Computing the Delaunay Triangulation

Outline:

I. Edge legalization

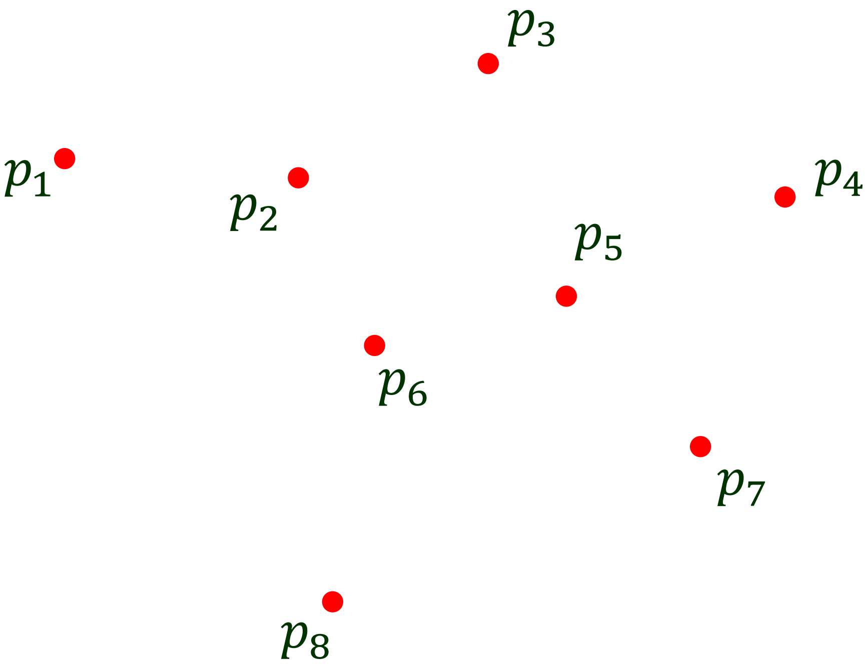
II. Correctness

III. Use of a trapezoidal map

IV. Analysis of storage and run time

I. The Construction Problem

Input: a set P of n points.



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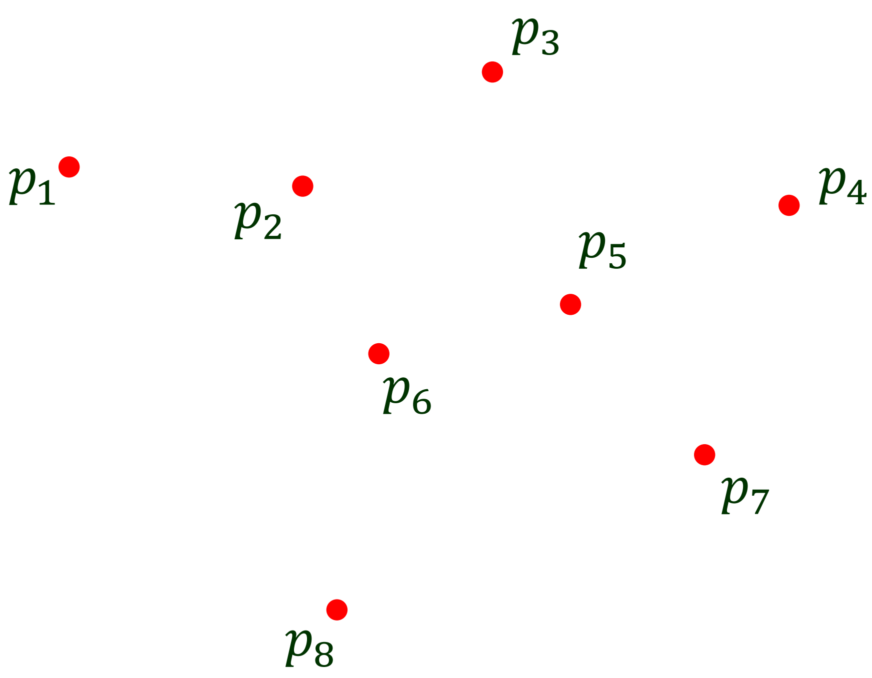
Algorithm 1

1) Compute the Voronoi Diagram $\mathrm{Vor}(P)$.



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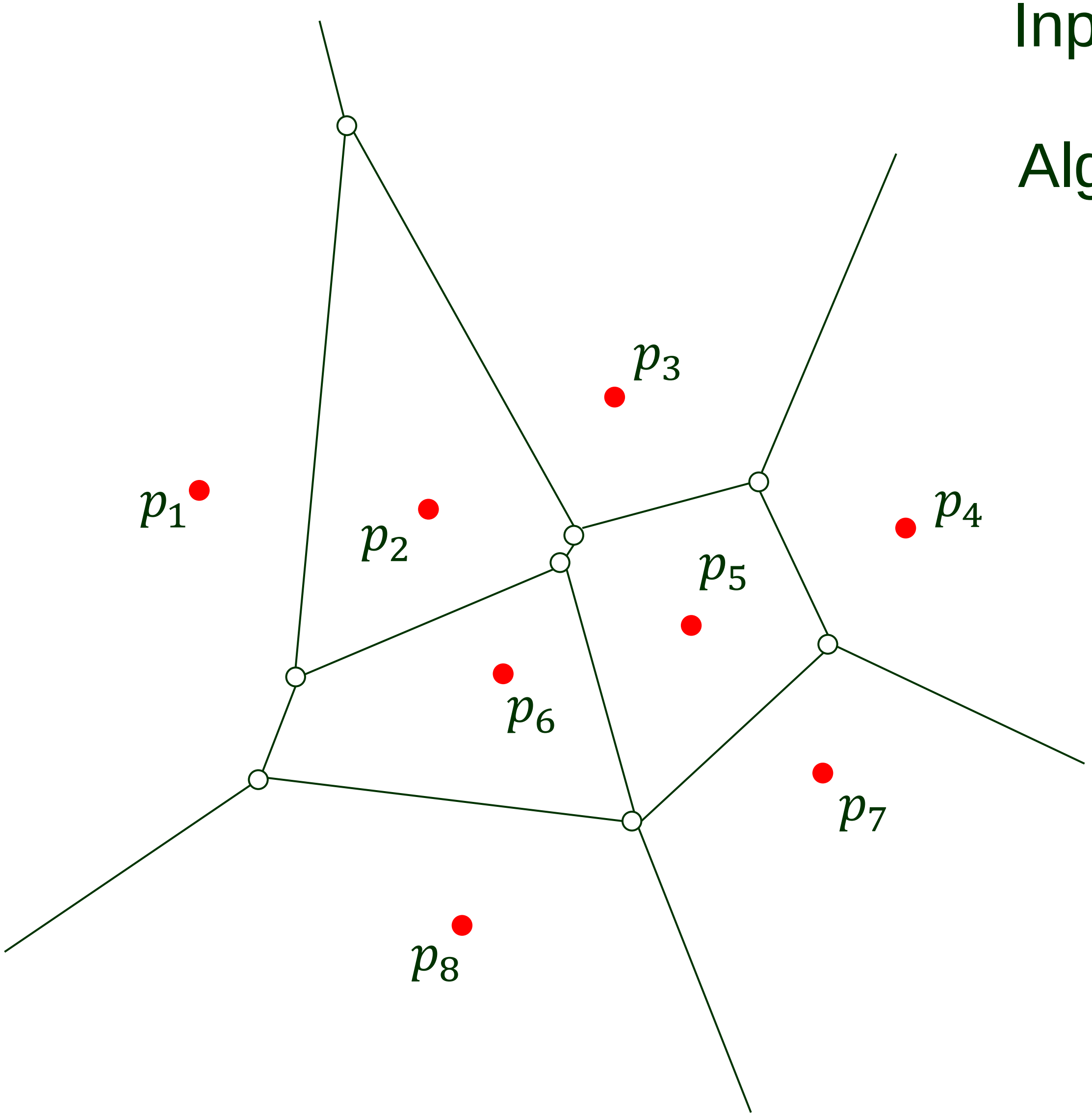

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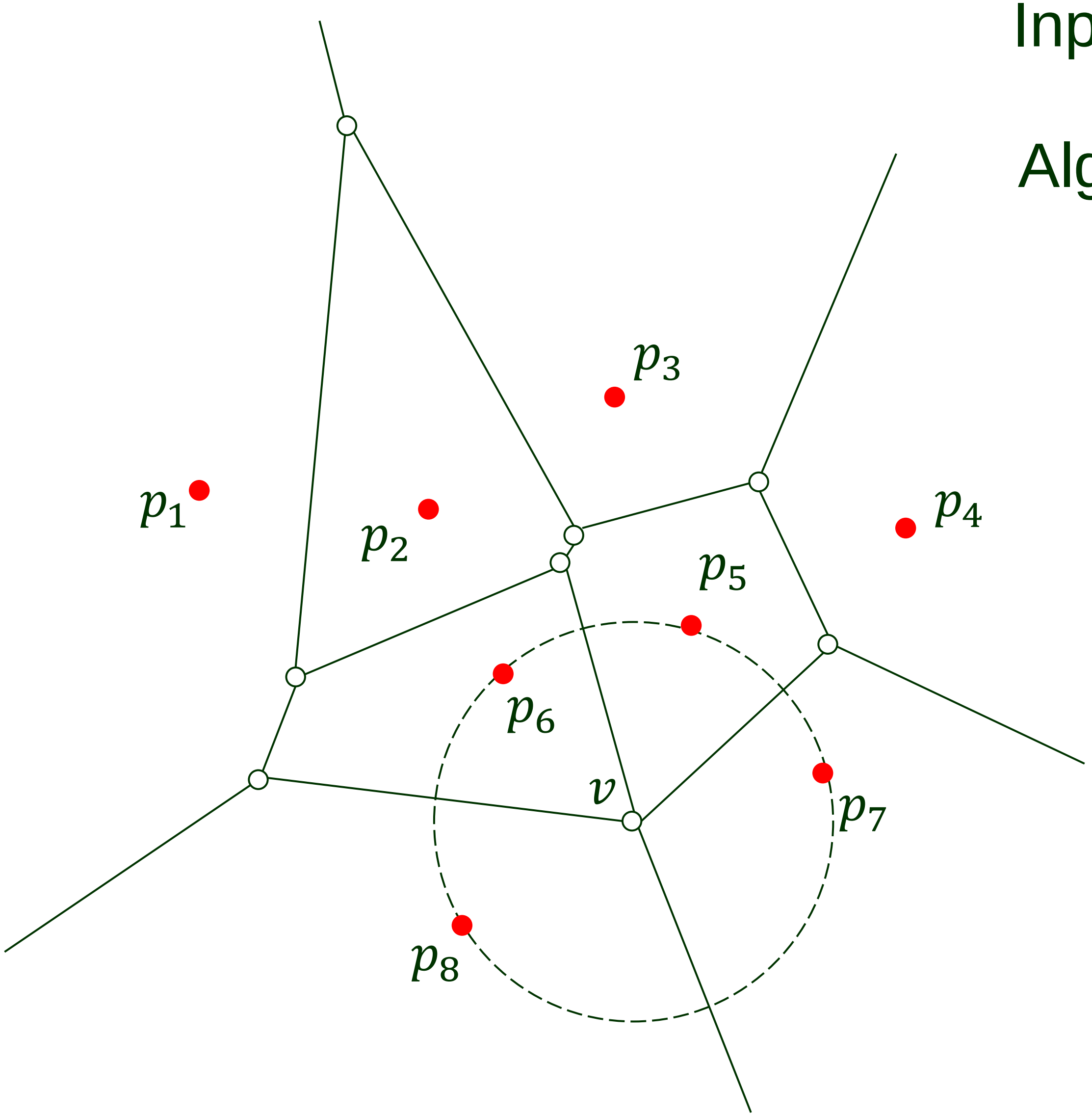

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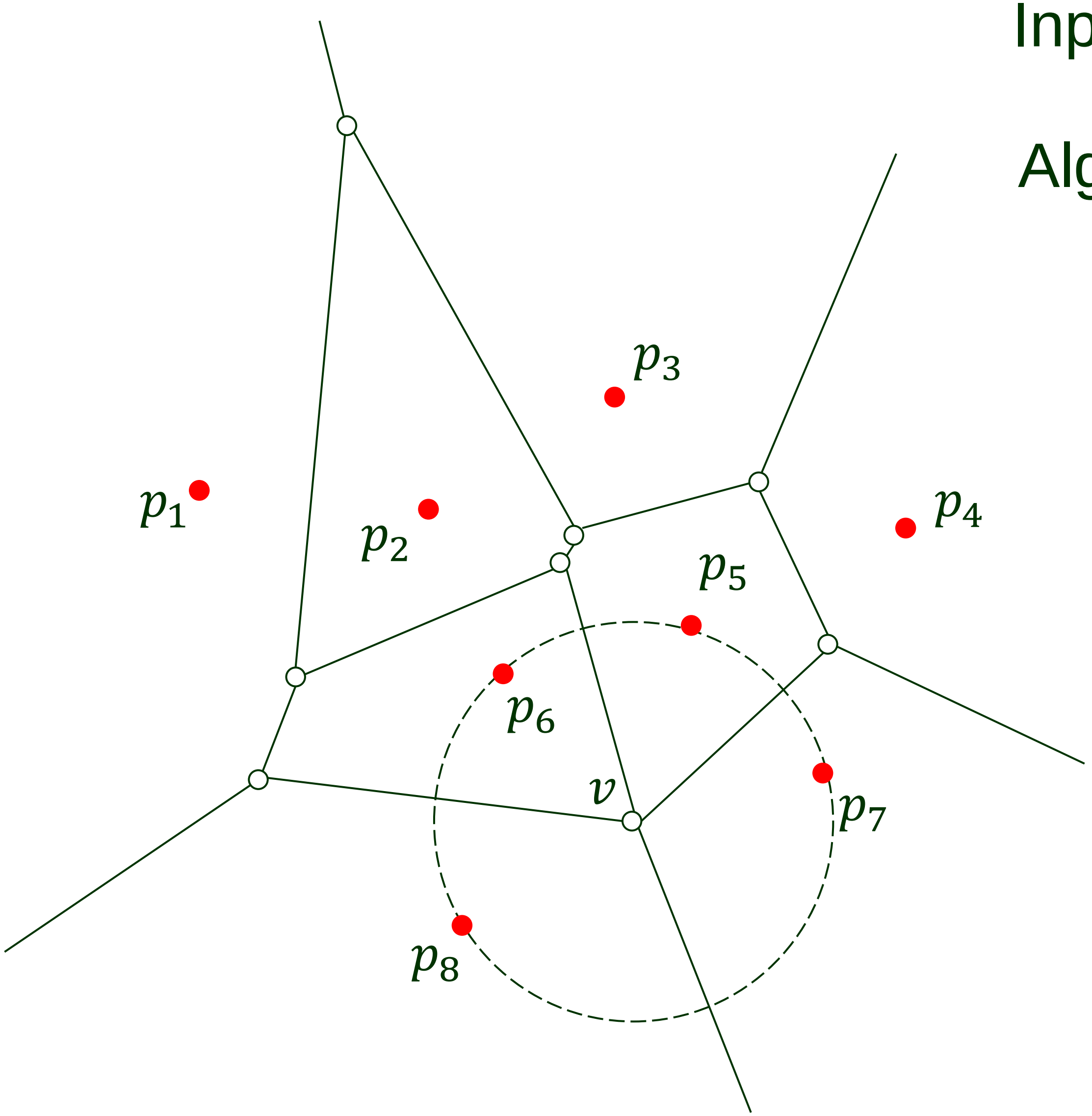

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2) Obtain DG(P).

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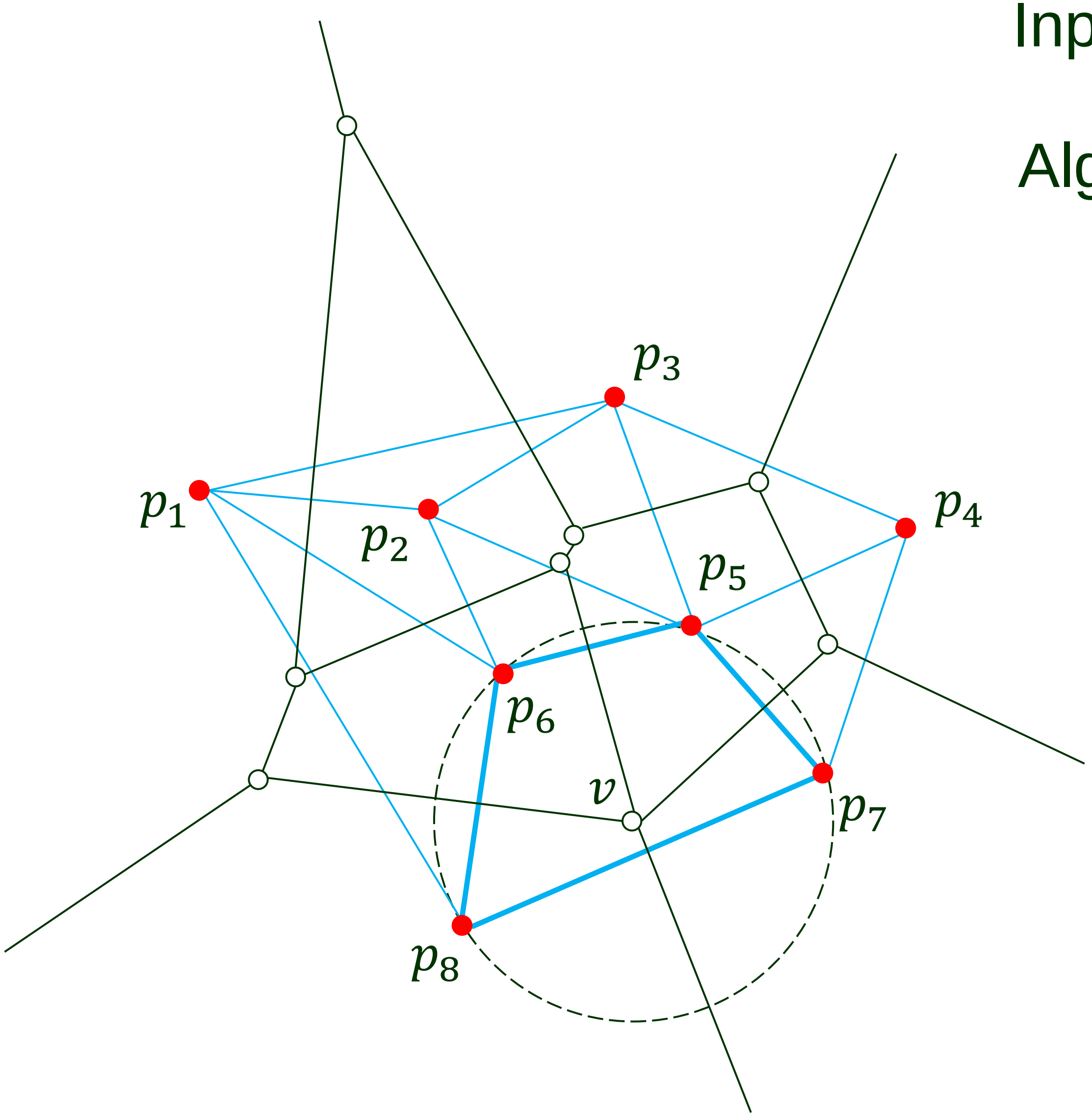

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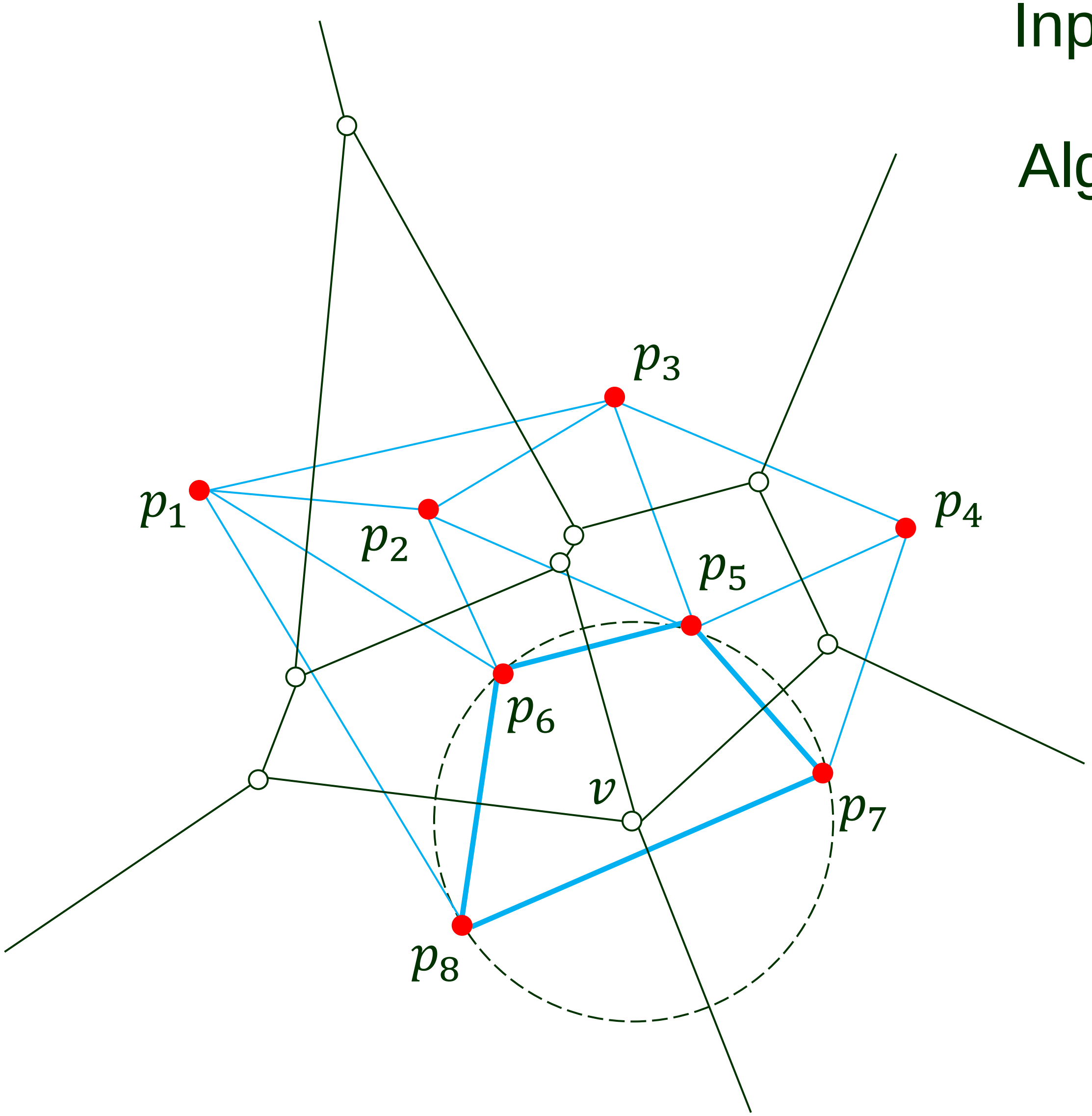

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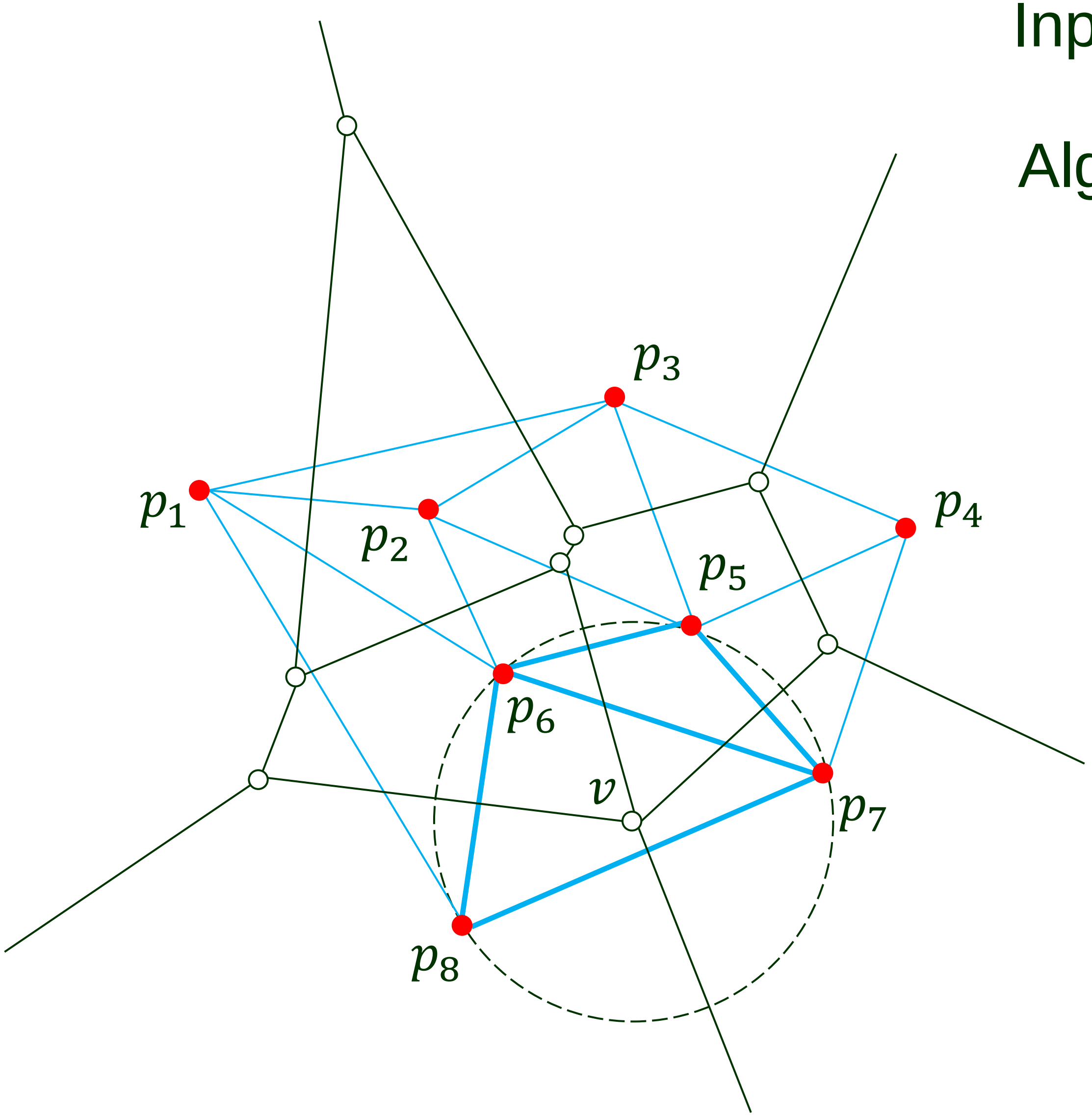

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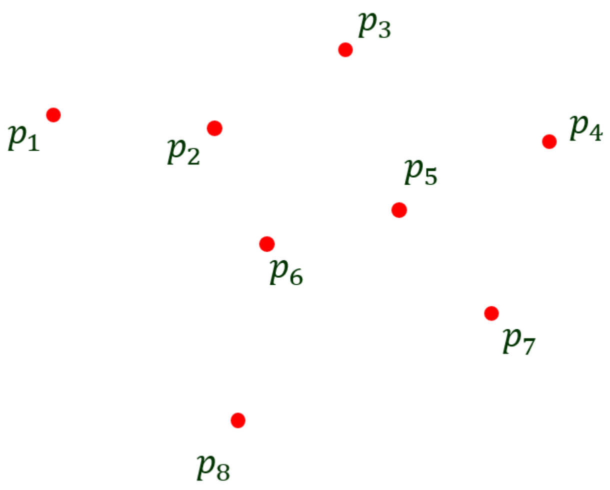

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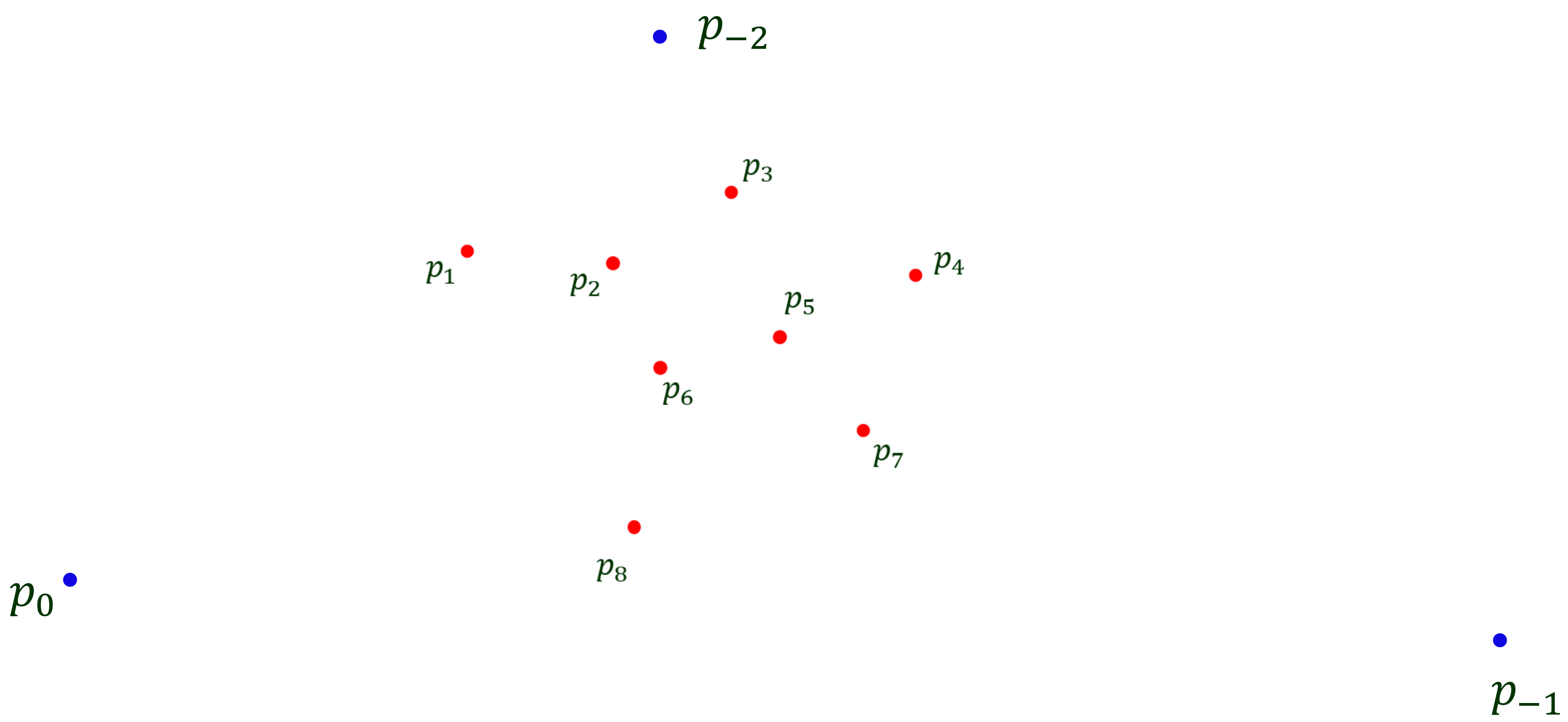

Randomized Incremental Construction

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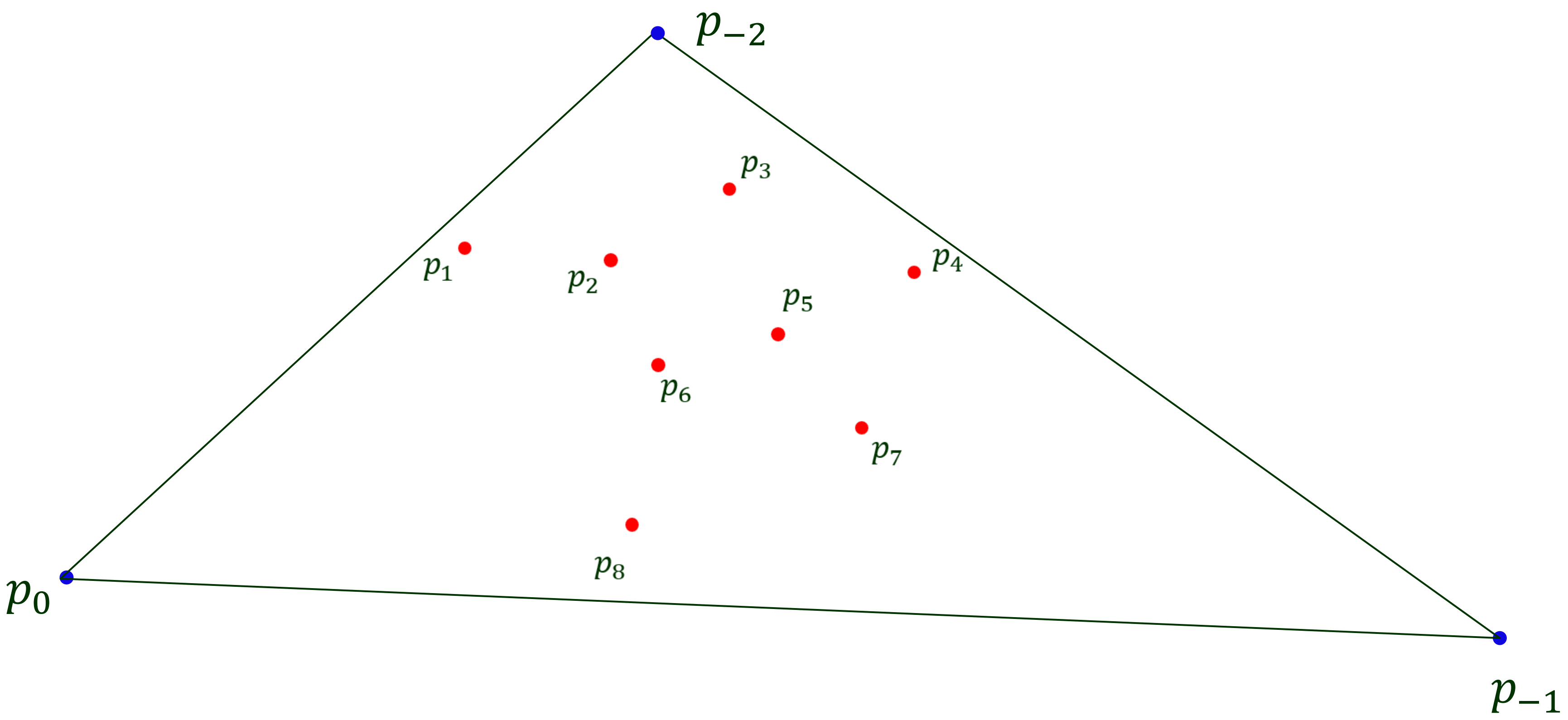
   Start with $\Delta\, p_0 p_{-1} p_{-2}$ containing all points from P in the interior.



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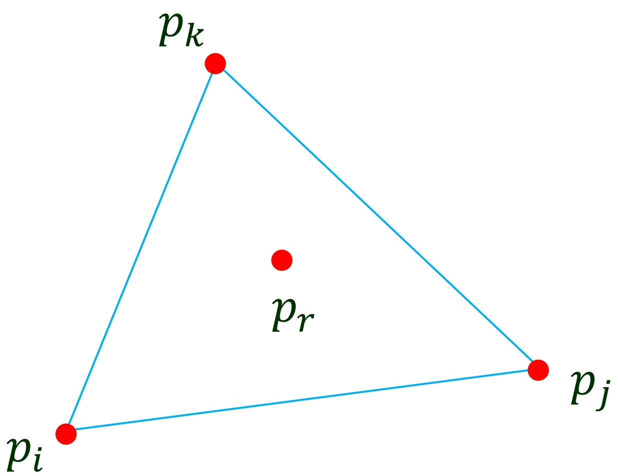

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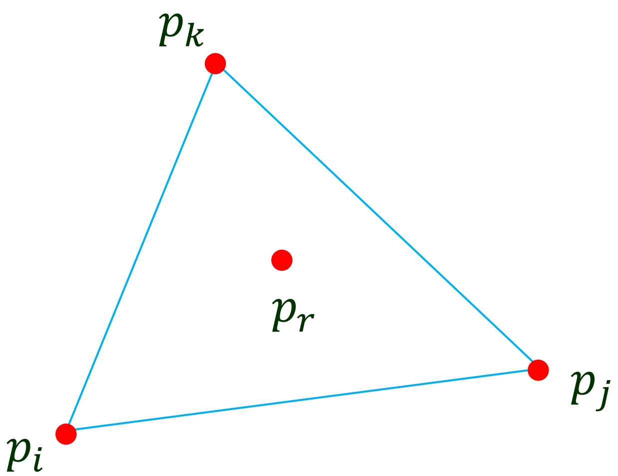

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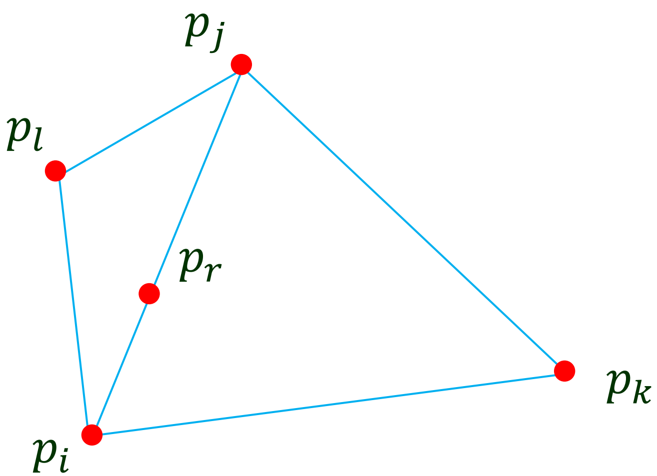
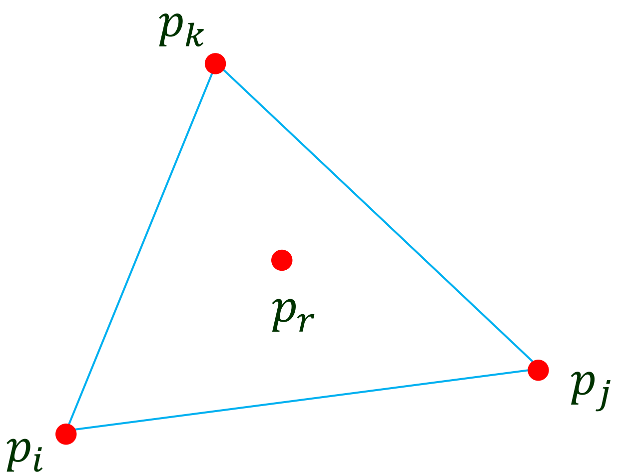
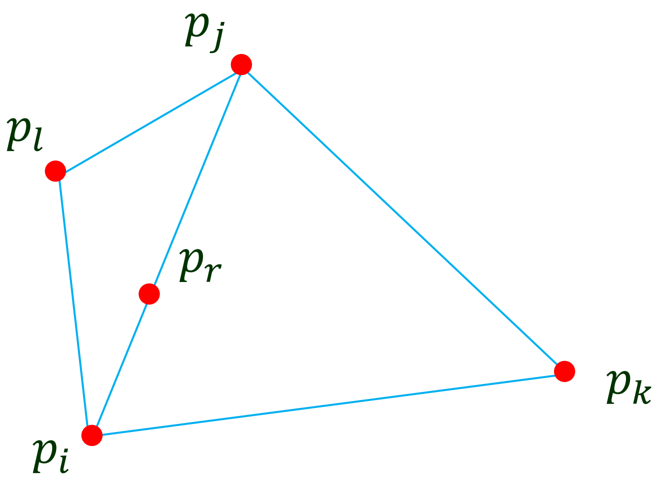


Case 2: p_r on an edge

Legal and Illegal Edges

- Add edges from p_r to the vertices of the containing triangle(s).





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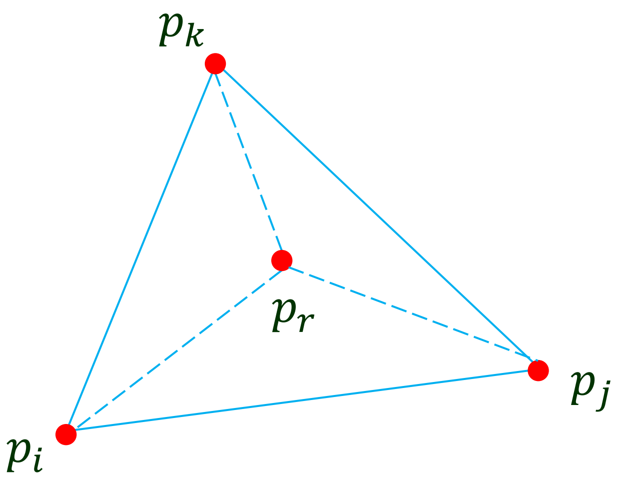

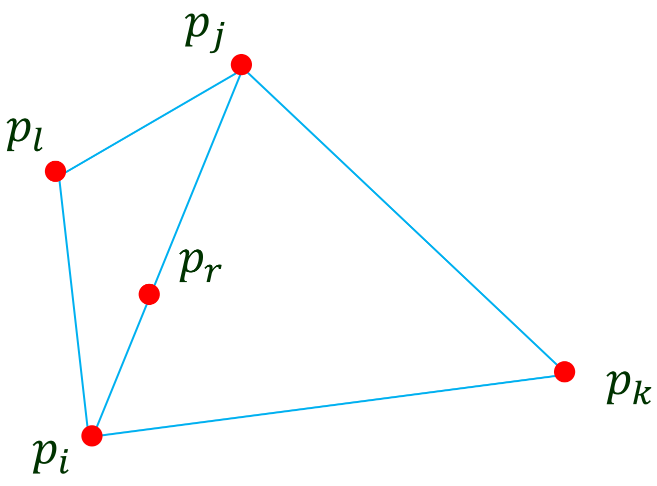

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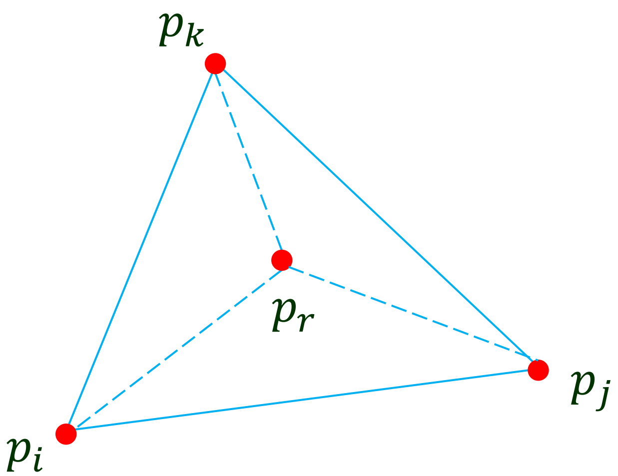

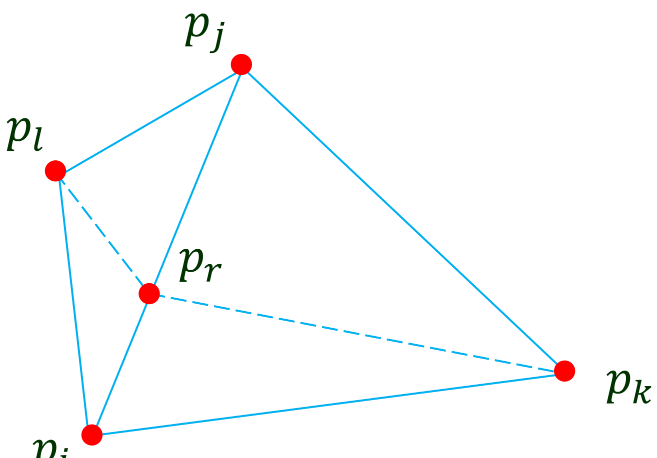

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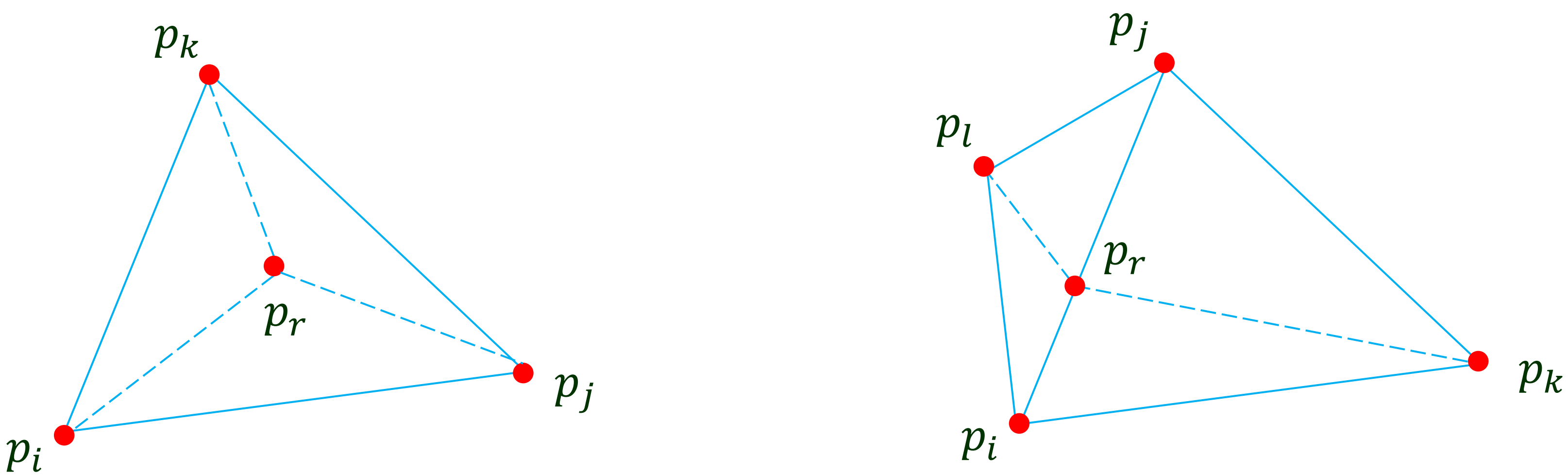

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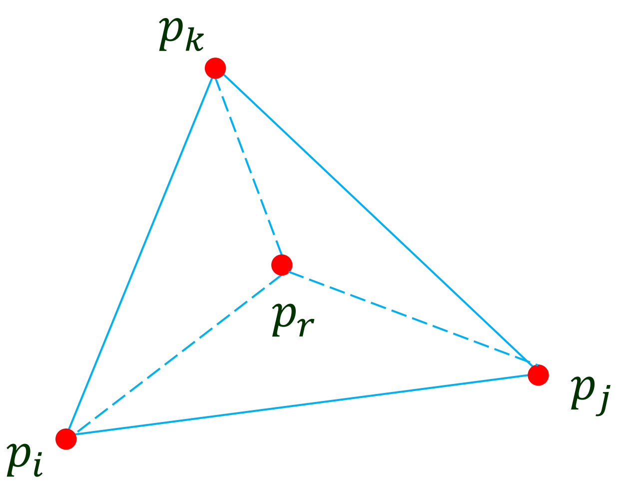

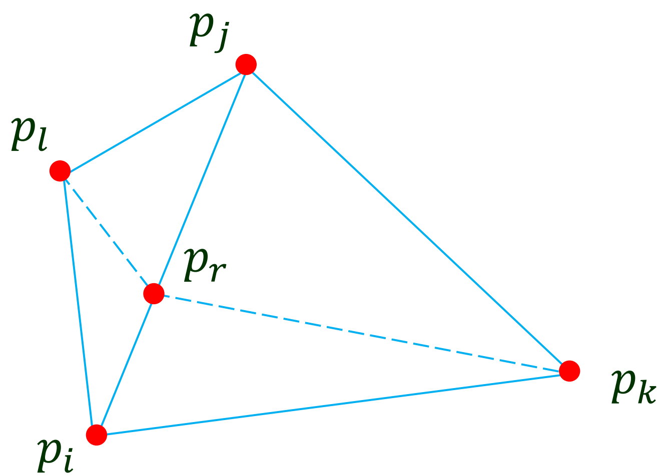

- Replace illegal edges by legal edges through edge flips.

Edge Legalization

LegalizeEdge($p_r, \overline{p_i p_j}, T$)

1. **if** $\overline{p_i p_j}$ is illegal
2. **then** let $\Delta\, p_i p_j p_k$ and $\Delta\, p_i p_j p_r$ be adjacent along $\overline{p_i p_j}$
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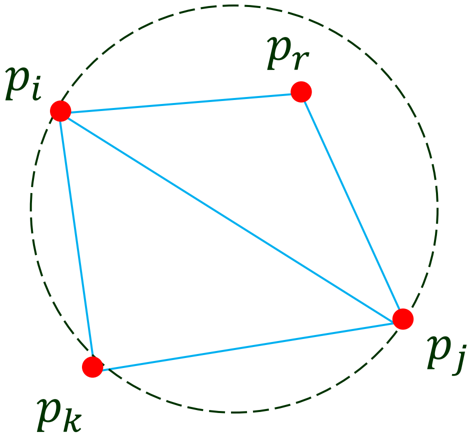

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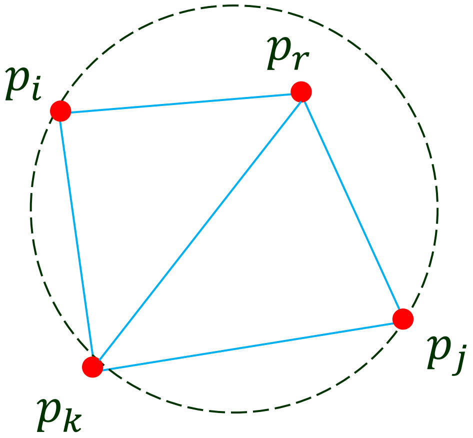

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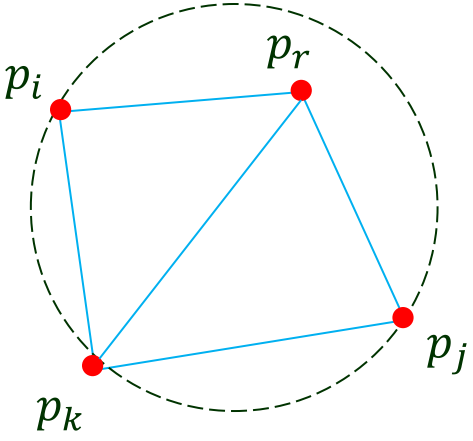

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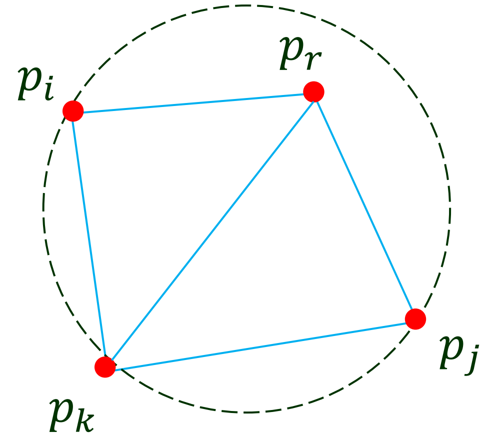

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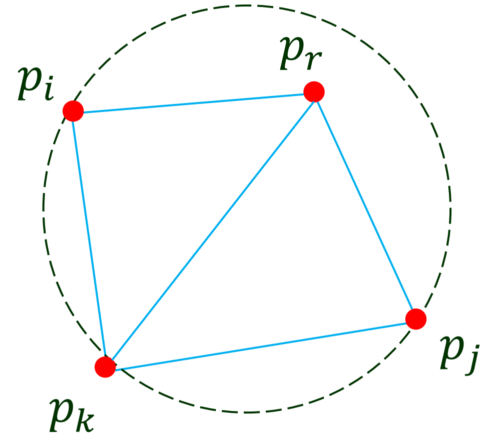

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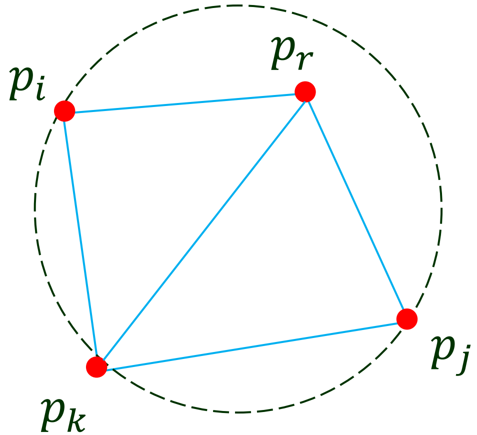

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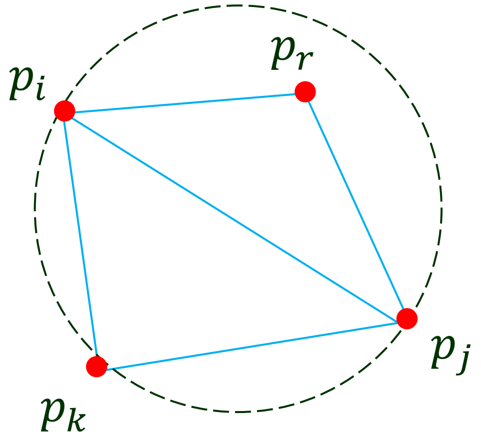

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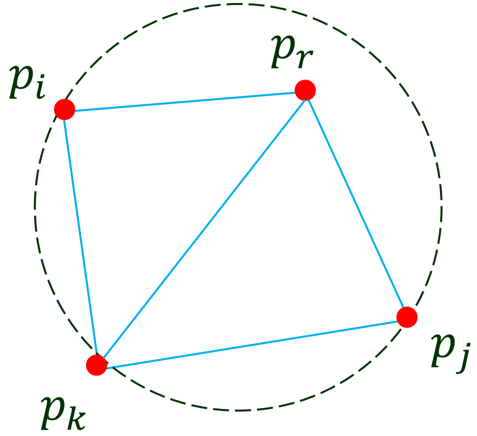

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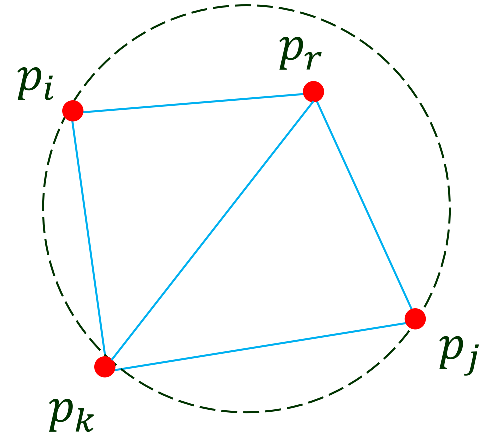

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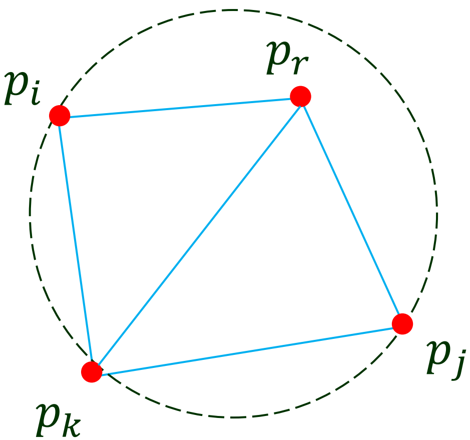

- Only the edges of the new triangles need to be checked.
- $\overline{p_r p_i}$ and $\overline{p_r p_j}$ are newly inserted and legal (to be shown).
- So check $\overline{p_i p_k}$ and $\overline{p_j p_k}$ opposing p_r, and recursively from there.

Iterations

```
Compute a random permutation p_1, p_2 ..., p_n
for r ← 1 to n
  do
    find Δ p_i p_j p_k ⊃ p_r in the current triangulation T
    if p_r lies in its interior
      then   // case 1
        add edges $\overline{p_r p_i}$, $\overline{p_r p_j}$, $\overline{p_r p_k}$
        LegalizeEdge(p_r, $\overline{p_i p_j}$, T)
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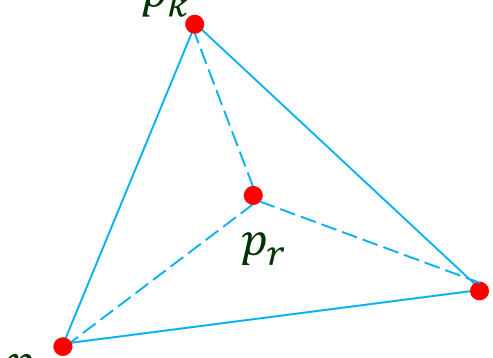

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for $r \leftarrow 1$ to n

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      else // case 2

        add edges $\overline{p_r p_k}$, $\overline{p_r p_l}$

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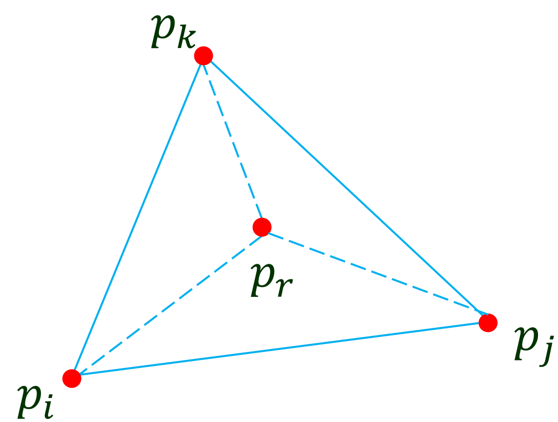

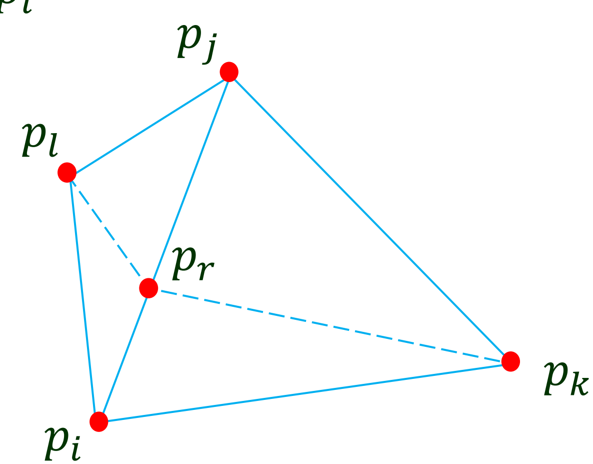

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Need to prove that no illegal edges remain after all calls to `LegalizeEdge`. Correctness is implied by the following:

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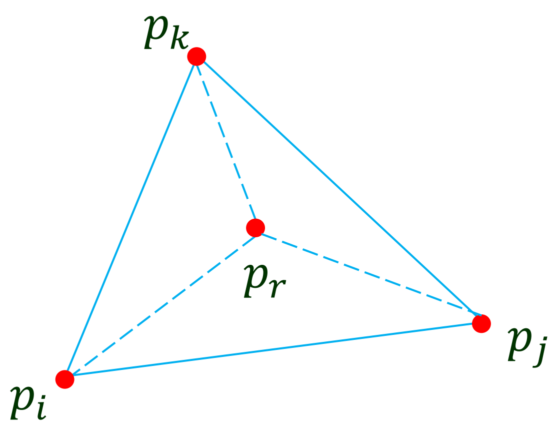

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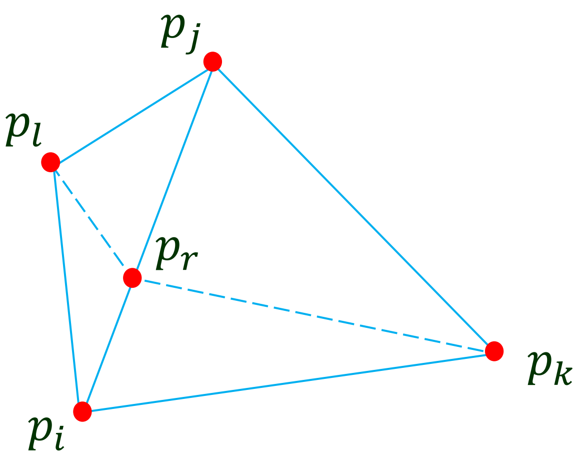

Case 2

Immediately Added Edges

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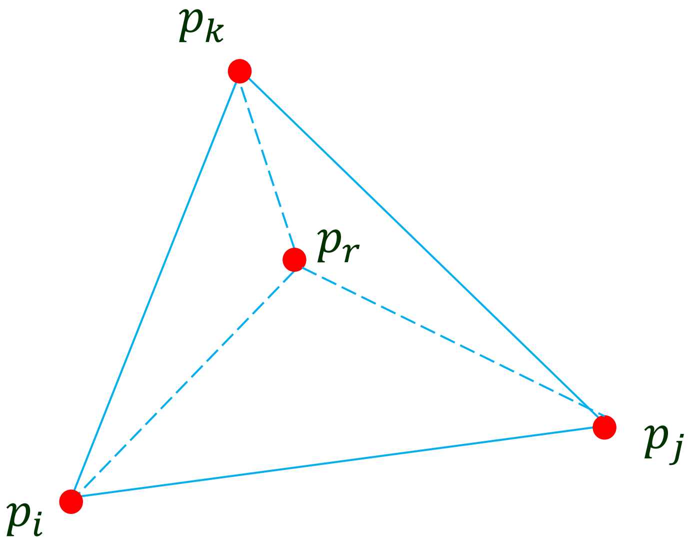
  $\Delta\, p_i p_j p_k$ is a triangle in $DG(\Omega \cup \{p_1, \ldots, p_{r-1}\})$.



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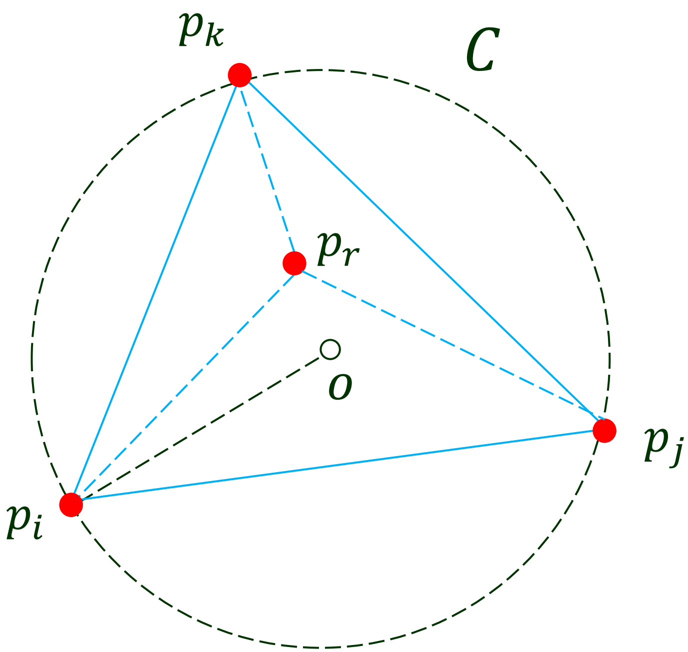

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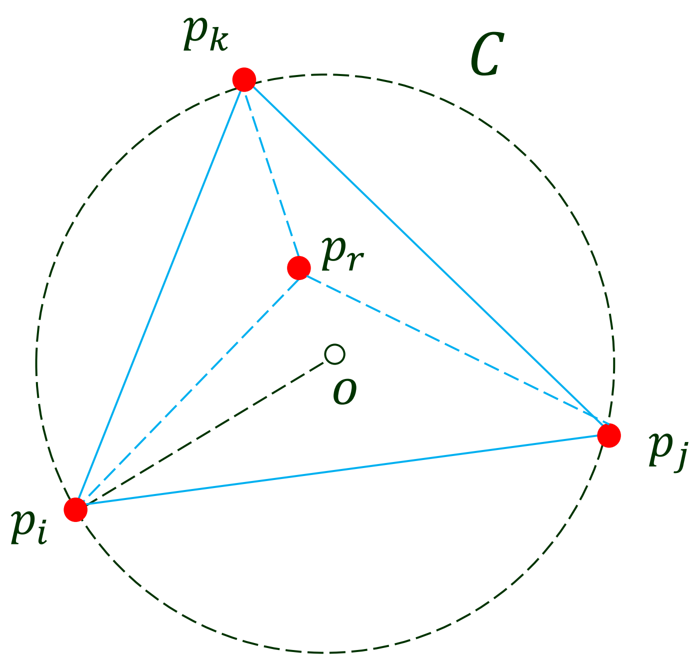

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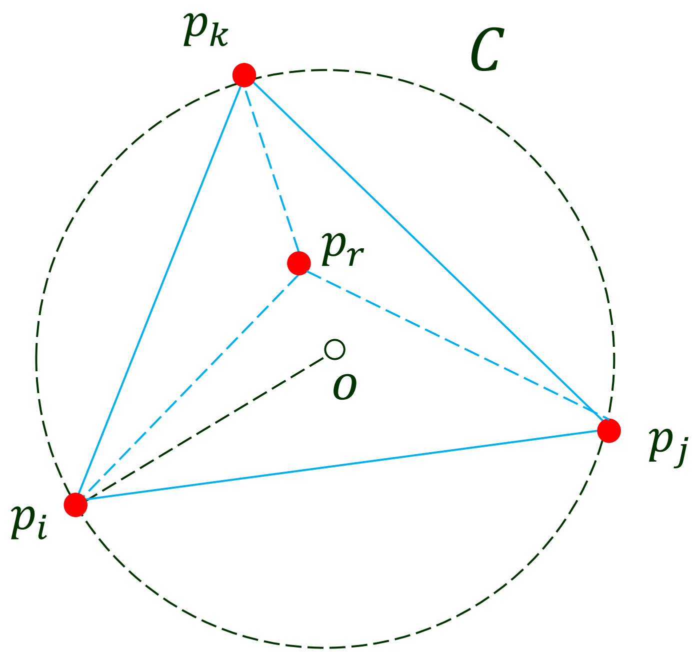

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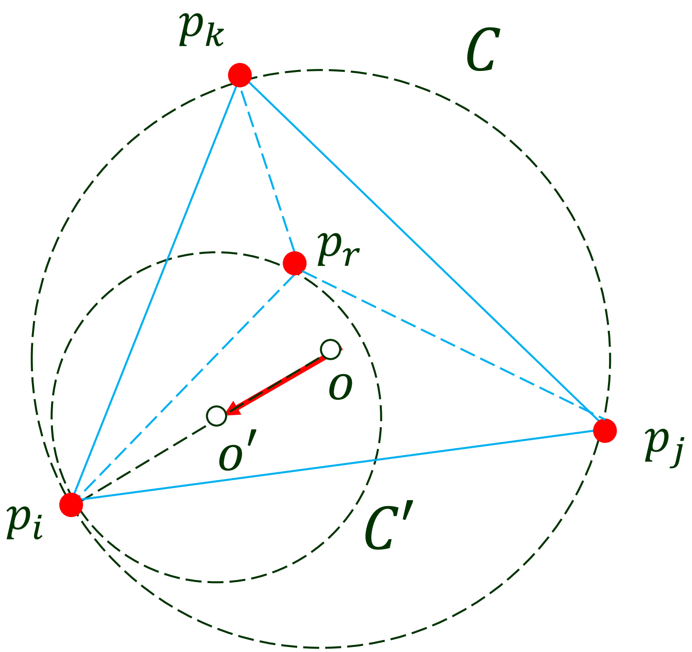

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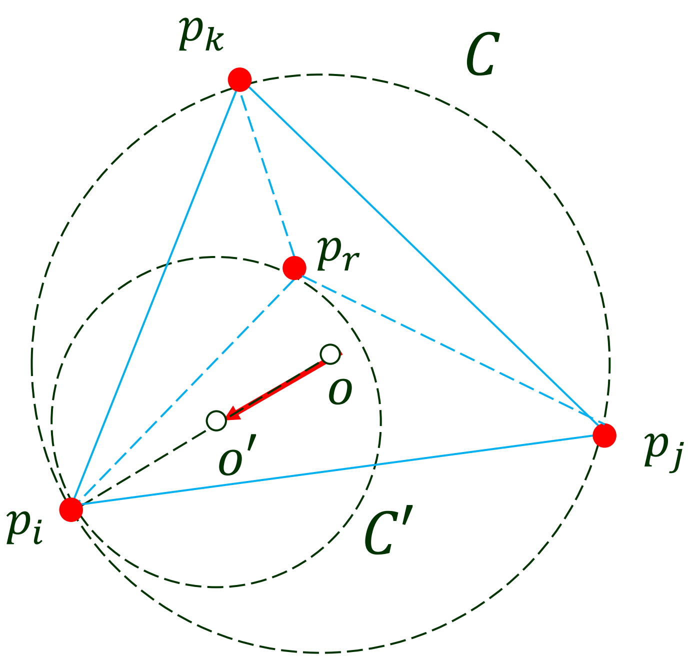

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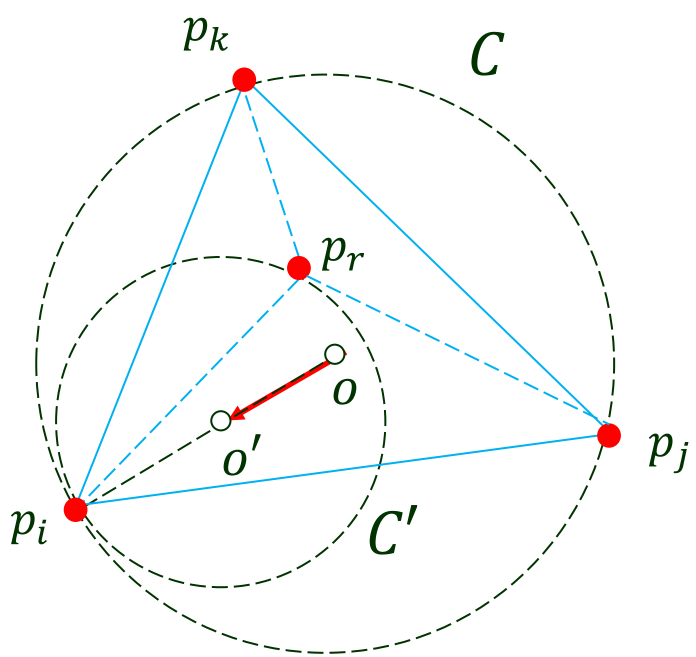

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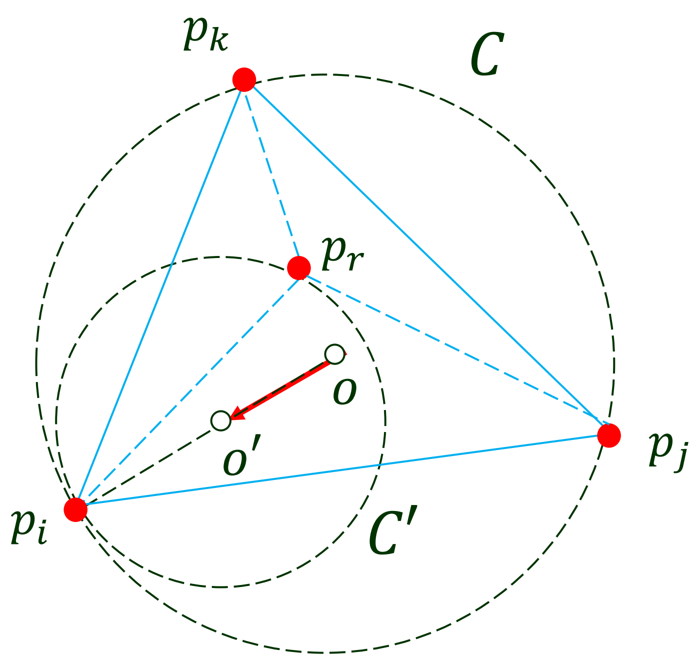

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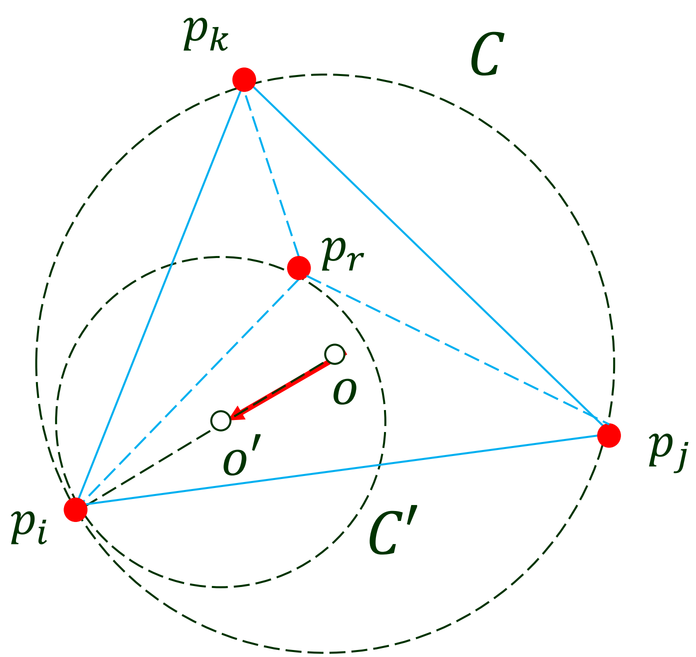

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Similar to Case 1.

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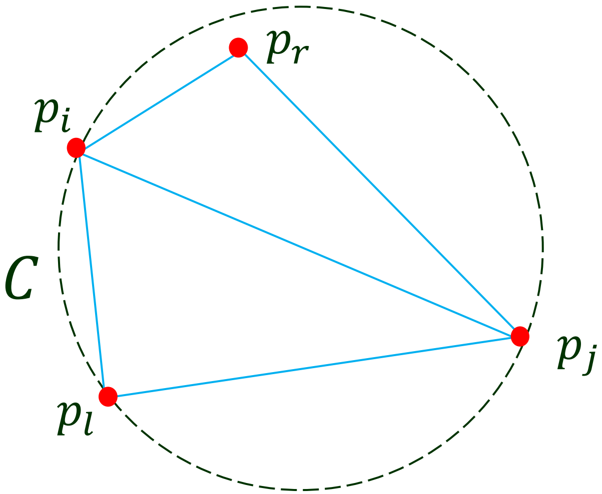

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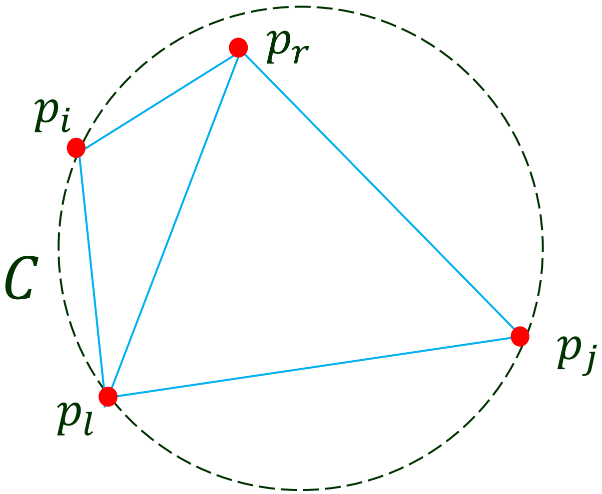

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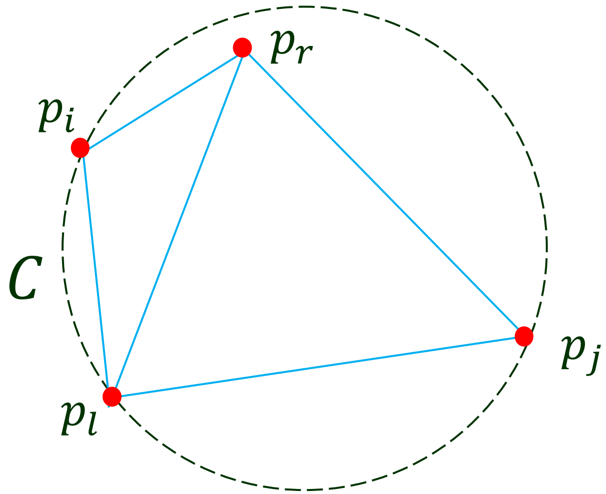

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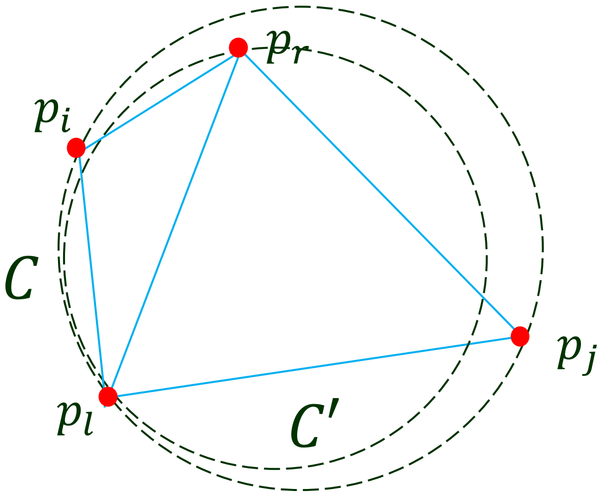

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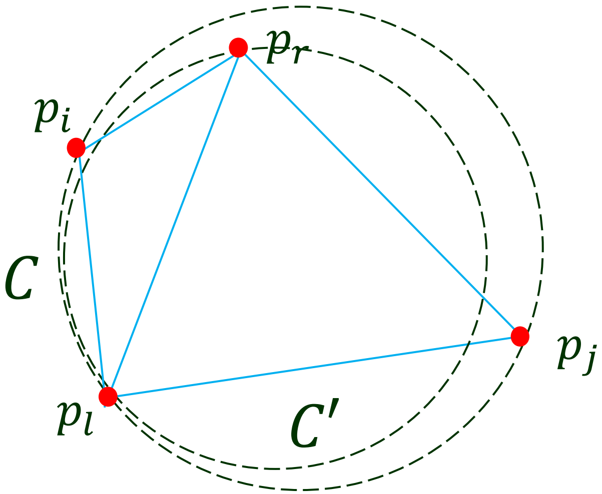

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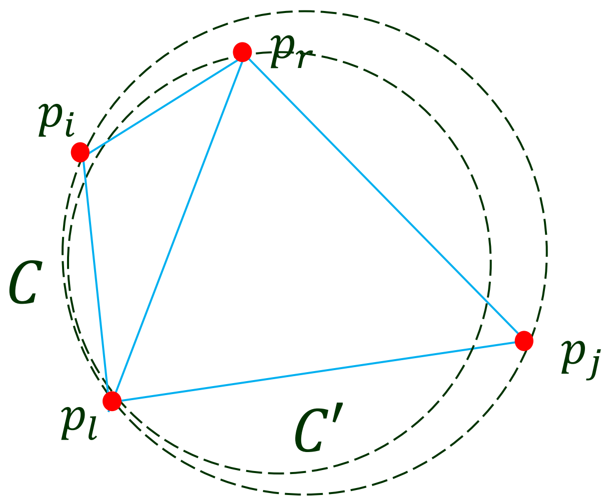

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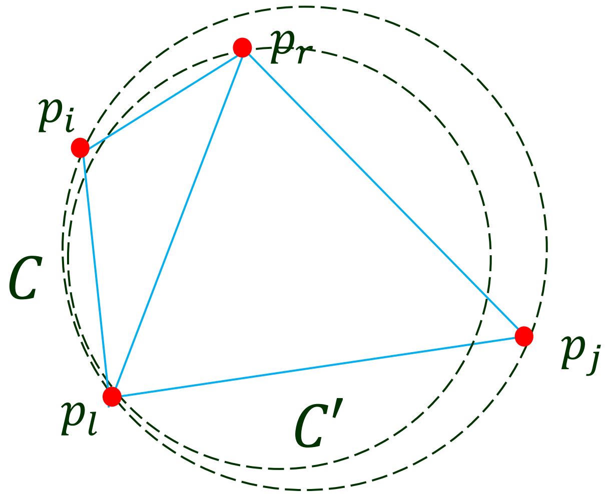

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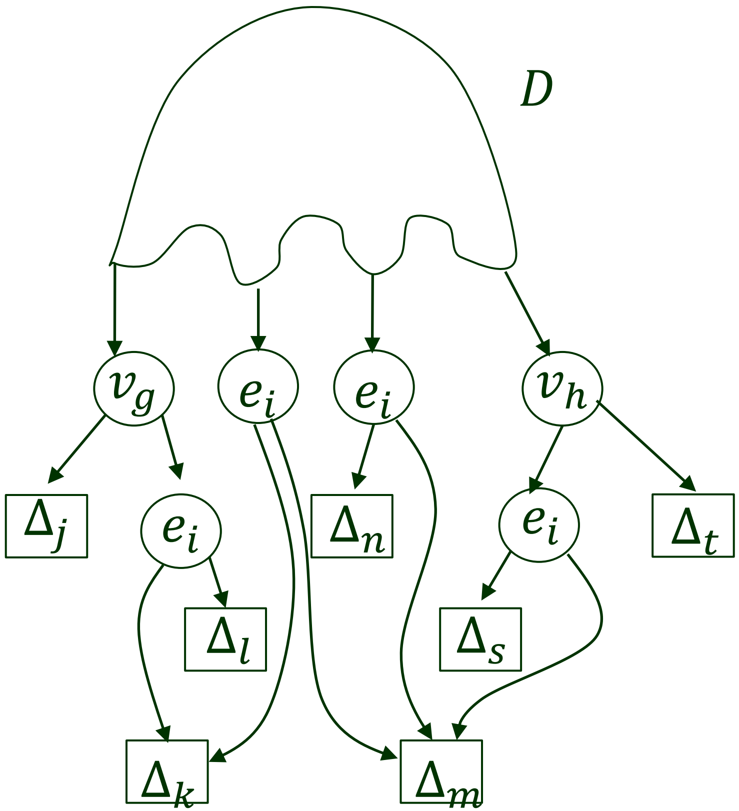
Trapezoidal map with only triangles no trapezoids.



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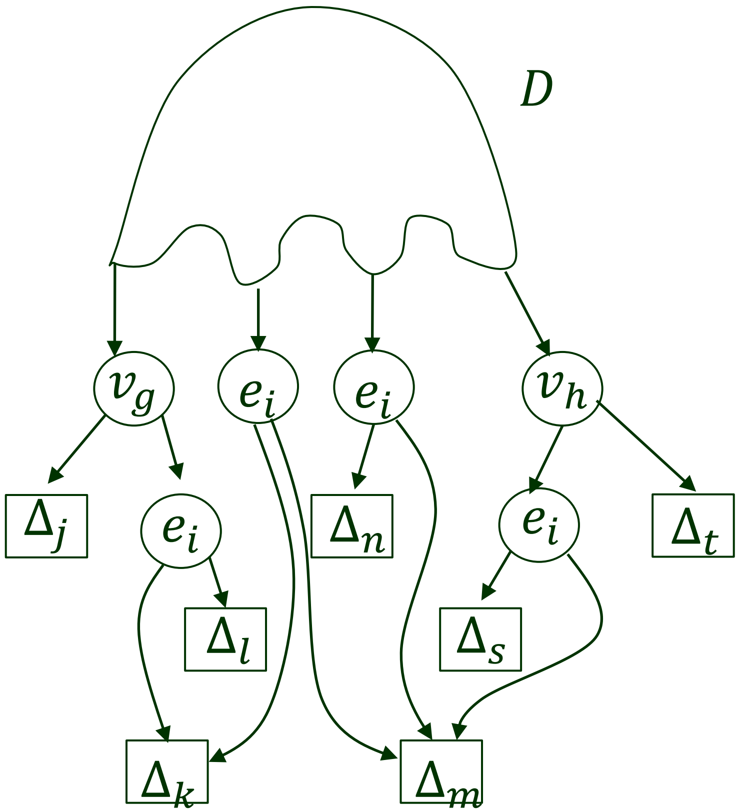

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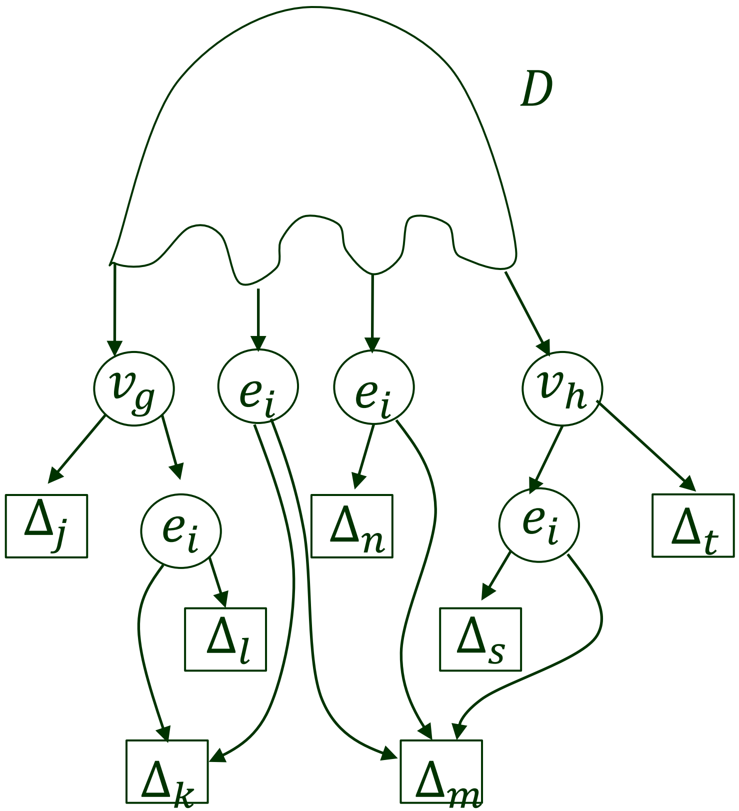

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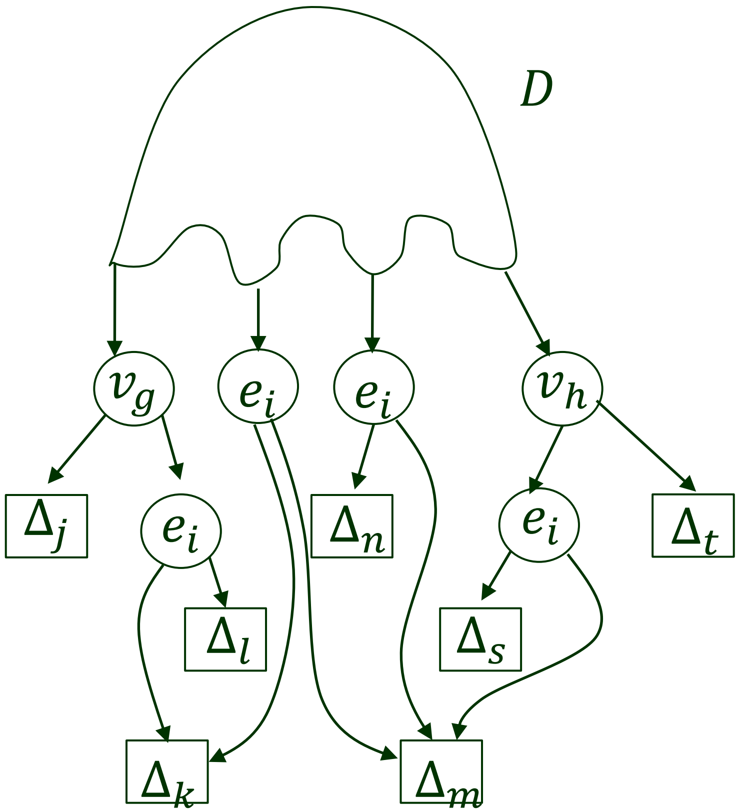
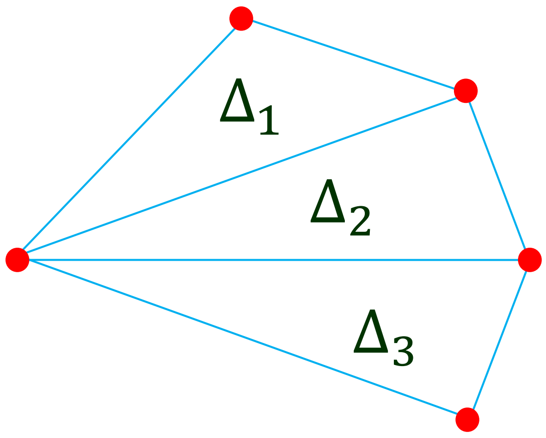
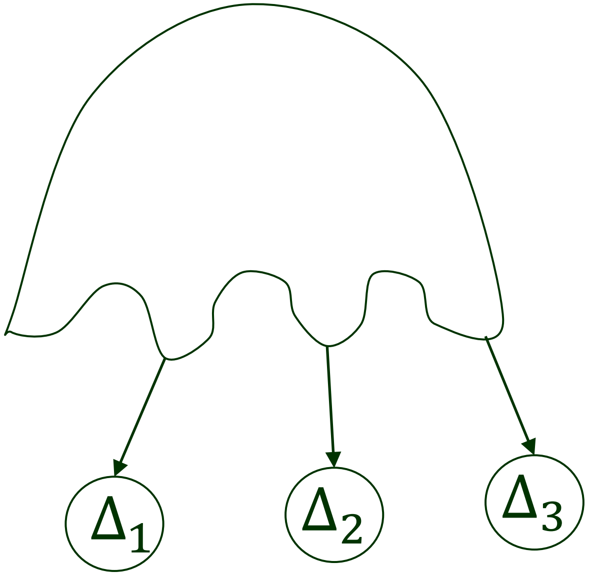


- Leaves: triangles of the current triangulation T.
- Internal nodes: triangles that existed before but have been destroyed.
- Initialized as a DAG with one node ($\Delta\, p_0 p_{-1} p_{-2}$).

Example





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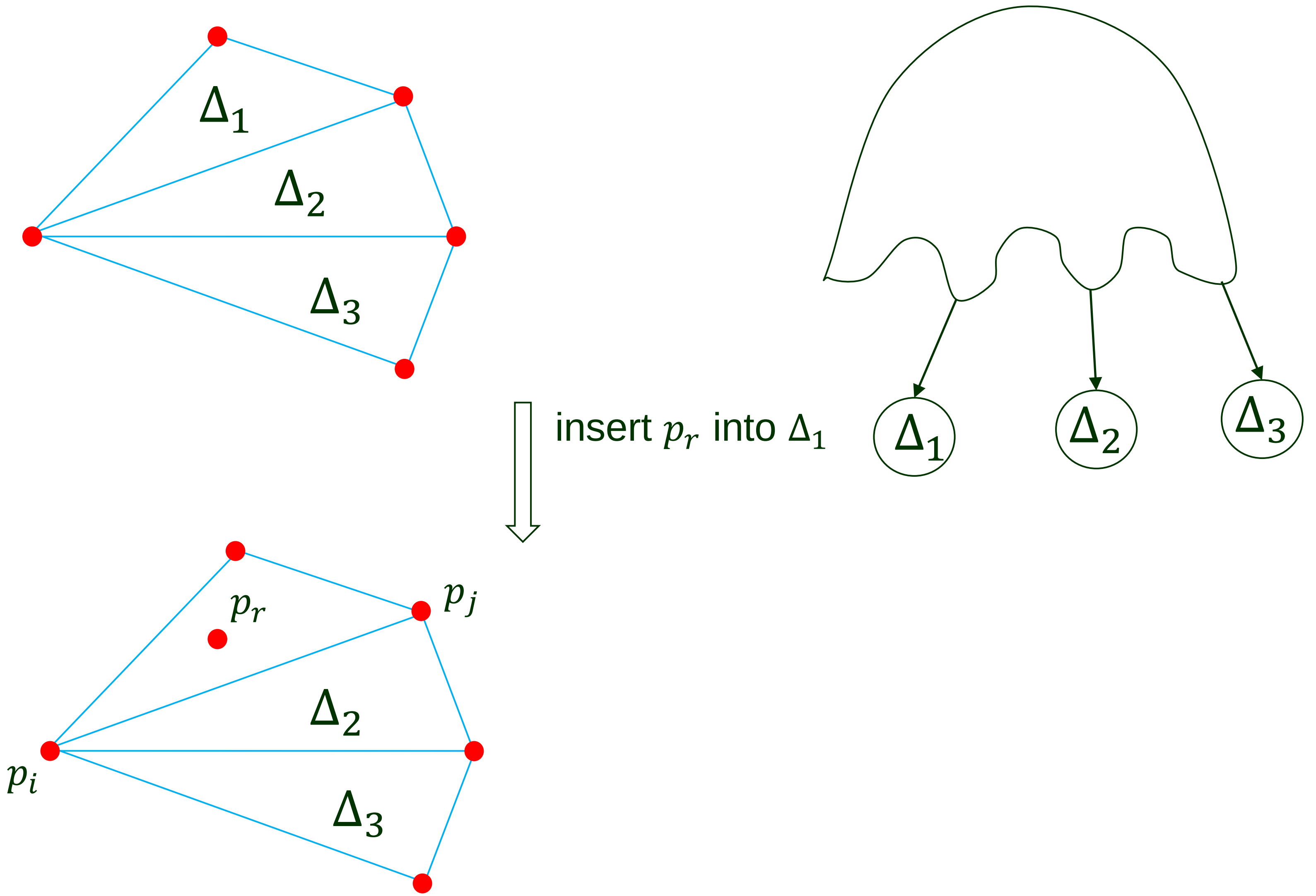
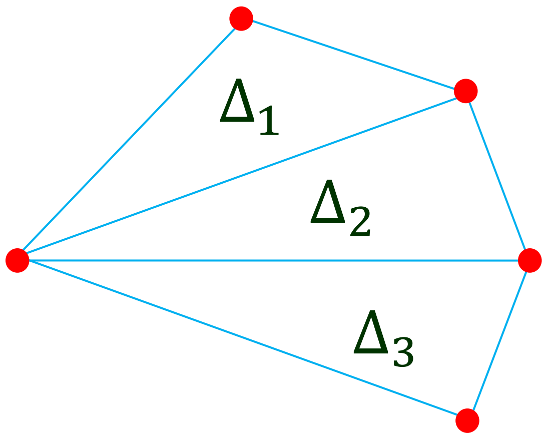
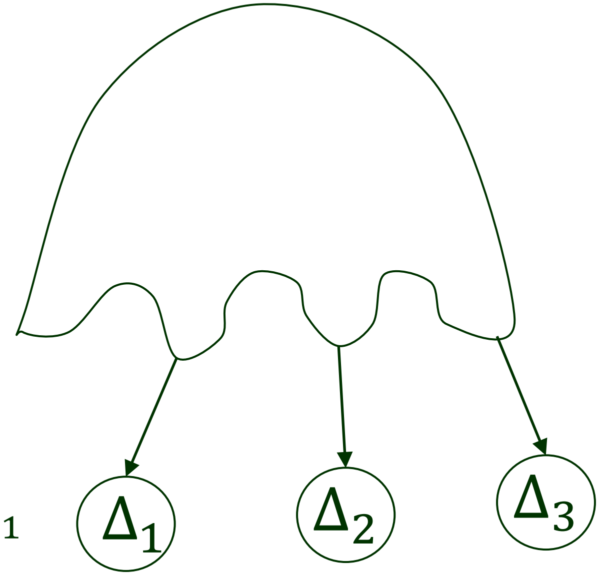


Δ_1
Δ_2
Δ_3
insert p_r into Δ_1
Δ_1
Δ_2
Δ_3
p_r
p_j
Δ_2
p_i
Δ_3

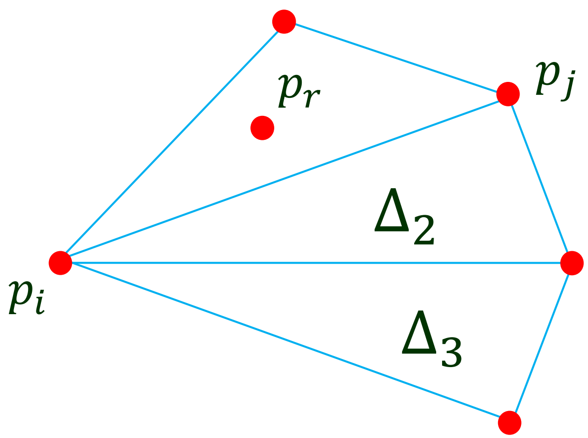
Example





insert p_r into Δ_1

split Δ_1



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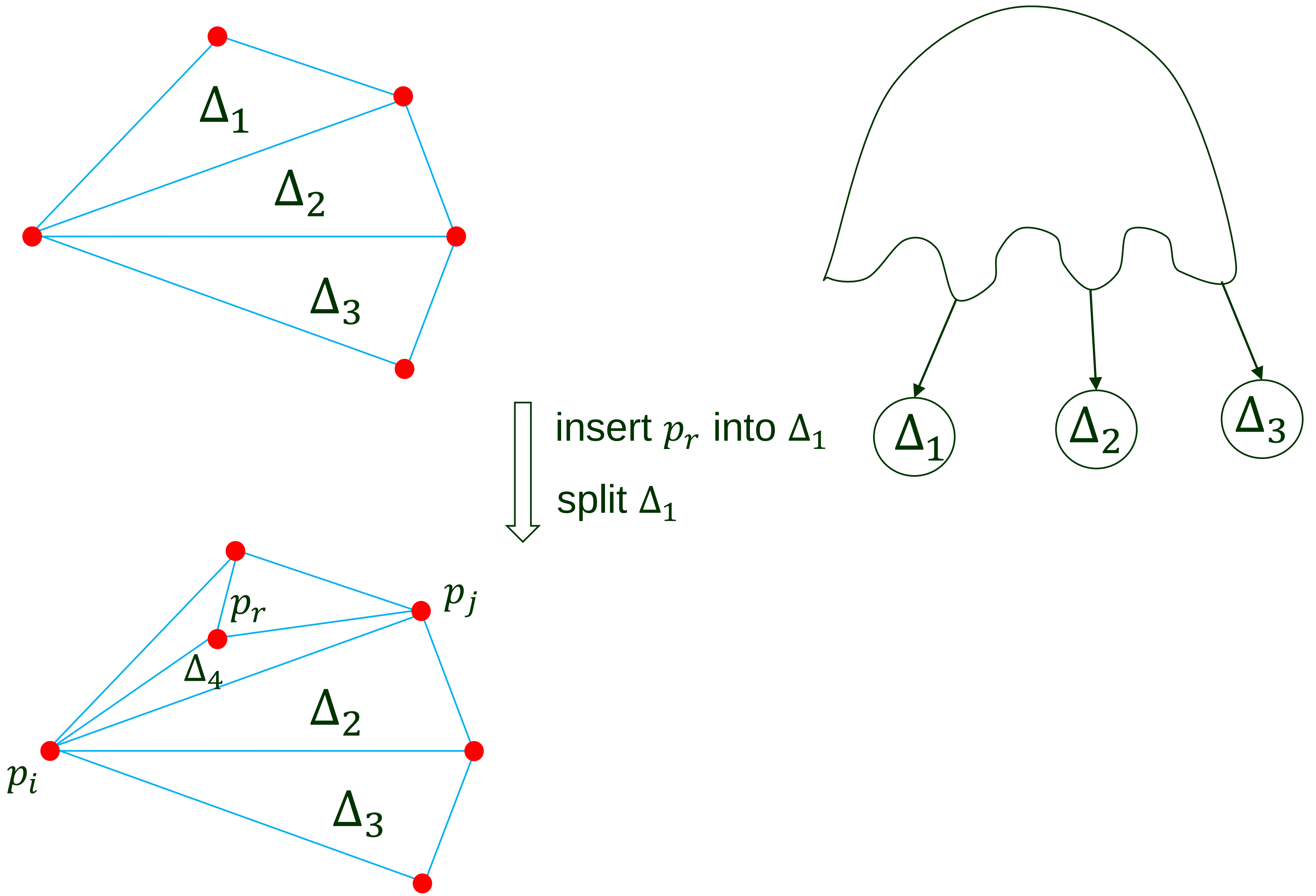

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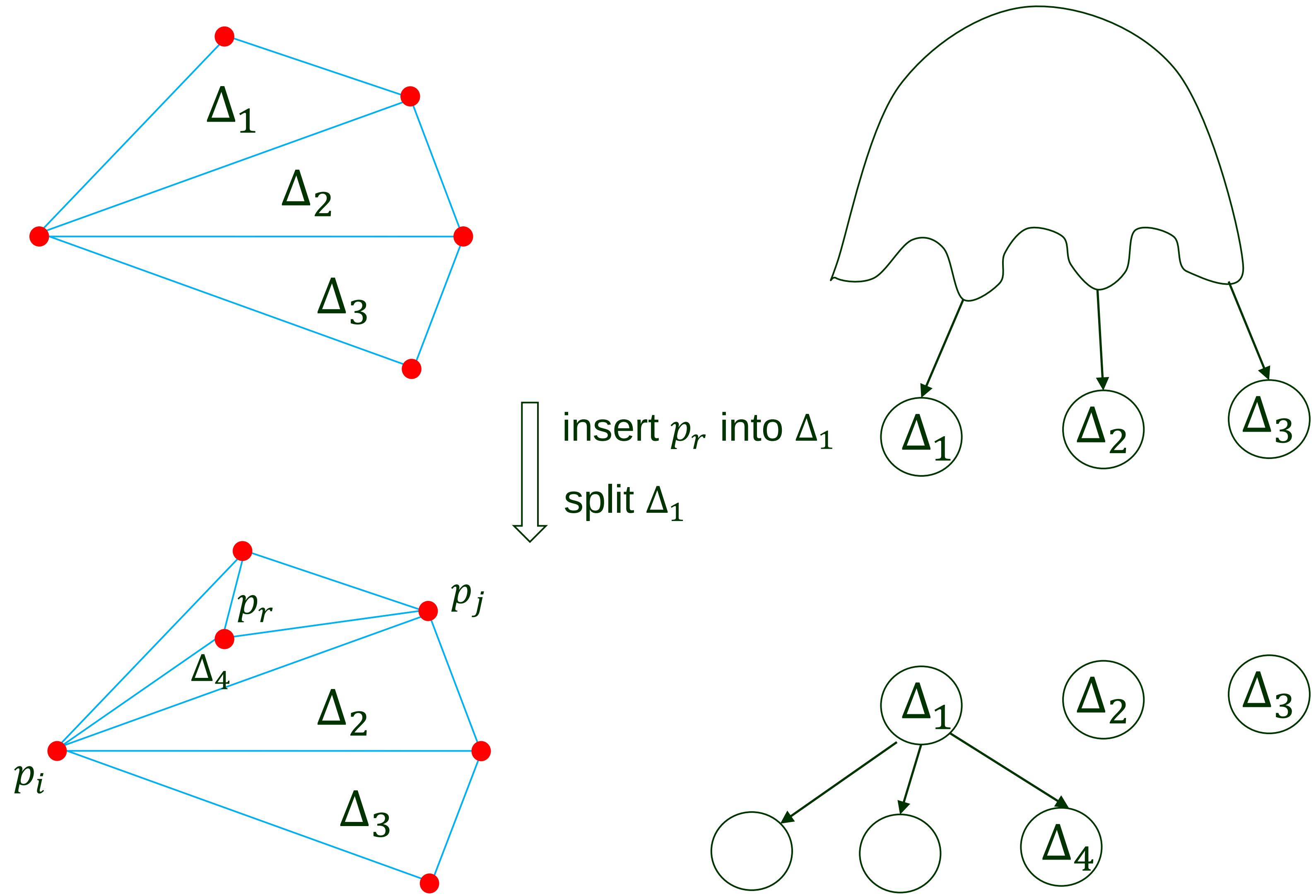
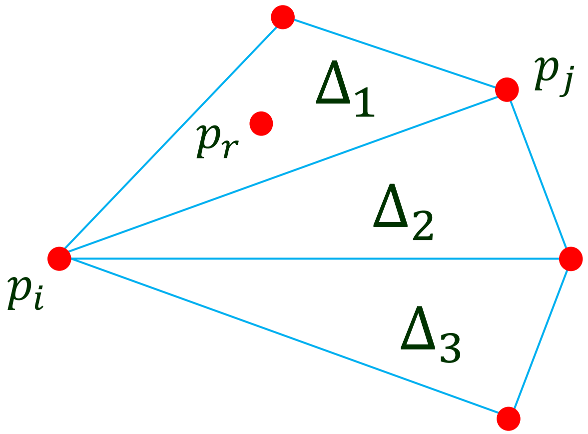


Insertion

Locate p_r in $DG(\{p_{-2}, p_{-1}, p_0, p_1, \ldots, p_{r-1}\})$



- Start at the root ($\Delta\, p_0 p_{-1} p_{-2}$) of D.

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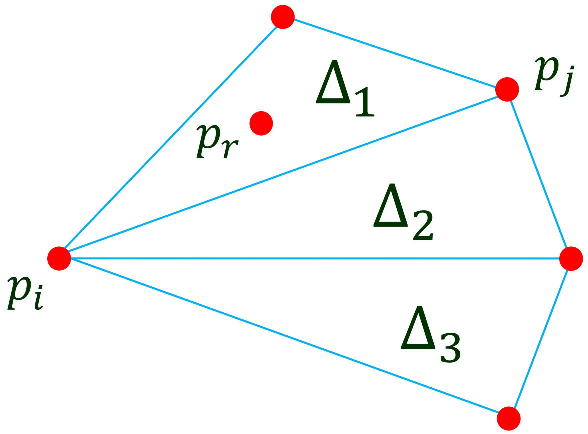

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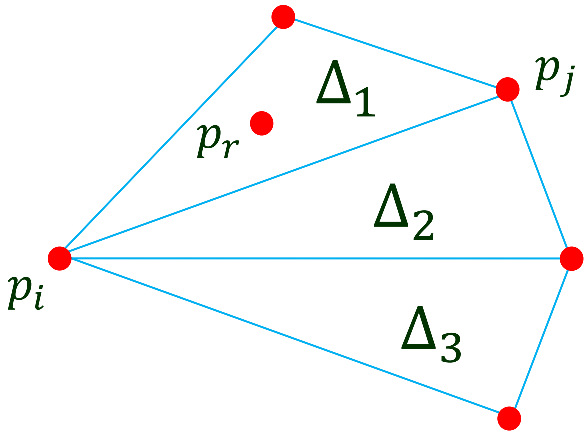

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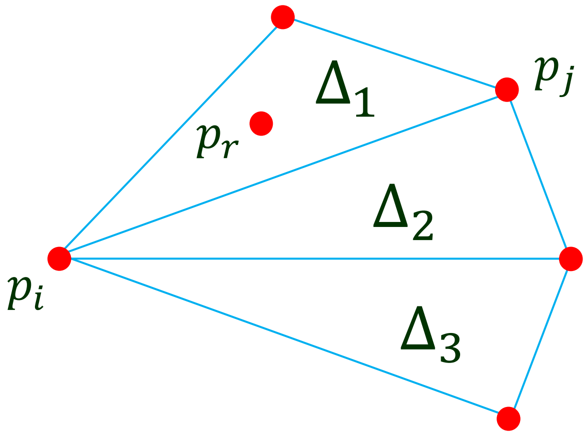

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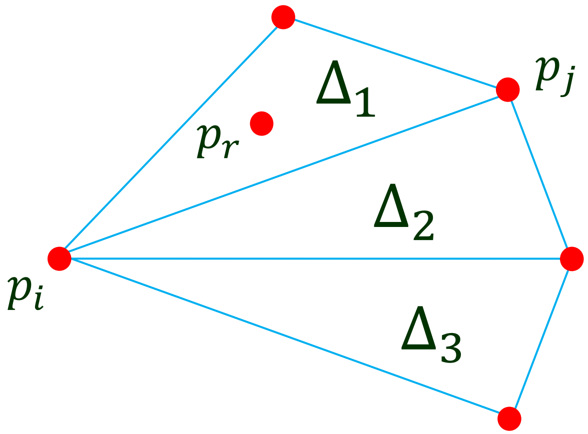

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- Repeat the above two steps to reach a leaf.

Insertion

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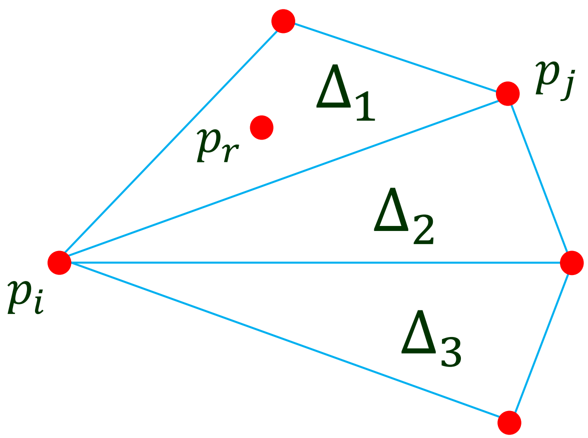

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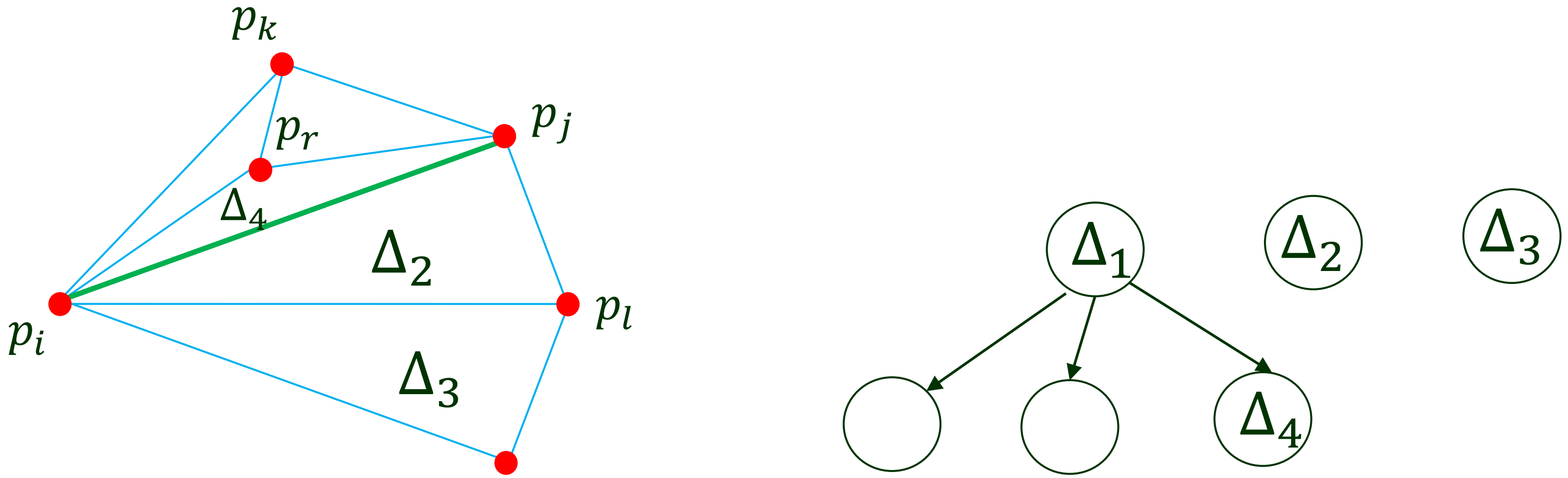
Time linear in

#nodes on the search path $=$ #triangles stored in D that contains p_r

Example (cont'd)



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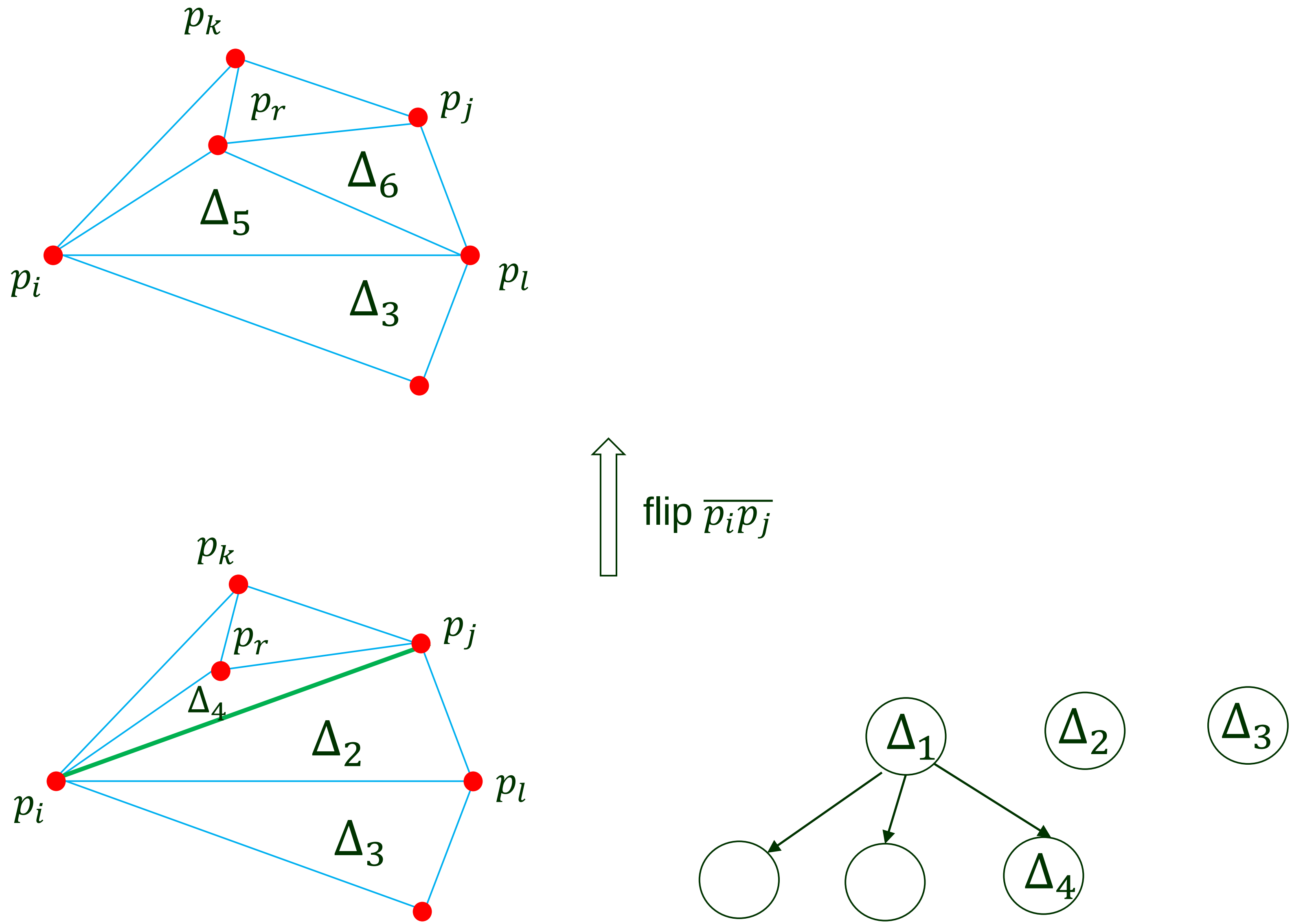

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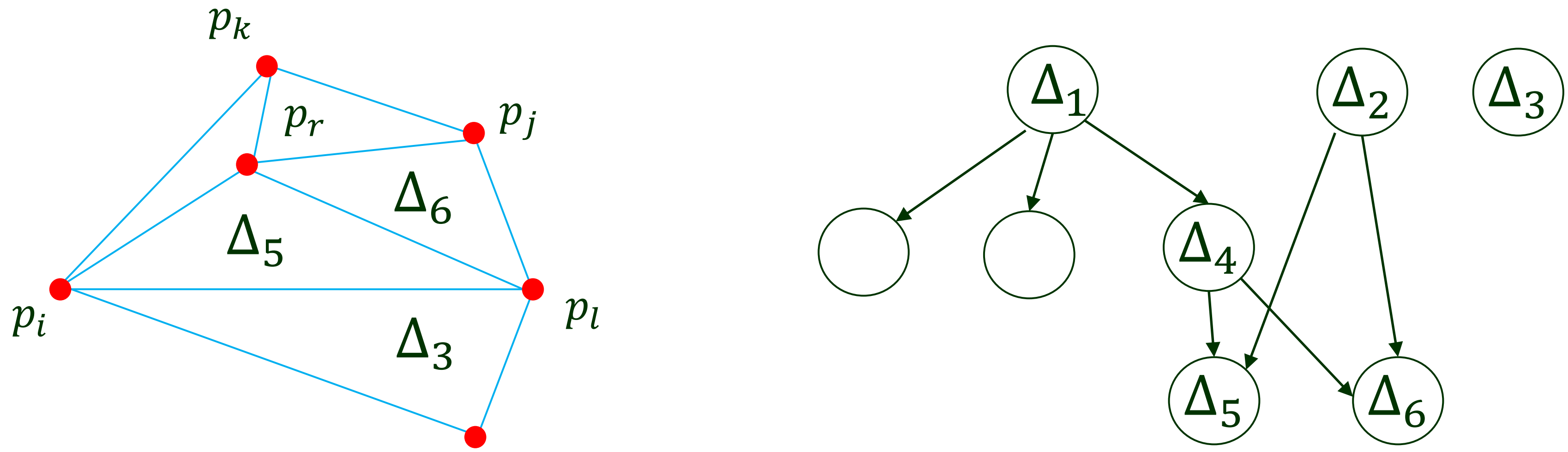
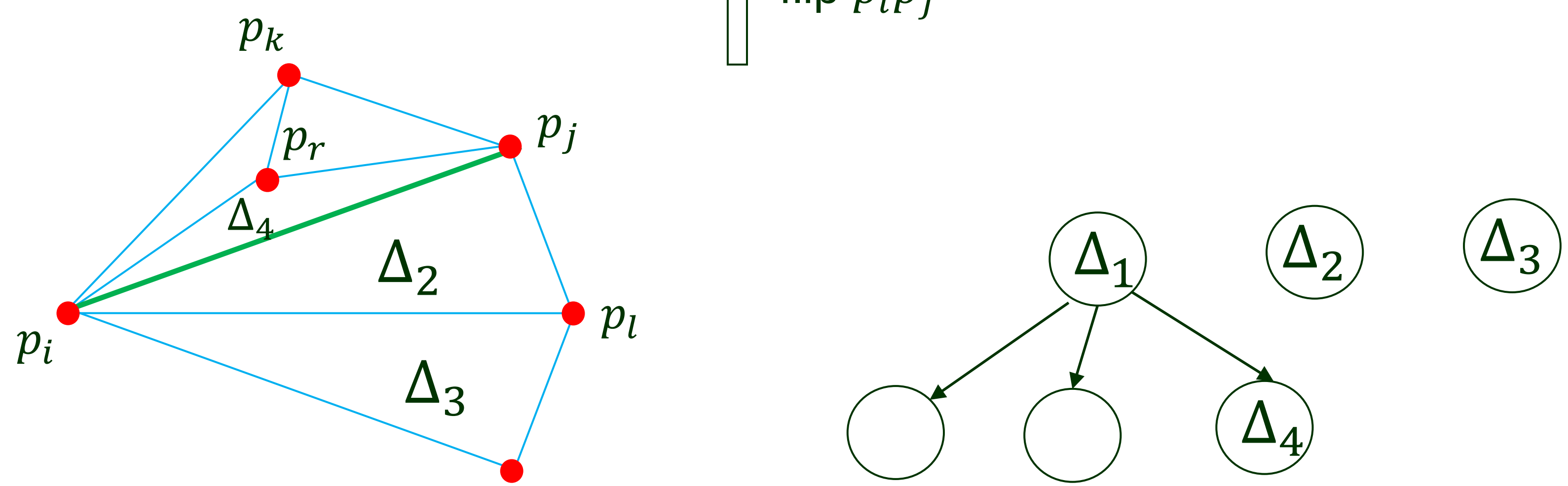


flip $\overline{p_i p_j}$



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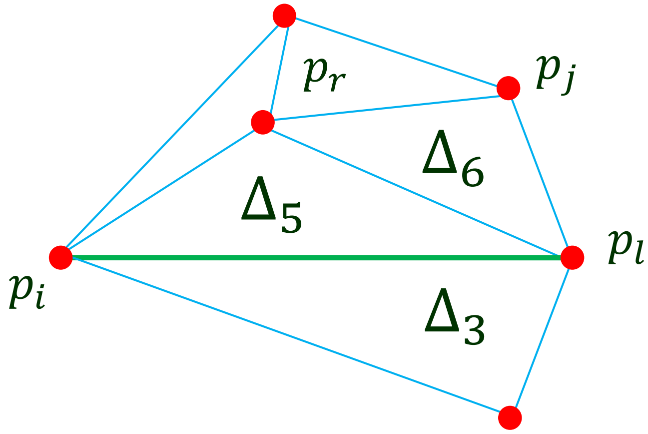

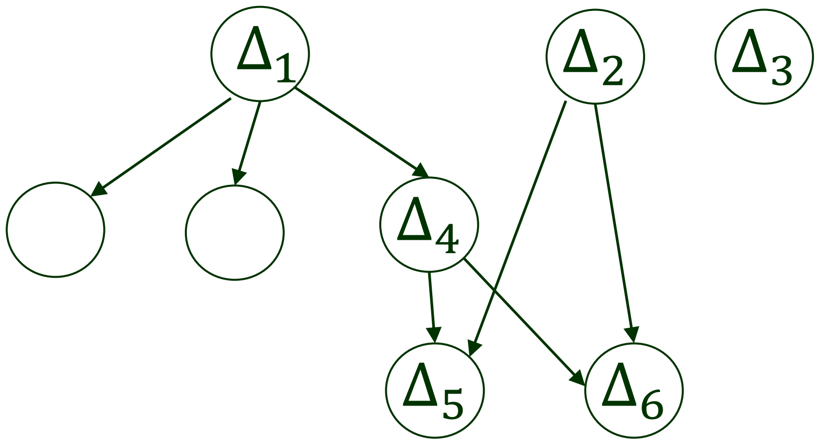

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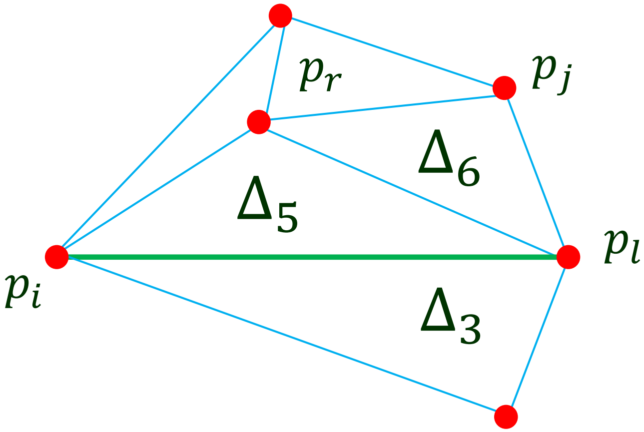
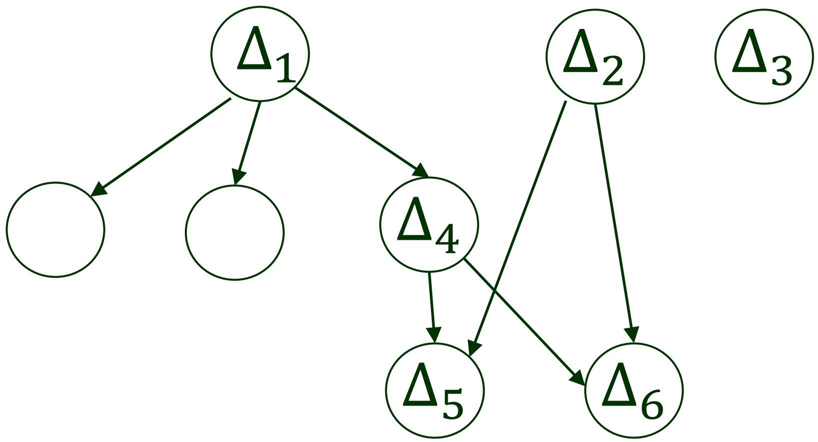
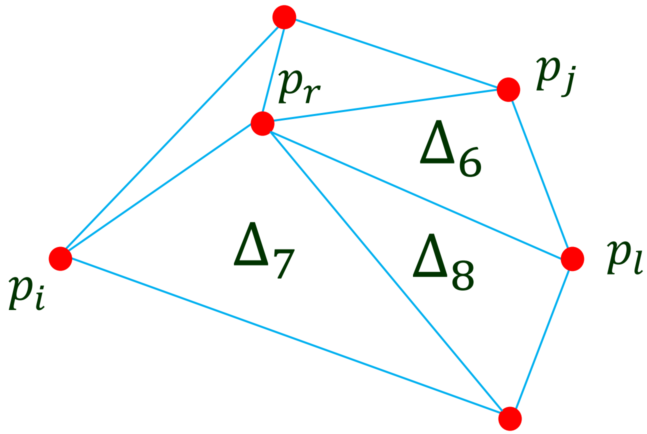
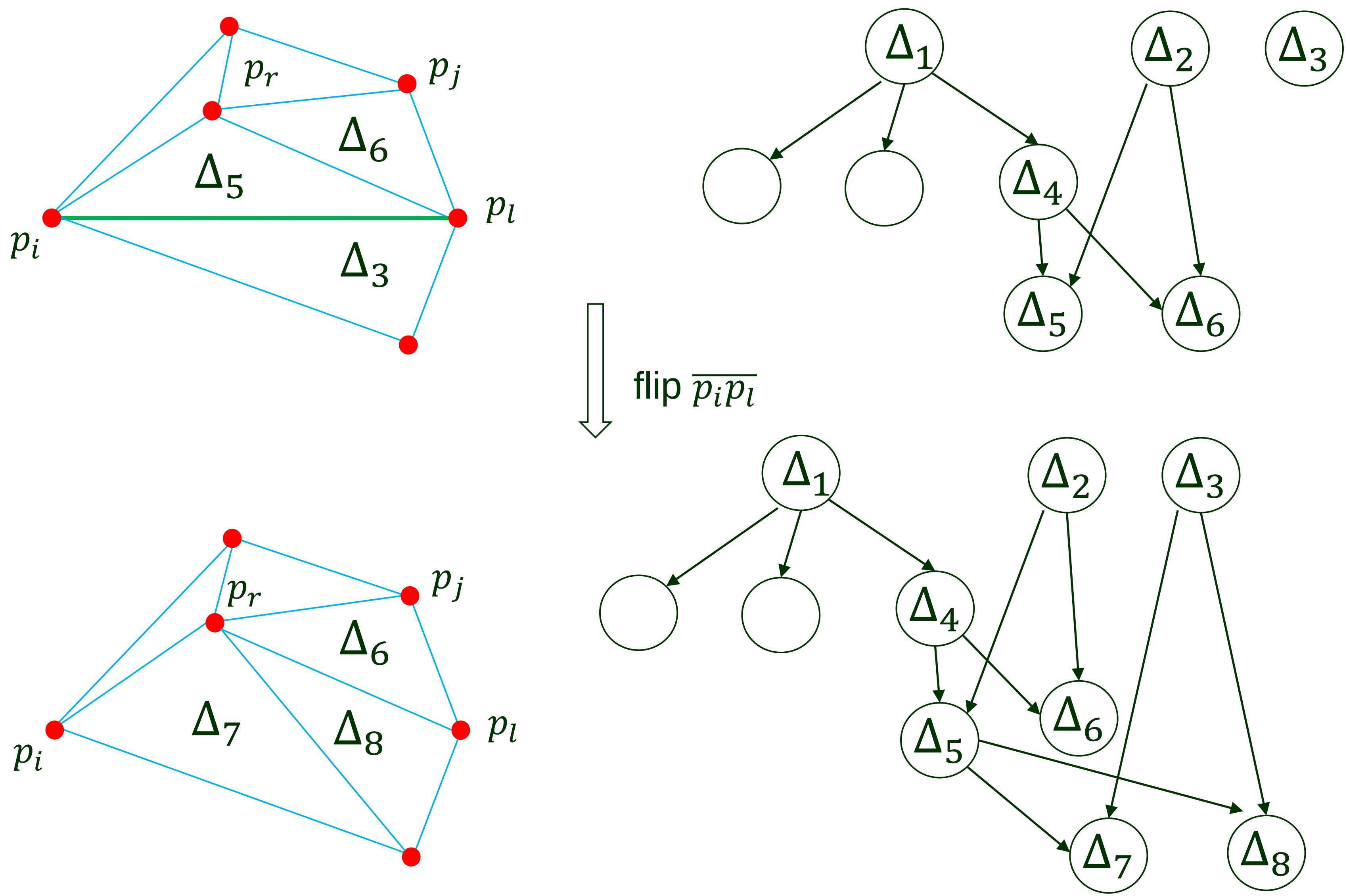




flip $\overline{p_i p_l}$

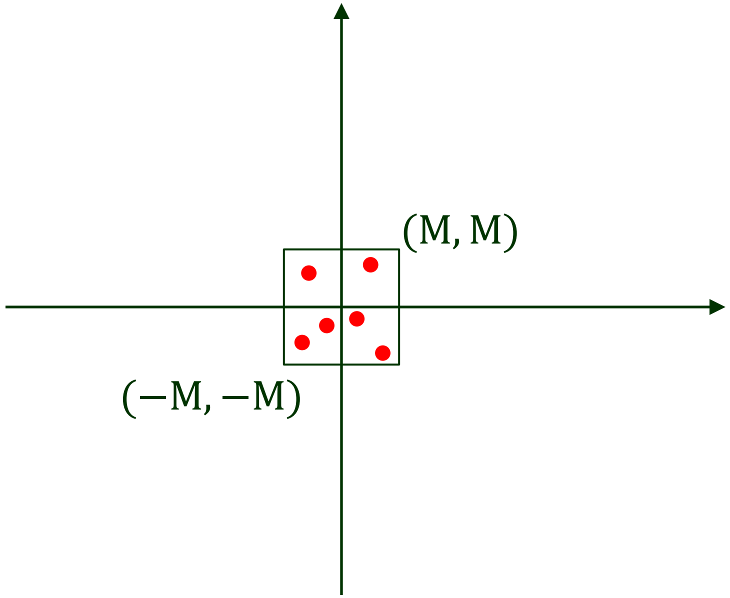


Example (finish)



Selecting p_{-2}, p_{-1}, p_0

$$M = \max_{\substack{p_i=(x_i,y_i)\\ 1\le i\le n}} \{|x_i|, |y_i|\}$$



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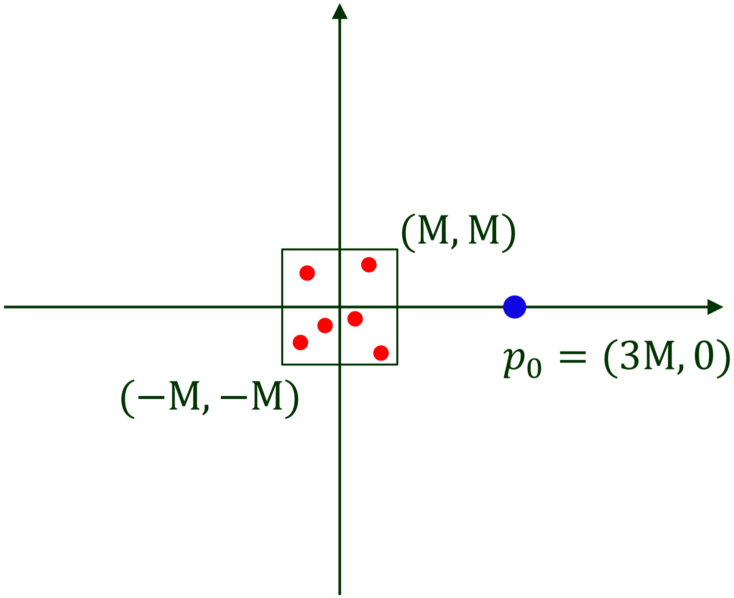

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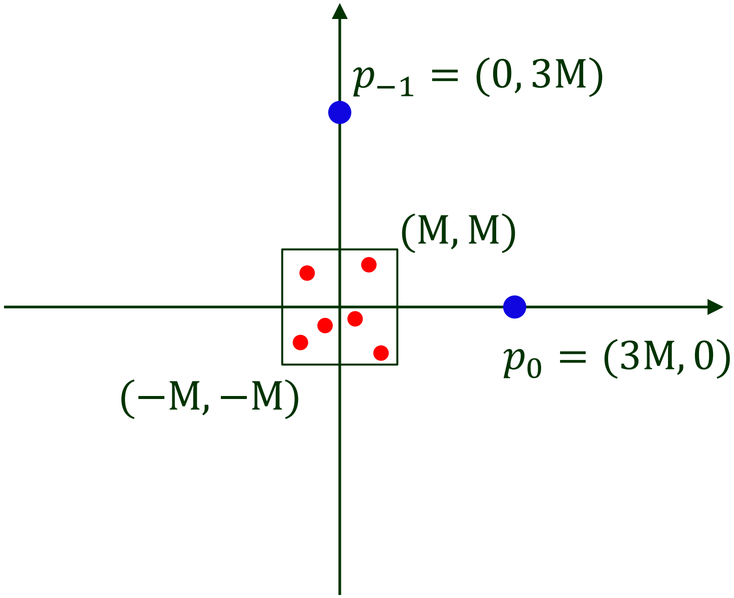

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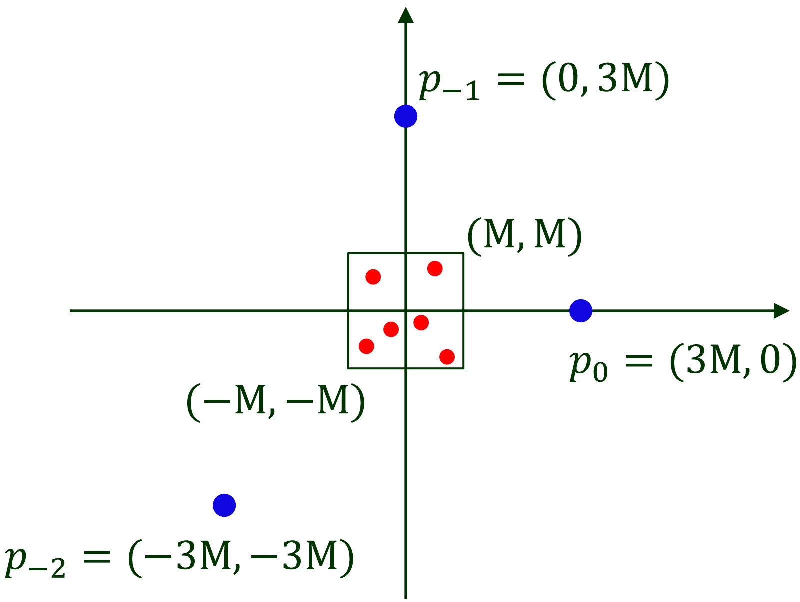

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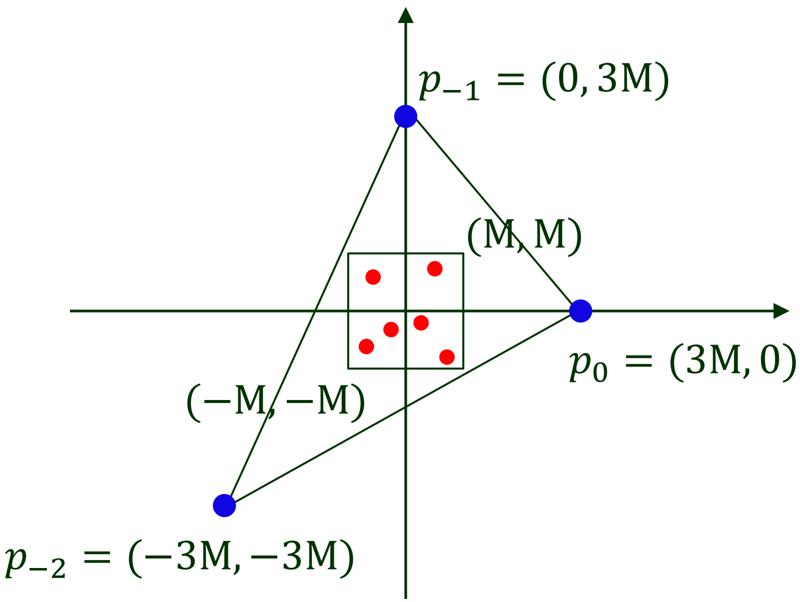

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$P_r = \{p_1, p_2, \ldots, p_r\}$ $\quad\quad$ $DG_r = DG(\{p_{-2}, p_{-1}, p_0, p_1, \ldots, p_r\})$

Lemma 2 Expected number of triangles created (and deleted) by the algorithm is $\leq 9n + 1$.

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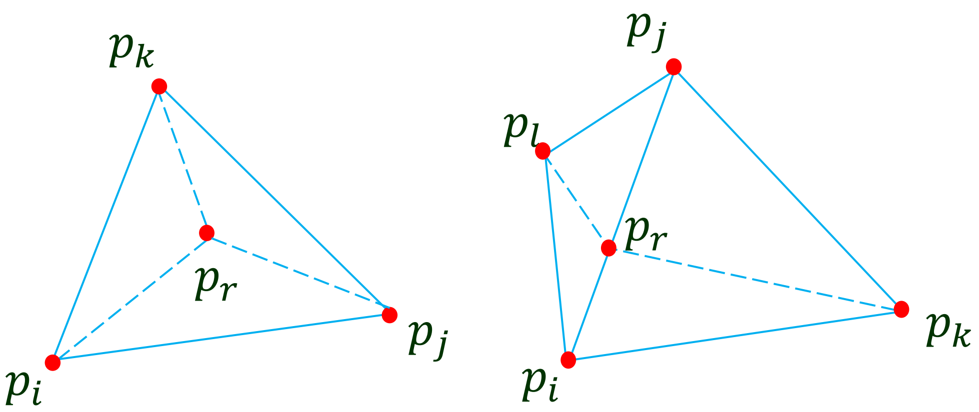

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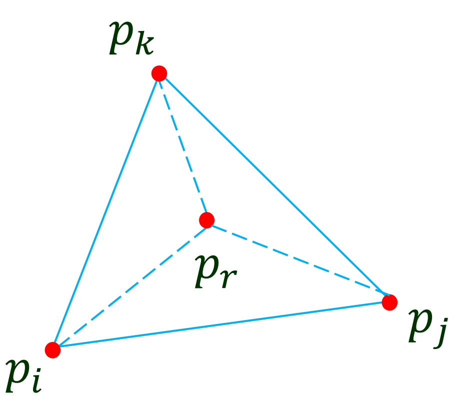

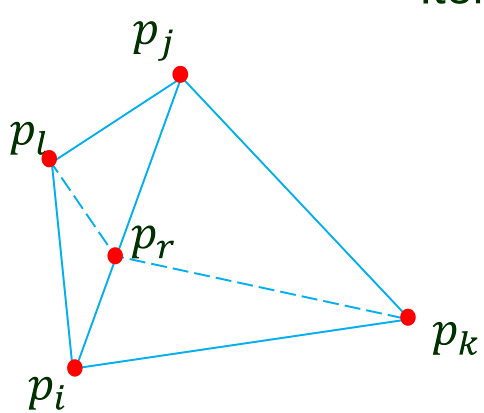

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- Split 1 or 2 triangles, creating 3 or 4 new ones, and the same number of edges.
- Every edge flipped in the subsequent `LegalizeEdge` results in creation of an edge adjacent to p_r and 2 new triangles bordering this edge.

Proof of Lemma 2 (cont'd)

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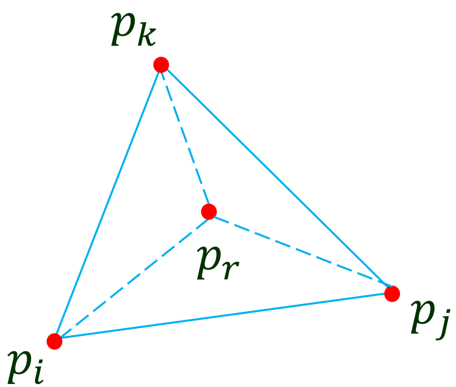

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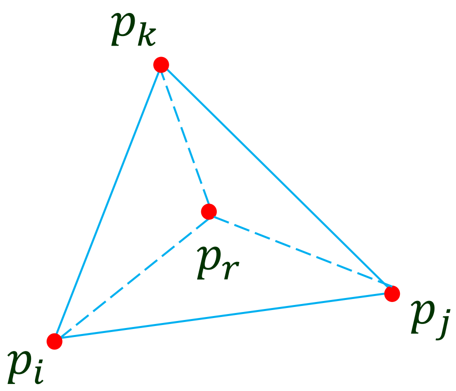

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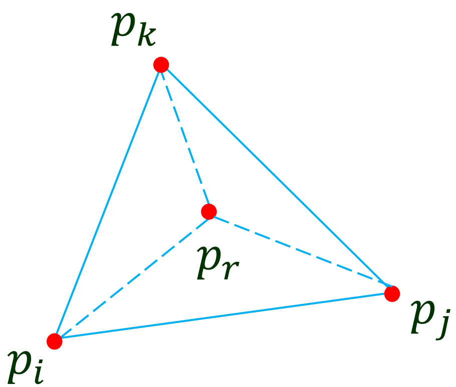

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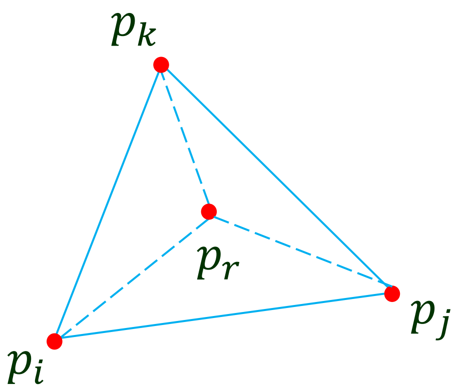

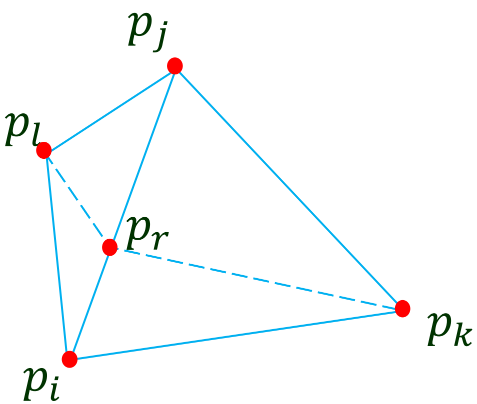

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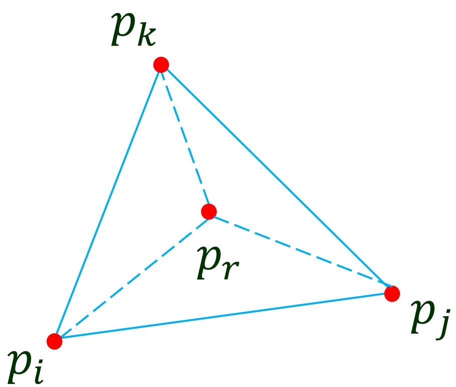

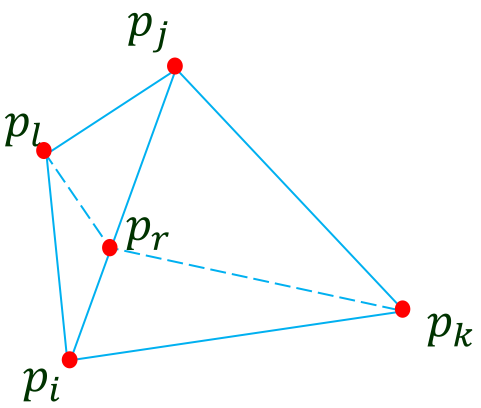

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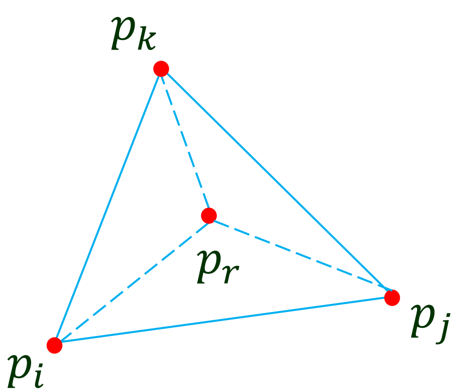

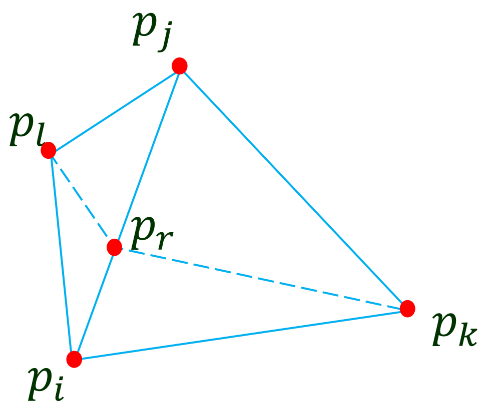

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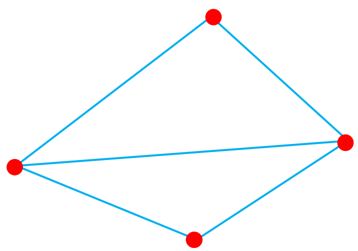

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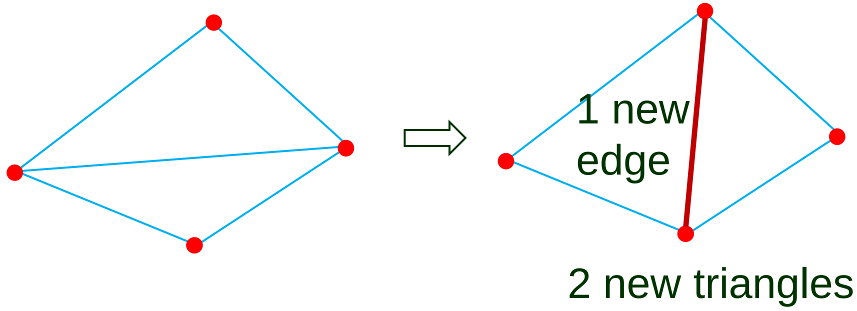

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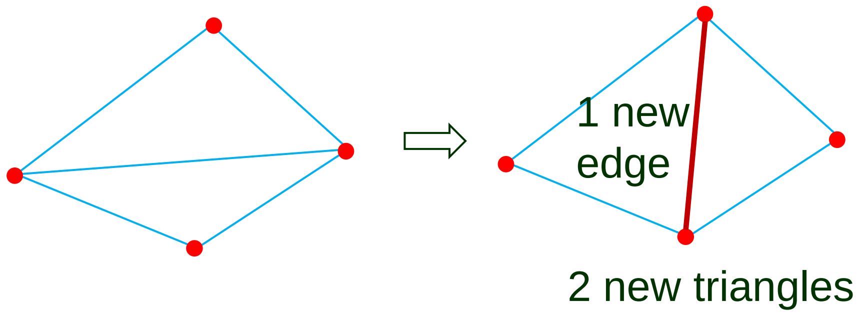

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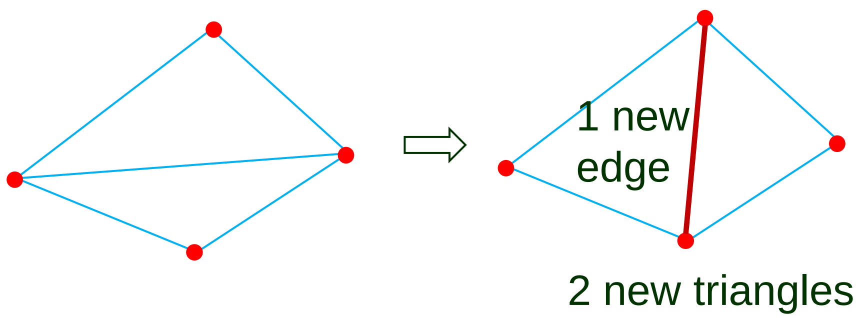

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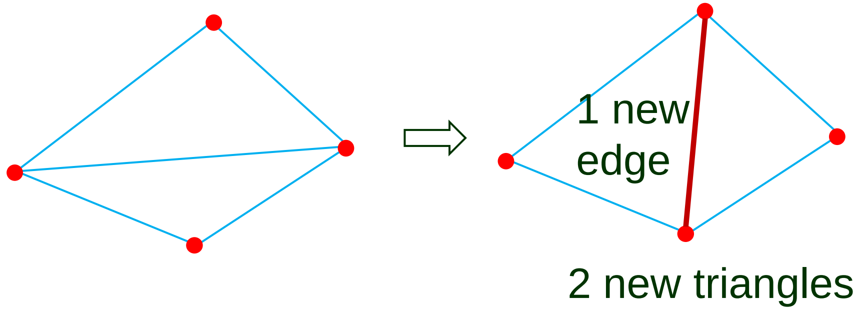

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#new triangles due to iteration r $\leq \max\{2(k-3)+3, 2(k-4)+4\}$

$$= 2k - 3$$

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Let $\deg(p_r, DG_r) = k$ be the degree of p_r in DG_r.

Apply *backward analysis* to determine its expected value.

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Three of the edges are $\overline{p_{-1}p_{-2}}$, $\overline{p_{-1}p_0}$, $\overline{p_{-2}p_0}$.

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$\Downarrow$

Expected degree of such a vertex is ≤ 6.

Proof of Lemma 2 (finish)

Expected # triangles created in step r

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$= 2E(\deg(p_r, DG_r)) - 3$

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$\Downarrow$ n insertion steps .

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$\Downarrow$ n insertion steps .

$\leq 9n$ triangles created

$\Downarrow$ Include $\Delta\, p_0 p_{-1} p_{-2}$

$\leq 9n + 1$ triangles.

Storage and Run Time

Theorem 3 $DG(P)$ can be computed in $O(n \log n)$ expected time using $O(n)$ expected storage.

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