Basics of Probability

Outline:

I. Acting under uncertainty

II. Probability model

III. Probability distribution

* Figures are from the textbook site or by the instructor.
** A small part of the notes are adapted from those by Dr. Jin Tian.
I. Acting Under Uncertainty
Drawbacks of Using Belief States

Keeping track of a belief state has several drawbacks:

- **Large belief state** full of unlikely possibilities because every possible explanation of the percept needs to be considered.

- **Correct contingent plan** growing arbitrarily large to handle every eventuality.

- **No successful plan** guaranteed sometimes yet action required.

There needs to be a way to compare the merits of plans that are not guaranteed.
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There needs to be a way to compare the merits of plans that are not guaranteed.

Make a rational decision depending on

- relative importance of various goals, and

- likelihood that, and degree to which, they will be achieved.
Intelligent behavior requires knowledge about the world.

Propositional and first-order logics are effective for representing and reasoning with categorical beliefs about the world.

Due to uncertainty, any logical sentence could be true, false or unknown.

Logical approach can break down.
Dental Diagnosis

Toothache $\Rightarrow$ Cavity
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// Wrong rule as some patients with toothaches
// have no cavity but gum disease, an abscess, …
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Toothache $\Rightarrow$ Cavity $\lor$ Gumproblem $\lor$ Abscess $\lor$ …

// We would have to add an almost unlimited list of possible problems.
Dental Diagnosis

$\text{Toothache} \Rightarrow \text{Cavity}$  // Wrong rule as some patients with toothaches have no cavity but gum disease, an abscess, …

$\text{Toothache} \Rightarrow \text{Cavity} \lor \text{Gumproblem} \lor \text{Abscess} \lor \cdots$  // We would have to add an almost unlimited list of possible problems.

$\text{Cavity} \Rightarrow \text{Toothache}$  // Still not right since not all cavities cause pain.
Dental Diagnosis

Toothache $\Rightarrow$ Cavity

$\Rightarrow$ Wrong rule as some patients with toothaches have no cavity but gum disease, an abscess, ...

Toothache $\Rightarrow$ Cavity $\lor$ Gumproblem $\lor$ Abscess $\lor$ ...

$\Rightarrow$ We would have to add an almost unlimited list of possible problems.

Cavity $\Rightarrow$ Toothache

$\Rightarrow$ Still not right since not all cavities cause pain.

Fix the rule by augment the LHS with all qualifications required for a cavity to cause a toothache!

- Too much work.
- Medical science has no complete domain theory.
- Not all the necessary tests have been or can be run on a patient.
Example of Reasoning Under Uncertainty

Beliefs

• If a patient has lung cancer, there is a 60% chance that an X-ray test will come back positive; and a 40% percent chance negative.

• If a patient does not have lung cancer, there is 2% percent chance that an X-ray test will come back positive; and a 98% percent chance negative.

• Population cancer rate is 1/1000.
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Probability theory provides a framework for representing and reasoning under uncertainty
II. Probability Basics

Probability deals with chance experiments that have a set $\Omega$ of distinct outcomes or possible worlds ($\Omega$ is called the *sample space*).
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- Roll two dice: $\Omega = \{6, (1,2), \ldots, (1,6), \ldots, (6,1), \ldots, (6,6)\}$
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- Toss a coin: $\{\text{head, tail}\}$
- Roll two dice: $\{(6, 1), \ldots, (1,6), \ldots, (6,6)\}$

$\omega \in \Omega$ is a sample point, possible world, or atomic event.

- mutually exclusive
- exhaustive
A *probability space* associates a probability $P(\omega)$ with every possible world $\omega$:

\begin{align*}
0 \leq P(\omega) &\leq 1 \\
\sum_{\omega \in \Omega} P(\omega) &\leq 1
\end{align*}
A *probability space* associates a probability $P(\omega)$ with every possible world $\omega$:

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- Roll two dice: $\Omega = \{(1,1), (1,2), \ldots, (6,6)\}$
  - If both dice are fair, then each possible world has probability $1/36$.
  - If the dice are loaded, then the worlds will have uneven probabilities that still sum to 1.
Probability Model

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  $$P(a) = P(2) + P(4) + P(6) = 1/2$$

- $b$ is the event that the results of rolling two dice sum to 10.
  $$P(\text{Total} = 10) = P(b) = P((4,6)) + P((5,5)) + P((6,4)) = 1/12$$
Prior vs. Posterior Probabilities

*Prior* (or *unconditional*) *probabilities* are in the absence of any other information.

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The *conditional* (or *posterior*) *probability* of an event (i.e., proposition) \(a\) given an event \(b\) with \(P(b) > 0\) is

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The prior of \( a \) is 1/6, and the posterior of \( a \) given \( b \) is 1/3.
Go to a dentist for a regular checkup. Consider the prior probability:

\[ P(\text{cavity}) = 0.2 \]
Prior vs. Posterior (cont’d)

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• If the dentist finds no cavities, we would not want to conclude the above posterior:

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The *product rule*: 
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The product rule:

\[ P(a \land b) = P(a | b)P(b) \]
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Random Variable

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A probability space \( P \) induces a probability distribution for any random variable \( X \).

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P(X = x) = \sum_{\omega: X(\omega) = x} P(\omega)
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$$ P(\text{Even} = \text{true}) = P(2) + P(4) + P(6) = 1/2 $$
Use of Proportional Logic

For a Boolean random variable $A$, we abbreviate proportions $A = true$ and $A = false$ as $a$ and $\neg a$, respectively.

// The probability that the patient has a cavity, given that // she is a teenager with no toothache, is 0.1.
$P(cavity \mid \neg toothache \land teen) = 0.1$

$P(cavity \mid \neg toothache, teen) = 0.1$
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Often in AI applications, random variables are basic elements.

♦ The sample points are the values of a set of random variables.

♦ A possible world is an assignment of exactly one value to every random variable.
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Often in AI applications, random variables are basic elements.

♦ The sample points are the values of a set of random variables.

♦ A possible world is an assignment of exactly one value to every random variable.

Example 4 distinct atomic events (or 4 possible worlds):

\begin{align*}
\text{cavity} \land \text{toothache} \\
\neg \text{cavity} \land \text{toothache} \\
\text{cavity} \land \neg \text{toothache} \\
\neg \text{cavity} \land \neg \text{toothache}
\end{align*}
III. Probability Distribution

- Probabilities of all the possible values of a random variable.

\[
P(\text{Weather} = \text{sun}) = 0.6 \\
P(\text{Weather} = \text{rain}) = 0.1 \\
P(\text{Weather} = \text{cloud}) = 0.29 \\
P(\text{Weather} = \text{snow}) = 0.01
\]
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Abbreviated as

\[ P(\text{Weather}) = \langle 0.6, 0.1, 0.29, 0.01 \rangle \]
III. Probability Distribution

- Probabilities of all the possible values of a random variable.

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\[ P(Weather = snow) = 0.01 \]

Abbreviated as:

**Probability distribution:** \( P(Weather) = \langle 0.6, 0.1, 0.29, 0.01 \rangle \)
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Probability distribution: \[ \mathbf{P}(\text{Weather}) = \langle 0.6, 0.1, 0.29, 0.01 \rangle \]

\( \mathbf{P}(X \mid Y) \) gives the values of \( P(X = x_i \mid Y = y_j) \) for all \( i,j \).
Joint Probability Distribution

Joint probability distribution gives the probability of every atomic event on a set of RVs (i.e., every combination of their values).

\[ P(\text{Weather, Cavity}) \] is a $4 \times 2$ matrix.

<table>
<thead>
<tr>
<th>Weather</th>
<th>sun</th>
<th>rain</th>
<th>cloud</th>
<th>snow</th>
</tr>
</thead>
<tbody>
<tr>
<td>true</td>
<td>0.144</td>
<td>0.02</td>
<td>0.016</td>
<td>0.02</td>
</tr>
<tr>
<td>false</td>
<td>0.576</td>
<td>0.08</td>
<td>0.064</td>
<td>0.08</td>
</tr>
</tbody>
</table>
Concise P Notation

\[
P(\text{Weather}, \text{Cavity}) = P(\text{Weather} \mid \text{Cavity}) \ P(\text{Cavity})
\]

\[
\{ \text{sun, rain, cloud, snow} \} \ \{ \text{true, false} \}
\]

replaces 8 equations (corresponding to 8 possible worlds)

\[
P(W = \text{sun} \land C = \text{true}) = P(W = \text{sun} \mid C = \text{true})P(C = \text{true})
\]
\[
P(W = \text{rain} \land C = \text{true}) = P(W = \text{rain} \mid C = \text{true})P(C = \text{true})
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Kolmogorov’s Axioms

1. \( 0 \leq P(\omega) \leq 1 \) for every world \( \omega \in \Omega \).

2. \( \sum_{\omega \in \Omega} P(\omega) = 1 \)

3. \( P(a \lor b) = P(a) + P(b) - P(a \land b) \) for any two propositions \( a, b \).

The rest of probability theory are built up from these axioms.
Kolmogorov’s Axioms

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Inconsistent Beliefs

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<tr>
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<th>Agent 1’s belief</th>
<th>Agent 2 bets</th>
<th>Agent 1 bets</th>
<th>Agent 1 payoffs for each outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.4</td>
<td>$4 on $a$</td>
<td>$6 on $\neg a$</td>
<td>$-6$ $-6$ $4$ $4$</td>
</tr>
<tr>
<td>$b$</td>
<td>0.3</td>
<td>$3 on $b$</td>
<td>$7 on $\neg b$</td>
<td>$-7$ $3$ $-7$ $3$</td>
</tr>
<tr>
<td>$a \lor b$</td>
<td>0.8</td>
<td>$2 on $\neg (a \lor b)$</td>
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<tr>
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Agent 2 will lose $4 if $\neg a$

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\[\text{Agent 1 payoffs for each outcome:} \begin{array}{ccccc} a, b & a, \neg b & \neg a, b & \neg a, \neg b \\ -6 & -6 & 4 & 4 \\ -7 & 3 & -7 & 3 \\ 2 & 2 & 2 & -8 \\ -11 & -1 & -1 & -1 \end{array}\]
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<td></td>
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<td>$4 on $a</td>
<td>$6 on $\neg$\neg$a = a $a$</td>
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<td>$a$</td>
<td>0.4</td>
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<td>$-6$ $-6$ $4$ $4$</td>
</tr>
<tr>
<td>$b$</td>
<td>0.3</td>
<td>$3 on $b$</td>
<td>$7 on $\neg$\neg$b$ $b$</td>
<td>$-7$ $3$ $-7$ $3$</td>
</tr>
<tr>
<td>$a \lor b$</td>
<td>0.8</td>
<td>$2 on $\neg(a \lor b)$</td>
<td>$8 on $a \lor b$</td>
<td>$2$ $2$ $2$ $-8$</td>
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Agent 2 will lose $4 if $\neg a$

Agent 1 will lose $6 if $\neg \neg a = a$
Inconsistent Beliefs

Agent 1 will lose $6 if $\neg \neg a = a$.

Agent 2 will lose $4 if $\neg a$.

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Agent 1 has inconsistent beliefs since

$$0.8 = P(a \lor b) > P(a) + P(b) = 0.7$$
Inconsistent Beliefs

- Agent 1 has inconsistent beliefs since
  
  \[ 0.8 = P(a \lor b) > P(a) + P(b) = 0.7 \]

- Agent 2 can devise a set of bets to guarantee a win.

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Agent 2 will lose $4 if \(\neg a\)  
Agent 1 will lose $6 if \(\neg \neg a = a\)
Inconsistent Beliefs

- Agent 1 has inconsistent beliefs since
  \[0.8 = P(a \lor b) > P(a) + P(b) = 0.7\]

- Agent 2 can devise a set of bets to guarantee a win.

No rational agent can have beliefs that violate the axioms of probability (following De Finetti’s theorem).

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Agent 2 will lose $4 if $\neg a$ and $6 if $\neg \neg a = a$.
Agent 1 has inconsistent beliefs since

\[ 0.8 = P(a \lor b) > P(a) + P(b) = 0.7 \]

Agent 2 can devise a set of bets to guarantee a win.

No rational agent can have beliefs that violate the axioms of probability (following De Finetti’s theorem).

For more on basics of probability, see [https://faculty.sites.iastate.edu/jia/files/inline-files/probability.pdf](https://faculty.sites.iastate.edu/jia/files/inline-files/probability.pdf).