Yan-Bin Jia, Iowa State University

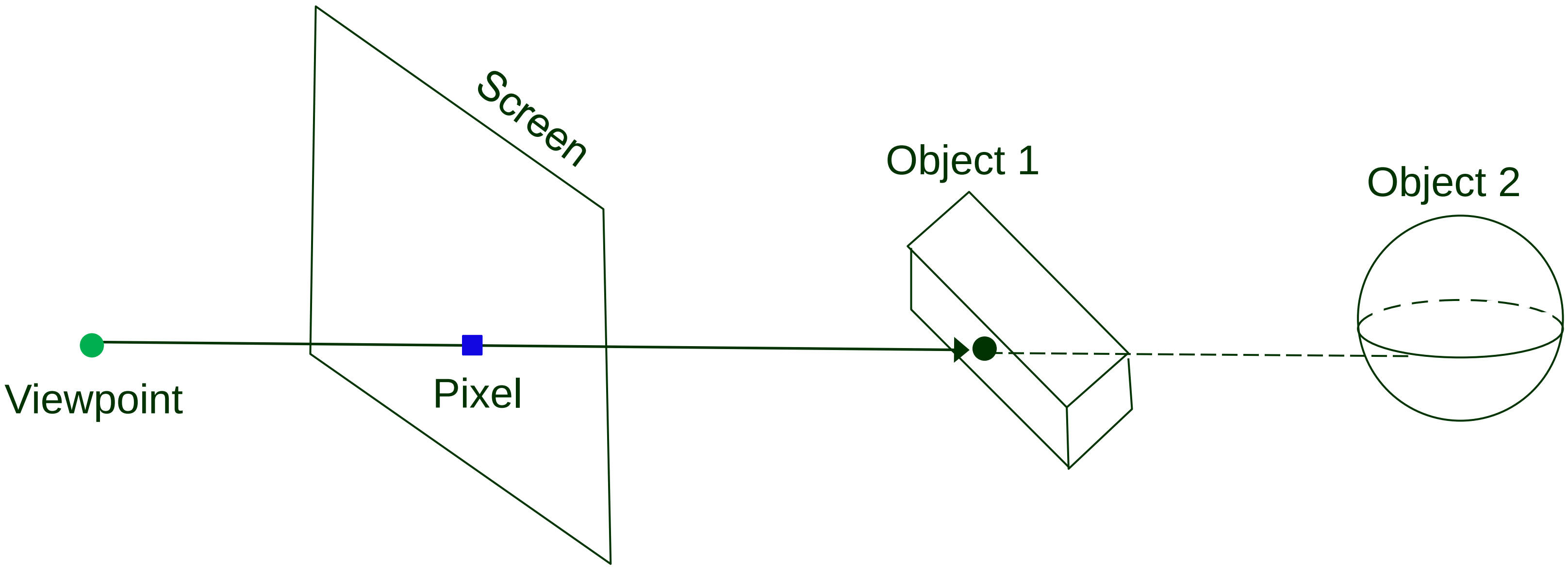
Discrepancy Computation

Outline:

I. Half-plane discrepancy

II. Extrema of discrepancy and computation

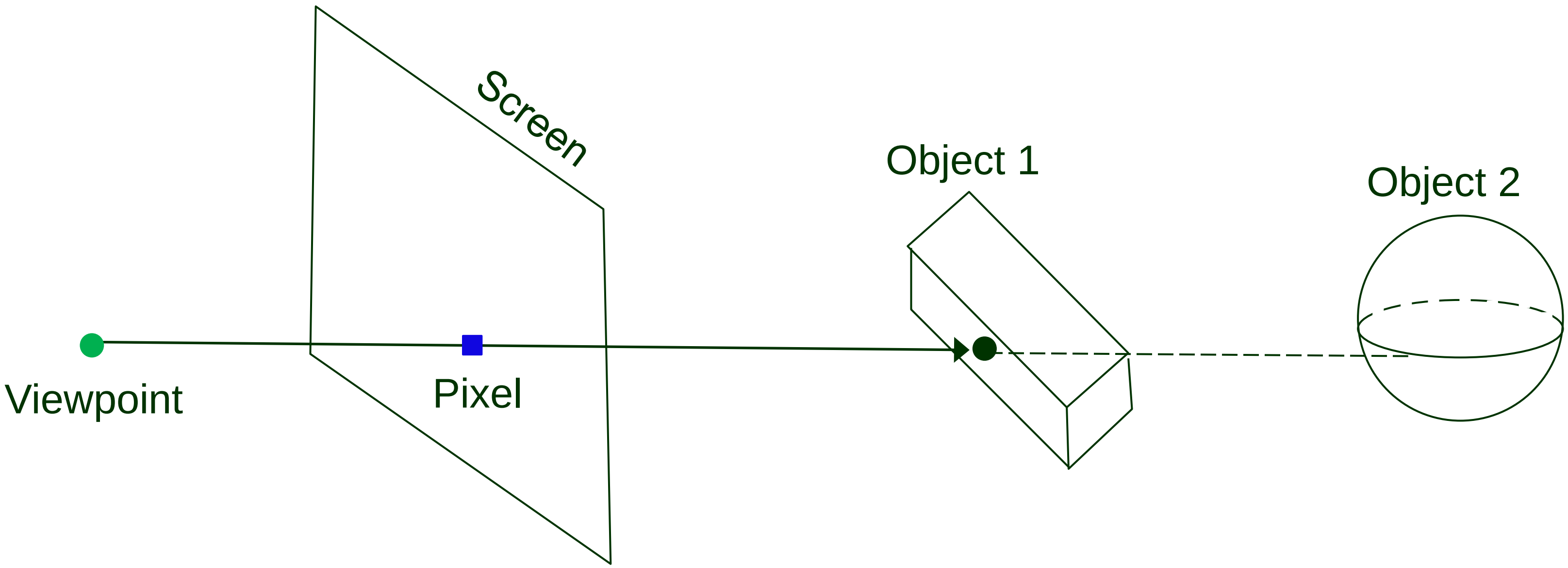
Rendering



Rendering

Determine

- for every pixel on the screen which object is visible



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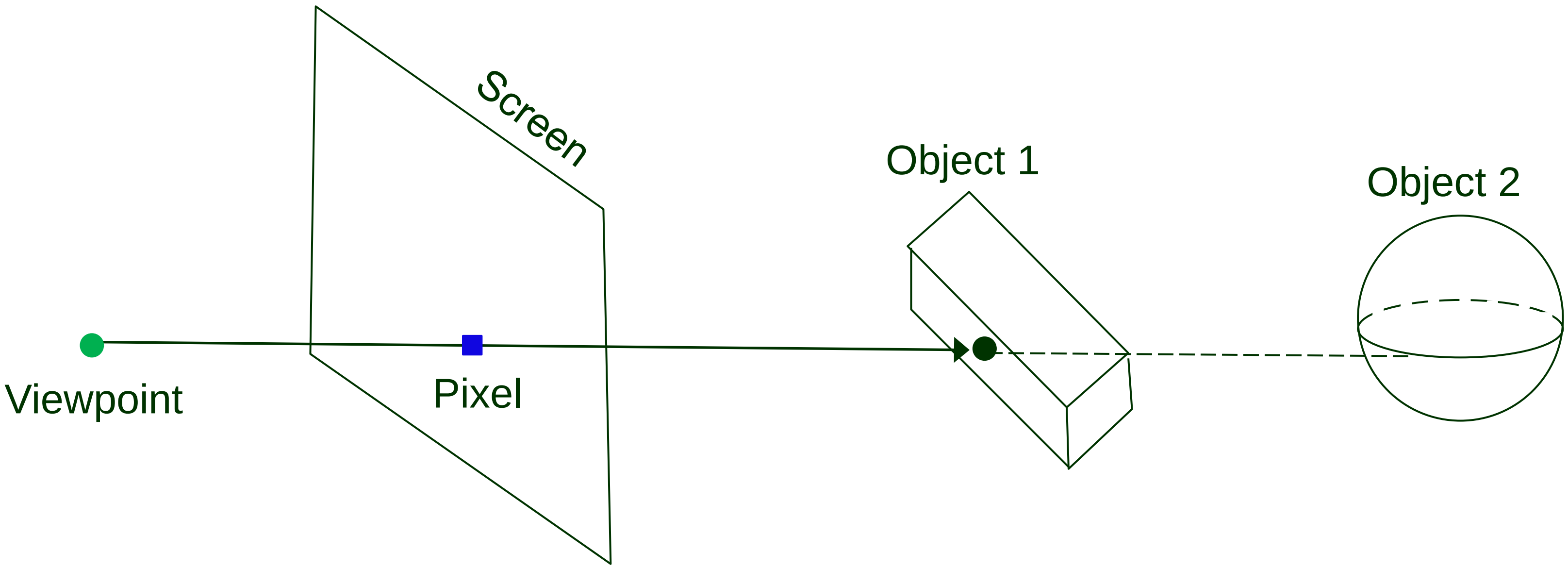
Determine

- for every pixel on the screen which object is visible
- at that pixel the *intensity of light* emitted by the object in the direction of the viewpoint



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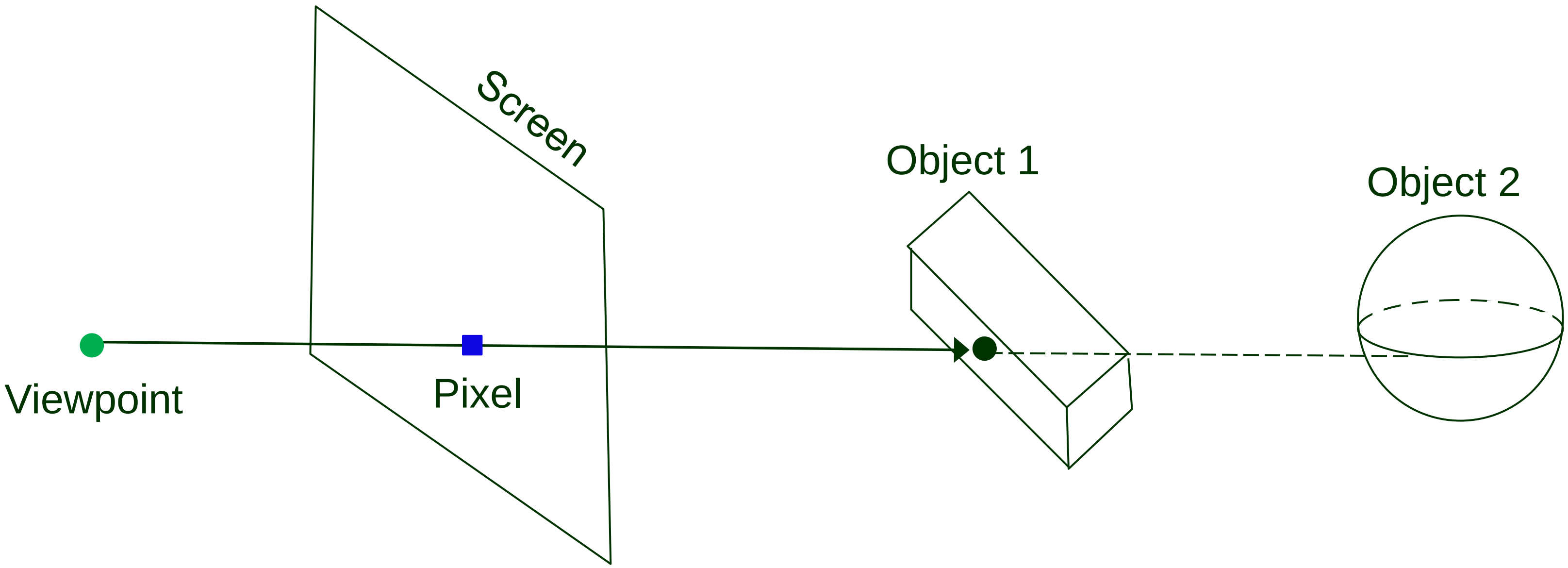
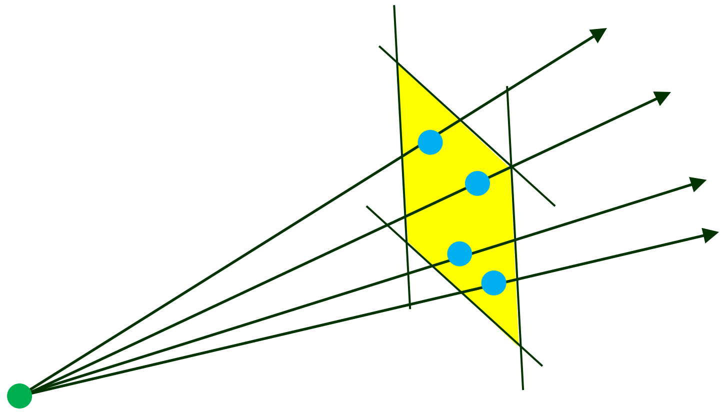


Ray tracing: Shoot a ray through each pixel. First object hit is visible.

How to Determine Light Intensity?

Shoot multiple rays per pixel.



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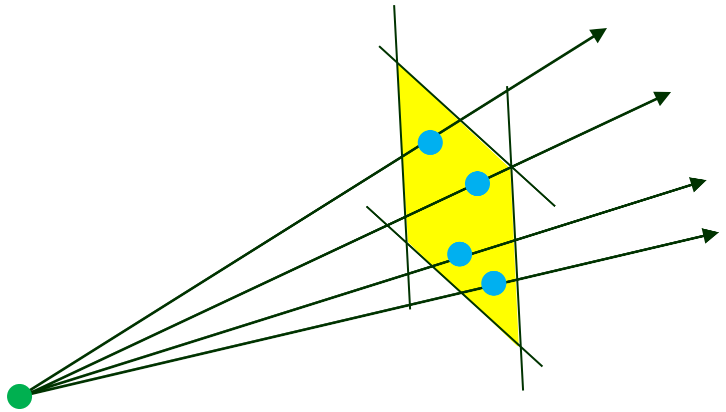

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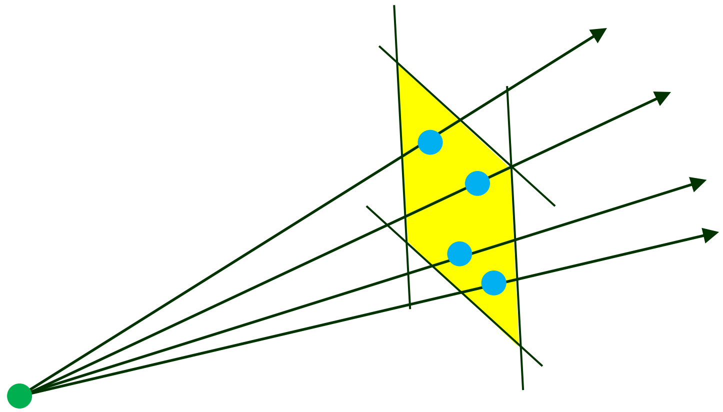

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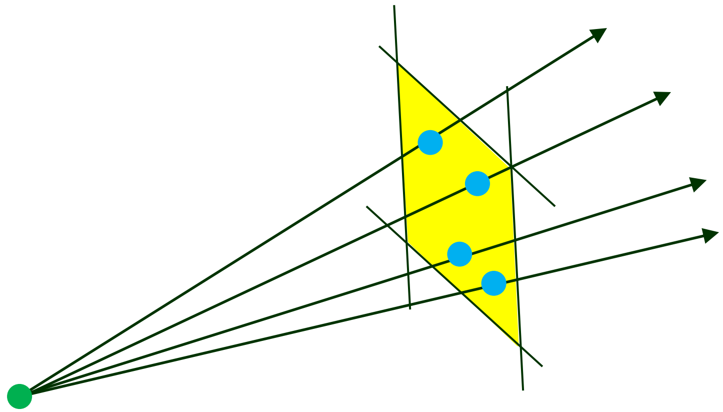

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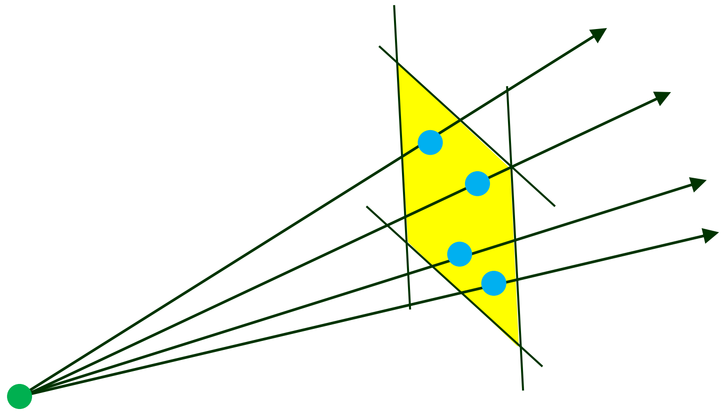

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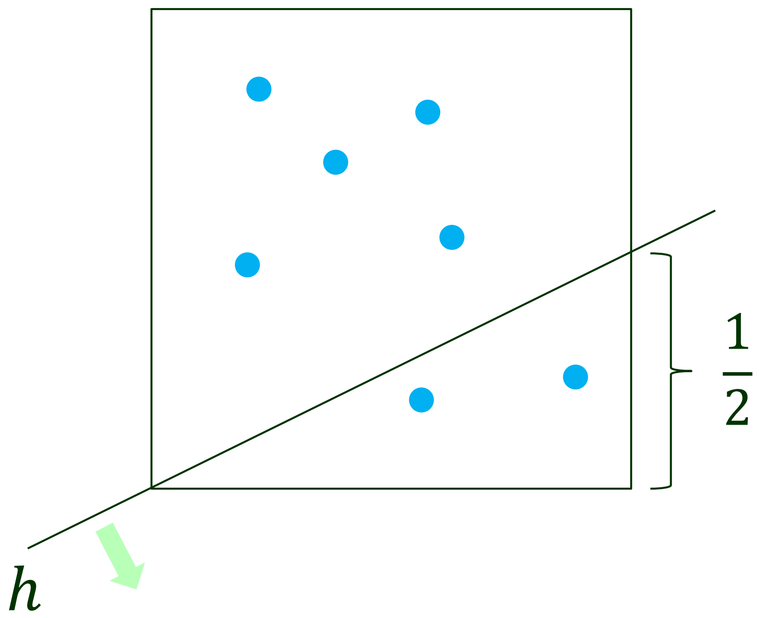
Discrepancy of the sample set:

| % of hits on an object – % of visible pixel area for it |

Discrepancy Measures

Unit square U: $[0, 1] \times [0,1]$



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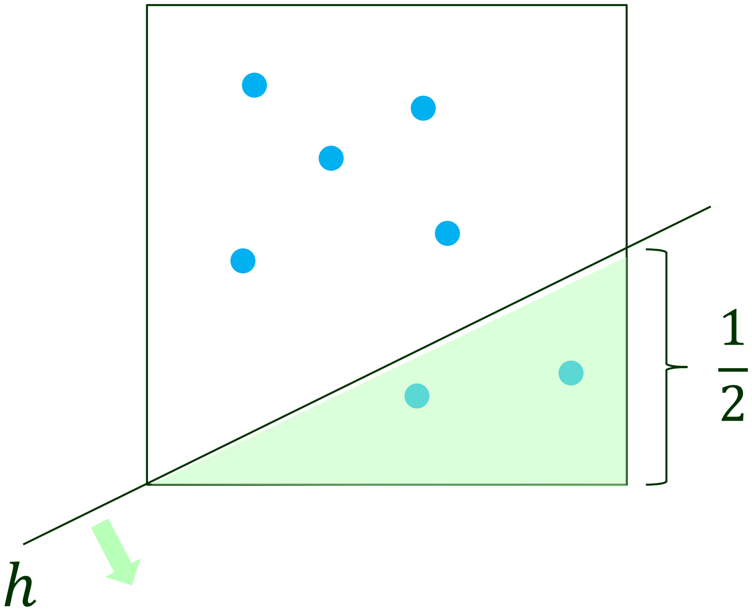

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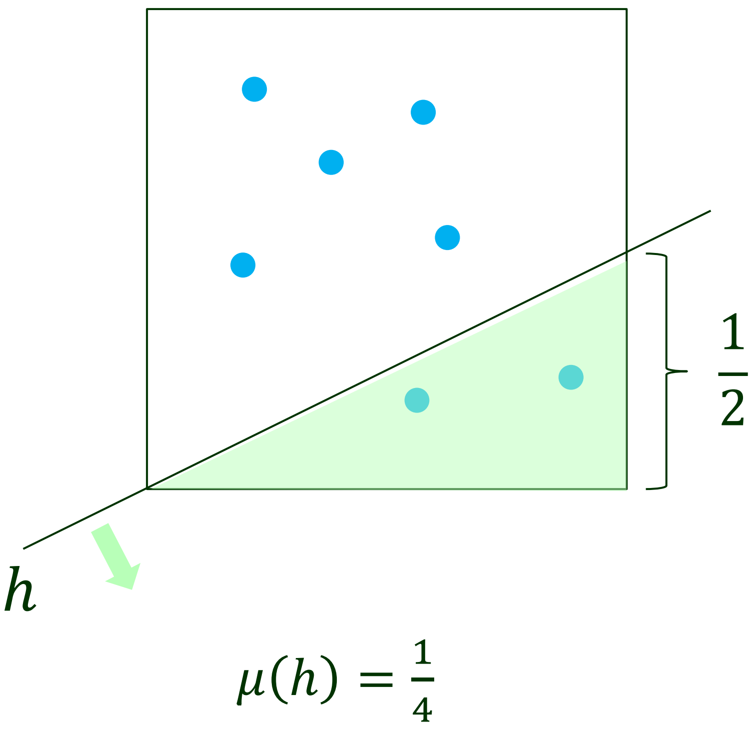

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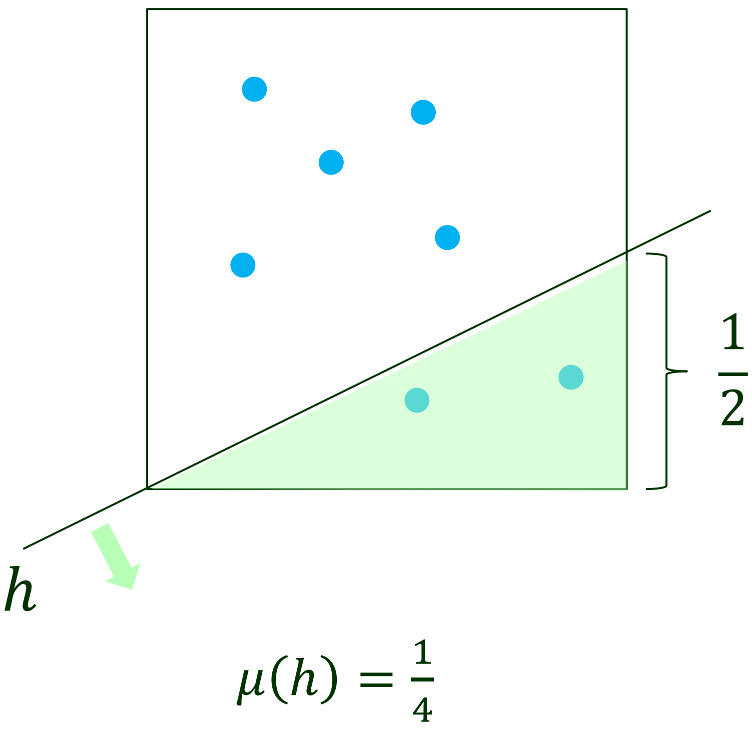

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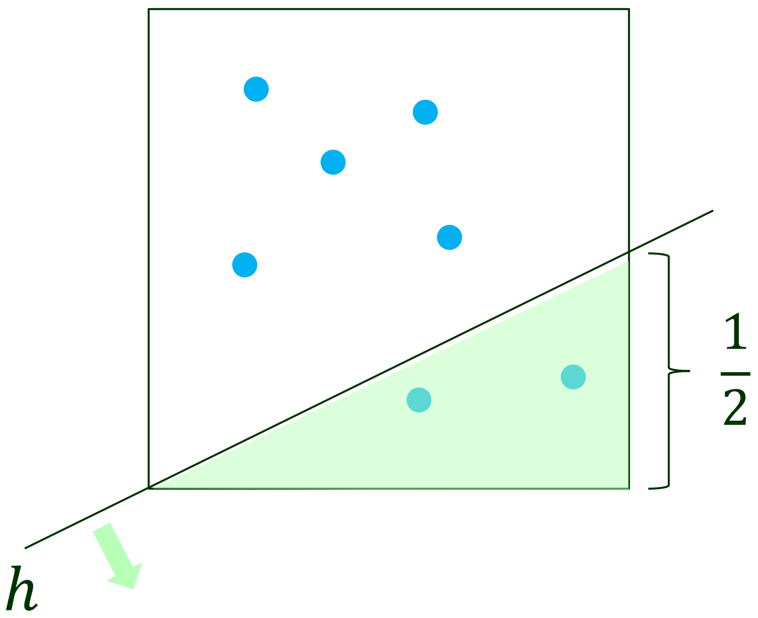

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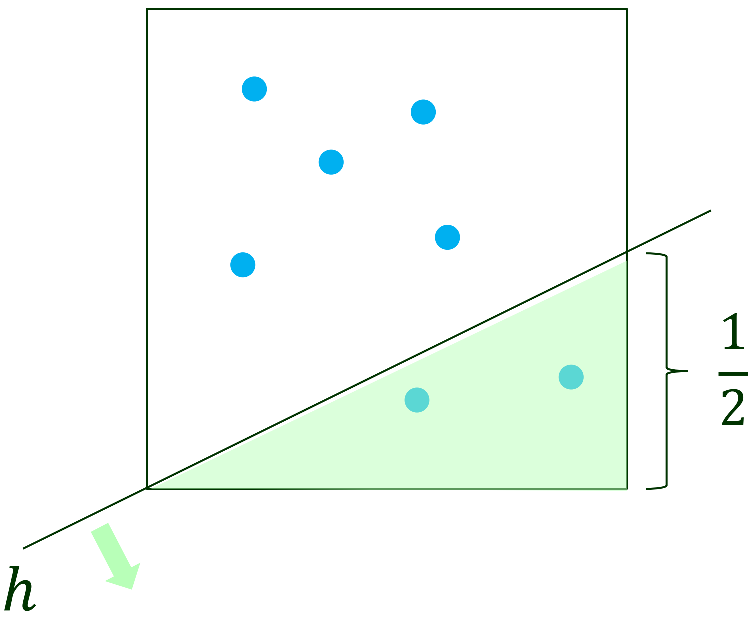

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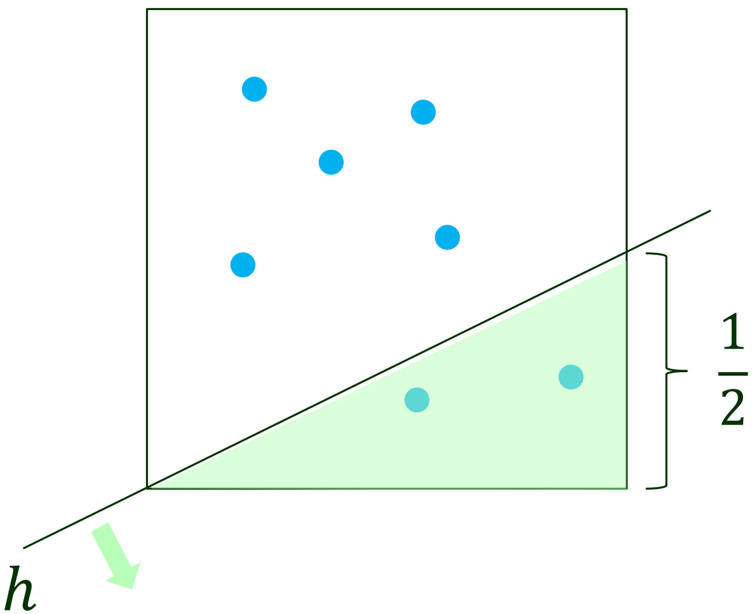

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The *maximum* discrepancy of any closed half-plane.

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- Check every half-plane that has *local maximum* discrepancy.

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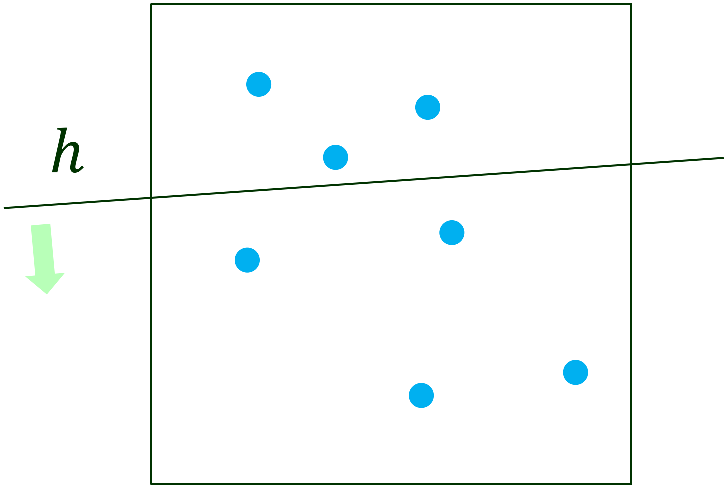
- Pick one of them that has the largest value of these local maxima.

II. Half-Plane Achieving Local Maximum

Suppose h does not contain a point on its boundary.



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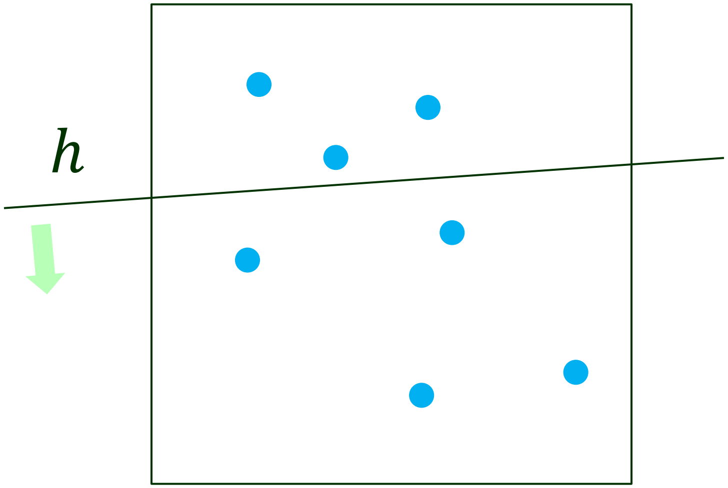

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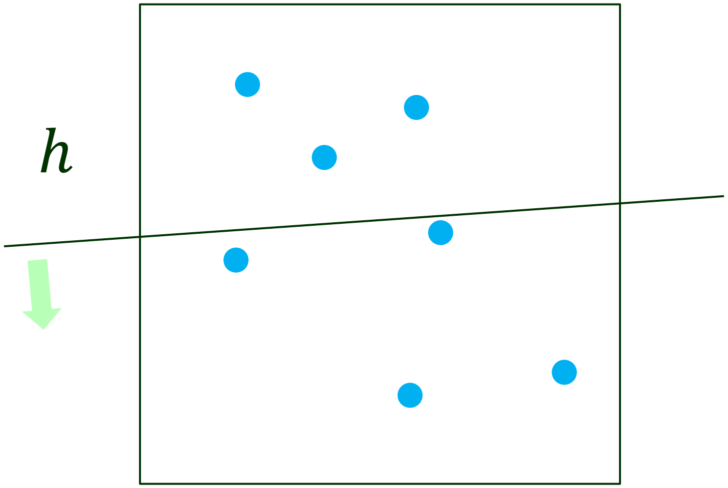

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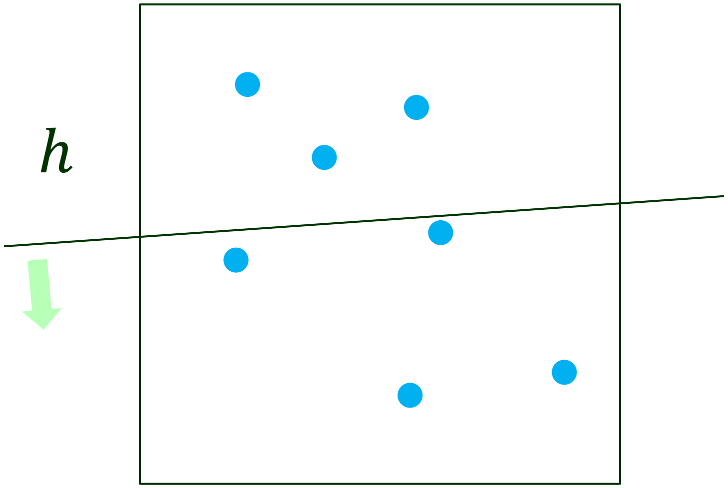

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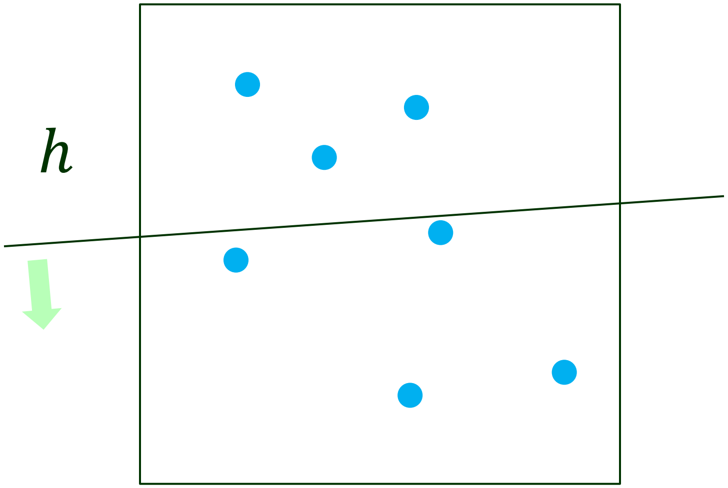

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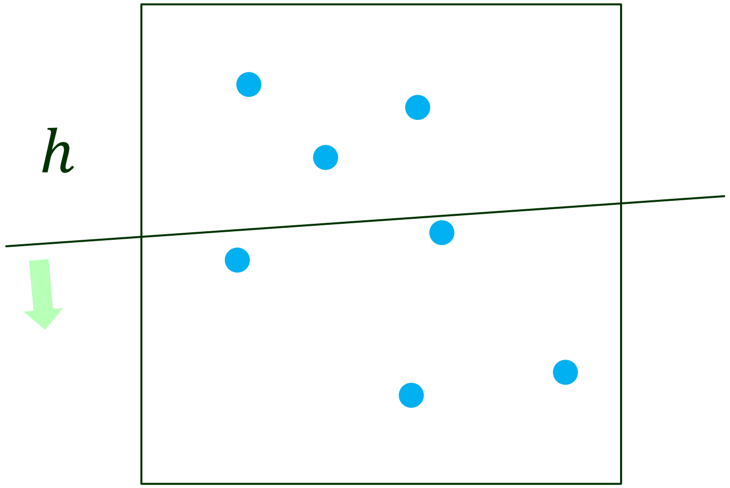

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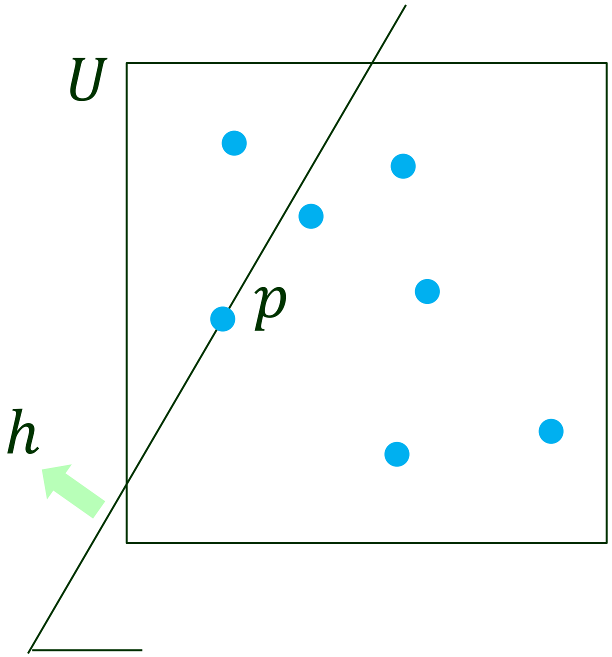
Claim A half-plane of local maximum must have a point on its boundary.

Half-Plane Rotation

Consider a maximizing half-plane h with exactly one point p from S on its boundary line.



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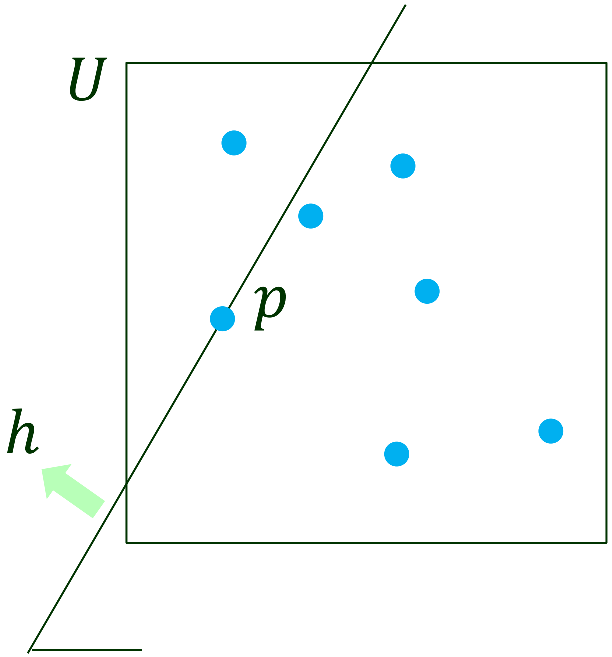

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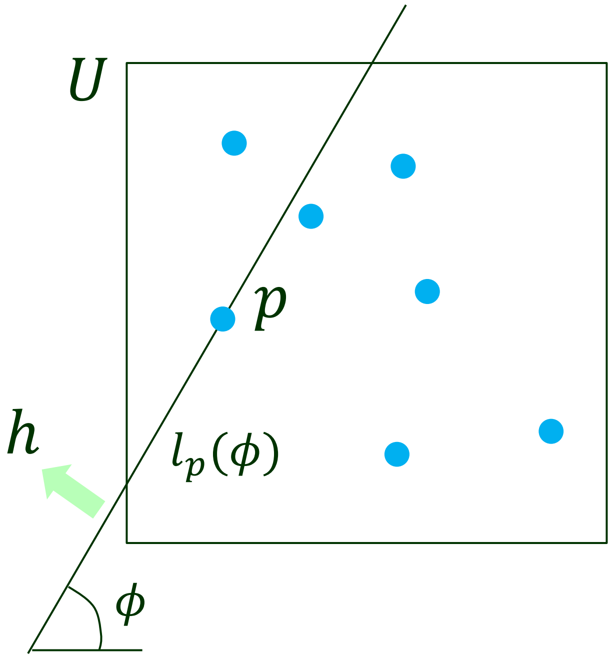

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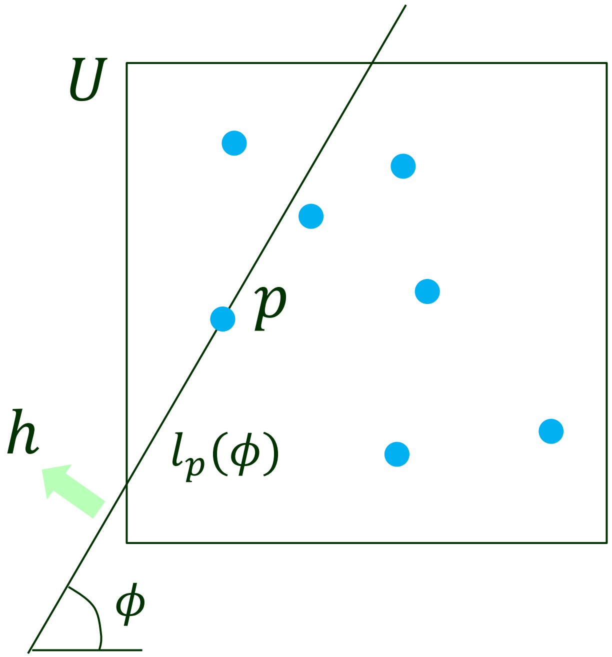

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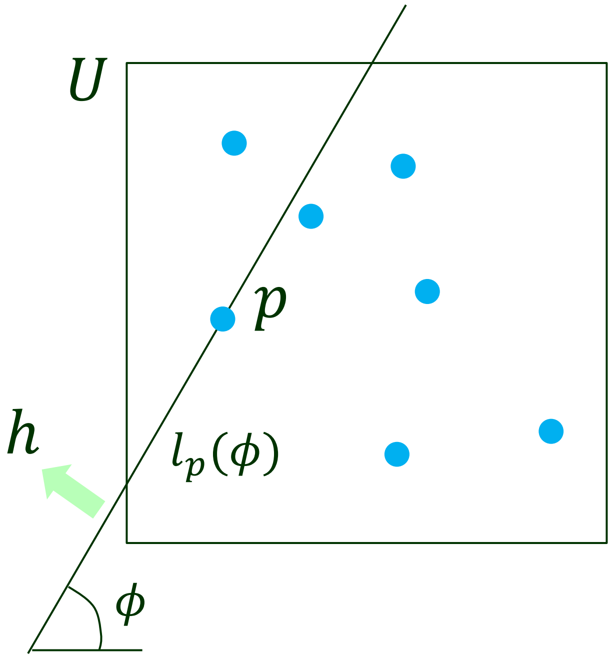

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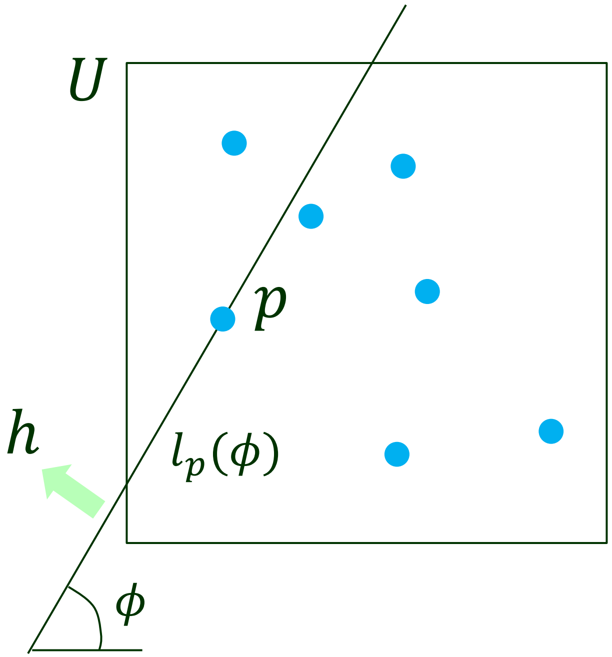

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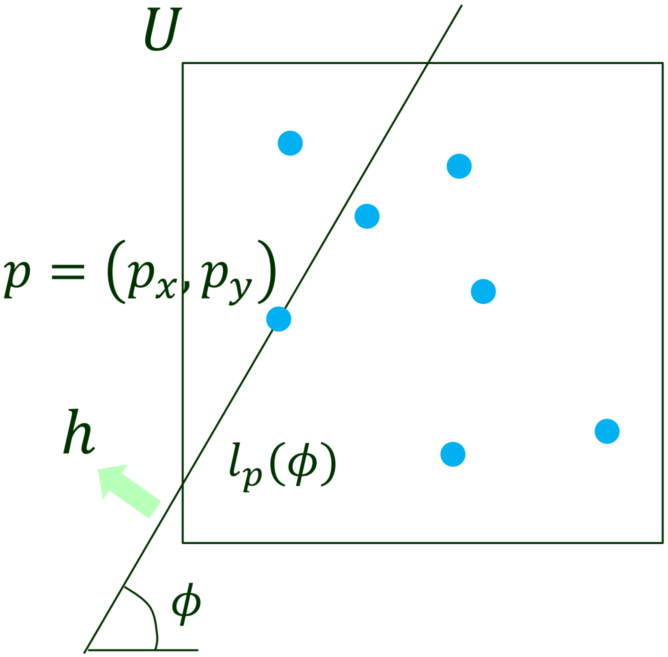
Extrema of this kind are checked in $O(1)$ time.

Half-Plane Rotation (cont'd)



- In between, it intersects the same two edges of U, resulting in a *continuous* measure function $\mu\big(h_p(\phi)\big)$.

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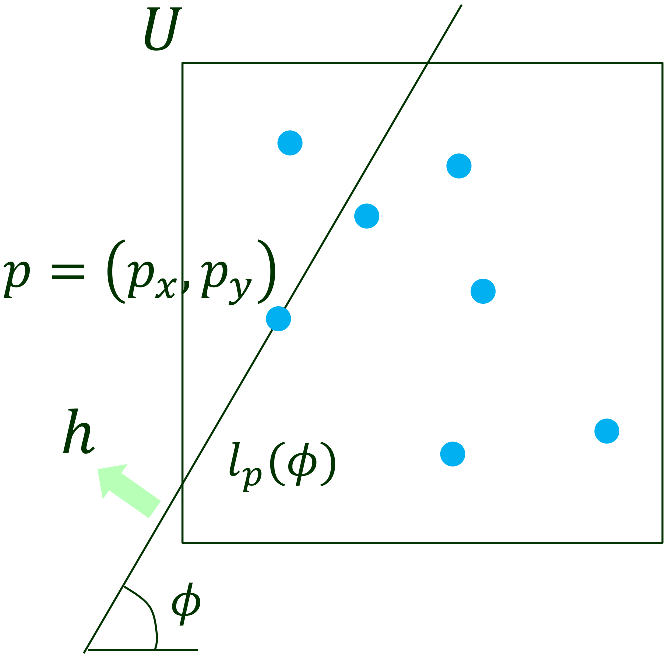

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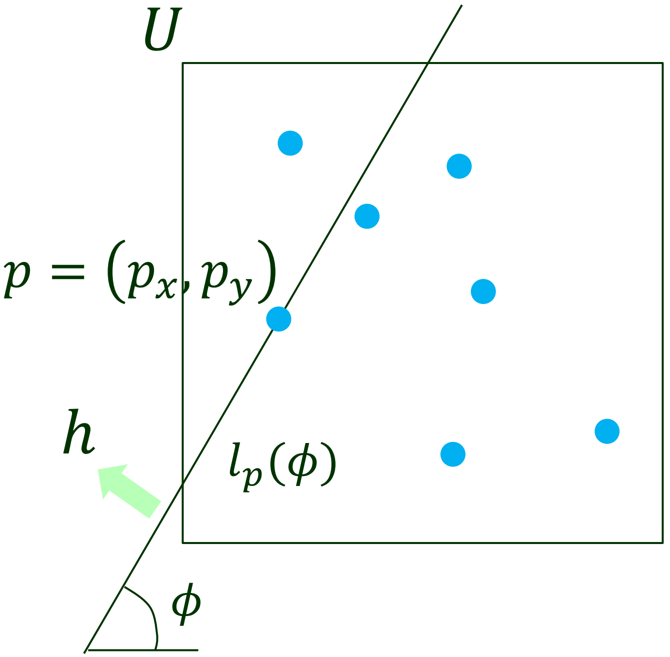

e.g., top & left boundaries intersected:

$$\mu\big(h_p(\phi)\big) = \frac{1}{2}\left(1 - p_y + p_x \tan\phi\right)\left(p_x + \frac{1-p_y}{\tan\phi}\right)$$

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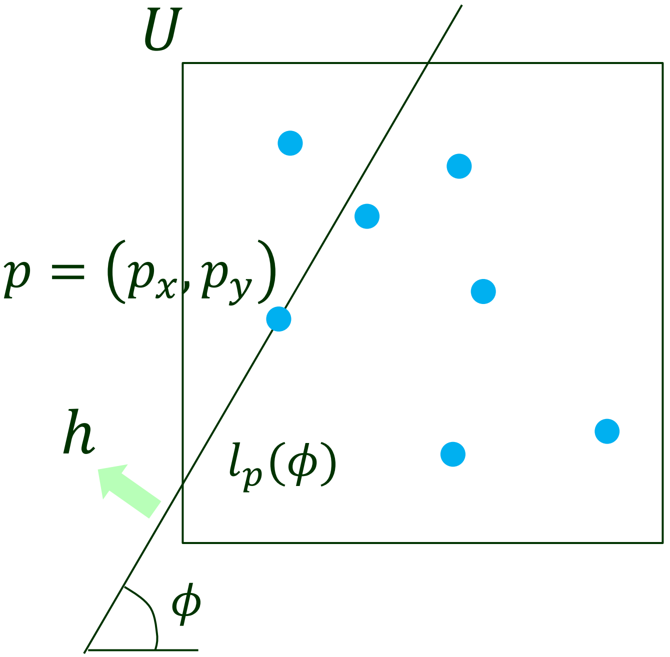

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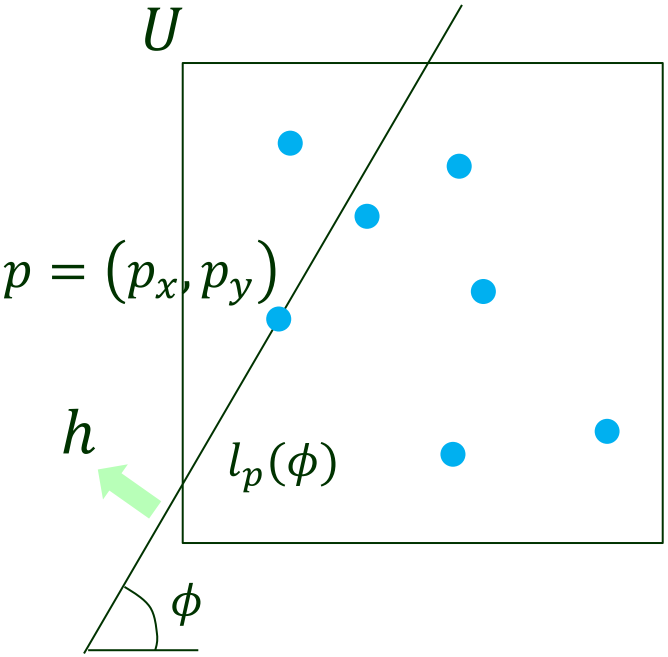

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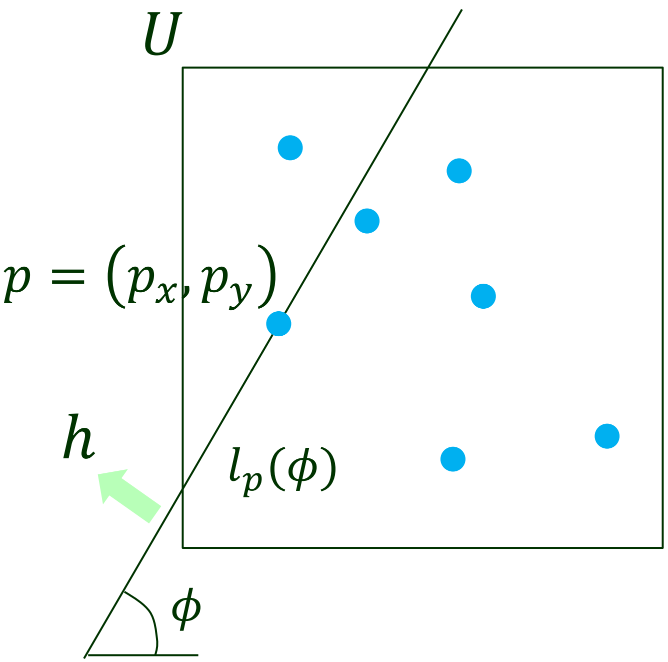

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This estimate is conservative since the line will hit another point before it completes a rotation of 2π.

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Lemma If a half-plane h achieves the maximum discrepancy, then its boundary line must pass through either

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Theorem The half-plane discrepancy of a set S of n points in the unit square can be computed in $O(n^2)$ time.