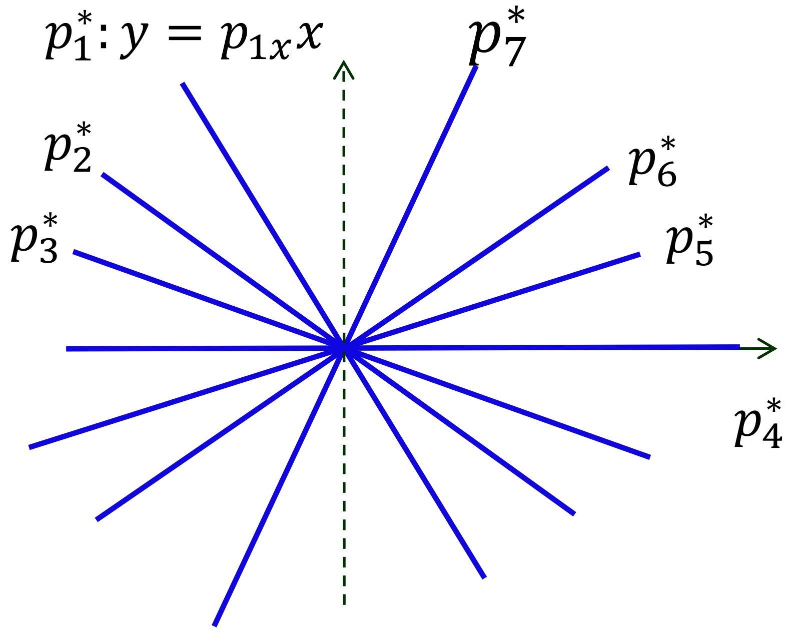




Dual of the x-axis

x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$

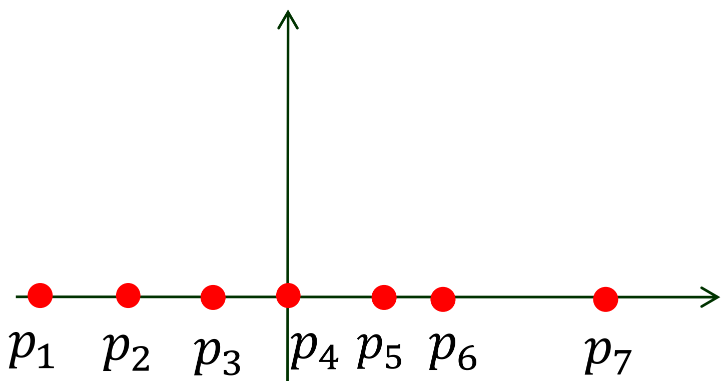


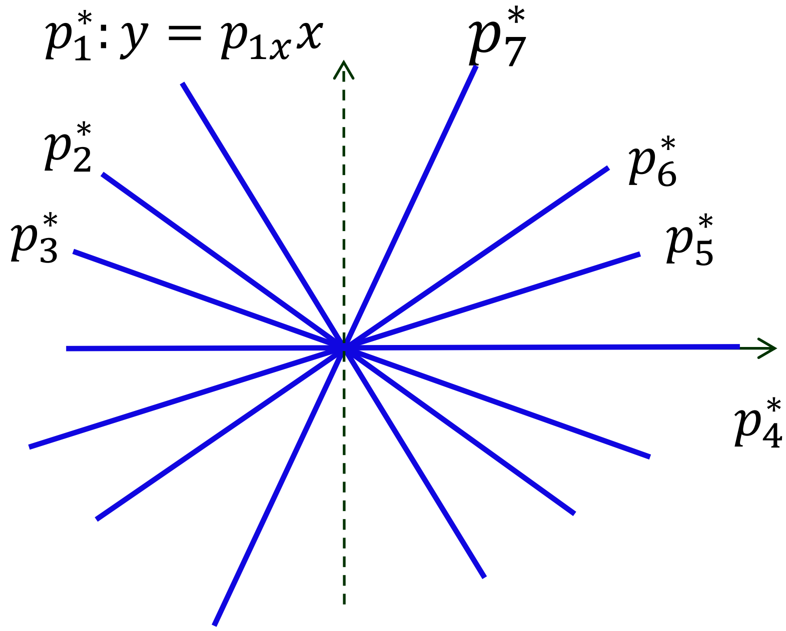


Dual of the x-axis

x-axis $= \{(k, 0)\} \longmapsto \{y = kx\}$

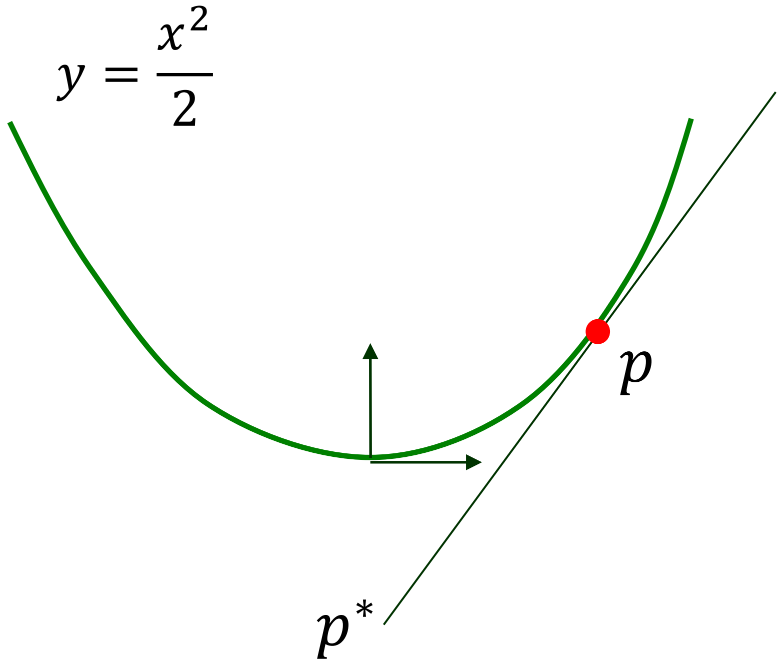
Entire plane except for the y-axis but including the origin!





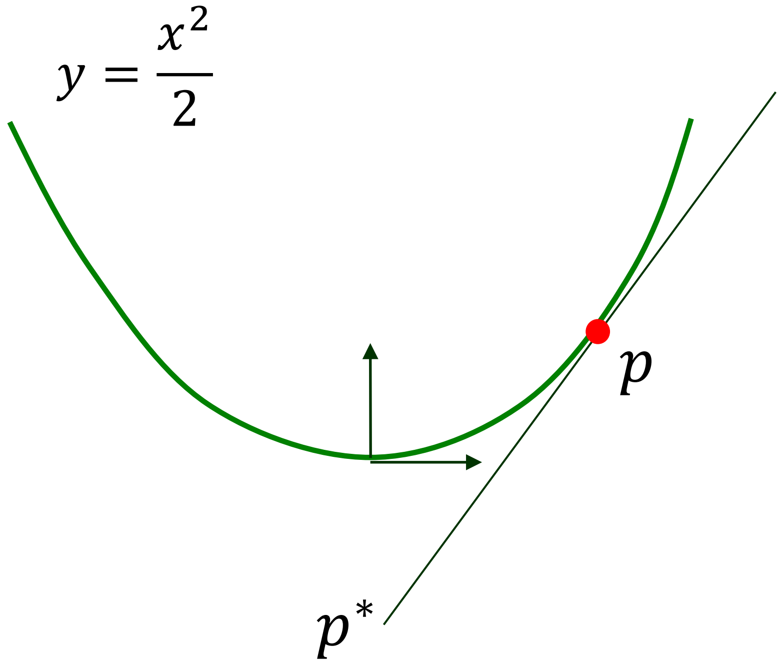
III. Relation to a Parabola

Claim The dual p^* of p on the parabola $y = \frac{x^2}{2}$ is the tangent line at p.



III. Relation to a Parabola

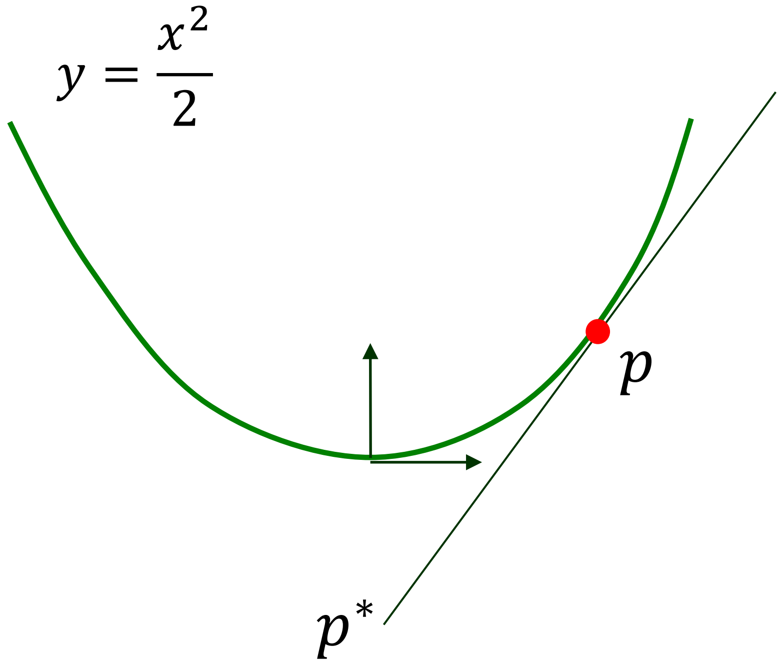
Claim The dual p^* of p on the parabola $y = \frac{x^2}{2}$ is the tangent line at p.



Proof Let $p = (p_x, p_y)$.

III. Relation to a Parabola

Claim The dual p^* of p on the parabola $y = \frac{x^2}{2}$ is the tangent line at p.

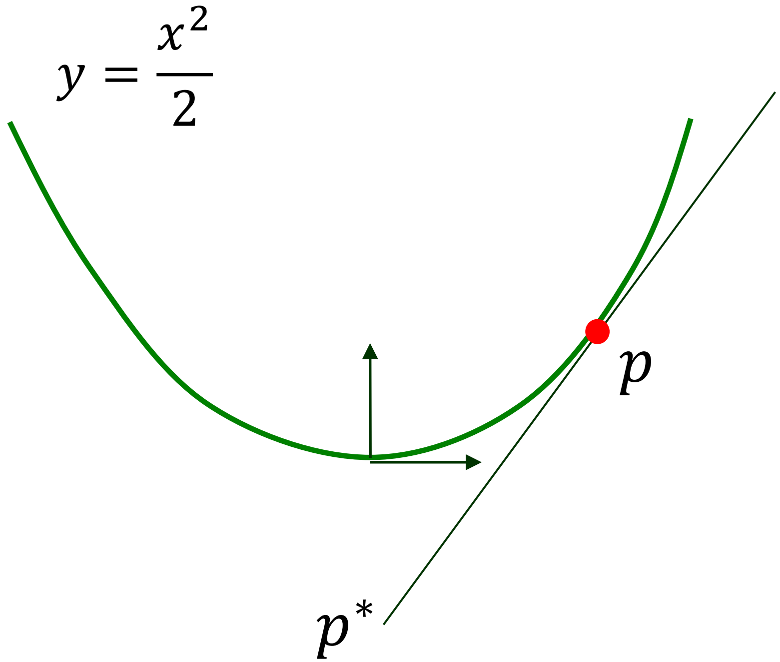


Proof Let $p = (p_x, p_y)$.

Slope of the tangent: $\frac{dy}{dx}|_{(p_x,p_y)} = p_x$.

III. Relation to a Parabola

Claim The dual p^* of p on the parabola $y = \frac{x^2}{2}$ is the tangent line at p.



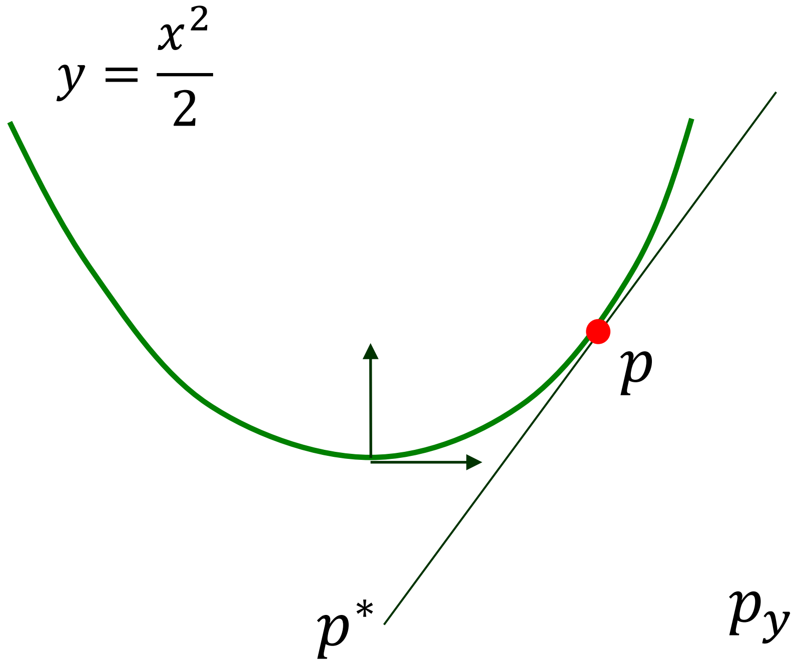
Proof Let $p = (p_x, p_y)$.

Slope of the tangent: $\frac{dy}{dx}|_{(p_x,p_y)} = p_x$.

But the dual is the line $y = p_x x - p_y$, which has the same slope.

III. Relation to a Parabola

Claim The dual p^* of p on the parabola $y = \frac{x^2}{2}$ is the tangent line at p.



Proof Let $p = (p_x, p_y)$.

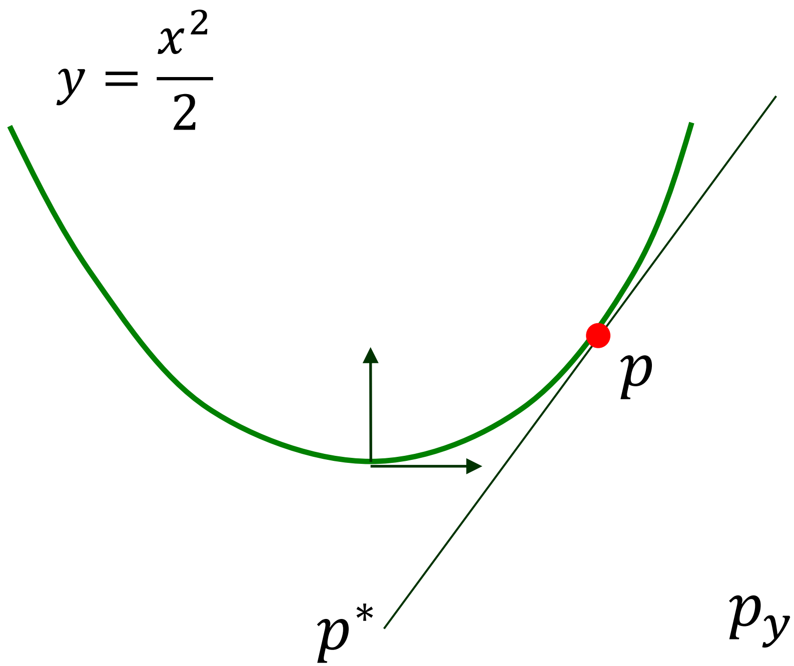
Slope of the tangent: $\frac{dy}{dx}\big|_{(p_x, p_y)} = p_x$.

But the dual is the line $y = p_x x - p_y$, which has the same slope.

$$p_y = p_x^2/2 \implies p_y = p_x p_x - p_y$$

III. Relation to a Parabola

Claim The dual p^* of p on the parabola $y = \frac{x^2}{2}$ is the tangent line at p.



Proof Let $p = (p_x, p_y)$.

Slope of the tangent: $\frac{dy}{dx}|_{(p_x,p_y)} = p_x$.

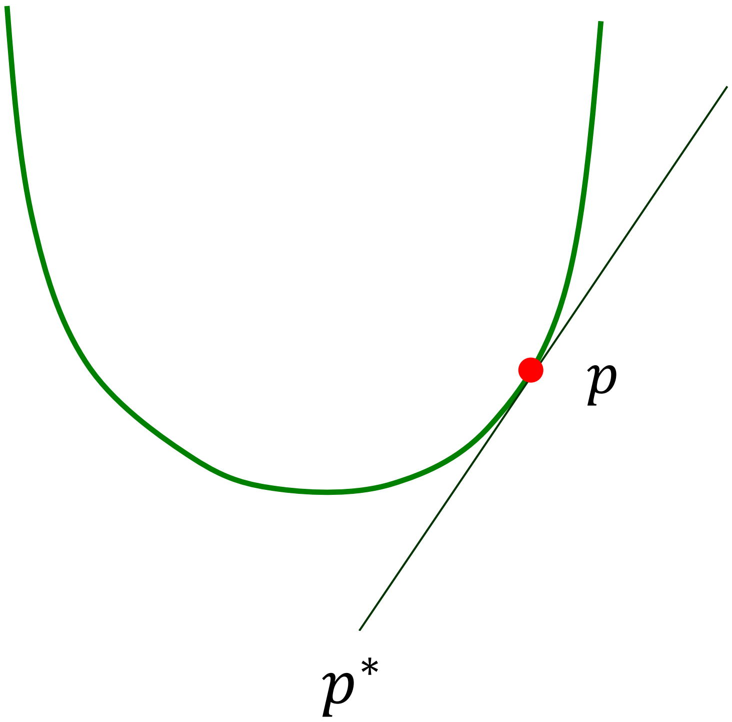
But the dual is the line $y = p_x x - p_y$, which has the same slope.

$$p_y = p_x^2/2 \implies p_y = p_x p_x - p_y$$

Thus, $p = (p_x, p_y)$ lies on the dual line. □

Dual of a Point off the Parabola

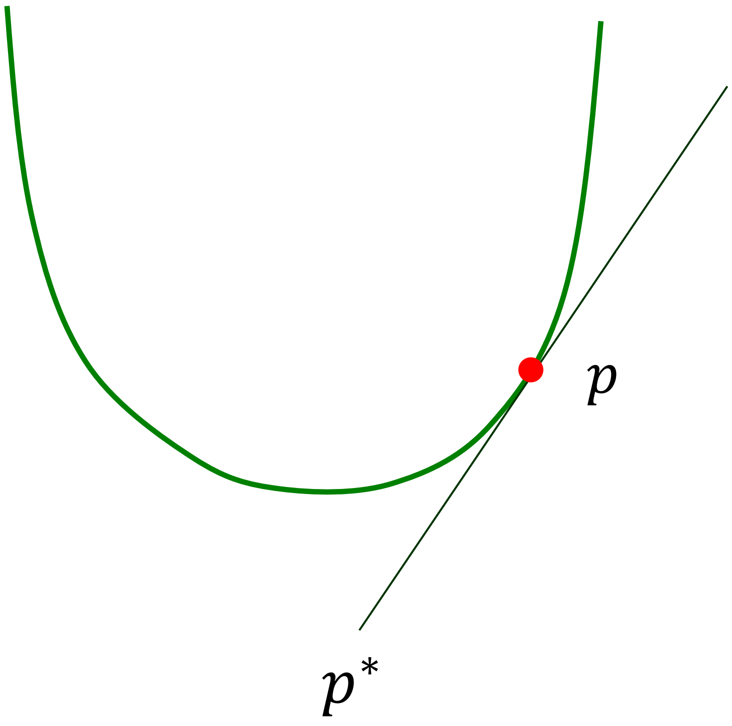
$$y = \frac{x^2}{2}$$



Dual of a Point off the Parabola

$$y = \frac{x^2}{2}$$

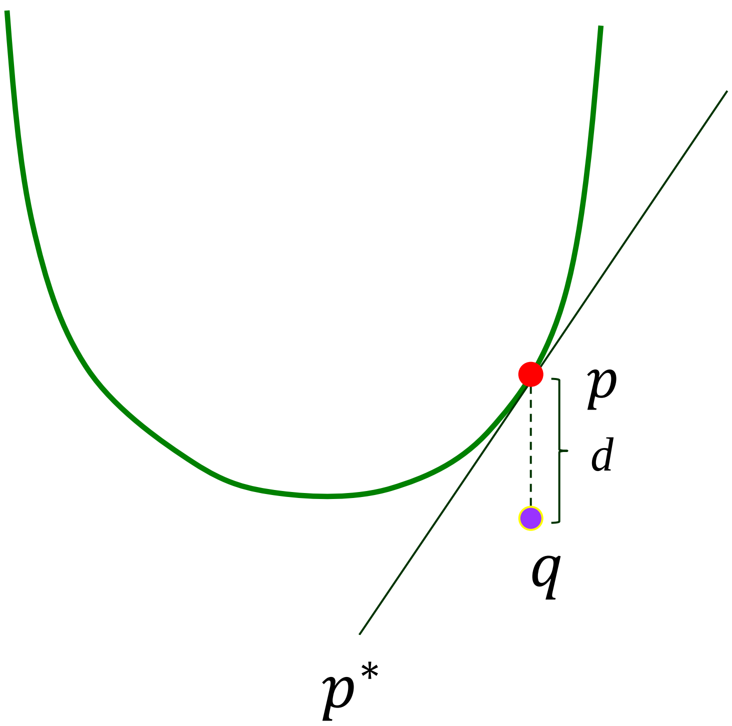
$$p = (p_x, p_y) \Longrightarrow p^*: y = p_x x - p_y$$



Dual of a Point off the Parabola

$$y = \frac{x^2}{2}$$

$$p = (p_x, p_y) \Longrightarrow p^*: y = p_x x - p_y$$



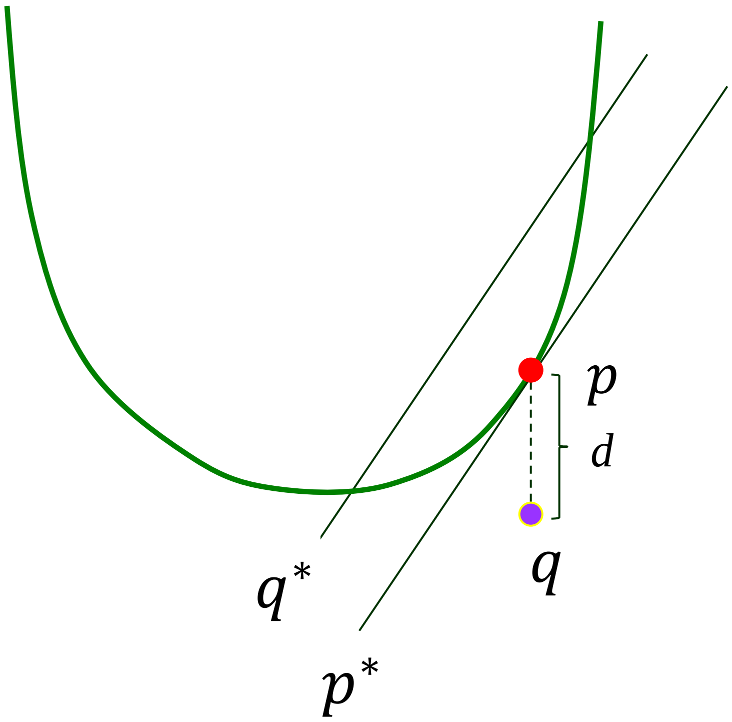
$$q = (p_x, p_y - d)$$

Dual of a Point off the Parabola

$$y = \frac{x^2}{2}$$

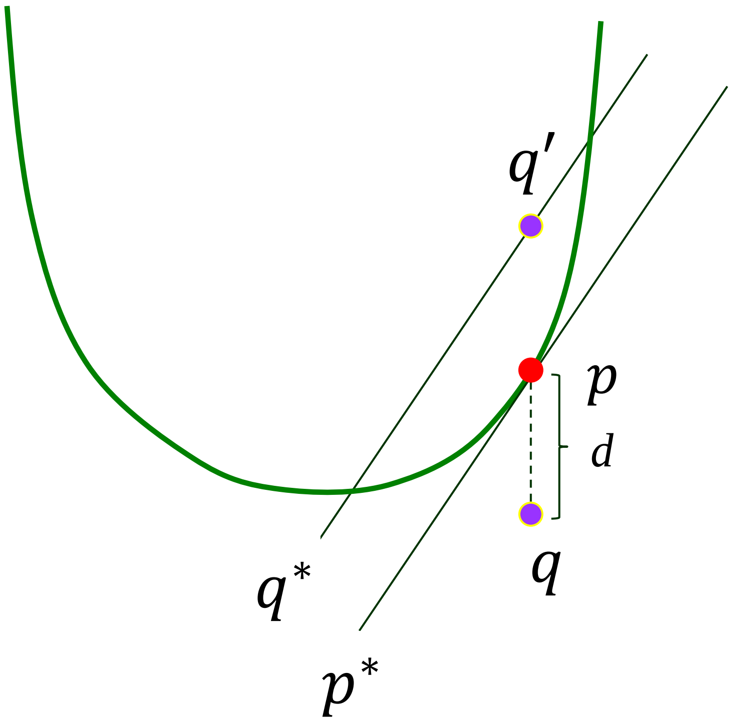
$$p = (p_x, p_y) \Longrightarrow p^*: y = p_x x - p_y$$

$$q = (p_x, p_y - d) \Longrightarrow q^*: y = p_x x - p_y + d$$



Dual of a Point off the Parabola

$$y = \frac{x^2}{2}$$



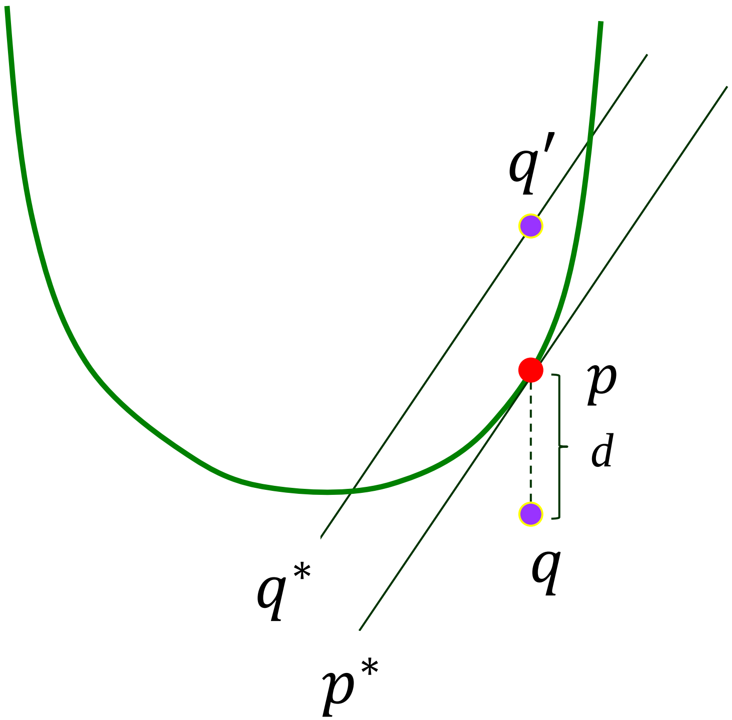
$p = (p_x, p_y) \Longrightarrow p^*: y = p_x x - p_y$

$q = (p_x, p_y - d) \Longrightarrow q^*: y = p_x x - p_y + d$

$q' = (p_x, p_x p_x - p_y + d)$ // q' on q^*

Dual of a Point off the Parabola

$$y = \frac{x^2}{2}$$



$$p = (p_x, p_y) \Longrightarrow p^*: y = p_x x - p_y$$

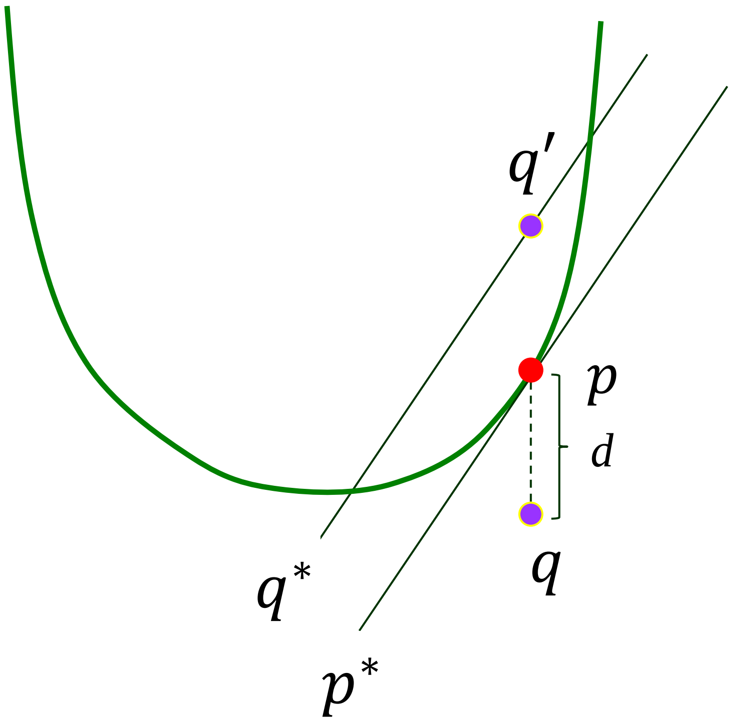
$$q = (p_x, p_y - d) \Longrightarrow q^*: y = p_x x - p_y + d$$

$$q' = (p_x, p_x p_x - p_y + d)$$ // q' on q^*

$$= (p_x, p_x p_x - \frac{1}{2} p_x^2 + d)$$

Dual of a Point off the Parabola

$$y = \frac{x^2}{2}$$



$$p = (p_x, p_y) \Longrightarrow p^*: y = p_x x - p_y$$

$$q = (p_x, p_y - d) \Longrightarrow q^*: y = p_x x - p_y + d$$

$$q' = (p_x, p_x p_x - p_y + d) \quad // \ q' \text{ on } q^*$$

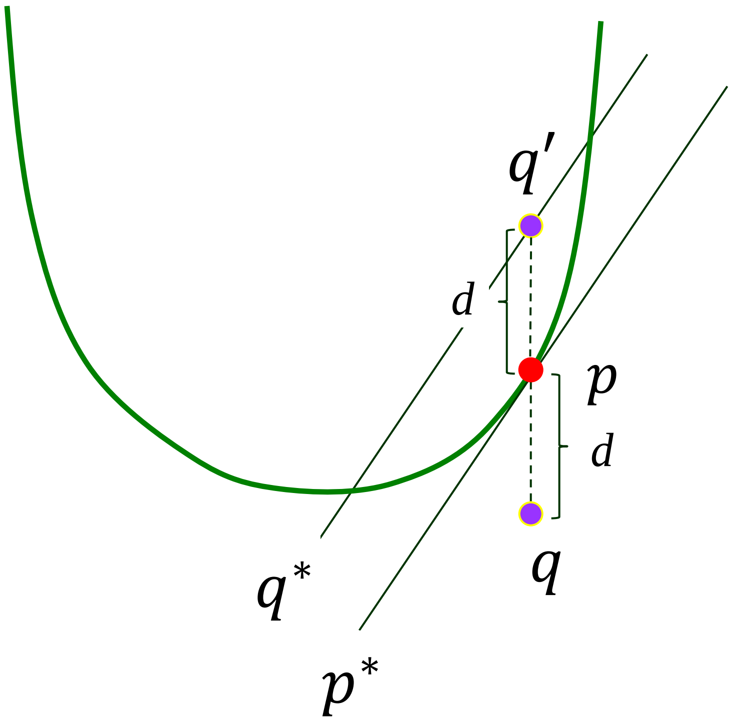
$$= (p_x, p_x p_x - \frac{1}{2} p_x^2 + d)$$

$$= \left(p_x, \frac{1}{2} p_x^2 + d\right) = (p_x, p_y + d)$$

Dual of a Point off the Parabola

$$y = \frac{x^2}{2}$$

$$p = (p_x, p_y) \Longrightarrow p^*: y = p_x x - p_y$$



$$q = (p_x, p_y - d) \Longrightarrow q^*: y = p_x x - p_y + d$$

$$q' = (p_x, p_x p_x - p_y + d) \quad \text{// } q' \text{ on } q^*$$

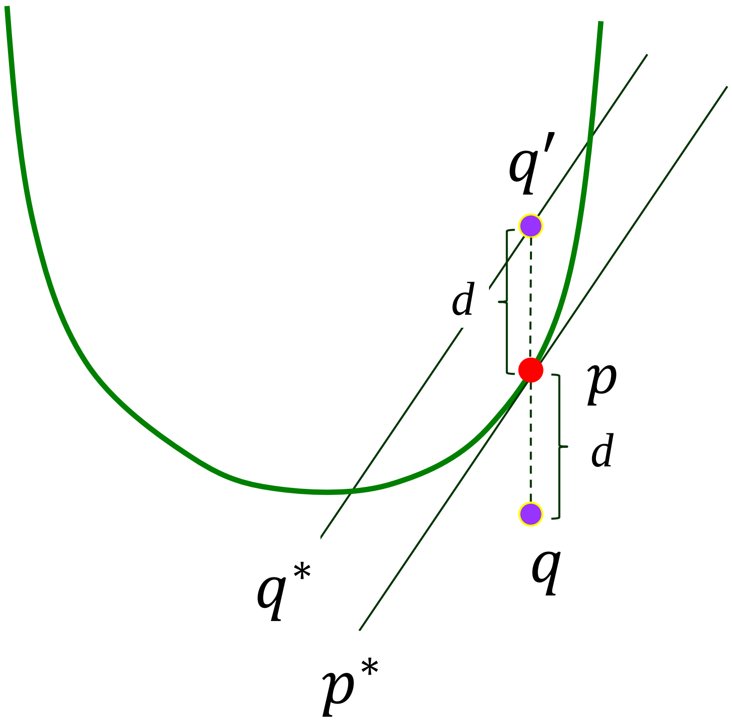
$$= \left(p_x, p_x p_x - \frac{1}{2} p_x^2 + d\right)$$

$$= \left(p_x, \frac{1}{2} p_x^2 + d\right) = (p_x, p_y + d)$$

$$q' - p = p - q$$

Dual of a Point off the Parabola

$$y = \frac{x^2}{2}$$



$$p = (p_x, p_y) \Longrightarrow p^*: y = p_x x - p_y$$

$$q = (p_x, p_y - d) \Longrightarrow q^*: y = p_x x - p_y + d$$

$$q' = (p_x, p_x p_x - p_y + d) \quad // \; q' \text{ on } q^*$$

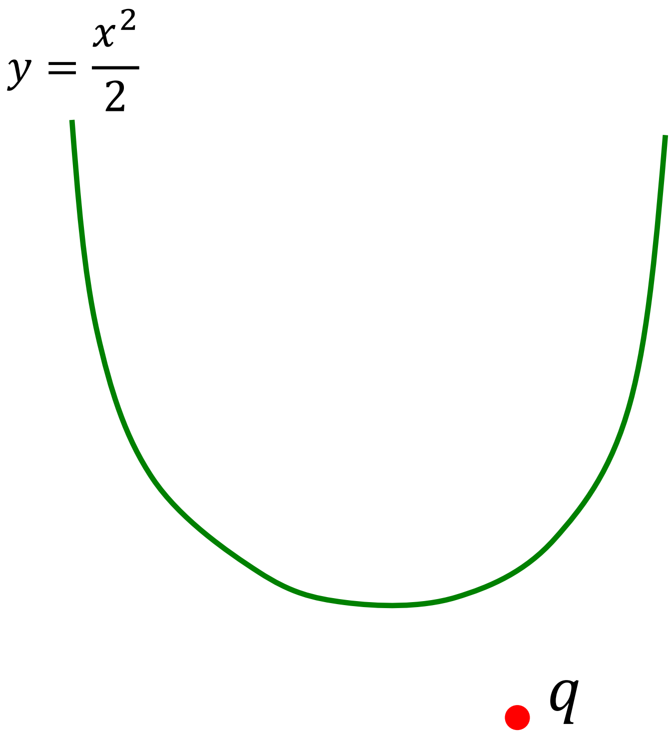
$$= \left(p_x, p_x p_x - \frac{1}{2} p_x^2 + d\right)$$

$$= \left(p_x, \frac{1}{2} p_x^2 + d\right) = (p_x, p_y + d)$$

$$q' - p = p - q$$

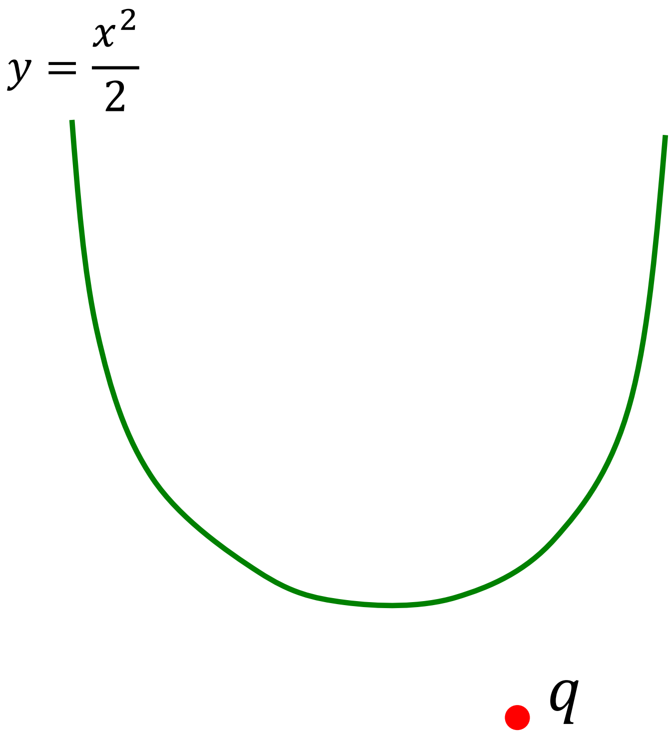
The dual line $q^* \parallel p^*$ and it passes through q'.

Dual Line From Two Tangents



Construct the dual line q^* of q without measuring distances:

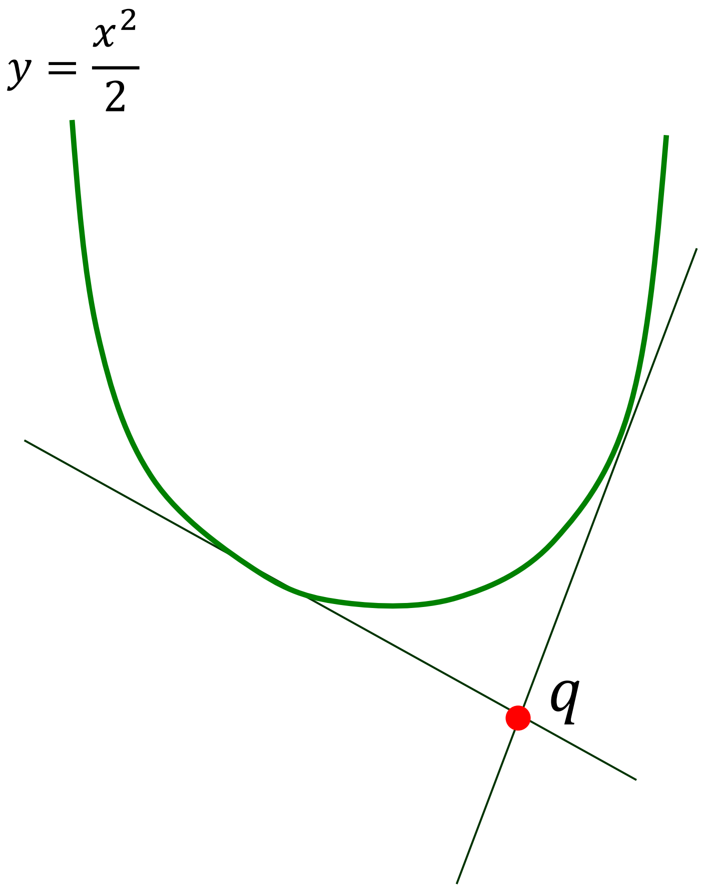
Dual Line From Two Tangents



Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

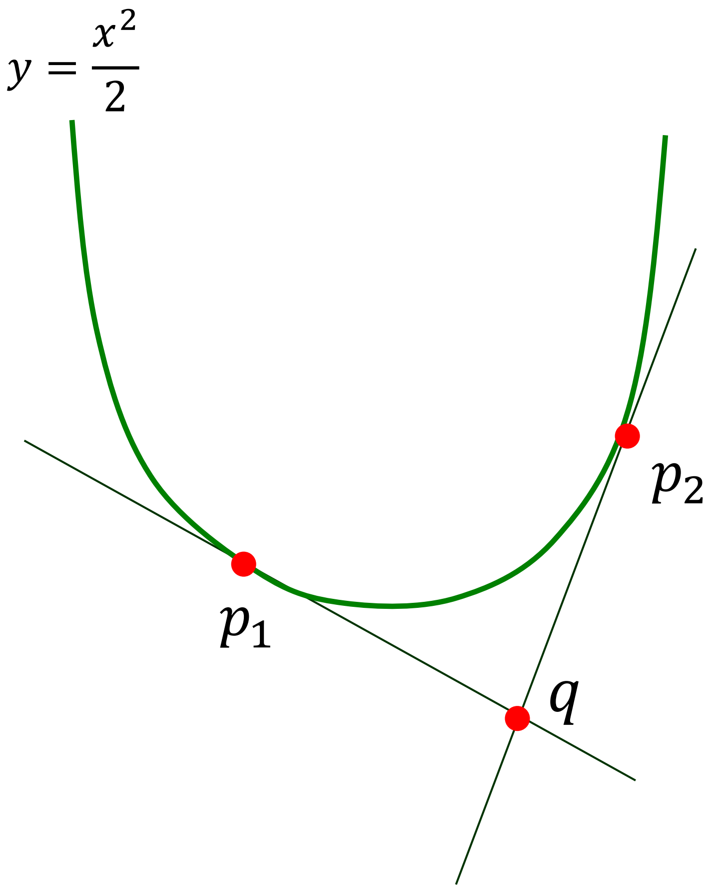
Dual Line From Two Tangents



Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

Dual Line From Two Tangents

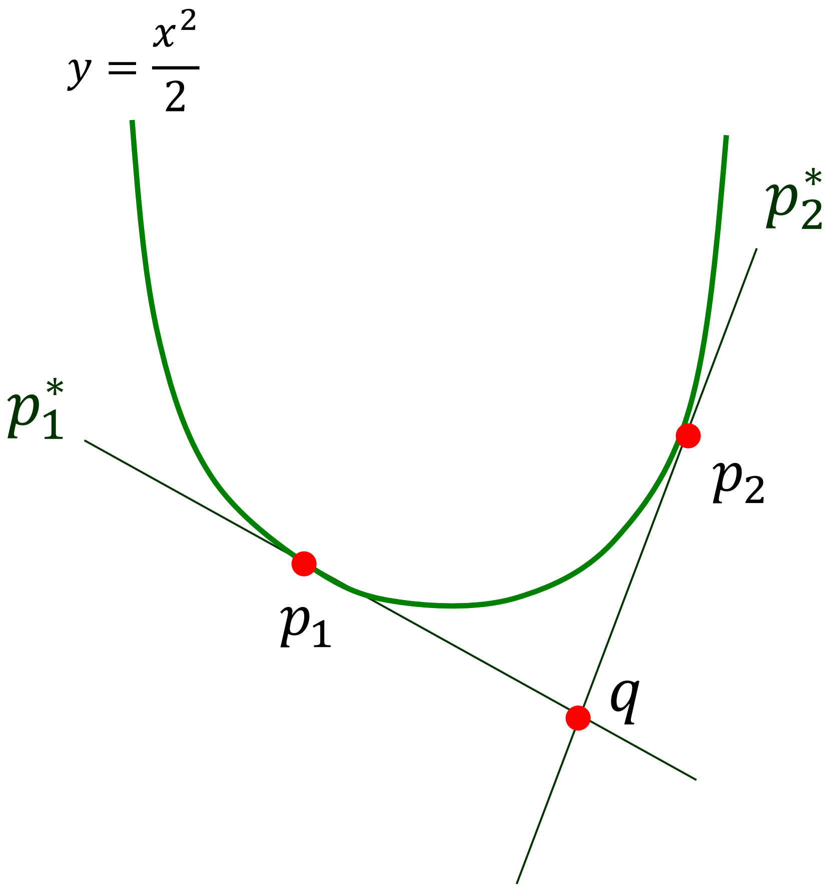


Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

Dual Line From Two Tangents



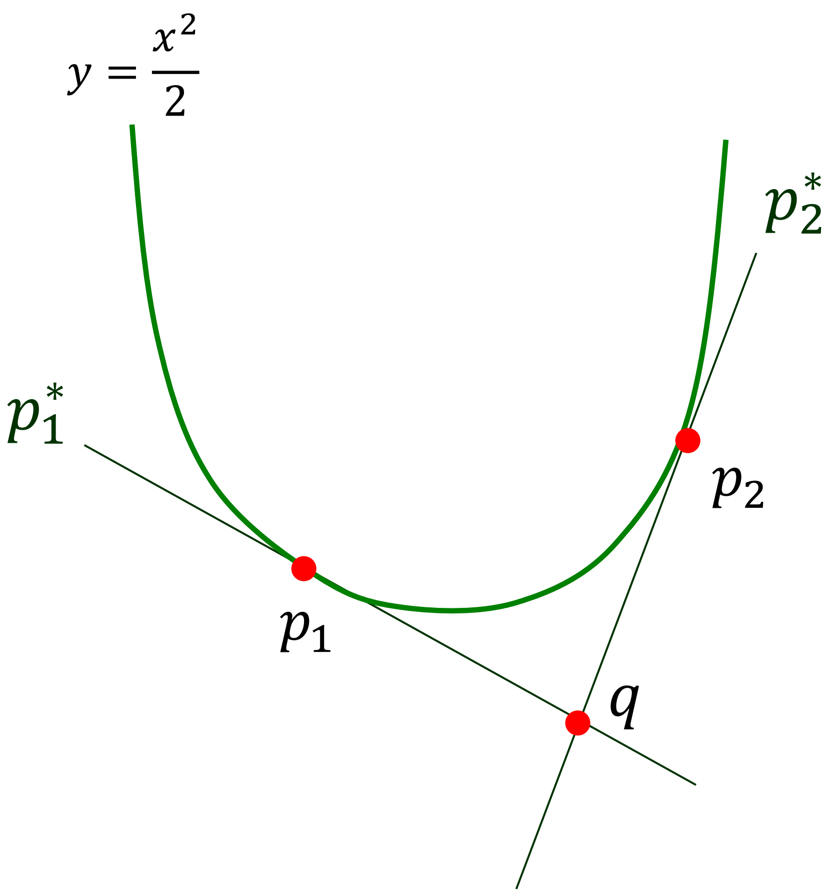
Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

The two tangent lines are p_1^* and p_2^*.

Dual Line From Two Tangents



Construct the dual line q^* of q without measuring distances:

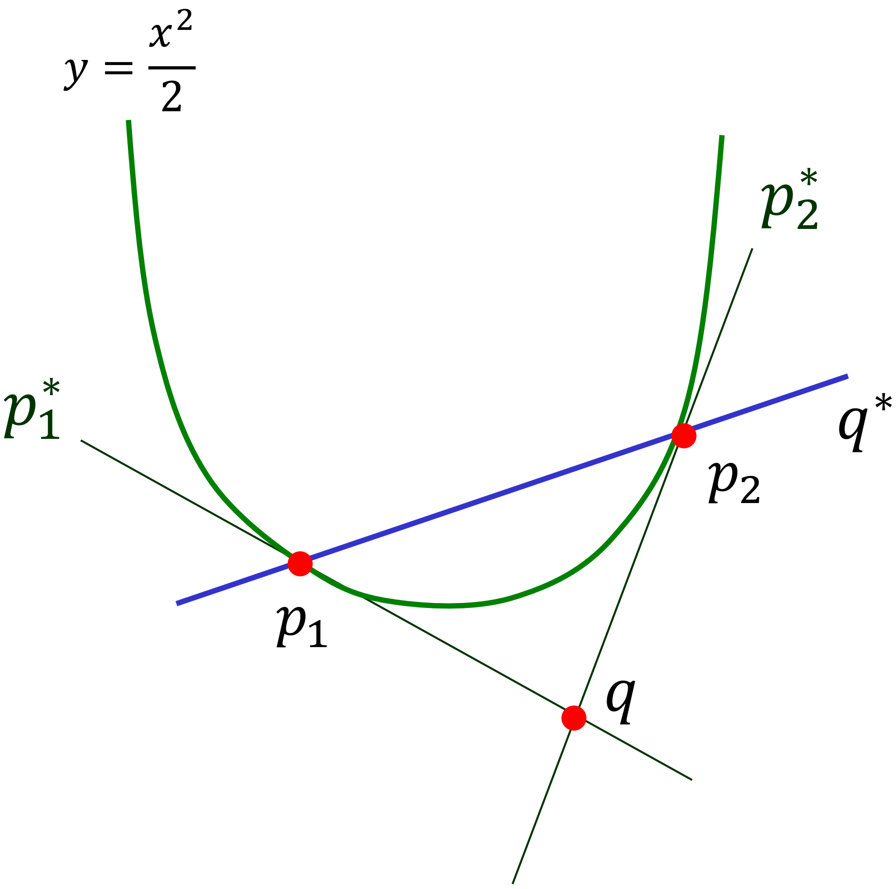
1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

3) q^* is the line through p_1 and p_2.

The two tangent lines are p_1^* and p_2^*.

Dual Line From Two Tangents



Construct the dual line q^* of q without measuring distances:

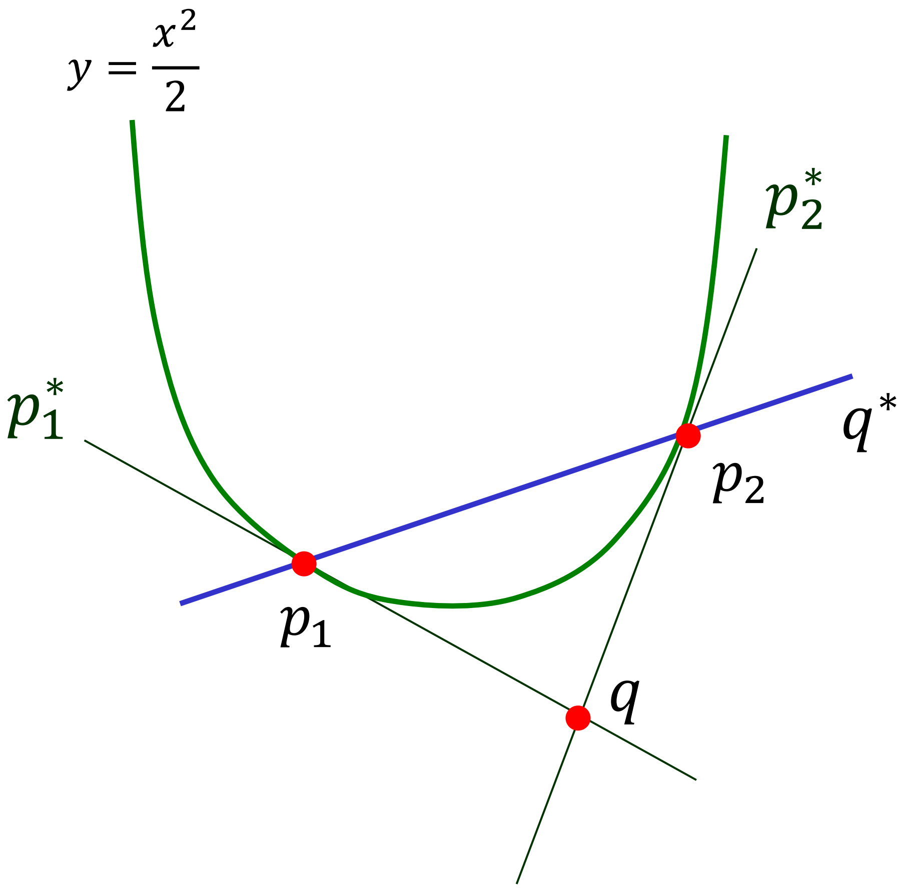
1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

3) q^* is the line through p_1 and p_2.

The two tangent lines are p_1^* and p_2^*.

Dual Line From Two Tangents



Construct the dual line q^* of q without measuring distances:

1) Through q draw two tangent lines to the parabola.

2) Let p_1 and p_2 be the points of tangency, respectively.

3) q^* is the line through p_1 and p_2.

The two tangent lines are p_1^* and p_2^*.

p_1^* and p_2^* intersect at q. $\Leftrightarrow$ q^* passes through p_1 and p_2.

Duality in Higher Dimensions

Point $p = (p_1, p_2, ..., p_d)$

Duality in Higher Dimensions

Point $p = (p_1, p_2, \ldots, p_d)$

$$\Downarrow$$

Hyperplane p^*: $x_d = p_1 x_1 + p_2 x_2 + \cdots + p_{d-1} x_{d-1} - p_d$

Duality in Higher Dimensions

Point $p = (p_1, p_2, \ldots, p_d)$

$\Downarrow$

Hyperplane p^*: $x_d = p_1 x_1 + p_2 x_2 + \cdots + p_{d-1} x_{d-1} - p_d$

Hyperplane h: $x_d = a_1 x_1 + a_2 x_2 + \cdots + a_{d-1} x_{d-1} + a_d$

Duality in Higher Dimensions

Point $p = (p_1, p_2, \ldots, p_d)$

$\Downarrow$

Hyperplane p^*: $x_d = p_1 x_1 + p_2 x_2 + \cdots + p_{d-1} x_{d-1} - p_d$

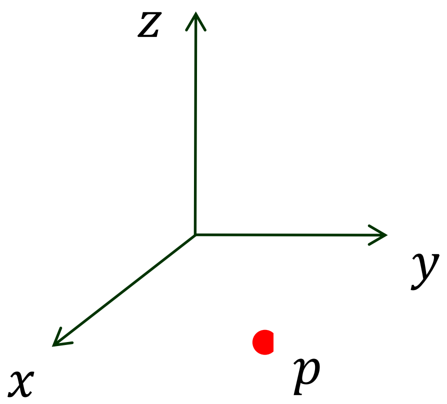
Hyperplane h: $x_d = a_1 x_1 + a_2 x_2 + \cdots + a_{d-1} x_{d-1} + a_d$

$\Downarrow$

Point $h^*: a = (a_1, a_2, \ldots, -a_d)$

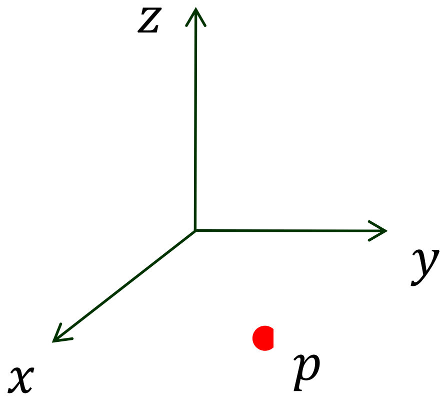
IV. Inversion

Point $p = (p_x, p_y)$



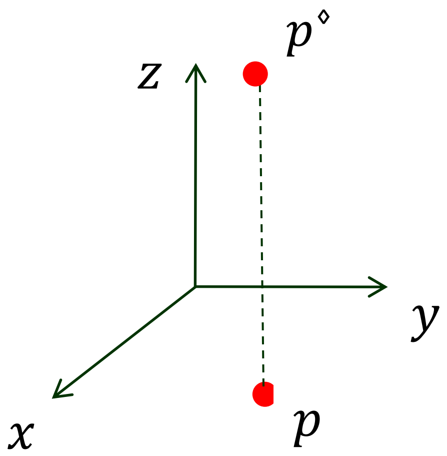
IV. Inversion

Point $p = (p_x, p_y)$ $\Longrightarrow$ Point $p^{\diamond} = (p_x, p_y, p_x^2 + p_y^2)$



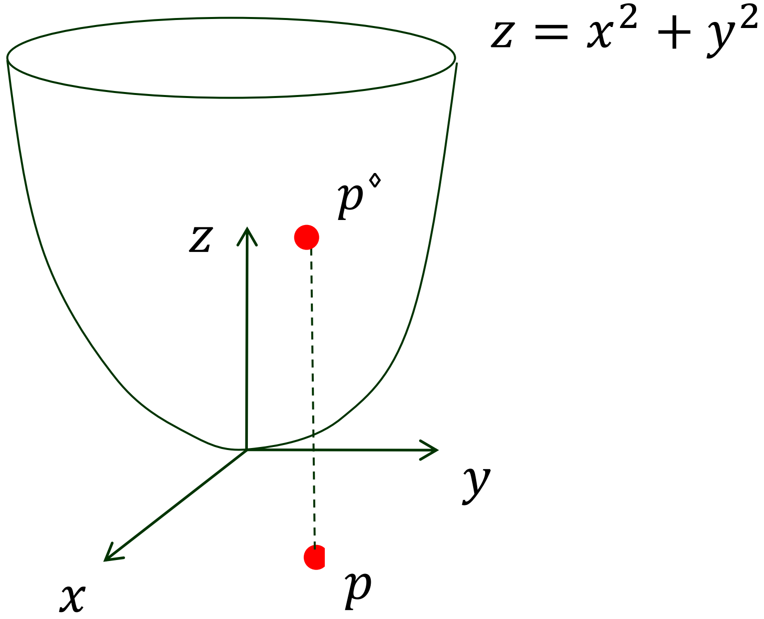
IV. Inversion

Point $p = (p_x, p_y)$ $\Longrightarrow$ Point $p^{\diamond} = (p_x, p_y, p_x^2 + p_y^2)$



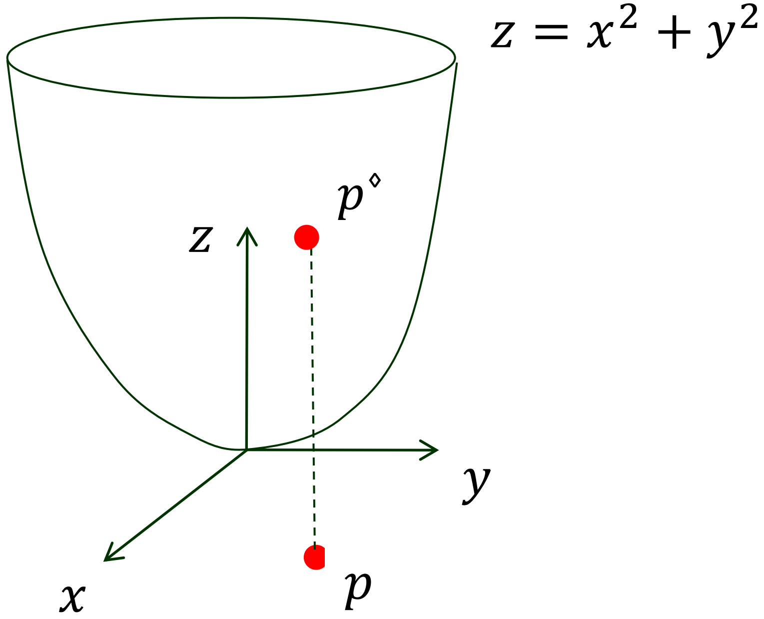
IV. Inversion

Point $p = (p_x, p_y)$ $\Longrightarrow$ Point $p^{\diamond} = (p_x, p_y, p_x^2 + p_y^2)$



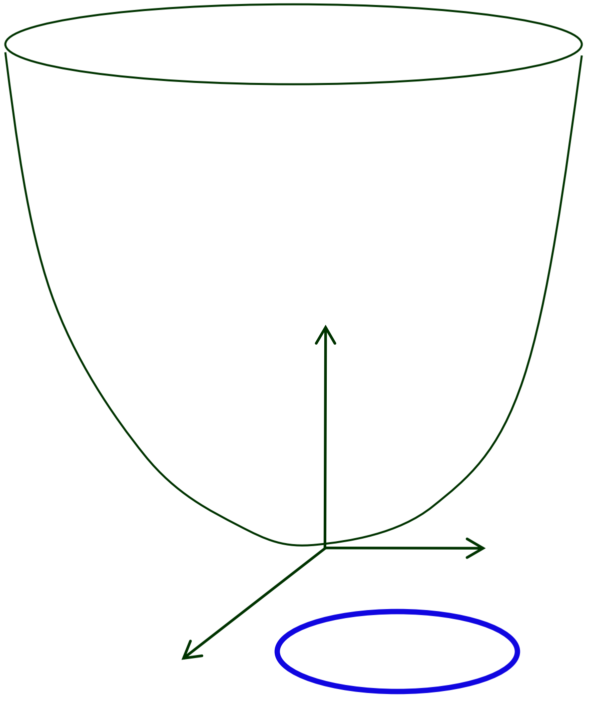
IV. Inversion

Point $p = (p_x, p_y)$ $\Longrightarrow$ Point $p^\diamond = (p_x, p_y, p_x^2 + p_y^2)$



The point p is lifted to the unit paraboloid to become $p^\diamond$.

Image of a Circle

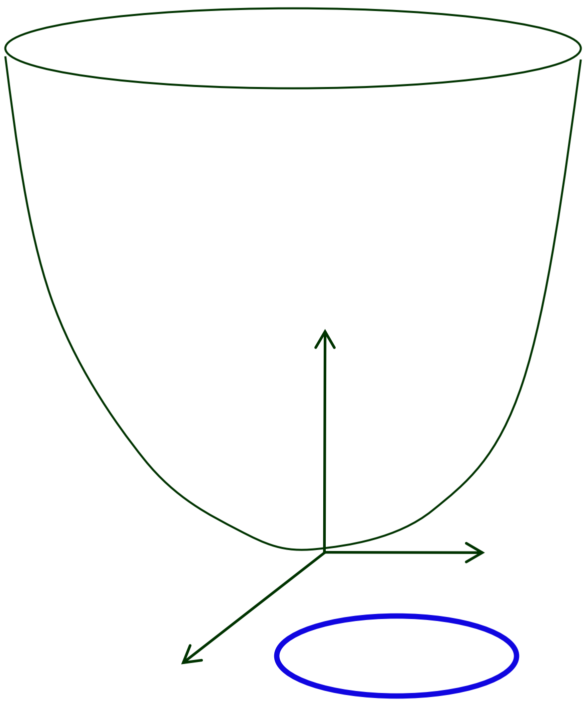


$C: (x-a)^2 + (y-b)^2 = r^2$

Image of a Circle

Image of a point on the circle C has z-coordinate

$$z = x^2 + y^2$$



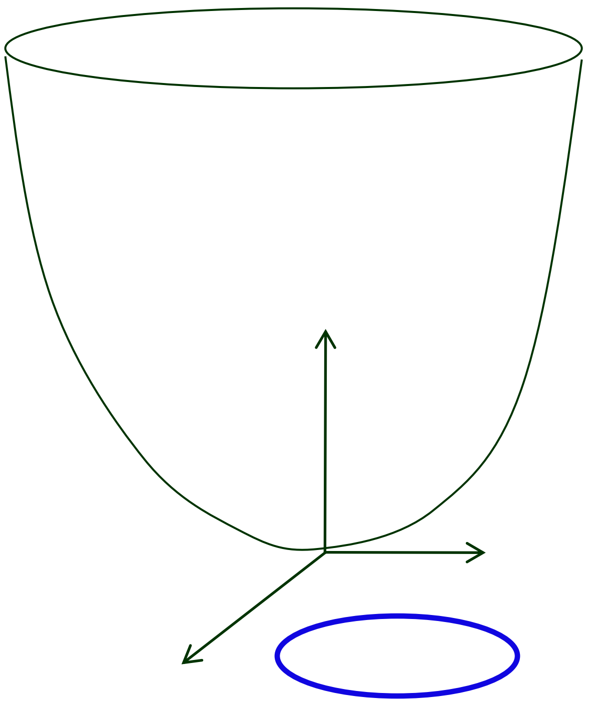
$$C: (x-a)^2 + (y-b)^2 = r^2$$

Image of a Circle

Image of a point on the circle C has z-coordinate

$$z = x^2 + y^2$$

$$= 2ax + 2ay - a^2 - b^2 + r^2$$



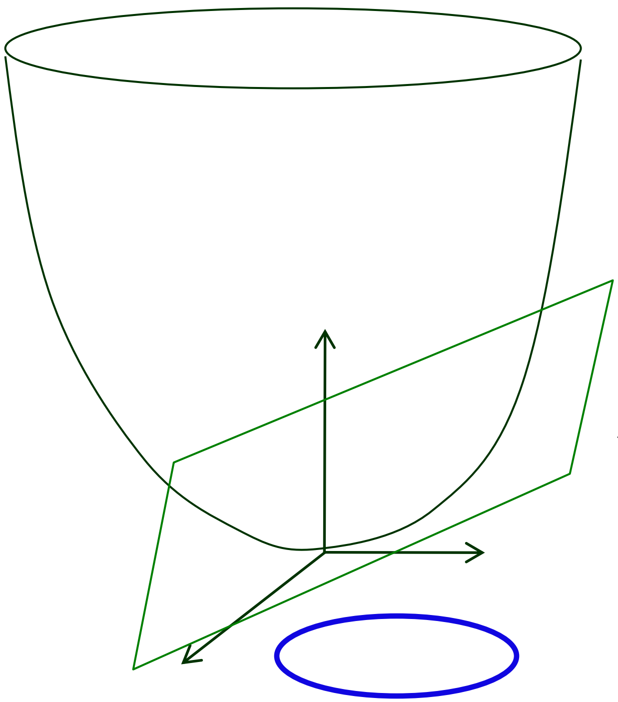
$C: (x-a)^2 + (y-b)^2 = r^2$

Image of a Circle

Image of a point on the circle C has z-coordinate

$$z = x^2 + y^2$$

$$= 2ax + 2ay - a^2 - b^2 + r^2$$



plane

$P: z = 2ax + 2ay - a^2 - b^2 + r^2$

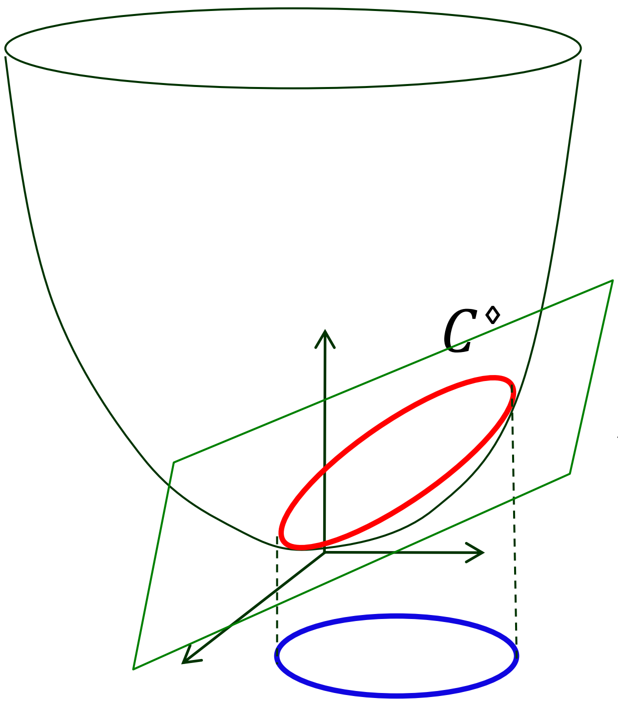
$C: (x - a)^2 + (y - b)^2 = r^2$

Image of a Circle

Image of a point on the circle C has z-coordinate

$$z = x^2 + y^2$$

$$= 2ax + 2ay - a^2 - b^2 + r^2$$



plane

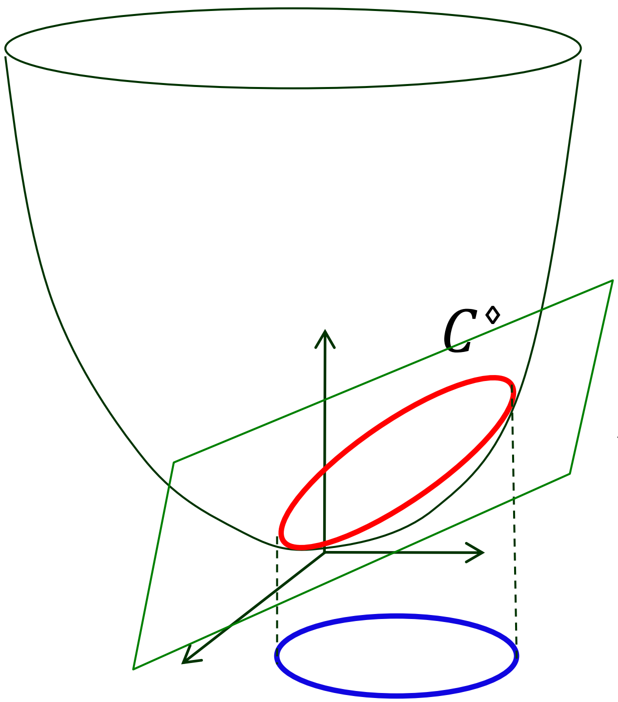
$P: z = 2ax + 2ay - a^2 - b^2 + r^2$

$C: (x - a)^2 + (y - b)^2 = r^2$

Image of a Circle

Image of a point on the circle C has z-coordinate

$$z = x^2 + y^2$$
$$= 2ax + 2ay - a^2 - b^2 + r^2$$



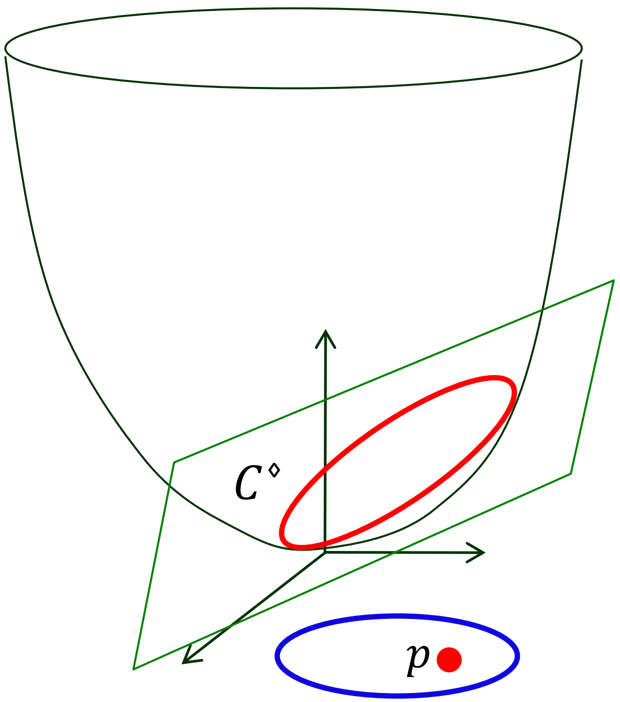
plane

$P: z = 2ax + 2ay - a^2 - b^2 + r^2$

$C: (x-a)^2 + (y-b)^2 = r^2$

$C^\diamond$ is the intersection of the plane P with the unit paraboloid.

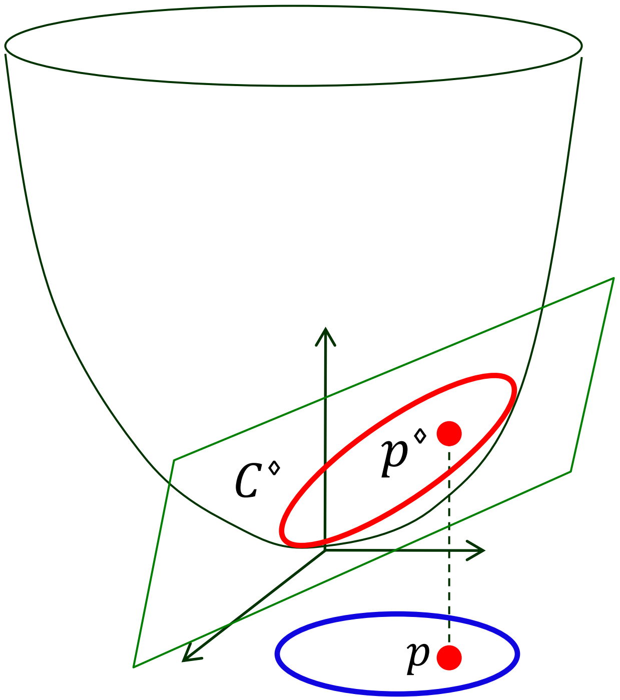
Inside/Outside ⇔ Below/Above



$P: z = 2ax + 2ay - a^2 - b^2 + r^2$

$C: (x-a)^2 + (y-b)^2 = r^2$

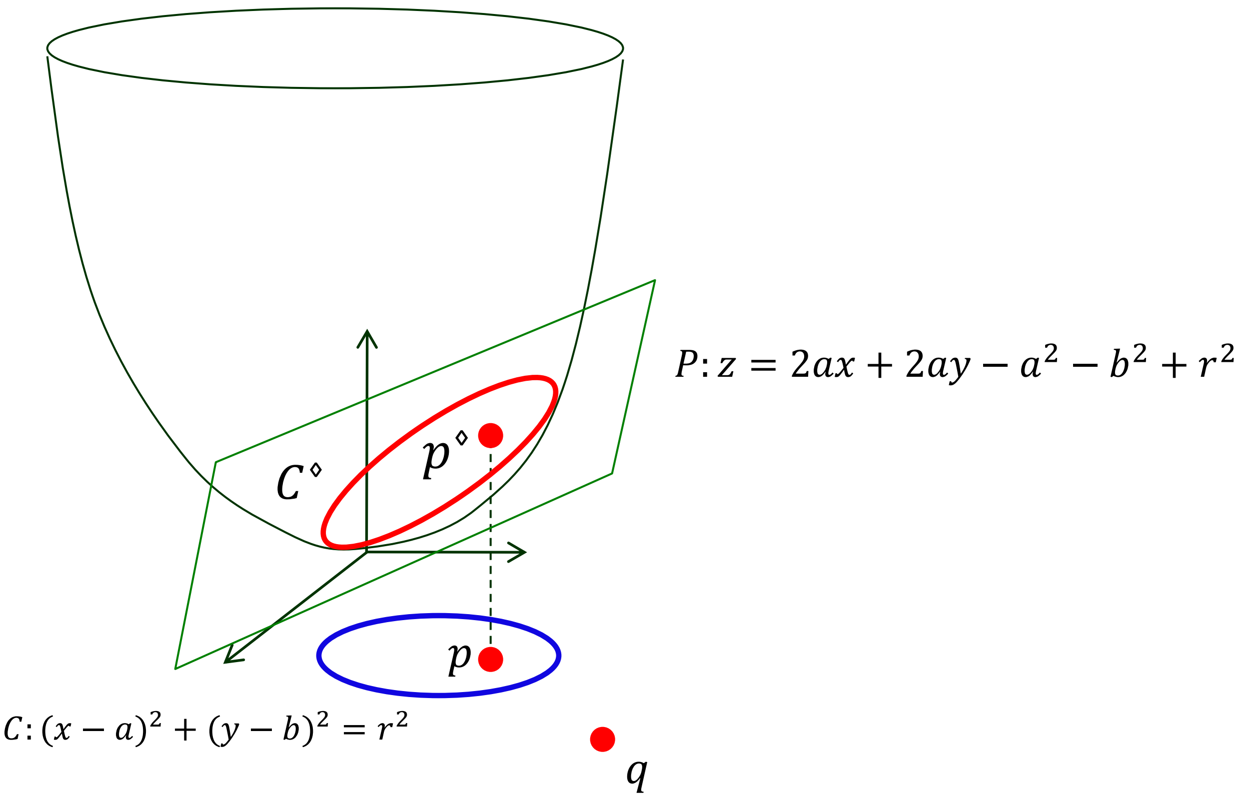
Inside/Outside ⇔ Below/Above



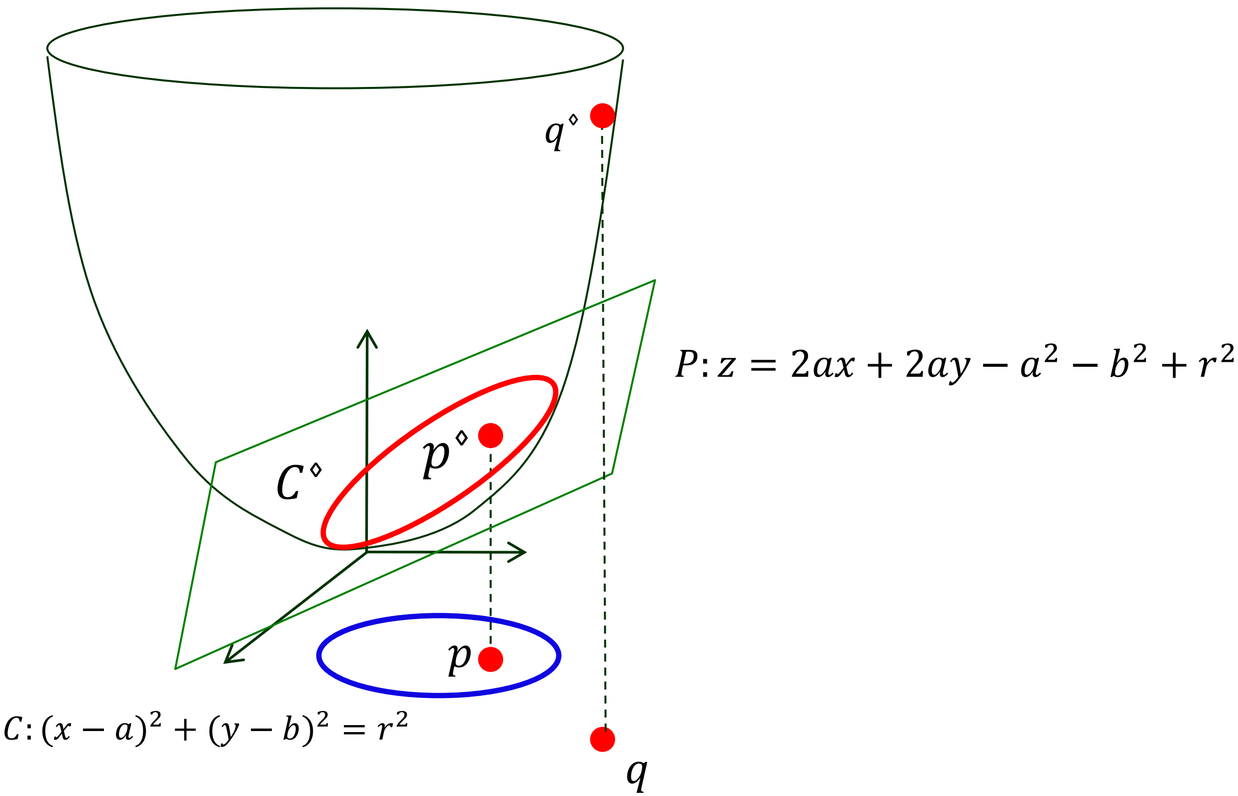
$P: z = 2ax + 2ay - a^2 - b^2 + r^2$

$C: (x-a)^2 + (y-b)^2 = r^2$

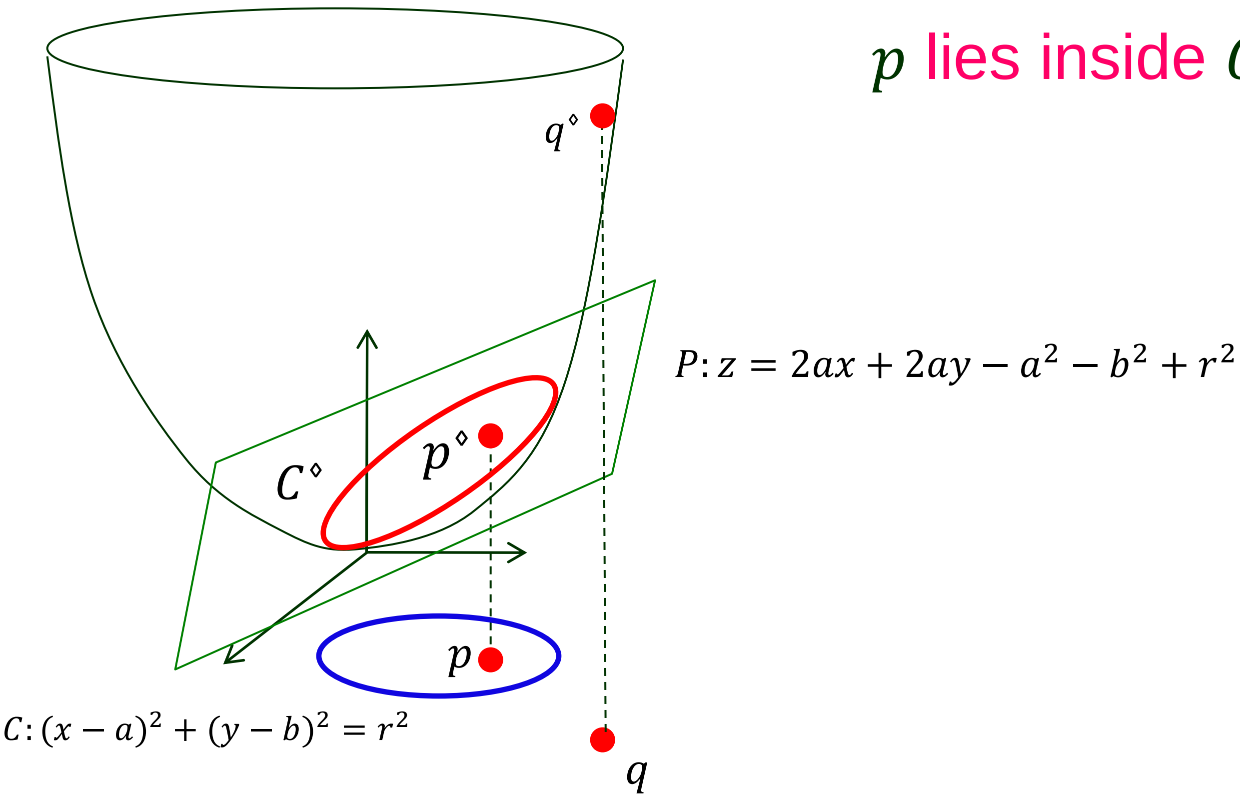
Inside/Outside ⇔ Below/Above



Inside/Outside ⇔ Below/Above



Inside/Outside ⇔ Below/Above



p lies inside C iff $p^\diamond$ is below $C^\diamond$.